Sparse Recovery on Euclidean Jordan Algebras

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Abstract

This paper is concerned with the problem of sparse recovery on Euclidean Jordan algebra (SREJA), which includes the sparse signal recovery problem and the low-rank symmetric matrix recovery problem as special cases. We introduce the notions of restricted isometry property (RIP), null space property (NSP), and s-goodness for linear transformations in s-SREJA, all of which provide sufficient conditions for s-sparse recovery via the nuclear norm minimization on Euclidean Jordan algebra. Moreover, we show that both the s-goodness and the NSP are necessary and sufficient conditions for exact s-sparse recovery via the nuclear norm minimization on Euclidean Jordan algebra. Applying these characteristic properties, we establish the exact and stable recovery results for splving SREJA problems via nuclear norm minimization.

Keywords: Sparse recovery on Euclidean Jordan algebra, nuclear norm minimization, restricted isometry property, null space property, s-goodness, exact and stable recovery

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1 Introduction

The *sparse recovery on Euclidean Jordan algebra* (SREJA) is the problem of recovering a sparse (low-rank) element of a Euclidean Jordan algebra from a number of linear measurements (see Section 2 for more details). The problem of SREJA includes the problems of sparse signal recovery (SSR) and low-rank symmetric matrix recovery (LMR) as special cases. Since $\mathbb{R}^n$ and $\mathbb{S}^n$ are the two simplest Euclidean Jordan Algebras (other Euclidean Jordan algebras include, e.g, the Lorenz space $\mathbb{L}^n$ and the Hermitian Space $\mathbb{H}^n$), the problem SPEJA is a non-trivial generalization of SSR and LMR. The study of SPEJA may disclose more essential properties of the sparse recovery problem, as conjectured by Recht, Fazel, and Parrilo [25], which is the main purpose of this study.

SREJA is generally NP-hard since SSR is a well-known NP-hard problem. In the terminology of compressed sensing (CS), SSR is also called *the cardinality minimization problem, or the l_0-minimization problem*, see the papers by Donoho [10] and Candès, Romberg and Tao [5, 7]. In particular, Candès and Tao [7] introduced a restricted isometry property (RIP) of a sensing matrix which guarantees to recover a sparse solution of SSR by ℓ_1-norm minimization. After that, several other sparse recovery conditions were introduced, including the null space properties (NSPs) [9] and the s-goodness [17, 18]. The recovery conditions for SSR is important since they open the door for efficient solution of SSR, which has wide applications in signal and image processing, statistics, computer vision, system identification, and control. For more details, see the survey papers [2, 24] and a new monograph [11]. Recently, recovery conditions on RIP, NSP and s-goodness for SSR have also been successfully extended to the case of LMR, see, [21, 25, 26, 27]. Recht, Fazel, and Parrilo [25] provided a certain RIP condition on the linear transformation of LMR, which guarantees that the minimum nuclear norm solution is a minimum rank solution, they also presented an analysis on the parallels between the cardinality minimization and rank minimization. Recht, Xu, and Hassibi [27] gave the NSP condition for LMR, which is also discussed by Oymak et al. in [23]. Note that the NSP is both necessary and sufficient for exactly recovering a low-rank matrix via nuclear norm minimization problem. Recently, Chandrasekaran, Recht, Parrilo, and Willsky [8] indicated that a fixed s-rank matrix

X_0 can be recovered if and only if the null space of $\mathcal{A}$ does not intersect the tangent cone of the nuclear norm ball at X_0. Kong, Tunçel, and Xiu [21] extended the results of [8], studied the concept of s-goodness for the sensing matrix in SSR and the linear transformations in LMR, and established the equivalence of s-goodness and the NSP in the matrix setting.

The study of SREJA is also motivated by the recent development of optimization techniques on Euclidean Jordan algebras, which provide a foundation for solving SREJA via convex relaxation. For details on Euclidean Jordan algebra optimization, see a survey paper [31] and the papers [13, 14, 20, 22, 29, 30], to name a few. In particular, Recht, Fazel, and Parrilo [25] mentioned the power of Jordan-algebraic approach and asked whether similar results can be obtained in the more general framework of Euclidean Jordan algebras. Moreover, since SSR and LMR are two specific classes of sparse descriptions, they also asked whether there are other kinds of easy-to-describe parametric models that are amenable to exact solutions via convex optimizations techniques.

This paper will deal with the following mathematical model of SREJA:

$$\min \ \operatorname{rank}(x), \quad \text{s.t.} \ \ \mathcal{A}x = b, \tag{1}$$

where rank (x) is the rank of an element x in a Euclidean Jordan algebra $\mathcal{V}$(see details in Section 2), $\mathcal{A} : \mathcal{V} \to \mathbb{R}^m$ is a linear transformation (operator), and $b \in \mathbb{R}^m$. We define the convex relaxation of SREJA as *nuclear norm minimization on Euclidean Jordan algebra (NNMEJA)*:

$$\min \ \|x\|_* \quad \text{s.t.} \ \ \mathcal{A}x = b, \tag{2}$$

where $\|x\|_*$ is the nuclear norm of an element $x \in \mathcal{V}$. Assume the solutions to (1) and (2) exist. We will study the properties of rank (x) and $\|x\|_*$ based on the necessary concepts in Euclidean Jordan algebra. Then, we introduce and study the three important recovery conditions related to the restricted isometry property (RIP), the null space property (NSP) and s-goodness. We show that RIP provides a sufficient condition for every s-sparse vector to be the unique solution to NNMEJA, while both s-goodness and NSP are necessary and sufficient conditions for recovering the low-rank solution exactly via NNMEJA. Employing the proposed conditions, we give the exact and stable recovery results via NNMEJA. These results extend the similar results from SSR and LMR to the general setting of Euclidean Jordan algebras, and we therefore give affirmative answers to the open questions in [25].

This paper is organized as follows. In Section 2, we briefly review some concepts and results on Euclidean Jordan algebras. Then we develop some properties of the rank and nuclear norm operators which are useful in providing the sparse recovery condition for SREJA via NNMEJA. In Section 3, we introduce the restricted isometry property (RIP), null space property (NSP) and s-goodness properties for linear transformations in SREJA, and discuss their close connections. In Section 4, applying the proposed the recovery conditions, we establish the exact and stable recovery results for SREJA via NNMEJA. Section 5 concludes this paper with some remarks.

2 Preliminaries

We review some necessary concepts and results on Euclidean Jordan algebras. In particular, we develop some topological properties of the rank and nuclear norm, which are useful in our analysis on the sparse recovery condition via convex optimization problem in the setting of Euclidean Jordan algebra. Details of basic concepts on Euclidean Jordan algebras can be found in Koecher's lecture notes [19] and the monograph by Faraut and Korányi [12].

2.1 Euclidean Jordan algebras

Let V be a n-dimensional vector space over $\mathbb{R}$ and $(x,s) \mapsto x \circ s : V \times V \to V$ be a bilinear mapping. We call $(V, \circ)$ a *Jordan algebra* iff the bilinear mapping satisfies the following conditions:

(i) $x \circ s = s \circ x$ for all $x, s \in V$,

(ii) $x \circ (x^2 \circ s) = x^2 \circ (x \circ s)$ for all $x, s \in V$,

where $x^2 := x \circ x$ and $x \circ s$ is the *Jordan product* of x and s. In general, there may exist $x, s, z \in V$, such that $(x \circ s) \circ z \neq x \circ (s \circ z)$. We call an element e the *identity* element if and only if $z \circ e = e \circ z = z$ for all $z \in V$. A Jordan algebra $(V, \circ)$ with an identity element e is called a *Euclidean Jordan algebra*, denoted by $\mathcal{V} := (V, \langle\cdot,\cdot\rangle, \circ)$, if and only if there is an inner product, $\langle\cdot,\cdot\rangle$, such that

$$\langle x \circ s, z\rangle = \langle x, s \circ z\rangle \quad \text{for all } x, s, z \in V.$$

Given a Euclidean Jordan algebra $\mathcal{V}$, define *the set of squares* as $K := \{z^2 : z \in \mathcal{V}\}$. It is known by Theorem III 2.1 in [12] that K is the *symmetric cone*, i.e., K is a closed, convex, homogeneous and self-dual cone.

For $z \in \mathcal{V}$, the *degree* of z denoted by $\deg(z)$ is the smallest positive integer k, such that the set $\{e, z, z^2, \cdots, z^k\}$ is linearly dependent. The *rank* of $\mathcal{V}$ is defined as $\max\{\deg(z) : z \in \mathcal{V}\}$. In this paper, r will denote the rank of the underlying Euclidean Jordan algebra. An element $q \in \mathcal{V}$ is an *idempotent* if and only if $q^2 = q$, it is called *primitive* if and only if it is nonzero and cannot be written as a sum of two nonzero idempotents. A *complete system of orthogonal idempotents* is a finite set $\{q_1, q_2, \cdots, q_k\}$ of idempotents where $q_i \circ q_j = 0$ for all $i \neq j$, and $q_1 + q_2 + \cdots + q_k = e$. A *Jordan frame* is a complete system of orthogonal primitive idempotents in $\mathcal{V}$. Note that the number of elements in every Jordan frame is r.

We state below the spectral decomposition theorem for the elements on a Euclidean Jordan algebra.

Theorem 2.1 *(Spectral Decomposition Type II (Theorem III.1.2, [12])) Let $\mathcal{V}$ be a Euclidean Jordan algebra with rank r. Then for $z \in \mathcal{V}$ there exist a Jordan frame $\{q_1, q_2, \cdots, q_r\}$ and real numbers $\lambda_1(z) \geq \lambda_2(z) \geq \cdots \geq \lambda_r(z)$, such that*

$$z = \lambda_1(z)q_1 + \lambda_2(z)q_2 + \cdots + \lambda_r(z)q_r. \tag{3}$$

The numbers $\lambda_i(z)$ $(i \in \{1, 2, \cdots, r\})$ are the eigenvalues of z. We call (3) the spectral decomposition (or the spectral expansion) of z.

Observe that the Jordan frame $\{q_1, q_2, \cdots, q_r\}$ in (3) depends on z. We do not write this dependence explicitly for the simplicity of notation (the same for $\{e_1, e_2, \cdots, e_{\bar{r}}\}$ below). Let $\mathcal{C}(z)$ be the set consisting of all Jordan frames in the spectral decomposition of z. Let *the* *spectrum* $\sigma(z)$ be the set of all eigenvalues of z. Then $\sigma(z) = \{\mu_1(z), \mu_2(z), \cdots, \mu_{\bar{r}}(z)\}$ and for each $\mu_i(z) \in \sigma(z)$, denoting $N_i(z) := \{j : \lambda_j(z) = \mu_i(z)\}$ we obtain that $e_i = \sum_{j \in N_i(z)} q_j$ and e_i is idempotent but may not be primitive. By Theorem III.1.1 in [12], $\{e_1, e_2, \cdots, e_{\bar{r}}\}$ is a unique complete system of orthogonal idempotents, such that

$$z = \mu_1(z)e_1 + \cdots + \mu_k(z)e_{\bar{r}}.$$

Let $g : \mathbb{R} \to \mathbb{R}$ be a real-valued function. Define the vector-valued function $G : \mathcal{V} \to \mathcal{V}$ as

$$G(z) := \sum_{i=1}^{r} g(\lambda_i(z))q_i = g(\lambda_1(z))q_1 + g(\lambda_2(z))q_2 + \cdots + g(\lambda_r(z))q_r, \tag{4}$$

which is a *Löwner operator*. In particular, letting $t_+ := \max\{0, t\}$, $t_- := \min\{0, t\}$ $(t \in \mathbb{R})$, we respectively define

$$\Pi_K(z) := z_+ := \sum_{i=1}^{r} (\lambda_i(z))_+ q_i, \quad z_- := \sum_{i=1}^{r} (\lambda_i(z))_- q_i.$$

In words, z_+ is the *metric projection* of z onto K, and z_- is the *metric projection* of z onto $-K$, where the norm is defined by $\|z\|_{\mathcal{V}} := \sqrt{\langle z, z\rangle}$. Note that $z \in K$ $(z \in \mathrm{int}(K))$ if and only if $\lambda_i(z) \geq 0$ $(\lambda_i(z) > 0)$ $\forall i \in \{1, 2, \cdots, r\}$, where $\mathrm{int}(K)$ denotes the *interior* of K. It is easy to verify that

$$z_+ \in K, \quad -z_- \in K, \quad z = z_+ - z_-. \tag{5}$$

Next, we recall the Peirce decomposition on the space $\mathcal{V} = (V, \langle\cdot,\cdot\rangle, \circ)$. Let $q \in \mathcal{V}$ be a nonzero idempotent. Then $\mathcal{V}$ is the orthogonal direct sum of $V(q, 0), V(q, \frac{1}{2})$ and $V(q, 1)$, where

$$V(q, \varepsilon) := \{x \in \mathcal{V} : q \circ x = \varepsilon x\}, \quad \varepsilon \in \left\{0, \frac{1}{2}, 1\right\}.$$

This is called the *Peirce decomposition* of $\mathcal{V}$ with respect to the nonzero idempotent q. Fix a Jordan frame $\{q_1, q_2, \cdots, q_r\}$. Defining the following subspaces for $i, j \in \{1, 2, \cdots, r\}$,

$$V_{ii} := \{x \in \mathcal{V} : x \circ q_i = x\} \text{ and } V_{ij} := \left\{x \in \mathcal{V} : x \circ q_i = \frac{1}{2}x = x \circ q_j\right\}, \ i \neq j,$$

we have the Peirce decomposition theorem as follows.

Theorem 2.2 *(Theorem IV.2.1, [12]) Let $\{q_1, q_2, \cdots, q_r\}$ be a given Jordan frame in a Euclidean Jordan algebra $\mathcal{V}$ of rank r. Then $\mathcal{V}$ is the orthogonal direct sum of spaces V_{ij} $(i \leq j)$. Furthermore,*

(i) $V_{ij} \circ V_{ij} \subseteq V_{ii} + V_{jj}$;

(ii) $V_{ij} \circ V_{jk} \subseteq V_{ik}$, *if* $i \neq k$;

(iii) $V_{ij} \circ V_{kl} = \{0\}$, *if* $\{i, j\} \cap \{k, l\} = \emptyset$.

Based on Theorems 2.1 and 2.2, we will introduce a decomposition result for the space $\mathcal{V}$ with respect to a point $z \in \mathcal{V}$. First, we need the following two important operators. For each $z \in \mathcal{V}$, we define the *Lyapunov transformation* $\mathbb{L}(z) : \mathcal{V} \to \mathcal{V}$ by $\mathbb{L}(z)x = z \circ x$ for all $x \in \mathcal{V}$, which is a symmetric self-adjoint operator in the sense that $\langle \mathbb{L}(z)x, s\rangle = \langle x, \mathbb{L}(z)s\rangle$ for all $x, s \in \mathcal{V}$. The operator $\mathcal{Q}(z) := 2\mathbb{L}^2(z) - \mathbb{L}(z^2)$ is called the *quadratic representation* of z. We say two elements $x, s \in \mathcal{V}$ *operator commute* if and only if $\mathbb{L}(x)\mathbb{L}(s) = \mathbb{L}(s)\mathbb{L}(x)$. By Lemma X.2.2 in [12], two elements x, s *operator commute* if and only if they share a common Jordan frame. In the matrix algebra of Hermitian matrices, this corresponds to two matrices admitting a simultaneous diagonalization with respect to an orthogonal basis.

2.2 Rank and Nuclear Norm

We first recall the definitions of the rank and the nuclear norm of an element z in $\mathcal{V}$. Let $z = \sum_{i=1}^{r} \lambda_i(z) q_i$ with the eigenvalue vector $\lambda(z)$. It is known that the *inertia* of z is defined by

$$\mathrm{In}\,(z) := (I^+(z), I^-(z), I^0(z)),$$

where $I^+(z)$, $I^-(z)$, and $I^0(z)$ are, respectively, the number of eigenvalues of z which are positive, negative, and zero, counting multiplicities. We define the *rank* of z by

$$\mathrm{rank}\,(z) := I^+(z) + I^-(z).$$

Clearly, $\mathrm{rank}\,(z) = \|\lambda(z)\|_0$, i.e., the rank of z is the l_0-norm of its eigenvalue vector $\lambda(z)$. Beside $\|z\|_{\mathcal{V}}$, we specify another important norm of z, *the nuclear norm* $\|z\|_*$, by the l_1-norm of its eigenvalue vector $\lambda(z)$, while $\|z\|_{\mathcal{V}}$ is equal to l_2-norm of $\lambda(z)$. Thus,

$$\|z\|_{\mathcal{V}} = \|\lambda(z)\|_2, \quad \|z\|_* = \|\lambda(z)\|_1.$$

Now, we present some rank inequalities, which are useful in the analysis of recovery conditions.

Lemma 2.3 *For any $x \geq 0$ and $y \geq 0$,*

$$\mathrm{rank}\,(x \pm y) \leq \mathrm{rank}\,(x) + \mathrm{rank}\,(y). \tag{6}$$

The equality holds in (6) when x and y operator commute and $x \circ y = 0$.

Proof. Case 1 If x and y operator commute, then it is easy to verify that (6) holds.

Case 2 If x and y do not operator commute, then there exits $d > 0$ such that $\mathcal{Q}_d(x)$ and $\mathcal{Q}_d(y)$ operator commute by Lemma 1 in [16]. Note that $\mathcal{Q}_d(x) \geq 0$ and $\mathcal{Q}_d(y) \geq 0$. Thus,

$$\mathrm{rank}\,(\mathcal{Q}_d(x) \pm \mathcal{Q}_d(y)) \leq \mathrm{rank}\,(\mathcal{Q}_d(x)) + \mathrm{rank}\,(\mathcal{Q}_d(y)).$$

Since $\mathrm{In}\,(\mathcal{Q}_d(x)) = \mathrm{In}\,(x)$ and $\mathrm{In}\,(\mathcal{Q}_d(y)) = \mathrm{In}\,(y)$ by Theorem 11 in [15], (6) holds.

It is easy to verify that the equality holds in (6) when x and y operator commute and $x \circ y = 0$. $\square$

Theorem 2.4 *For any $x, y \in V$,*

$$\mathrm{rank}\,(x \pm y) \leq \mathrm{rank}\,(x) + \mathrm{rank}\,(y). \tag{7}$$

The equality holds in (7) when x and y operator commute and $x \circ y = 0$

Proof. From Lemma 2.3, direct calculation yields

$$\begin{aligned}
\mathrm{rank}\,(x+y) &= \mathrm{rank}\,(x^+ - x^- + y^+ - y^-) \\
&= \mathrm{rank}\,(x^+ + y^+ - (x^- + y^-)) \\
&\leq \mathrm{rank}\,(x^+ + y^+) + \mathrm{rank}\,(x^- + y^-)
\end{aligned}$$

$$
\begin{aligned}
&\leq \operatorname{rank}(x^+) + \operatorname{rank}(y^+) + \operatorname{rank}(x^-) + \operatorname{rank}(y^-) \\
&= (\operatorname{rank}(x^+) + \operatorname{rank}(x^-)) + (\operatorname{rank}(y^+) + \operatorname{rank}(y^-)) \\
&= \operatorname{rank}(x) + \operatorname{rank}(y).
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
\operatorname{rank}(x-y) &= \operatorname{rank}(x^+ - x^- - y^+ + y^-) \\
&= \operatorname{rank}(x^+ + y^- - (x^- + y^+)) \\
&\leq \operatorname{rank}(x^+ + y^-) + \operatorname{rank}(x^- + y^+) \\
&\leq \operatorname{rank}(x^+) + \operatorname{rank}(y^-) + \operatorname{rank}(x^-) + \operatorname{rank}(y^+) \\
&= (\operatorname{rank}(x^+) + \operatorname{rank}(x^-)) + (\operatorname{rank}(y^+) + \operatorname{rank}(y^-)) \\
&= \operatorname{rank}(x) + \operatorname{rank}(y).
\end{aligned}
$$

It is easy to verify that the equality holds in (7) when x and y operator commute and $x \circ y = 0$.

□

Theorem 2.5 *Fixing a Jordan frame* $\{e_1, e_2, \ldots, e_r\}$. *Let* $c = \sum_1^k e_i$ *with* $k < r$. *For any* $x \in V$, $x = u + v + w$, *where* $u \in V(c, 1)$, $v \in V(c, \frac{1}{2})$, *and* $w \in V(c, 0)$. *Then*

$$
\operatorname{rank}(u+w) = \operatorname{rank}(u) + \operatorname{rank}(w), \qquad \operatorname{rank}(u+v) \leq 2k \;\text{ and }\; \operatorname{rank}(v) \leq 2k. \tag{8}
$$

Proof. It is obvious that $\operatorname{rank}(u+w) = \operatorname{rank}(u) + \operatorname{rank}(w)$. We prove the rank inequalities in two cases.

Case 1 When u is invertible in $V(c, 1)$.

$$
\begin{aligned}
\operatorname{rank}(u+v) &= \operatorname{rank}(u+v+0) \\
&= \operatorname{rank}(u) + \operatorname{rank}(0 - P_v(u^{-1})) \\
&= k + \operatorname{rank}(P_v(u^{-1})) \\
&\leq k + \operatorname{rank}(u^{-1}) \\
&= k + k = 2k.
\end{aligned}
$$

The above inequality is from Theorem 11 in [15].

Case 2 When u is not invertible in $V(c,1)$. Since π and γ are lower semicontinuous on V by Theorem 10 in [15], we have $\operatorname{rank}(u+v) \leq \operatorname{rank}(\bar{u}+v)$, where $\bar{u}$ is in some neighborhood of u and invertible in $V(c,1)$. Hence, $\operatorname{rank}(u+v) \leq 2k$. Similarly, $\operatorname{rank}(v) \leq \operatorname{rank}(\varepsilon c+v) \leq 2k$, with a small $\varepsilon > 0$. $\square$

The following theorem states a easily-checked sufficient condition for the nuclear norm equality between two elements.

Theorem 2.6 *Let c be an idempotent. For any $u \in V(c,1)$ and $w \in V(c,0)$, it holds*

$$\|u+w\|_* = \|u\|_* + \|w\|_*.$$

Proof By assumptions that $u \in V(c,1), w \in V(c,0)$ and c is an idempotent, there exists a Jordan frame $\{e_1, e_2, \cdots, e_k, e_{k+1}, \cdots, e_r\}$ such that

$$c = \sum_{i=1}^{k} e_i, \quad u = \sum_{i=1}^{k} \lambda_i(u) e_i, \quad \text{and} \quad w = \sum_{i=k+1}^{r} \lambda_i(w) e_i,$$

where k is the rank of subalgebra $V(c,1)$. Thus, $u + w = \sum_{i=1}^{k} \lambda_i(u)e_i + \sum_{i=k+1}^{r} \lambda_i(w)e_i$, and hence the conclusion holds immediately. $\square$

We end this section with the subdifferential property of the nuclear norm. Here we need recall *the spectral function* $F : \mathcal{V} \to \mathbb{R}$ generated by a function $f : \mathbb{R}^r \to \mathbb{R}$ if $F(z) = f(\lambda(z))$ where $\lambda(z)$ is the eigenvalue vector of z.

Theorem 2.7 *Let $w = \sum_{i=1}^{r} \lambda_i(w) c_i$ be its spectral decomposition where $\lambda_i(w) \neq 0$ for $i \in J := \{1, 2, \cdots, s\}$. The subdifferential of the nuclear norm $\|\cdot\|_*$ is given as*

$$\partial \|w\|_* = \left\{ z : z = \sum_{i \in J} \operatorname{sign}(\lambda_i(w)) c_i + \sum_{i \in \bar{J}} \lambda_i(z) e_i \quad \text{with} \quad \lambda_i(z) \in [-1,1] \right\},$$

where $\{c_1, c_2, \cdots, c_s, e_{s+1}, \cdots, e_r\}$ is any Jordan frame commonly shared by z and w.

Proof Note that $\|z\|_*$ is the spectral function generated by $\|\lambda(z)\|_1$. By Corollary 4.4.3 in [1], we know that $\partial\|w\|_* = \{z : \lambda(z) \in \partial\|\lambda(w)\|_1, z \text{ and } w \text{ operate commute}\}$. The desired conclusion follows from the definition of operate commute and the subdifferential of the l_1-norm. $\square$

3 Recovery Conditions

Applying the concepts and results in the previous section, we will present the sparse recovery conditions for SREJA via NNMEJA. We mainly focus on the restricted isometry property (RIP), null space property (NSP), and s-goodness properties. We will discuss their intimate connections in the setting of Euclidean Jordan algebra. In what follows, we say element $z \in \mathcal{V}$ is *s-sparse (s-rank)* if the rank of z is no more than s.

3.1 Restricted isometry property

We define the *s-restricted isometry constant (RIC)* δ_s of a linear transformation $\mathcal{A}$ as the smallest constant such that the following holds for all s-sparse (s-rank) element $z \in \mathcal{V}$,

$$(1 - \delta_s)\|z\|_{\mathcal{V}}^2 \leq \|\mathcal{A}z\|^2 \leq (1 + \delta_s)\|z\|_{\mathcal{V}}^2,$$

where $\|z\|_{\mathcal{V}} := \sqrt{\langle z, z\rangle}$ is the *norm* of z induced by inner product in a Euclidean Jordan algebra $\mathcal{V}$, which is equal to the l_2-norm of the vector of singular values of z, $\|\cdot\|$ is the norm in $\mathbb{R}^m$. It is easy to see that $\delta_s \leq \delta_t$ for $s < t$. For $s + s' \leq r$, the s, s'- *restricted orthogonality constant* $\theta_{s,s'}$ of $\mathcal{A}$ is the smallest constant which satisfies

$$|\langle \mathcal{A}x, \mathcal{A}y\rangle| \leq \theta_{s,s'}\|x\|_{\mathcal{V}}\|y\|_{\mathcal{V}},$$

for all s-sparse x and s'-sparse y such that x and y operator commute and $x \circ y = 0$.

Lemma 3.1 *Let $\mathcal{A} : \mathcal{V} \to \mathbb{R}^m$ be a linear transformation and $b \in \mathbb{R}^m$. If all nonzero elements in the null space of $\mathcal{A}$ are at least $(2s + 1)$-sparse, then any s-sparse solution to $\mathcal{A}x = b$ is unique.*

Proof We prove the conclusion by contradiction. Suppose that x_0 and x^* are two s-sparse solutions to SREJA. That is, $\mathcal{A}x_0 = \mathcal{A}x^*$ with $\mathrm{rank}(x_0) \leq s$ and $\mathrm{rank}(x^*) \leq s$. Then $\mathrm{rank}(x_0 - x^*) \leq \mathrm{rank}(x_0) + \mathrm{rank}(x^*) \leq 2s$ by Theorem 2.5, and $\mathcal{A}(x_0 - x^*) = 0$. This is a contradiction with the assumption. $\square$

We state the RIP condition under which the solution to SREJA is unique.

Theorem 3.2 *Let $\mathcal{A} : \mathcal{V} \to \mathbb{R}^m$ be a linear transformation, $x_0 \in \mathcal{V}$ and $b \in \mathbb{R}^m$ such that $\mathcal{A}x_0 = b$. Suppose rank(x_0) $= s$ and $s < r$. If $\delta_{2s} < 1$, then x_0 is the unique solution to SREJA.*

Proof Suppose there is another s-sparse solution $x^* \neq x_0$ to SREJA. That is, $\mathcal{A}x^* = b$ with $\text{rank}(x^*) = s$. Let $h = x^* - x_0$. Clearly, $\text{rank}(h) \leq \text{rank}(x_0) + \text{rank}(x^*) = 2s$ by Theorem 2.5, and $\mathcal{A}h = 0$. These together with the RIP condition yields $0 = \|\mathcal{A}h\|^2 \geq (1 - \delta_{2s})\|h\|_{\mathcal{V}}^2 > 0$. This is a contradiction since $h \neq 0$ and $1 - \delta_{2s} > 0$. $\square$

We are ready to give our RIP recovery result for exact SREJA via its convex relaxation. The following proof is inspired by those in [3, 6, 25], but our proof is more general and we employ the Peirce decomposition of a element in $\mathcal{V}$ and the Euclidean Jordan algebraic technique results in Section 2.

Theorem 3.3 *Let $\mathcal{A} : \mathcal{V} \to \mathbb{R}^m$ be a linear transformation, $x_0 \in \mathcal{V}$ and $b \in \mathbb{R}^m$ such that $\mathcal{A}x_0 = b$. Suppose rank(x_0) $= s$ and $1 \leq s < r$. If $\delta_{3s} + \sqrt{2}\delta_{4s} < 1$ or $\delta_{4s} < \sqrt{2} - 1$, then x_0 is the common unique solution to SREJA and its convex relaxation NNMEJA.*

Proof From assumptions and Theorem 3.2, we obtain that x_0 is the unique solution to SREJA. We remain to show that x_0 is also the unique solution to NNMEJA. Suppose there is another solution $x \neq x_0$ to NNMEJA. Take $h = x - x_0$ and let $x_0 = \sum_{i=1}^{s} \lambda_i(x)c_i$ with $c = \sum_{i=1}^{s} c_i$, and $x = u + v + w$ where $u \in V(c, 1)$, $v \in V(c, \frac{1}{2})$ and $w \in V(c, 0)$. Since $x_0 \in V(c, 1)$, from Theorem 2.6, we obtain that

$$\|x_0 + w\|_* = \|x_0\|_* + \|w\|_*. \tag{9}$$

Since $V(c, 0)$ forms a Euclidean Jordan algebra with rank $r - s$, without loss of generality, let $w = \sum_{i=s+1}^{r} w_i e_i$ be the spectral decomposition of w in $V(c, 0)$ with $|w_{s+1}| \geq |w_{s+2}| \geq \cdots \geq |w_r|$. Thus, we can decompose w as a sum of $h_{T_i} (i = 1, 2, \cdots, \lceil \frac{r}{s} \rceil)$ where

$$h_{T_i} := \sum_{j=is+1}^{\min\{r,(i+1)s\}} w_j e_j.$$

Then, h_{T_1} is the part of w corresponding to the s largest absolute eigenvalues, and h_{T_2} is the part of w corresponding to the next s largest absolute eigenvalues, and so on. Clearly, $h_{T_i} \in V(c, 0)$, $\text{rank}(h_{T_i}) \leq s$, and $h_{T_i}, v, u - x_0$ are all orthogonal one another.

As the common approaches in CS [3, 6], we proceed the proof in two steps. The first step shows that $\|\sum_{j\geq 2} h_{T_j}\|_{\mathcal{V}}$ is essentially bounded by $\|u - x_0 + v\|_*$. The second shows that $h = 0$, and hence a contradiction.

Step 1: From the above decomposition, we easily obtain that for $j \geq 2$,

$$\|h_{T_j}\|_{\mathcal{V}} \leq s^{1/2}\|h_{T_j}\|_\infty \leq s^{-1/2}\|h_{T_{j-1}}\|_*,$$

where $\|h_{T_j}\|_\infty$ is the spectral (operator) norm of $h_{T_j} \in \mathcal{V}$, i.e., its largest absolute eigenvalue. Then, it follows

$$\sum_{j\geq 2}\|h_{T_j}\|_{\mathcal{V}} \leq s^{-1/2}\sum_{j\geq 2}\|h_{T_{j-1}}\|_* \leq s^{-1/2}\|w\|_*. \tag{10}$$

This yields

$$\|w - h_{T_1}\|_{\mathcal{V}} = \left\|\sum_{j\geq 2} h_{T_j}\right\|_{\mathcal{V}} \leq s^{-1/2}\|w\|_*. \tag{11}$$

Moreover, since x is a solution to NNMEJA, we have

$$\begin{aligned}
\|x_0\|_* &\geq \|x\|_* = \|x_0 + h\|_* = \|u + v + w\|_* \\
&= \|x_0 + w + u + v - x_0\|_* \\
&\geq \|x_0 + w\|_* - \|u - x_0 + v\|_* \\
&= \|x_0\|_* + \|w\|_* - \|u - x_0 + v\|_*.
\end{aligned}$$

Therefore,

$$\|w\|_* \leq \|u - x_0 + v\|_*. \tag{12}$$

It holds by (11) and (12) that

$$\|w - h_{T_1}\|_{\mathcal{V}} \leq s^{-1/2}\|w\|_* \leq s^{-1/2}\|u - x_0 + v\|_*. \tag{13}$$

Step 2: We will prove $h = 0$ by claiming $u - x_0 + v + h_{T_1} = 0$. Let $h_{T_0} := u - x_0 + v$. Then

$$h_{T_0\cup T_1} = u - x_0 + v + h_{T_1}.$$

By Theorem 2.5, we get rank$(h_{T_0\cup T_1})\le$rank(h_{T_0})+rank$(h_{T_1})\le 3s$. Notice that $\mathcal{A}h=0$ and

$$\begin{aligned}\|\mathcal{A}(h_{T_0\cup T_1})\|^2 &= \langle \mathcal{A}(h_{T_0\cup T_1}), \mathcal{A}(h-h_{(T_0\cup T_1)^C})\rangle\\ &= \langle \mathcal{A}(h_{T_0\cup T_1}), \mathcal{A}h\rangle - \sum_{j\ge 2}\langle \mathcal{A}(h_{T_0\cup T_1}), \mathcal{A}h_{T_j}\rangle.\end{aligned}$$

Moreover, $\|\mathcal{A}(h_{T_0\cup T_1})\|^2\ge(1-\delta_{3s})\|h_{T_0\cup T_1}\|_{\mathcal{V}}^2$, and for $j\ge 2$,

$$-\langle \mathcal{A}(h_{T_0\cup T_1}), \mathcal{A}h_{T_j}\rangle\le \delta_{4s}\|h_{T_0\cup T_1}\|_{\mathcal{V}}\|h_{T_j}\|_{\mathcal{V}}.$$

Thus, by direct calculations, we obtain that

$$\begin{aligned}(1-\delta_{3s})\|h_{T_0\cup T_1}\|_{\mathcal{V}}^2 &\le \|\mathcal{A}h_{T_0\cup T_1}\|^2\\ &\le \sum_{j\ge 2}\delta_{4s}\|h_{T_0\cup T_1}\|_{\mathcal{V}}\|h_{T_j}\|_{\mathcal{V}}\\ &= \delta_{4s}\|h_{T_0\cup T_1}\|_{\mathcal{V}}\sum_{j\ge 2}\|h_{T_j}\|_{\mathcal{V}}\\ &\le \delta_{4s}\|h_{T_0\cup T_1}\|_{\mathcal{V}}s^{-1/2}\|w\|_* \quad (by\ \ (10))\\ &\le \delta_{4s}\|h_{T_0\cup T_1}\|_{\mathcal{V}}s^{-1/2}\|u-x_0+v\|_* \quad (by\ \ (12))\\ &\le \delta_{4s}\|h_{T_0\cup T_1}\|_{\mathcal{V}}s^{-1/2}\sqrt{2s}\|h_{T_0}\|_{\mathcal{V}} \quad (by\ \ \mathrm{rank}(h_{T_0})\le 2s)\\ &\le \sqrt{2}\delta_{4s}\|h_{T_0\cup T_1}\|_{\mathcal{V}}^2.\end{aligned}$$

Therefore, $1-\delta_{3s}\le\sqrt{2}\delta_{4s}$, i.e., $\sqrt{2}\delta_{4s}+\delta_{3s}\ge 1$, a contradiction. Then, $h_{T_0\cup T_1}=0$ and hence $h_{T_j}=0$. Thus, $h=0$ and we complete the proof. □

The above theorem states that the restricted isometry property provides a sufficient condition for every s-sparse vector to be the unique solution to NNMEJA. In the next subsection, we will consider necessary and sufficient conditions for the exact sparse recovery, such as the null space property (NSP) and s-goodness.

3.2 Null space property

We begin with the definition of the null space property (NSP). We say $\mathcal{A}$ *satisfies NSP (of order s)* if for every nonzero element $w\in$ Null$(\mathcal{A})$ with the spectral decomposition $w=\sum_{i=1}^r\lambda_i(w)c_i$ and for any index set $J\subseteq\{1,2,\cdots,r\}$ with $|J|=s$, then

$$\sum_{i\in J}|\lambda_i(w)|<\sum_{i\in \bar{J}}|\lambda_i(w)|,$$

where $\text{Null}(\mathcal{A}) = \{x : \mathcal{A}x = 0\}$ denote the null space of $\mathcal{A}$. In the cases of sparse vector recovery and low-rank matrix recovery, it is well known that the NSP property has been attracted much attention, see, e.g., [9, 23, 24, 26, 27] and references therein. The following theorem says that NSP is a necessary and sufficient condition for the sparse recovery in Euclidean Jordan algebra.

Theorem 3.4 *Let $\mathcal{A} : \mathcal{V} \to \mathbb{R}^m$ be a linear transformation, and $1 \leq s < r$. Then $\mathcal{A}$ satisfies NSP if and only if for any $x_0 \in \mathcal{V}$ with rank(x_0) $= s$, NNMEJA problem with $b = \mathcal{A}x_0$ has an unique solution and it is given by $x = x_0$.*

Proof We first show the "If" part. Consider any $h \neq 0$ and $h \in \text{Null}(\mathcal{A})$, write its spectral decomposition $h = \sum_{i=1}^{r} \lambda_i(h)c_i$. For any index set $J \subseteq \{1, 2, \cdots, r\}$ with $|J| = s$, we define $h_J = \sum_{i\in J} \lambda_i(h)c_i$ and $h_{J^c} = \sum_{i\in J^c} \lambda_i(h)c_i$. Clearly, $\mathcal{A}h_J + \mathcal{A}h_{J^c} = 0$. Then consider the NNMEJA problem:

$$\min \|x\|_* \ \text{ s.t. } \ \mathcal{A}x = \mathcal{A}h_J. \tag{14}$$

Thus from the assumption that for any $x_0 \in \mathcal{V}$ with $\text{rank}(x_0) = s$, NNMEJA problem with $b = \mathcal{A}x_0$ has an unique solution given by $x = x_0$, we obtain that h_J is the unique solution to problem (14) because of $\text{rank}(h_J) = s$. Noting that $\mathcal{A}h_J = \mathcal{A}(-h_{J^c})$ and $h \neq 0$, we immediately get $\|h_{J^c}\|_* > \|h_J\|_*$. That is, $\sum_{i\in J} |\lambda_i(h)| < \sum_{i\in \bar{J}} |\lambda_i(h)|$, and hence $\mathcal{A}$ satisfies NSP.

We next show the "Only if" part. Suppose that $\mathcal{A}$ satisfies NSP. Letting $x_0 \in \mathcal{V}$ with $\text{rank}(x_0) = s$, we need to show NNMEJA problem with $b = \mathcal{A}x_0$ has an unique solution and it is given by $x = x_0$. If not, we assume there is another solution to NNMEJA, $x \neq x_0$. Clearly, $\|x\|_* \leq \|x_0\|_*$. Set $h = x_0 - x$. Then $\|x_0 - h\|_* \leq \|x_0\|_*$. From Theorem 3.6.4 in [1], we obtain that

$$\|x_0 - h\|_* \geq \|\lambda(x_0) - \lambda(h)\|_1. \tag{15}$$

Since $\text{rank}(x_0) = s$, $\lambda(x_0)$ is s-sparse and $|J| = s$ if we take $J := \text{supp}(\lambda(x_0))$. Similarly, we decompose $\lambda(h)$ as $\lambda(h) = (\lambda(h))_J + (\lambda(h))_{J^c}$. Thus, we

$$\|\lambda(x_0) - \lambda(h)\|_1 = \|\lambda(x_0) - (\lambda(h))_J\|_1 + \|(\lambda(h))_{J^c}\|_1. \tag{16}$$

Noting that $\mathcal{A}$ satisfies NSP and $\mathcal{A}h = 0$, we have $\|(\lambda(h))_J\|_1 < \|(\lambda(h))_{J^c}\|_1$. Then,

$$\|\lambda(x_0) - (\lambda(h))_J\|_1 + \|(\lambda(h))_{J^c}\|_1 > \|\lambda(x_0) - (\lambda(h))_J\|_1 + \|(\lambda(h))_J\|_1 \geq \|\lambda(x_0)\|_1.$$

This together with (15) and (16) yields

$$\|x_0\|_* \geq \|x_0 - h\|_* > \|\lambda(x_0)\|_1 = \|x_0\|_*.$$

Hence, we have a contradiction. Thus, we get x_0 is the unique solution to NNMEJA problem.

□

3.3 S-goodness Property

We first discuss some concepts related to s-goodness of the linear transformation in SREJA. They are extensions of those given in the $\mathbb{R}^n$ setting [17]. In what follows, for a vector $y \in \mathbb{R}^p$, let $\|\cdot\|_d$ be the *dual norm* of $\|\cdot\|$ specified by $\|y\|_d := \max_v\{\langle v, y\rangle : \|v\| \leq 1\}$. In particular, $\|\cdot\|_\infty$ is the dual norm of $\|\cdot\|_1$ for a vector. We denote by X^T the *transpose* of X. For a linear transformation $\mathcal{A} : \mathbb{R}^{m\times n} \to \mathbb{R}^p$, we denote by $\mathcal{A}^* : \mathbb{R}^p \to \mathbb{R}^{m\times n}$ the *adjoint* of $\mathcal{A}$.

Definition 3.5 *Let $\mathcal{A} : \mathcal{V} \to \mathbb{R}^m$ be a linear transformation and s be an integer, $s \in \{1, 2, \cdots, r\}$. We say that $\mathcal{A}$ is s-good, if for every s-rank element $w \in \mathcal{V}$, w is the unique optimal solution to the optimization problem*

$$\min_{x\in\mathcal{V}}\{\|x\|_* : \mathcal{A}x = \mathcal{A}w\}. \tag{17}$$

To characterize s-goodness we need to introduce the two useful *s-goodness constants*: *G*-numbers γ_s and $\hat{\gamma}_s$.

Definition 3.6 *Let $\mathcal{A} : \mathcal{V} \to \mathbb{R}^m$ be a linear transformation, $\beta \in [0, \infty]$ and s be an integer, $s \in \{1, 2, \cdots, r\}$. We define the s-goodness constants, G-numbers $\gamma_s(\mathcal{A}, \beta)$ and $\hat{\gamma}_s(\mathcal{A}, \beta)$, as follows:*

(i) G-number $\gamma_s(\mathcal{A}, \beta)$ is the infimum of $\gamma \geq 0$ such that for every element $x \in \mathcal{V}$ with its spectral decomposition $x = \sum_{i=1}^s \lambda_i(x)c_i$ and $\lambda_i(x) \in \{-1, 1\}$ (i.e., s nonzero singular values,

equal to -1 or 1), there exists a vector $y \in \mathbb{R}^m$ such that $\|y\|_d \leq \beta$, and $\mathcal{A}^*y$ and x operate commute where $\mathcal{A}^*y = \sum_{i=1}^r \lambda_i(\mathcal{A}^*y)c_i$ with

$$\lambda_i(\mathcal{A}^*y) \begin{cases} = \lambda_i(x), & \textit{if } \ \lambda_i(x) \neq 0, \\ \in [-\gamma, \gamma], & \textit{if } \ \sigma_i(x) = 0, \end{cases} \qquad i \in \{1, 2, \cdots, r\}.$$

If there does not exist such y for some x as above, we take $\gamma_s(\mathcal{A}, \beta) = \infty$.

(ii) G-number $\hat{\gamma}_s(\mathcal{A}, \beta)$ is the infimum of $\gamma \geq 0$ such that for every element $x \in \mathcal{V}$ with its spectral decomposition $x = \sum_{i=1}^s \lambda_i(x)c_i$ and $\lambda_i(x) \in \{-1, 1\}$ (i.e., s nonzero singular values, equal to -1 or 1), there exists a vector $y \in \mathbb{R}^m$ such that $\|y\|_d \leq \beta$, and $\mathcal{A}^*y$ and x operate commute where $\mathcal{A}^*y = \sum_{i=1}^r \lambda_i(\mathcal{A}^*y)c_i$ with

$$\|\mathcal{A}^*y - x\| \leq \gamma. \tag{18}$$

If there does not exist such y for some x as above, we take $\gamma_s(\mathcal{A}, \beta) = \infty$. In that case, we write $\gamma_s(\mathcal{A})$, $\hat{\gamma}_s(\mathcal{A})$ instead of $\gamma_s(\mathcal{A}, \infty)$, $\hat{\gamma}_s(\mathcal{A}, \infty)$, respectively.

G-numbers of $\mathcal{A}$ preserve some nice properties. Clearly, for any nonsingular matrix $B \in \mathbb{R}^{m\times m}$, G-numbers $\gamma_s(\mathcal{A}, \beta)$ and $\hat{\gamma}_s(\mathcal{A}, \beta)$ are equal to $\gamma_s(B\mathcal{A}, \beta)$ and $\hat{\gamma}_s(B\mathcal{A}, \beta)$, respectively. At the same time, it is not hard to see that G-numbers $\gamma_s(\mathcal{A}, \beta), \hat{\gamma}_s(\mathcal{A}, \beta)$ are not only increasing in s, but also convex nonincreasing functions of β, see [21] for more details. We below observe the relationship between the G-numbers $\gamma_s(\mathcal{A}, \beta)$ and $\hat{\gamma}_s(\mathcal{A}, \beta)$. We omit its proof for brevity, for details, see the similar argument as in the proof of Theorem 1 in [17] and Proposition 2.5 in [21].

Proposition 3.7 *Let $\mathcal{A} : \mathcal{V} \to \mathbb{R}^m$ be a linear transformation, $\beta \in [0, \infty]$ and s be an integer, $s \in \{1, 2, \cdots, r\}$. Then we have*

$$\gamma := \gamma_s(\mathcal{A}, \beta) < 1 \qquad \Rightarrow \quad \hat{\gamma}_s(\mathcal{A}, \frac{1}{1+\gamma}\beta) = \frac{\gamma}{1+\gamma} < \frac{1}{2}; \tag{19}$$

$$\hat{\gamma}_s := \hat{\gamma}_s(\mathcal{A}, \beta) < \frac{1}{2} \quad \Rightarrow \qquad \gamma_s(\mathcal{A}, \frac{1}{1-\hat{\gamma}}) = \frac{\hat{\gamma}}{1-\hat{\gamma}} < 1. \tag{20}$$

We are ready to give the following characterization result of s-goodness of a linear transformation $\mathcal{A}$ via G-number $\gamma_s(\mathcal{A})$, which explains the importance of $\gamma_s(\mathcal{A})$ in SREJA. In the case of SSR, it reduces to Theorem 1 in [17].

Theorem 3.8 *Let $\mathcal{A} : \mathcal{V} \to \mathbb{R}^m$ be a linear transformation, and s be an integer, $s \in \{1, 2, \cdots, r\}$. Then the following statements are equivalent.*

i) $\mathcal{A}$ *is s-good.*

ii) $\gamma_s(\mathcal{A}) < 1$.

iii) $\hat{\gamma}_s(\mathcal{A}) < 1/2$.

Proof. In terms of G-numbers $\hat{\gamma}_s(\mathcal{A}), \hat{\gamma}_s(\mathcal{A})$, we directly obtain the following equivalence of (ii) and (iii) from Proposition 3.7. We only need to show that $\mathcal{A}$ is s-good if and only if $\gamma_s(\mathcal{A}) < 1$. Suppose that $\mathcal{A}$ is s-good, and let us prove that $\gamma_s(\mathcal{A}) < 1$. Let $w \in \mathcal{V}$ be an element of rank s, $s \in \{1, 2, \cdots, r\}$, write $w = \sum_{i=1}^{s} \lambda_i(w) c_i$ as a spectral decomposition where $\lambda_i(w) \neq 0$ for $i \in J := \{1, 2, \cdots, s\}$. By the definition of s-goodness of $\mathcal{A}$, w is the unique solution to the optimization problem (17). From the first optimality conditions, we obtain that for certain $y \in \mathbb{R}^m$ the function $f_y(x) = \|x\|_* - y^T[\mathcal{A}x - \mathcal{A}w]$ attains its minimum over $x \in \mathcal{V}$ at $x = w$. So, $0 \in \partial f_y(w)$, or $\mathcal{A}^* y \in \partial \|w\|_*$. From Theorem 2.7, it follows that there exists a Jordan frame $\{c_1, \cdots, c_s, c_{s+1}, \cdots, c_r\}$ such that $\mathcal{A}^* y = \sum_{i=1}^{r} \lambda_i(\mathcal{A}^* y) c_i$ where

$$\lambda_i(\mathcal{A}^* y) \begin{cases} = \operatorname{sign}(\lambda_i(w)), & \text{if } \ i \in J, \\ \in [-1, 1], & \text{if } \ i \in \bar{J}, \end{cases}$$

where $\bar{J} = \{1, 2, \cdots, r\} \setminus J$. Therefore, the linear optimization problem

$$\min_{y,\gamma} \left\{ \gamma : \lambda_i(\mathcal{A}^* y) \begin{cases} = \operatorname{sign}(\lambda_i(w)), & \text{if } \ i \in J, \\ \in [-\gamma, \gamma], & \text{if } \ i \in \bar{J}, \end{cases} \right\} \tag{21}$$

has optimal value no more than 1. We will show that the optimal value is less than 1. In fact, assuming that the optimal value equals 1, there should exist Lagrange multipliers $\{\mu_i : i \in J\}$, $\{\nu_i^+ \geq 0 : i \in \bar{J}\}$ and $\{\nu_i^- \geq 0 : i \in \bar{J}\}$ such that the function

$$L(\gamma, y) := \gamma + \sum_{i \in \bar{J}} [(\nu_i^+(\lambda_i(\mathcal{A}^* y) - \gamma)) + (\nu_i^+(\lambda_i(-\mathcal{A}^* y) - \gamma))] + \sum_{i \in J} \mu_i [\lambda_i(\mathcal{A}^* y) - \operatorname{sign}(\lambda_i(w))]$$

has unconstrained minimum in γ, y equal to 1 (see, e.g., [28]). For the Jordan frame $\{c_1, \cdots, c_s, c_{s+1}, \cdots, c_r\}$, setting $d = \sum_{i \in J} \mu_i c_i + \sum_{i \in \bar{J}} (\nu_i^+ + \nu_i^-) c_i$ and $\langle \mathcal{A} d, y \rangle = \langle d, \mathcal{A}^* y \rangle = \sum_{i \in \bar{J}} [(\nu_i^+ - \nu_i^+) \lambda_i(\mathcal{A}^* y)] +$

$\sum_{i\in J}\mu_i\lambda_i(\mathcal{A}^*y)$, we reformulate the above function as

$$L(\gamma,y)=\gamma-\gamma\sum_{i\in\bar{J}}(\nu_i^+ +\nu_i^-)+\langle \mathcal{A}d,y\rangle-\sum_{i\in J}\mu_i\mathrm{sign}(\lambda_i(w)).$$

This claims that

1. $\sum_{i\in\bar{J}}(\nu_i^+ +\nu_i^-)=1$,

2. $\sum_{i\in J}\mu_i\mathrm{sign}(\lambda_i(w))=-1$,

3. $\mathcal{A}d=0$.

Clearly, d and w share the Jordan frame $\{c_1,\cdots,c_s,c_{s+1},\cdots,c_r\}$. Thus, for small enough $t>0$, the matrices $x_t:=w+td$ are feasible to (17). From the above equations, we easily obtain that $\|x_t\|_*=\|w\|_*$ for all small enough t. Noting that w is the unique optimal solution to (17), we have $x_t=w$, which means that $\mu_i=0$ for $i\in J$. This is a contradiction, and hence the desired conclusion holds.

We next prove that $\mathcal{A}$ is s-good if $\gamma_s(\mathcal{A})<1$. That is, letting w be s-rank, we should prove that w is the unique optimal solution to (17). Without loss of generality, let w be an element of rank $s'\neq 0$ and $w=\sum_{i=1}^{s'}\lambda_i(w)c_i$. It follows $\gamma_{s'}(\mathcal{A})\le\gamma_s<1$. By the definition of $\gamma_s(\mathcal{A})$, there exists $y\in\mathbb{R}^m$ $\|y\|_d\le\beta$, and $\mathcal{A}^*y$ and x operate commute where $\mathcal{A}^*y=\sum_{i=1}^r\lambda_i(\mathcal{A}^*y)c_i=\sum_{i=1}^{s'}\mathrm{sign}(\lambda_i(w))c_i+\sum_{i=s'+1}^r\lambda_i(\mathcal{A}^*y)c_i$ with $\lambda_i(\mathcal{A}^*y)\in[-\gamma_{s'}(\mathcal{A}),\gamma_{s'}(\mathcal{A})]$ for $i\in\{s'+1,\cdots,r\}$. Then, $\mathcal{A}^*y$ and x share the Jordan frame $\{c_1,\cdots,c_{s'},c_{s'+1},\cdots,c_r\}$. The function

$$f(x)=\|x\|_*-y^T[\mathcal{A}x-\mathcal{A}w]=\|x\|_*-\langle\mathcal{A}^*y,x\rangle+\|w\|_*$$

becomes the objective of (17) on its feasible set. Note that $\langle\mathcal{A}^*y,x\rangle\le\|x\|_*$ by $\|\mathcal{A}^*y\|\le 1$ and the definition of dual norm. So, $f(x)\ge\|x\|_*-\|x\|_*+\|w\|_*=\|w\|_*$ and this function attains its unconstrained minimum in x at $x=w$. Hence $x=w$ is an optimal solution to (17). It remains to show that this optimal solution is unique. Let z be another optimal solution to the problem. Then $f(z)-f(w)=\|z\|_*-y^T\mathcal{A}z=\|z\|_*-\langle\mathcal{A}^*y,z\rangle=0$. This together with the fact $\|\mathcal{A}^*y\|\le 1$ derives that there exists a Jordan frame $\{e_1,\cdots,e_{s'},e_{s'+1},\cdots,e_r\}$ for $\mathcal{A}^*y$ and z such that

$$\mathcal{A}^*y=\sum_{i=1}^r\lambda_i(\mathcal{A}^*y)e_i,\quad z=\sum_{i=1}^r\lambda_i(z)e_i,$$

where $\lambda_i(z) = 0$ if $\lambda_i(\mathcal{A}^* y) \neq \pm 1$. Thus, for $\lambda_i(\mathcal{A}^* y) = 0 (i \in \{s'+1, \cdots, r\}$, we must have $\lambda_i(z) = \lambda_i(w) = 0$. By the two Jordan frames of $\mathcal{A}^* y$ as above, $\sum_{i=1}^{s'} c_i = \sum_{i=1}^{s'} e_i$. Then, $V(\sum_{i=1}^{s'} c_i, 1) = V(\sum_{i=1}^{s'} e_i, 1)$. So, $w, z \in V(\sum_{i=1}^{s'} c_i, 1)$ because of $w = \sum_{i=1}^{s'} \lambda_i(w) c_i$ and $z = \sum_{i=1}^{s'} r\lambda_i(z) e_i$. Thus, $z - w \in V(\sum_{i=1}^{s'} c_i, 1)$ and the rank of $z - w$ is no more than $s' \leq s$. Since $r_s(\mathcal{A}) < 1$, there exists $\tilde{y}$ such that

$$\sigma_i(\mathcal{A}^* \tilde{y}) \begin{cases} = \operatorname{sign}(\lambda_i(z-w)), & \text{if } \ \lambda_i(z-w) \neq 0, \\ \\ \in (-1, 1), & \text{if } \ \sigma_i(z-w) = 0. \end{cases}$$

Therefore, $0 = \tilde{y}^T \mathcal{A}(z-w) = \langle \mathcal{A}^* \tilde{y}, z - w \rangle = \|z - w\|_*$. Then $z = w$. The proof is completed. $\square$

3.4 S-goodness, NSP and RIP

This section deals with the connections between s-goodness, the null space property (NSP) and the restricted isometry property (RIP). We begin with giving the equivalence between NSP and G-number $\hat{\gamma}_s(\mathcal{A}) < 1/2$.

Proposition 3.9 *For the linear transformation $\mathcal{A}$, $\hat{\gamma}_s(\mathcal{A}) < 1/2$ if and only if $\mathcal{A}$ satisfies NSP.*

Proof. We first give an equivalent representation of the G-number $\hat{\gamma}_s(\mathcal{A}, \beta)$. We define a compact convex set first:

$$P_s := \{z \in \mathcal{V} : \|z\|_* \leq s, \|z\| \leq 1\}.$$

Let $B_\beta := \{y \in \mathbb{R}^m : \|y\|_d \leq \beta\}$ and $B := \{x \in \mathcal{V} : \|x\| \leq 1\}$. By definition, $\hat{\gamma}_s(\mathcal{A}, \beta)$ is the smallest γ such that the closed convex set $C_{\gamma,\beta} := \mathcal{A}^* B_\beta + \gamma B$ contains all elements with s nonzero eigenvalues, all equal to -1 or 1. Equivalently, $C_{\gamma,\beta}$ contains the convex hull of these elements, namely, P_s. Note that γ satisfies the inclusion $P_s \subseteq C_{\gamma,\beta}$ if and only if for every $x \in \mathcal{V}$,

$$\begin{aligned} \max_{z \in P_s} \langle z, x \rangle \leq \max_{y \in C_{\gamma,\beta}} \langle y, x \rangle &= \max_{y \in \mathbb{R}^m, w \in \mathcal{V}} \{\langle x, \mathcal{A}^* y \rangle + \gamma \langle x, w \rangle : \|y\|_d \leq \beta, \|w\| \leq 1\} \\ &= \beta \|\mathcal{A} x\| + \gamma \|x\|_*. \end{aligned} \tag{22}$$

For the above, we adopt the convention that whenever $\beta = +\infty$, $\beta\|\mathcal{A}x\|$ is defined to be $+\infty$ or 0 depending on whether $\|\mathcal{A}x\| > 0$ or $\|\mathcal{A}x\| = 0$. Thus, $P_s \subseteq C_{\gamma,\beta}$ if and only if $\max_{z \in P_s}\{\langle z, x \rangle -$

$\beta\|\mathcal{A}x\|\} \leq \gamma\|x\|_*$. Using the homogeneity of this last relation with respect to x, the above is equivalent to

$$\max_{z,x}\{\langle z, x\rangle - \beta\|\mathcal{A}x\| : z \in P_s, \|x\|_* \leq 1\} \leq \gamma.$$

Therefore, we obtain $\hat{\gamma}_s(\mathcal{A}, \beta) = \max_{z,x}\{\langle z, x\rangle - \beta\|\mathcal{A}x\| : z \in P_s, \|x\|_* \leq 1\}$. Furthermore,

$$\hat{\gamma}_s(\mathcal{A}) = \max_{z,x}\{\langle z, x\rangle : z \in P_s, \|x\|_* \leq 1, \mathcal{A}x = 0\}. \tag{23}$$

For $x \in \mathcal{V}$ with $\mathcal{A}x = 0$, let $x = \sum_{i=1}^{r} \lambda_i(x)c_i$ with $|\lambda_1(x)| \geq |\lambda_2(x)| \geq \cdots \geq |\lambda_r(x)|$. Then we obtain the sum of the s largest eigenvalues of x as

$$\|x\|_{s,*} := \max_{z\in P_s}\langle z, x\rangle.$$

From (23), we immediately obtain that $\hat{\gamma}_s(\mathcal{A})$ is the best upper bound on $\|x\|_{s,*}$ of elements $x \in \mathrm{Null}(\mathcal{A})$ such that $\|x\|_* \leq 1$. Therefore, $\hat{\gamma}_s(\mathcal{A}) < 1/2$ implies that the maximum of $\|\cdot\|_{s,*}$-norms of $x \in \mathrm{Null}(\mathcal{A})$ with $\|x\|_* = 1$ is less than 1/2. That is, $\sum_{i=1}^{s} |\lambda_i(x)| < 1/2 \sum_{i=1}^{r} |\lambda_i(x)|$. Thus, $\sum_{i=1}^{s} |\lambda_i(x)| < \sum_{i=s+1}^{r} |\lambda_i(x)|$ and hence $\mathcal{A}$ satisfies NSP. It is easy to see that $\mathcal{A}$ satisfies NSP means $\hat{\gamma}_s(\mathcal{A}) < 1/2$. Thus, we obtain the desired results. □

We next consider the connection between restricted isometry constants and G-number of the linear transformation in SREJA. It is well known that, for a nonsingular transformation $B \in \mathbb{R}^{m\times m}$, the RIP constants of $\mathcal{A}$ and $B\mathcal{A}$ can be very different, as shown by Zhang in[32] for the special $\mathbb{R}^n$ case. However, the s-goodness properties of $\mathcal{A}$ and $B\mathcal{A}$ are always same for a nonsingular matrix $B \in \mathbb{R}^{m\times p}$ as in subsection 3.3.

Theorem 3.10 $\delta_{4s} < \sqrt{2} - 1 \Rightarrow \mathcal{A}$ *satisfies NSP* $\Leftrightarrow \hat{\gamma}_s(\mathcal{A}) < 1/2 \Leftrightarrow \gamma_s(\mathcal{A}) < 1 \Leftrightarrow \mathcal{A}$ *is s-good.*

Proof. It holds from Theorem 3.3, Proposition 3.9 and Theorems 3.8. □

The above theorem says that both s-goodness and NSP are necessary and sufficient conditions for recovering the low-rank solution exactly via nuclear norm minimization in Euclidean Jordan algebra, which are weaker than RIP.

4 Exact and Stable Recovery

This section study the exact and stable recovery results via NNMEJA under some condition based on the concepts given in previous section. We first state RIP implications for sparse recovery, which extend Theorems 1.1 and 1.2 in [3, 6]. We omit the proof, since it is similar to that of Theorem 3.3. Let $w = \sum_{i=1}^{r} \lambda_i(w)e_i$, where $\lambda(w) = (\lambda_1(w), \cdots, \lambda_r(w))^T$ with its eigenvalues $|\lambda_1(w)| \geq \cdots \geq |\lambda_r(w)|$. Set $w^s := \sum_{i=1}^{s} \lambda_i(w)c_i$. Clearly, in terms of nuclear norm, w^s stands for the *best s-rank approximation* of w.

Theorem 4.1 *Let $\mathcal{A} : \mathcal{V} \to \mathbb{R}^m$ be a linear transformation, s be an integer, $s \in \{1, 2, \cdots, r\}$. Let w be a element in $\mathcal{V}$ such that $\|\mathcal{A}w - b\| \leq \varepsilon$ with $\varepsilon \geq 0$ and let x be the optimal solution to the problem $\min_x\{\|x\|_* : \|\mathcal{A}x - b\| \leq \varepsilon\}$. If $\delta_{3s} + \sqrt{2}\delta_{4s} < 1$ or $\delta_{4s} < \sqrt{2} - 1$, then it holds that*

$$\|x - w\|_{\mathcal{V}} \quad \leq \quad C_1\|w - w^s\|_* + C_0\varepsilon, \tag{24}$$

for some constants $C_0 > 0, C_1 > 0$. In particular, when w is s-rank and $\mathcal{A}w = b$ without noise, the recovery is exact $x = w$.

Here and below, we mainly consider the exact and stable s-sparse recovery results based on G-numbers $\gamma_s(\mathcal{A})$ and $\hat{\gamma}_s(\mathcal{A})$, which are responsible for s-goodness of a linear transformation $\mathcal{A}$. Observe that the definition of s-goodness of a linear transformation $\mathcal{A}$ tells that whenever the observation b in the following

$$\hat{w} \in \mathrm{argmin}_x\{\|x\|_* : \|\mathcal{A}x - b\| \leq \varepsilon\} \tag{25}$$

is noiseless and comes from a s-rank element $w : b = \mathcal{A}w$, w should be the unique optimal solution of the above optimization problem (25) where ε is set to 0. This establishes a sufficient condition for the precise SREJA of an s-rank element w in the "ideal case" when there is no measurement error and the optimization problem (17) is solved exactly.

Theorem 4.2 *Let $\mathcal{A} : \mathcal{V} \to \mathbb{R}^m$ be a linear transformation, s be an integer, $s \in \{1, 2, \cdots, r\}$. Let w be s-rank and $\mathcal{A}w = b$. If $\hat{\gamma}_s(\mathcal{A}) < 1/2$, then w is the unique solution to SREJA (1), i.e., the solution to SREJA can be exactly recovered from Problem (17).*

Proof. By the definition of s-goodness of a linear transformation $\mathcal{A}$, the assumption that $\mathcal{A}w = b$ and $\text{rank}(w) \leq s$ implies that w is the unique solution to problem (17). We remain to show that w is the unique solution to problem (1). Suppose there is an another solution z to problem (1). Then $\mathcal{A}w = \mathcal{A}z = b$. By the s-goodness of $\mathcal{A}$, the problem $\min\{\|x\|_* : \mathcal{A}x = \mathcal{A}w\} \approx \min\{\|x\|_* : \mathcal{A}x = \mathcal{A}z\}$ has a unique solution, hence $z = w$ and a contradiction. This completes the proof. $\square$

We continue to show that the same quantities $\hat{\gamma}_s(\mathcal{A})$ $(\gamma_s(\mathcal{A}))$ control the error of low-rank recovery in the case when the element $w \in \mathcal{V}$ is not s-rank and the problem (17) is not solved exactly.

Theorem 4.3 *Let $\mathcal{A} : \mathcal{V} \to \mathbb{R}^m$ be a linear transformation, s be an integer, $s \in \{1, 2, \cdots, r\}$, and $\hat{\gamma}_s(\mathcal{A}) < 1/2$. Let w be a element such that $\mathcal{A}w = b$. Let x be a υ-optimal approximate solution to the problem (17), meaning that*

$$\mathcal{A}x = \mathcal{A}w \quad \text{and} \quad \|x\|_* \leq \text{Opt}\,(\mathcal{A}w) + \upsilon,$$

where Opt($\mathcal{A}w$) is the optimal value of (17). Then

$$\|x - w\|_* \leq \frac{\upsilon + 2\|w - w^s\|_*}{1 - 2\hat{\gamma}_s(\mathcal{A})}.$$

Proof. Let $h := w - x$ and its spectral decomposition $h = \sum_{i=1}^r \lambda_i(h) c_i$. By assumption, $\|x\|_* = \|w - h\|_* \leq \|w\|_* + \upsilon$. By Theorem 3.6.4 in [1], $\|w - h\|_* \geq \|\lambda(w) - \lambda(h)\|_1$. We define $W := \sum_{i=1}^r \lambda_i(w) c_i$ and $X := W - h$. That is, $X = \sum_{i=1}^r (\lambda_i(w) - \lambda_i(h)) c_i$. Clearly, $\|X\|_* \leq \|x\|_*$ and $\|W\|_* = \|w\|_*$. From $\mathcal{A}h = 0$ and $\|x\|_* \leq \|w\|_* + \upsilon$, we obtain

$$\mathcal{A}X = \mathcal{A}W \quad \text{and} \quad \|X\|_* \leq \|W\|_* + \upsilon.$$

It is easy to verify that $h^s = W^s - X^s$ and $\|h^s\|_* \leq \|h\|_{s,*}$. Along with the fact $\mathcal{A}h = 0$ and (23), this derives

$$\|h^s\|_* \leq \|h\|_{s,*} \leq \hat{\gamma}_s(\mathcal{A}) \|h\|_*. \tag{26}$$

On the other hand, noting that $\|W\|_* + \upsilon = \|w\|_* + \upsilon \geq \|w - h\|_* = \|x\|_* \geq \|X\|_* = \|W - h\|_*$, we obtain

$$\|w\|_* + \upsilon \quad \geq \quad \|W - h\|_*$$

$$\begin{aligned}
&\geq \|W^s-(h-h^s)\|_*-\|W-W^s-h^s\|_* \\
&= \|W^s\|_*+\|h-h^s\|_*-\|W-W^s\|_*-\|h^s\|_* \\
&= \|w^s\|_*+\|h-h^s\|_*-\|w-w^s\|_*-\|h^s\|_*, \qquad (27)
\end{aligned}$$

where the first equality holds from Theorem 2.6. This is equivalent to

$$\|h-h^s\|_* \leq \|h^s\|_*+2\|w-w^s\|_*+\upsilon.$$

Therefore, we obtain

$$\begin{aligned}
\|h\|_* \leq \|h^s\|_*+\|h-h^s\|_* &\leq 2\|h^s\|_*+2\|w-w^s\|_*+\upsilon \\
&\leq 2\hat{\gamma}_s(\mathcal{A})\|h\|_*+2\|w-w^s\|_*+\upsilon.
\end{aligned}$$

Since $\hat{\gamma}_s(\mathcal{A})<1/2$, we get the desired conclusion. □

We next consider approximate solutions x to the problem

$$\mathrm{Opt}\,(b)=\min_{x\in\mathcal{V}}\{\|x\|_*:\|\mathcal{A}x-b\|\leq\varepsilon\} \qquad (28)$$

where $\varepsilon\geq 0$ and

$$b=\mathcal{A}w+\zeta, \quad \zeta\in\mathbb{R}^m$$

with $\|\zeta\|\leq\varepsilon$. We will show that in the "non-ideal case", when w is "nearly s-rank" and (28) is solved to near-optimality, the error of the SREJA via NNMEJA remains "under control", which is governed by $\hat{\gamma}_s(\mathcal{A},\beta)$ with a finite β.

Theorem 4.4 *Let $\mathcal{A}:\mathcal{V}\to\mathbb{R}^m$ be a linear transformation, and s be an integer, $s\in\{1,2,\cdots,r\}$, and let $\beta\in[0,\infty]$ such that $\hat{\gamma}:=\hat{\gamma}_s(\mathcal{A},\beta)<1/2$. Let $\varepsilon\geq 0$ and let w and b in (28) be such that $\|\mathcal{A}w-b\|\leq\varepsilon$, and let w^s be defined in the beginning of this section. Let x be a (ϑ,υ)-optimal approximate solution to the problem (28), meaning that*

$$\|\mathcal{A}x-b\|\leq\vartheta \;\text{ and }\; \|x\|_*\leq \mathrm{Opt}\,(b)+\upsilon,$$

then

$$\|x-w\|_* \leq \frac{2\beta(\vartheta+\varepsilon)+2\|w-w^s\|_*+\upsilon}{1-2\hat{\gamma}} \qquad (29)$$

Proof. Note that w is a feasible solution to (28). Let $h = w - x$. As in the proof or Theorem 4.3, we obtain that $\|h^{(s)}\|_* \le \|h\|_{s,*}$ and

$$\|h\|_* \le 2\|h^{(s)}\|_* + 2\|w - w^s\|_* + \upsilon.$$

Employing (22) in the proof of Proposition 3.9, we derive

$$\|h\|_{s,*} \le \beta\|\mathcal{A}h\| + \hat{\gamma}\|h\|_* \le \beta(\vartheta + \varepsilon) + \hat{\gamma}\|h\|_*, \tag{30}$$

where the last inequality holds by $\|\mathcal{A}h\| = \|\mathcal{A}w - b + b - \mathcal{A}x\| \le \|\mathcal{A}w - b\| + \|b - \mathcal{A}x\|$. Combining with the above inequalities, we obtain

$$\|h\|_* \le 2\beta(\vartheta + \varepsilon) + 2\hat{\gamma}\|h\|_* + 2\|w - w^s\|_* + \upsilon.$$

The desired conclusion holds from the assumption $\hat{\gamma} < 1/2$. □

Theorem 4.4 tells that the error bound (29) for imperfect low-rank recovery can be bounded in terms of $\hat{\gamma}_s(\mathcal{A}, \beta), \beta$, measurement error ε, "s-tail" $\|w - w^s\|_*$ and the inaccuracy (ϑ, υ) to which the estimate solves the program (28).

5 Concluding remarks

In this paper, we studied the restricted isometry property, null space property, and s-goodness for linear transformations in the sparse recovery on Euclidean Jordan algebra (SREJA). The Jordan-algebraic techniques are extensively employed in this study. We showed that both s-goodness and NSP are necessary and sufficient conditions for exact s-sparse recovery via its convex relaxation NNMEJA, which are weaker than the RIP conditions. Applying the characteristic properties of the proposed conditions, we establish the stable SREJA results via NNMEJA. Our results affirmatively answered the open questions proposed by Recht, Fazel, and Parrilo in [25].

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