Direct and exact methods for the synthesis of discrete time PID controllers

Stefania Cuoghi and Lorenzo Ntogramatzidis.

***Abstract*—This paper presents a new set of formulae for the design of discrete proportional-integral-derivative (PID) controllers under requirements on steady-state performance and robustness specifications, such as phase and the gain margins, as well as the gain crossover frequency. The proposed technique has the advantage of avoiding trial-and-error procedures or approximations connected to an *a posteriori* discretisation. This method can also be implemented as a graphical design procedure in the Nyquist plane. The plot of suitable regions can be used to check *a priori* if the problem leads to feasible values of the PID parameters.**

***Index Terms*—Feedback control, discrete-time PID design, stability margins, steady-state performance.**

I. Introduction

It is a commonly accepted fact that PID controllers are by far the most utilised compensators in control engineering. Indeed, according to recent and independent estimates, of all compensators employed in industrial process control, those belonging to the family of PID controllers are used in 95-97% of all cases [13], [15].

Over the years, countless tuning techniques have been proposed for PID controllers, which differ in complexity, flexibility, and amount of knowledge required on the mathematical model of the process to be controlled, see [27], [28]. Many surveys and textbooks have been entirely devoted to these techniques, see e.g. [1], [2], [5], [7], [16], [18], [19], [22] and the references therein.

In recent times, a new stream of literature flourished on the design techniques for PID controllers under frequency domain specifications, see [9], [11], [12], [21], [23] and the references cited therein. In particular, much effort has been devoted to the computation of the parameters of the PID controllers that guarantee desired values of the gain/phase margins and of the crossover frequency. Specifications on the stability margins have always been extensively used in feedback control system design to ensure a robust control system. Specifications on phase margin and gain crossover frequency are also frequently considered, since these two parameters together often serve as a performance measure of the control system, [20].

In the last fifteen years, several design methods have been proposed to tackle specifications on the two stability margins and on the gain crossover frequency, whose purpose is to overcome the trial-and-error intrinsic nature of classical control methods based on Bode and Nichols diagrams, see e.g. [3], [8], [9], [12], [13], [21].

More recently, a further design method was proposed which is based on simple closed-form formulae, often referred to as *inversion formulae*, which explicitly express the parameters of the PID controller in terms of the design specifications given by the phase/gain margins and the corresponding crossover frequencies, [17]. This approach was first presented for lead, lag and lead-lag (notch) networks, [14], [24]. In particular, in [24] it was shown that this approach has an important graphical counterpart that enables the design to be carried out on Nyquist and Nichols plots, but which, differently from the classical graphical methods based on the same frequency response representations, leads to an exact geometric solution of the control problem. Indeed, via simple geometric constructions that can be performed with a compass and a ruler, the parameters of the lead-lag network can be determined in finite terms, without resorting to trial-and-error procedures. In [17], [25] it was shown that simple inversion formulae could be established for PID controllers for the computation of their parameters that *exactly* meet specifications on the steady-state performance, stability margins and crossover frequencies, without the need to resort to approximations for the transfer function of the plant.

Moreover, since in industrial applications discrete time is usually preferable to continuous time, being more directly connected to implementation, a goal of this paper is to show that a similar approach can also be established for discrete PID controllers, as introduced in [26].

The use of inversion formulae presents several advantages, compared with traditional design techniques. First, as aforementioned the desired phase/gain margins and crossover frequency can be achieved exactly, without the need for trial-and-error or approximations of the plant dynamics. Moreover, the explicit formulae presented here can be exploited for the self-tuning of the controller. Notice also that the inversion formulae presented here are straightforwardly implementable as MATLAB® routines. As shown in the numerical examples throughout the paper, the calculation of the parameters of the controller is carried out via standard manipulations on complex numbers, or, alternatively, with a straightforward geometric construction on Nyquist or Nichols plots.

Importantly, the inversion formulae that deliver the parameters of the PID controller as a function of the specifications only depend on the magnitude and argument of the frequency response of the system to be controlled at the desired crossover

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Partially supported by the Australian Research Council under the grant FT120100604.

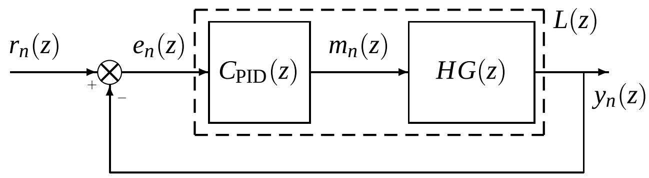

Fig. 1. Standard discrete-time feedback control architecture.

frequency. As such, this method can be used in conjunction with a graphical method based on *any* of the standard diagrams for the representation of the dynamics of the frequency response, e.g., the Bode, Nyquist or Nichols diagrams. The Inversion Formulae method for the design of discrete-time PID controllers and its graphical interpretation on the Nyquist plane are presented in Section II. In Section III the validity of the proposed method is investigated in the case of approximate knowledge of the plant. Finally, in Section IV, the case of proper discrete PID controllers is addressed. Conclusion and remarks end the paper.

II. Discrete PID controllers

Referring to the block scheme of Fig. 1, $HG(z)$ represents the transfer function of the discrete system to be controlled, which is given by the z-transform of a zero-order hold $H_0(s)$ and a continuous-time plant $G(s)$. Thus, we can write

$$HG(z) = \mathscr{Z}[H_0(s)\,G(s)],$$

where $\mathscr{Z}[\cdot]$ represents the z-transform, $H_0(s) = \frac{1-e^{-Ts}}{s}$ is the transfer function of a zero-order hold and T is the sampling period. Moreover, we denote by $C_{\text{PID}}(z)$ the transfer function of a discrete-time PID compensator which has the following structure:

$$C_{\text{PID}}(z) = K_p + K_d \frac{z-1}{z+1} + K_i \frac{z+1}{z-1}, \tag{1}$$

where the proportional and derivative gains K_p, K_d and the integral gain K_i are assumed to be real and positive. This transfer function has been obtained from the standard continuous time transfer function of a PID controller $C_{\text{PID}}^{\text{cont}}(s) = \overline{K}_p + s\overline{K}_d + \frac{\overline{K}_i}{s}$ using the bilinear transformation with pre-warping [20] described by the substitutions

$$s = \frac{\omega_1}{\tan\frac{\omega_1 T}{2}}\left(\frac{z-1}{z+1}\right) \quad \leftrightarrow \quad z = \frac{1+\frac{s}{\omega_1}\tan\frac{\omega_1 T}{2}}{1-\frac{s}{\omega_1}\tan\frac{\omega_1 T}{2}} \tag{2}$$

where ω_1 is the pre-warping frequency. The correspondence between the PID parameters in discrete and continuous time domains is the following:

$$K_d = \overline{K}_d \frac{\omega_1}{\tan\frac{\omega_1 T}{2}}, \qquad K_i = \overline{K}_i \frac{\tan\frac{\omega_1 T}{2}}{\omega_1}, \qquad \overline{K}_p = K_p.$$

Notice that the gains $\overline{K}_p$ and K_p are the same in the continuous and in the discrete time cases. The transfer function (1) can be rewritten as

$$C_{\text{PID}}(z) = K_p\left(1 + T_d\frac{z-1}{z+1} + \frac{1}{T_i}\frac{z+1}{z-1}\right), \tag{3}$$

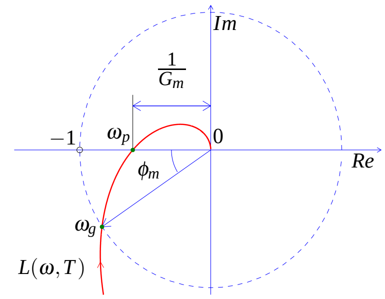

Fig. 2. Control design specifications.

where

$$T_i = \frac{K_p}{K_i} \quad \text{and} \quad T_d = \frac{K_d}{K_p} \tag{4}$$

are respectively the integral and derivative time constants of the discrete PID controller. The design of the three parameters of the controller can be obtained considering two different cases. In the first case the steady-state requirement leads to the imposition of the value of the integral term $K_i = K_p/T_i$. In this case the other two parameters can be used to meet specifications on the phase margin ϕ_m and the gain crossover frequency ω_g, see Fig. 2. As will be clarified in Remark 2.1, in most situations these specifications are satisfied if the loop gain frequency response $L(\omega,T) = C_{\text{PID}}(\omega,T)HG(\omega,T)$ at frequency ω_g verifies the following relations

$$\left|L\left(e^{j\omega_g T}\right)\right| = 1, \qquad \arg L\left(e^{j\omega_g T}\right) = \phi_m + \pi. \tag{5}$$

In the second case the pole at $z = 1$ of the discrete PID controller is sufficient to guarantee that the steady-state requirement is satisfied. In this case two of the three parameters can be used to meet design requirements on the phase margin and the gain crossover frequency specifications. The third parameter can be used to satisfy other requirements. Following the same approach taken in [17], in this paper we consider design specifications on the ratio T_d/T_i, to guarantee that the zeros of the PID controller are real, and on the imposition of the gain margin G_m. We now introduce some general concepts in order to solve the design problems considered in this paper using a graphical procedure. Let us start with some simple considerations on the Nyquist plane that will provide some insight into the graphical techniques presented in this paper. Loosely, in order to meet specifications on the phase margin ϕ_m and the gain crossover frequency ω_g, the loop gain frequency response $L(\omega,T)$ has to exactly pass through the point B of the unit circle with phase $\pi+\phi_m$ at frequency ω_g. Intuitively, this means that the point A of the frequency response of the discrete plant at frequency ω_g, i.e., $A = HG(e^{j\omega_g T})$, has to be brought to point $B = e^{j(\pi+\phi_m)}$ by multiplication with the frequency response of the controller at frequency ω_g. In other words, the equation $B = C_{\text{PID}}(\omega_g,T)\cdot A$ must be satisfied. Stated differently, the control problem considered in this paper is to determine $C_{\text{PID}}(\omega,T)$ such that $L(\omega_g,T) = e^{j(\pi+\phi_m)}$.

The set of all the points of the complex plane that can be brought to a desired point B using a PID controller, which will be referred to as the *admissible domain* for B, is not related to the structure of the plant, but only to the structure

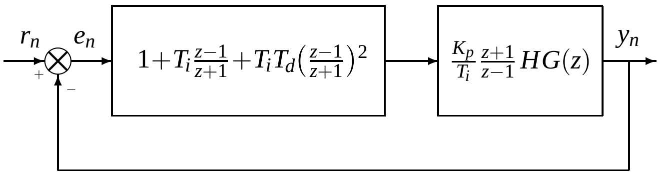

Fig. 3. Modified feedback structure with unity DC gain controller.

of the regulator and the position of point B. It is very useful to represent this domain graphically in order to verify *a priori* if the problem has a solution. If not, a different controller has to be considered.

Remark 2.1: As aforementioned, the design method based on the inversion formulae hinges on constraining the loop gain transfer function to cross a specific point of the Nyquist plane. In most situations, this goal is sufficient to guarantee that the specifications on the phase (or gain) margins and crossover frequencies are met. It is worth recalling, however, that this can be only rigorously ensured when

1) $L(z)=C_{\rm PID}(z)HG(z)$ has all poles in the unit circle and is strictly proper;
2) The Nyquist plot of $L(\omega,T)$ for $\omega\geq 0$ intersects the unit circle and the negative real semi-axis only once (except for the trivial intersection in the origin as $\omega\to\infty$).

The second of these requirement is the one which guarantees that the stability margins are univocally determined. In case of multiple intersections with the unit circle or with the negative real semi-axis, the application of the proposed method still guarantees that one intersection will correspond to point B, but this does not, in general, furnish reliable indication as to how far the Nyquist plot evolves from the critical point. If $L(z)$ has poles outside the unit circle, the closed-loop system can turn out to be unstable even in cases when the two margins are both univocally defined and strictly positive, by virtue of the Nyquist criterion. Notice that the second condition has to be checked *a posteriori*.

A. Imposition of the integral term $K_i>0$.

Let us consider the case of steady-state specifications that impose the value of the integral gain K_i. The other two degrees of freedom of the regulator can be imposed to meet specifications on the phase margin ϕ_m and the gain crossover frequency ω_g.

First, the factor $\frac{K_p}{T_i}\frac{z+1}{z-1}$ can be separated from the transfer function

$$\tilde{C}_{\rm PID}(z)\stackrel{\rm def}{=}1+T_i\frac{z-1}{z+1}+T_iT_d\left(\frac{z-1}{z+1}\right)^2, \tag{6}$$

and viewed as part of the plant. In this way, the part of the controller to be designed is $\tilde{C}_{\rm PID}(z)$, and the feedback scheme of Fig. 1 reduces to that of Fig. 3. Let

$$\widetilde{HG}(z)\stackrel{\rm def}{=}\frac{K_p}{T_i}\frac{z+1}{z-1}HG(z),$$

and the loop gain transfer function can be written as $L(z)=\tilde{C}_{\rm PID}(z)\widetilde{HG}(z)$. The frequency response of (6) for $\omega\in[0,\pi/T]$ and sampling period T can be written as

$$\tilde{C}_{\rm PID}(e^{j\omega T})=M(\omega,T)e^{j\varphi(\omega,T)}=P'(\omega,T)+jQ'(\omega,T), \tag{7}$$

where $M(\omega,T)$ and $\varphi(\omega,T)$ are the magnitude and the phase of $\tilde{C}_{\rm PID}(e^{j\omega T})$, and

$$P'(\omega,T)\stackrel{\rm def}{=}1-T_dT_i\tan^2\frac{\omega T}{2},\qquad Q'(\omega,T)\stackrel{\rm def}{=}T_i\tan\frac{\omega T}{2}.$$

We are ready to state the main result of this section.

Theorem 2.1: The equation $L(\omega_g,T)=e^{j(\pi+\phi_m)}$ holds with a suitable discrete PID controller (6) if and only if

$$0<\varphi_g<\pi \qquad \text{and} \qquad M_g\cos\varphi_g<1, \tag{8}$$

where

$$M_g\stackrel{\rm def}{=}M(\omega_g)=1/\ |\widetilde{HG}(e^{j\omega_gT})|\ , \tag{9}$$

$$\varphi_g\stackrel{\rm def}{=}\varphi(\omega_g)=\phi_m-\pi-\arg\widetilde{HG}(e^{j\omega_gT}). \tag{10}$$

If (8) are satisfied, the parameters of the PID controller $\tilde{C}_{\rm PID}(z)$ for which $L(\omega_g,T)=e^{j(\pi+\phi_m)}$ is satisfied are given by

$$\begin{aligned} T_i &= \frac{M_g\sin\varphi_g}{\tan\frac{\omega_gT}{2}},\\ T_d &= \frac{1-M_g\cos\varphi_g}{M_g\sin\varphi_g\tan\frac{\omega_gT}{2}},\\ K_p &= K_i\frac{M_g\sin\varphi_g}{\tan\frac{\omega_gT}{2}}. \end{aligned} \tag{11}$$

Proof: From (5), the controller (6) has to be designed in such a way that

$$\tilde{C}_{\rm PID}(e^{j\omega_gT})=M_ge^{j\varphi_g} \tag{12}$$

holds, where M_g and φ_g satisfy (9) and (10). By equating real and imaginary parts of both sides of (7) and (12) we get

$$M_g\cos\varphi_g=1-T_dT_i\tan^2\frac{\omega_gT}{2}, \tag{13}$$

$$M_g\sin\varphi_g=T_i\tan\frac{\omega_gT}{2}. \tag{14}$$

Solving (13) and (14) with respect to T_i and T_d we find (11). The conditions (8) are obtained by imposing that the parameters of the PID controller are all positive. □

Graphical interpretation. The phase margin specification defines the position of the point B that the loop gain frequency response has to cross, see Fig. 4. The admissible domain consisting of all the points $A=\widetilde{HG}(e^{j\omega_gT})$ that can be brought to $B=e^{j(\pi+\phi_m)}$ by (6) can be defined as

$$\mathscr{D}_B\stackrel{\rm def}{=}\left\{A\in\mathbb{C}\,\middle|\,\exists T_i,T_d>0,\exists\,\omega,T\geq 0:\ \tilde{C}_{\rm PID}(e^{j\omega T})\cdot A=B\right\}.$$

This domain can be represented by the shaded area shown in Fig. 4 and corresponds to the set of all the points $A=M_Ae^{j\varphi_A}$ that can be brought to point $B=e^{j\varphi_B}$ in such a way that $\varphi_g=\varphi_B-\varphi_A$ verifies (8). This representation can be obtained considering the graphical plot of admissible points $C=M_ge^{\varphi_g}$

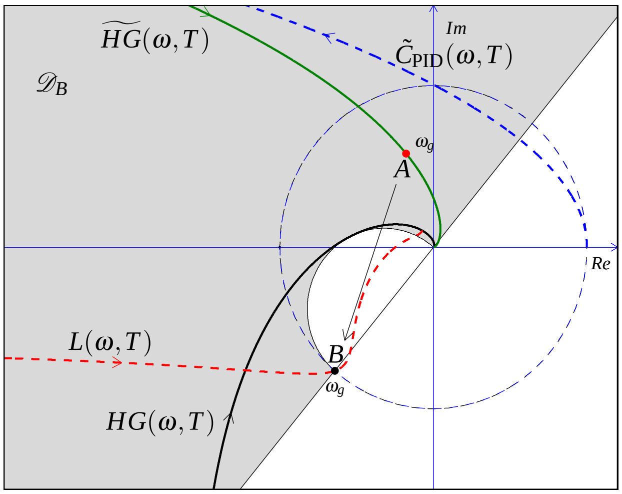

Fig. 4. Graphical solution of Example 2.1 on the Nyquist plane.

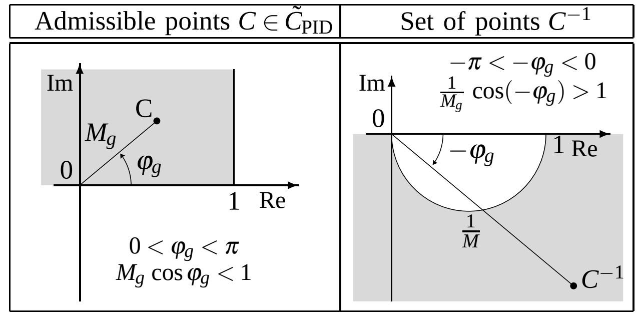

Fig. 5. Graphical representation of admissible of points C of $\tilde{C}_{\mathrm{PID}}(e^{j\omega T})$ and their inverse points $C^{-1} = \frac{1}{C}$.

with constraint (8) and their inverse points $C^{-1} = \frac{1}{M_g}e^{-\varphi_g}$, see the gray areas shown in Fig. 5. The domain $\mathscr{D}_B$ can be determined via a rotation of the set of points C^{-1} by an angle φ_B (i.e., $A = B/C$).

Thus, the problem admits a solution if and only if $A = \widetilde{HG}(e^{j\omega_g T})$ belongs to the considered admissible domain $\mathscr{D}_B$.

Example 2.1: Let us consider the discrete-time plant

$$HG(z) = \frac{0.015z^3 + 0.034z^2 - 0.029z - 0.0073}{z^4 - 3.03z^3 + 3.39z^2 - 1.66z + 0.301}$$

obtained from the discretisation of the continuous-time plant

$$G(s) = \frac{14(s+1)}{s(s+1.5)^2(s+3)}, \tag{15}$$

using a zero-order hold with the sampling period $T = 0.2$ sec. Design a PID compensator that meets the following specifications: acceleration constant $K_a = 2$; phase margin $\phi_m = 50°$; gain crossover frequency $\omega_g = 1.6$ rad/sec.

Solution: The steady-state requirement is satisfied by imposing the value of the integral constant $K_i = 0.097$. The desired magnitude and the phase of the controller (6) at frequency ω_g are $M_g = 1.65$ and $\varphi_g = 2.15$ rad, respectively. These values satisfy (8). Notice that (8) can be graphically verified drawing the admissible domain of (6) with respect to point $B = e^{j230°}$ and verifying that point $A = \widetilde{HG}(e^{j\omega_g T})$ belongs to this domain. Indeed, we see from Fig. 4 that in this example point A belongs to the shaded region representing the admissible domain. As a consequence, the problem admits a solution with a discrete PID controller as expected by virtue of Theorem 2.1. Indeed, using (11) we compute the time constants of the PID controller, and we obtain the values $T_i = 8.58$ sec and $T_d = 8.49$ sec which are indeed positive. Notice also that both conditions in Remark 2.1 are met. Thus, the closed loop system is asymptotically stable, and the constraints on the steady-state performance, on the phase margin and on the gain crossover frequency are satisfied.

Graphical representations of the Nyquist diagram and step response for different values of the sampling time T are plotted in Fig. 6. Notice that all the loop gain frequency responses (red curves in Fig. 6) exactly satisfy the design specifications.

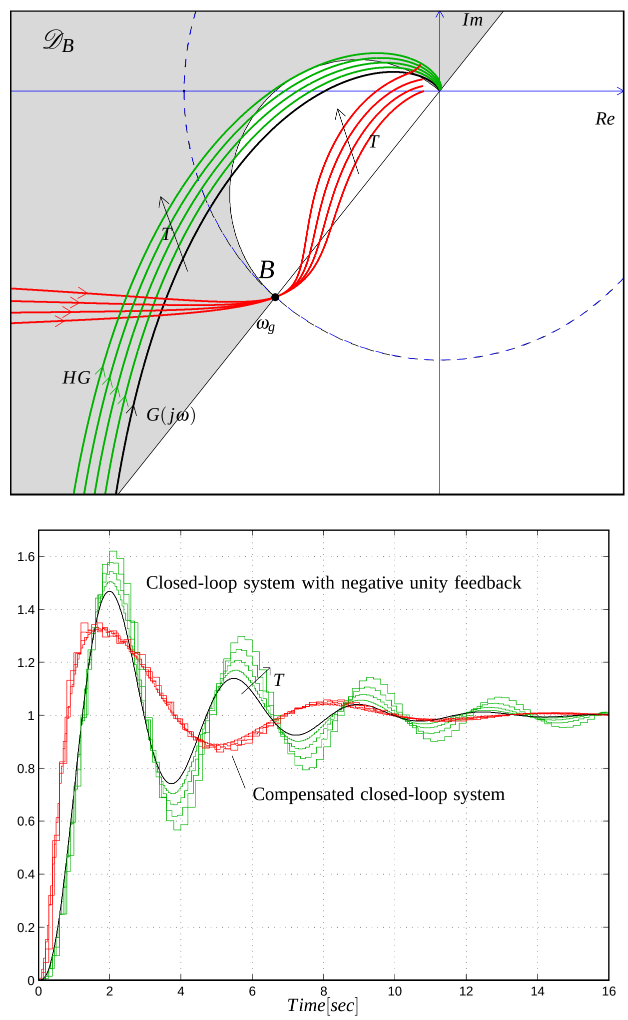

Fig. 6. Example 2.1: Nyquist diagram and step response of solutions on variation of sampling time $T = [0.05 : 0.05 : 0.2]$.

B. K_i unconstrained: Imposition of the ratio T_d/T_i

Let us consider the case in which the steady-state specifications do not constrain the value of the integral term K_i. In this case the three parameters of the controller can be designed to meet specifications on the phase margin ϕ_m, the gain crossover frequency ω_g and a third parameter. We first consider the case in which the ratio T_d/T_i is assigned. Values of $\sigma^{-1} = T_i/T_d \geq 4$ guarantee that the zeros of the discrete-time controller $C_{\rm PID}(z) = \tilde{N}(z)/\tilde{D}(z)$ are real, since the discriminant of quadratic equation $\tilde{N}(z) = 0$ is $\triangle = {T_i}^2 - 4T_iT_d$.[1] The following theorem is the discrete counterpart of a result in [2, p. 140]. The frequency response of (1) for $\omega \in [0, \pi/T]$ and sampling period T is

$$C_{\rm PID}(e^{j\omega T}) = N(\omega,T)e^{j\psi(\omega,T)} = P'' + jQ''(\omega,T), \qquad (16)$$

where

$$P'' = K_p, \qquad Q''(\omega,T) = K_pT_d \tan\frac{\omega T}{2} - \frac{1}{T_i}\frac{K_p}{\tan\frac{\omega T}{2}}.$$

Theorem 2.2: Referring to the system shown in Fig. 1, the equation $L(\omega_g,T) = e^{j(\pi+\phi_m)}$ holds with an admissible PID controller (1) – or (3) – if and only if

$$\psi_g \in (-\pi/2, \pi/2), \qquad (17)$$

where

$$\psi_g \stackrel{\rm def}{=} \psi(\omega_g) = \phi_m - \pi - \arg HG(e^{j\omega_g T}). \qquad (18)$$

If (17) is satisfied, the parameters of the PID controller $C_{\rm PID}(z)$ for which $L(\omega_g,T) = e^{j(\pi+\phi_m)}$ is satisfied are given by

$$\begin{aligned} K_p &= N_g \cos\psi_g, \\ T_i &= \frac{\tan\psi_g + \sqrt{\tan^2\psi_g + 4\sigma}}{2\sigma\tan\frac{\omega_g T}{2}}, \\ T_d &= T_i\,\sigma, \end{aligned} \qquad (19)$$

where

$$N_g \stackrel{\rm def}{=} N(\omega_g) = \frac{1}{|HG(e^{j\omega_g T})|}. \qquad (20)$$

Proof: From (5), the controller (3) has to be designed in such a way that

$$C_{\rm PID}(e^{j\omega_g T}) = N_g e^{j\psi_g} \qquad (21)$$

holds, where ψ_g satisfies (17). By equating real and imaginary parts of both sides of (16) and (21) we get

$$N_g \cos\psi_g = K_p, \qquad (22)$$

$$N_g \sin\psi_g = K_pT_d \tan\frac{\omega_g T}{2} - \frac{1}{T_i}\frac{K_p}{\tan\frac{\omega_g T}{2}}. \qquad (23)$$

Solving (22) and (23) with respect to T_i and T_d we find (19). The condition (17) is obtained by imposing that the parameters of the PID controller are all positive. □

[1] For a discussion on the role of (minimum and non minimum-phase) complex zeros in the discrete time we refer the readers to [6], [4], [10], [20].

C. K_i unconstrained: Imposition of the gain margin

In this case the third parameter of the controller is designed to assign the gain margin to a certain value G_m. This leads to a loop gain frequency response that passes through the point $(-1/G_m, 0)$ at the phase crossover frequency ω_p, i.e.,

$$\left|L\left(e^{j\omega_p T}\right)\right| = \frac{1}{G_m} \quad \text{and} \quad \arg L\left(e^{j\omega_p T}\right) = -\pi. \qquad (24)$$

Therefore, the three parameters of the controller must be calculated to simultaneously verify (5) and (24).

The frequency response of (3) at frequency ω_p can be expressed as

$$C_{\rm PID}(e^{j\omega_p T}) = K_p + j\left(K_pT_d\tan\frac{\omega_p T}{2} - \frac{1}{T_i}\frac{K_p}{\tan\frac{\omega_p T}{2}}\right), \qquad (25)$$

or it can be written in polar form as

$$C_{\rm PID}(e^{j\omega_p T}) = N_p e^{j\psi_p} = N_p\cos(\psi_p) + jN_p\sin(\psi_p), \qquad (26)$$

with $N_p = 1/(G_m|HG(e^{j\omega_p T})|)$ and $\psi_p = \pi - \arg HG(e^{j\omega_p T})$. By equating real and imaginary part of (25) and (26) we obtain

$$N_p\cos\psi_p = K_p, \qquad (27)$$

$$N_p \sin\psi_p = K_pT_d \tan\frac{\omega_p T}{2} - \frac{1}{T_i}\frac{K_p}{\tan\frac{\omega_p T}{2}}. \qquad (28)$$

These equations must be satisfied together with (22) and (23). Thus, the value of ω_p can be obtained as solution of the following equation

$$N_g\cos\psi_g = N_p\cos\psi_p. \qquad (29)$$

which follows directly by (22) and (27). By following the same argument used in the proof of Lemma 1 in [17], it is easy to see that when the transfer function of the plant is rational in z, (29) is a polynomial equation in ω_p, and therefore all its solutions within any given interval can be calculated with arbitrary precision.

Theorem 2.3: Referring to the system shown in Fig. 1, the parameters of PID controller $C_{\rm PID}(z)$ for which $L(\omega_g,T) = e^{j(\pi+\phi_m)}$ and $L(\omega_p,T) = \frac{1}{G_m}e^{-j\pi}$ are simultaneously satisfied are

$$K_p = N_g\cos\psi_g, \qquad (30)$$

$$T_i = \frac{\Omega_g^2 - \Omega_p^2}{\Omega_g\Omega_p\left(\Omega_p\tan\psi_g - \Omega_g\tan\psi_p\right)}, \qquad (31)$$

$$T_d = \frac{\Omega_g\tan\psi_g - \Omega_p\tan\psi_p}{\Omega_g^2 - \Omega_p^2}, \qquad (32)$$

where $\psi_g = \phi_m + \pi - \arg HG(e^{j\omega_g T})$, $N_g = 1/|HG(e^{j\omega_g T})|$, $\Omega_g = \tan\frac{\omega_g T}{2}$, $\Omega_p = \tan\frac{\omega_p T}{2}$ and ω_p is the solution of (29), with $N_p = 1/(G_m|HG(e^{j\omega_p T})|)$ and $\psi_p = \pi - \arg HG(e^{j\omega_p T})$. The solution leads to an admissible PID controller (i.e., one in which the three parameters are all non-negative) if and only if $\psi_g \in (-\pi/2, \pi/2)$ and ω_p verifies the following constraints

$$\begin{cases} \omega_p < \omega_g \\ \Omega_g\tan\psi_g > \Omega_p\tan\psi_p \\ \Omega_p\tan\psi_g > \Omega_g\tan\psi_p \end{cases} \quad \text{or} \quad \begin{cases} \omega_p > \omega_g \\ \Omega_g\tan\psi_g < \Omega_p\tan\psi_p \\ \Omega_p\tan\psi_g < \Omega_g\tan\psi_p \end{cases} \qquad (33)$$

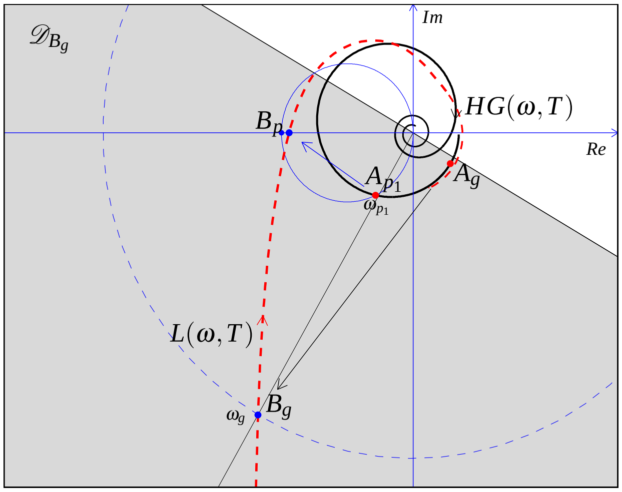

Fig. 7. Graphical solution of Example 2.2 realted to design specifications on the phase and gain margins, and the gain crossover frequency.

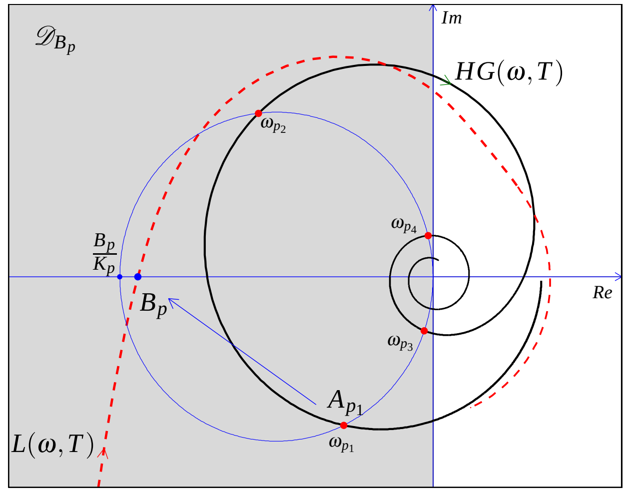

Fig. 8. Example 2.2: graphical solution of eq. (29) to determine $\omega_{pi} = (\omega_{p1},\, \omega_{p2},\, \omega_{p3},\, \omega_{p4},\, ...)$

Proof: (**Only if**). A necessary condition for the solution of the problem is that ω_p is a solution of (29). From (22) and (23), and from (27) and (28) we get

$$-\Omega_g T_i \tan\psi_g = 1-\Omega_g^2 T_i T_d, \qquad (34)$$

$$-\Omega_p T_i \tan\psi_p = 1-\Omega_p^2 T_i T_d. \qquad (35)$$

Solving (34) and (35) in T_i and T_d leads to (31) and (32), and K_p is positive only if $\psi_g \in (-\pi/2, \pi/2)$. Moreover, the constraints $T_i > 0$ and $T_d > 0$ are verified if (33) are satisfied. (**If**). It is a matter of straightforward substitution of (30-32) into (22), (23), (27) and (28). □

Graphical interpretation. The Nyquist plot of $C_{\rm PID}(e^{j\omega T}, K_p)$, which causes the loop gain frequency response $L(\omega,T)$ to pass through point $B_g = e^{j(\pi+\phi_m)}$ at the gain crossover frequency ω_g, is the straight line with modulus K_p given by (30), see [26]. An admissible controller exists if $A_g = HG(\omega_g, T) \in \mathscr{D}_{B_g}$. The loop gain frequency response $L(\omega,T)$ also passes through point $B_p = \frac{1}{G_m} e^{j\pi}$ at frequency ω_p (so that $L(\omega_p,T) = C_{\rm PID}(e^{j\omega_p T}, K_p) HG(\omega_p,T) = B_p$) if $A_p = HG(\omega_p,T) \in \mathscr{D}_{B_p}$, see Fig. 8, and only if

$$HG(\omega_p,T) = B_p / C_{\rm PID}(e^{j\omega_p T}, K_p).$$

Since the plot of $\frac{1}{C_{\rm PID}(e^{j\omega T},K_p)}$ is a circle having as diameter the segment $(0, 1/K_p)$, see [26], the plot of $B_p/C_{\rm PID}(e^{j\omega_p T}, K_p)$ is the circle with diameter $(0, B_p/K_p)$. It follows that the solutions of (29) are the frequencies ω_p that correspond to the intersection point of the Nyquist plot of $HG(\omega,T)$ with the circle whose diameter coincides with the segment $(0, B_p/K_p)$. The solution of the problem can be obtained by a controller which simultaneously brings points A_p to B_p and A_g to B_g. The graphical conditions $A_p \in \mathscr{D}_{B_p}$ and $A_g \in \mathscr{D}_{B_g}$ are necessary but not sufficient to ensure that all the parameters of the controller are non-negative.

Example 2.2: Let us consider the discrete-time plant

$$HG(z) = \frac{0.7z^4 - 1.59z^3 + 0.38z^2 + 1.23z - 0.71}{z^5 - 4.7z^4 + 8.9z^3 - 8.5z^2 + 4.1z - 0.78} 10^{-3} z^{-30}$$

obtained using the non-minimum phase plant

$$G(s) = \frac{s^3 - 3.7s^2 + s + 2.5}{s^5 + 6s^4 + 40s^3 + 43s^2 + 43s + 17} e^{-1.2s}, \qquad (36)$$

and a zero-order hold with a sampling period $T = 0.04$ sec. Design a discrete-time PID compensator $C_{\rm PID}(z)$ to meet the following design specifications: phase margin $\phi_m = 60°$, gain margin $G_m = 2.5$ and gain crossover frequency $\omega_g = 0.2$ rad/sec.

Solution: The values of the desired magnitude and phase of the controller at frequency ω_g are $N_g = 6.55$ and $\psi_g = -1.43 \in (-\pi/2, \pi/2)$, respectively. Notice that point $A_g = HG(e^{j\omega_g T})$ belongs to the domain $\mathscr{D}_{B_g}$, see Fig. 7. From (30) it follows that $K_p = 0.94$. The solutions of (29) are $\omega_{pi} \in \{0.62,\, 0.95,\, 2.41,\, 3.97 \ldots\}$ and correspond to the frequencies at the intersections points of $HG(\omega,T)$ with the circle whose diameter coincides with the segment $(0, B_p/K_p)$, see Fig. 8. One possible solution is obtained by bringing the point A_{p_1} at frequency ω_{p1} to point B_p. From (31) and (32) the other designed parameters are $T_i = 35.14$ sec and $T_d = 59.07$ sec. The corresponding loop gain frequency response $L(\omega,T)$ is represented by the dashed red line in Fig. 7.

III. Case of Approximate Knowledge of the Plant and Comparison with Other Methods

The discrete-time methods proposed in the previous sections are based on the exact knowledge of the transfer function of the plant to be controlled. However when such knowledge is not available, the methods may still be applied in conjunction with process models obtained by numerical or heuristic methods. These can be obtained by the discretisation of a continuous time model, or by a model obtained directly in the discrete time domain, see [1], [2], [9]. In the continuous time case, a comparison with other techniques in [9] has shown that given a first- or second-order plus delay approximation of the plant, the inversion formulae method enables the design requirements to be satisfied with a better accuracy, since the approximation is related only to the selected process model

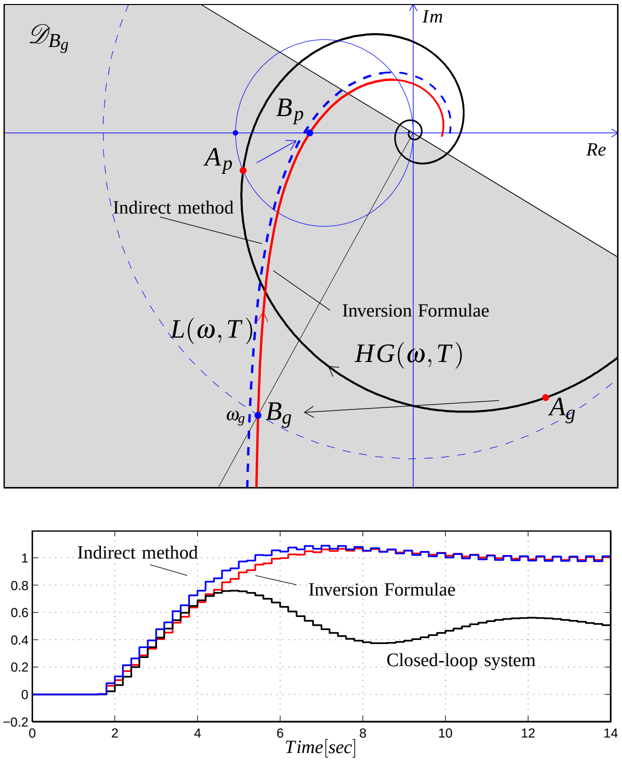

Fig. 9. Example 3.1: the plant $G(s) = \frac{1-s}{(1+s)^3}$ is approximatively known by the model $G_m(s) = \frac{e^{-1.58s}}{(1+1.00s)^2}$, the red and dashed blue lines denote the loop gain frequency responses and step responses obtained by Inversion Formulae and the indirect methods in [9].

design, see [17]. It must be noted that often the approximation of the response of a real plant to a first or second-order transfer function is far from being easy – and at times impossible – to obtain. However, in the situations in which it is possible to obtain an approximate knowledge of the plant, the same considerations presented in [17] can still be applied for the discrete time case as shown in the following example.

Example 3.1: The plant $G(s) = \frac{1-s}{(1+s)^3}$ is approximatively known by its second-order model $G_m(s) = \frac{e^{-1.58s}}{(1+1.00s)^2}$, see Table 5 in [9]. Using a sampling time $T = 0.2$ sec, design a discrete PID controller to satisfy the phase margin $\phi_m = 60°$, the gain margin $G_m = 3$ and the gain crossover frequency $\omega_g = 0.3$ rad/sec.

Solution: The discretisation of $G_m(s)$ leads to the transfer function

$$HG(z) = \frac{0.02z^2 + 2.04z + 1.22}{z^2 - 1.64z + 0.67} 10^{-2} z^{-8}.$$

The problem can be solved by the following discrete PID controller, obtained using the indirect method in [9]:

$$C_{\rm PID}(z) = \frac{3.99z^2 - 6.53z + 2.67}{z^2 - 1}. \qquad (37)$$

The phase margin obtained using the modelled system is 58.25°, the gain margin is 2.84, the gain crossover frequency is 0.33 rad/sec, see the dashed blue loop gain frequency response $HG_m(z)C_{\rm PID}(z)$ in Fig. 9. The effect of (37) on the real system, (characterised by $HG(z) = \mathscr{Z}[H_0(s)\,G(s)]$), leads to the phase margin 53.47°, the gain margin 3.03, and the gain crossover frequency 0.33 rad/sec. The discrete PID parameters obtained by (30-32) are $K_p = 0.58$, $T_i = 19.5$ sec and $T_d = 4.14$ sec. The corresponding closed loop system exactly satisfies the given design specifications, see the red loop gain frequency response in Fig. 9. The effect of the designed controller on the real system with $HG(z) = \mathscr{Z}[H_0(s)\,G(s)]$ leads to the phase margin $\phi_m = 53.76°$, the gain margin 2.86 and the gain crossover frequency 0.30 rad/sec.

Method	PID	case of model $G_m(s)$ ϕ_m	G_m	ω_g	case of $G(s)$ ϕ_m	G_m	ω_g
[9]	$\frac{3.99z^2-6.53z+2.67}{z^2-1}$	58.25°	2.84	0.33	53.47°	3.03	0.33
Inversion Formulae	$K_p = 0.58$, $T_i = 19.5$, $T_d = 4.14$	60°	3	0.3	53.76°	2.86	0.30

TABLE I
EXAMPLE 3.1: FREQUENCY RESULTS IN THE CASE OF APPROXIMATE KNOWLEDGE OF THE PLANT

IV. PROPER DISCRETE PID CONTROLLER

An important drawback of the structure connected with the controller (1), as pointed out in [20, p. 314], is that when the plant $HG(z)$ does not have a zero at $z = -1$ and z approaches -1, the derivative part $T_d\,\frac{z-1}{z+1}$ of the ideal PID controller $C_{\rm PID}(z)$ in (1), and thus $|HG(z)C_{\rm PID}(z)|$, will approach infinity. As a consequence, in the Nyquist plot the frequency response of the open loop system will probably encircle the critical point $(-1,0)$, and that means that the ideal PID controller could lead to instability. To avoid this critical case, the following structure of proper discrete PID controller can be used:

$$C'_{\rm PID}(z) = K_p\left(1 + \frac{1}{T_i}\frac{z+1}{z-1} + T_d\,\frac{z-1}{(z+1)+\tau_d\,(z-1)}\right), \qquad (38)$$

where the parameters K_p, T_i, T_d, τ_d are real and positive. The derivative term in (38) can be obtained from the derivative term $T_d\,\frac{z-1}{z+1}$ of the ideal PID in (1) by adding a zero at $z = -1$ and a pole at $z_p = \frac{\tau_d - 1}{\tau_d + 1}$. [2]

When τ_d is very small, the influence of the new pole z_p becomes negligible. Let us consider for the sake of argument the case in which the steady-state specifications lead to a sharp constraint on T_i and the proportional gain K_p. The factor $\frac{K_p}{T_i}\frac{z+1}{z-1}$ in (38) can be separated from the remaining part of the controller

$$\tilde{C}'_{\rm PID}(z) = 1 + T_i\,\frac{z-1}{z+1} + \frac{T_i\,T_d\,\left(\frac{z-1}{z+1}\right)^2}{1+\tau_d\,\frac{z-1}{z+1}} \qquad (39)$$

and considered as part of the plan. In this way, the loop gain transfer function can be expressed as $L(z) = \tilde{C}'_{\rm PID}(z)\,\tilde{HG}'(z)$

[2] The transfer function (38) can also be obtained from the standard continuous-time proper PID

$$\overline{C}_{\rm PID}(s) = \overline{K}_p\left(1 + \frac{1}{\overline{T}_i\,s} + \frac{\overline{T}_d\,s}{1+\overline{\tau}_d\,s}\right)$$

using the bilinear transformation with pre-warping described by the substitutions (2).

where $\tilde{HG}'(z) = \frac{K_p}{T_i}\frac{z+1}{z-1}HG(z)$. The inversion formulae for the proper version of the discrete PID controller considered here are given in the following result.

Theorem 4.1: The equation $L(\omega_g, T) = e^{j(\pi+\phi_m)}$ holds with (39) if and only if $0 < \phi_g < \pi$ and $M_g \cos\varphi_g < 1$, where

$$M_g \overset{\text{def}}{=} M(\omega_g) = 1/|\widetilde{HG}'(e^{j\,\omega_g T})|,$$
$$\varphi_g \overset{\text{def}}{=} \varphi(\omega_g) = \phi_m - \pi - \arg\widetilde{HG}'(e^{j\,\omega_g T}).$$

If $0 < \phi_g < \pi$ and $M_g \cos\varphi_g < 1$, the parameters of the PID controller $\tilde{C}_d'(z)$ for which $L(\omega_g, T) = e^{j(\pi+\phi_m)}$ is satisfied are given by

$$T_i = \frac{M_g \sin\varphi_g}{\Omega_g} - K_i\,\tau_d\,(1 - M_g \cos\varphi_g),$$
$$T_d = \frac{1+\tau_d^2\,\Omega_g^2}{\Omega_g\left(\frac{M_g \sin\varphi_g}{1-M_g \cos\varphi_g} - \tau_d\,\Omega_g\right)},$$
$$K_p = K_i\,\frac{M_g \sin\varphi_g}{\Omega_g} - \tau_d\,(1 - M_g \cos\varphi_g),$$

with $\Omega_g = \tan\frac{\omega_g T}{2}$, $T_i > 0$, $T_d > 0$, and

$$\tau_d < \frac{M_g \sin\varphi_g}{1 - M_g \cos\varphi_g}\,\frac{1}{\tan\frac{\omega_g T}{2}}.$$

The proof can be carried out along the same lines of the proof of [17, Theorem 5], and is omitted.

V. Concluding remarks

In this paper, new and simple formulae for the design of discrete PID controllers have been presented. These enable the control system to be designed to directly and exactly meet the design specifications on the steady-state accuracy, phase/gain margins and crossover frequencies. Numerical examples on the proposed analytical and graphical procedures confirm the efficacy of the methods. We have also shown how this method can be adapted to the case of PID controllers with an additional pole in the derivative term.

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