School of Mathematics and Statistics

GRAPHS THAT ARE CRITICAL WITH RESPECT TO MATCHING EXTENSION AND DIAMETER

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DEDICATION

To my parents for their unconditional love

and

my husband, Mr Watcharaphong, for his endless love.

CERTIFICATION

I certify that the work presented in this thesis is my own work and that all references are duly acknowledged. This thesis has not been submitted previously, in whole or in part, in respect of any academic award at Curtin University of Technology or elsewhere.

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SUMMARY

Let G be a simple connected graph on 2n vertices with a perfect matching. For $1 \le k \le n - 1$, G is said to be k-extendable if for every matching M of size k in G there is a perfect matching in G containing all the edges of M. A k-extendable graph G is said to be k-critical (k-minimal) if G+uv (G-uv) is not k-extendable for every non-adjacent (adjacent) pair of vertices u and v of G. The problem that arises is that of characterizing k-extendable, k-critical and k-minimal graphs.

In Chapter 2, we establish that $\delta(G) \ge \frac{1}{2}(n + k)$ is a sufficient condition for a bipartite graph G on 2n vertices to be k-extendable. For a graph G on 2n vertices with $\delta(G) \ge n + k - 1$, n - k even and $\frac{n}{2} \le k \le n - 2$, we prove that a necessary and sufficient condition for G to be k-extendable is that its independence number is at most n - k. We also establish that a k-extendable graph G of order 2n has $k + 1 \le \delta(G) \le n$ or $\delta(G) \ge 2k + 1$, $1 \le k \le n - 1$. Further, we establish the existence of a k-extendable graph G on 2n vertices with $\delta(G) = j$ for each integer $j \in [k + 1, n] \cup [2k + 1, 2n - 1]$. For $k = n - 1$ and $n - 2$, we completely characterize k-extendable graphs on 2n vertices. We conclude Chapter 2 with a variation of the concept of extendability to odd order graphs.

In Chapter 3, we establish a number of properties of k-critical graphs. These results include sufficient conditions

for k-extendable graphs to be k-critical. More specifically, we prove that for a k-extendable graph $G \neq K_{2n}$ on 2n vertices, $2 \le k \le n - 1$, if for every pair of non-adjacent vertices u and v of G there exists a dependent set S (a subset S of V(G) is dependent if the induced subgraph G[S] has at least one edge) of G-u-v such that $o(G-(S \cup \{u,v\})) = |S|$, then G is k-critical. Moreover, for k = 2 this sufficient condition is also a necessary condition for non-bipartite graphs. We also establish a necessary condition, in terms of the minimum degree, for k-critical graphs. In fact, we prove that if $G \neq K_{2n}$ is k-critical on 2n vertices, $1 \le k \le n - 1$, then

$$\delta(G) \le \begin{cases} n & \text{, for } n < 2k \\ n + 2\lfloor \frac{k-1}{2} \rfloor & \text{, for } n \ge 2k. \end{cases}$$

We conclude Chapter 3 by completely characterizing k-critical graphs on 2n vertices for k = 1, n - 1 and n - 2.

Chapter 4 contains results on k-minimal graphs. These results include necessary and sufficient conditions for k-extendable graphs to be k-minimal. More specifically, we prove that for a k-extendable graph G on 2n vertices, $1 \le k \le n - 1$, the following are equivalent:

- G is minimal
- for every edge e = uv of G there exists a matching M of size k in G-e such that $V(M) \cap \{u,v\} = \phi$ and for every perfect matching F in G containing M, $e \in F$.
- for every edge e = uv of G there exists a vertex set S of G-u-v such that: $|M(S)| \ge k$; $o(G-e-S) = |S| - 2k +$

2; and u and v belong to different odd components of G-e-S, where M(S) denotes a maximum matching in G[S].

We also establish a necessary condition, in terms of minimum degree, for k-minimal and k-minimal bipartite graphs. In fact, we prove that a k-minimal graph $G \neq K_{2n}$ on 2n vertices, $1 \leq k \leq n - 1$, has minimum degree at most $n + k - 1$. For a k-minimal bipartite graph $G \neq K_{n,n}$, $1 \leq k \leq n - 3$, we show that $\delta(G) < \frac{1}{2}(n + k)$. The last section of this chapter contains a characterization of k-minimal graphs on 2n vertices for $k = n - 1$ and $n - 2$.

Let G be a simple undirected graph with edge set E(G) and diameter k. G is said to be strongly t-edge-critical or simply (k,t)-critical if for every $E' \subseteq E(G)$, $G-E'$ has diameter greater than k if and only if $|E'| \geq t$. The problem that arises is that of characterizing (k,t)-critical graphs. (k,1)-critical graphs have been studied by many authors. P. Kys conjectured that there is no (k,t)-critical graph for $k \geq 2$, $t \geq 2$. Denote the class of (k,t)-critical graphs by $\mathcal{G}(k,t)$. In Chapter 5, we prove that $\mathcal{G}(k,t) = \phi$ for $k \geq 2$ and $t \geq 3$ and for $t \geq 2$, $\mathcal{G}(k,t) = \phi$ when $k =$ 4 and 5.

Chapter 1 provides the notation, terminology, general concepts and the problems concerning extendability graphs and (k,t)-critical graphs.

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CONTENTS

CHAPTER 1

INTRODUCTION

The early development of graph theory evolved from the consideration of puzzles such as the Königsberg Bridge Problem and the Knight's Tour Problem (see Biggs et.al. (1976)). In 1736 the Swiss mathematician, Leonhard Euler, published his solution to the Königsberg Bridge Problem and introduced the concept of what is now called Eulerian graphs. These graphs arise in many applications including: determining optimal routes for service vehicles such as mail delivery, refuse collection and street sweeping; drawing graphs and maps using mechanical plotters. The Knight's Tour Problem gave rise to the study of Hamiltonian graphs (graphs which contain a cycle passing through each vertex exactly once). Hamiltonian graphs play an important role in modern day problems such as the Travelling Salesman Problem which has many applications (see Lawler et.al.(1985)).

Despite its humble beginning in 1736, graph theory has emerged as an important and rapidly growing branch of mathematics that is rich in theory and application. The reason for this growth is that the very simple structure of a graph, a collection of points called vertices connected by lines called edges, makes it a very useful tool in mathematical modelling. Indeed, many real-world systems can be viewed as a collection of components some of which interact. For example: a chemical compound is a collection of atoms some of which are bonded; a communication network is a collection of centers some of which communicate directly; an assembly production line is a collection of work stations organized in some order. Such systems can be conveniently modelled by a graph in which the vertices represent the components and the edges represent the interaction between components. For a more detailed account of graph theory applications we refer to the papers of Caccetta and Vijayan (1987) and Caccetta (1989, 1993).

In many applications, the problem that arises is that of constructing a graph which satisfies certain properties (representing the requirements of the system) and which is optimal according to some performance measure such as cost. The properties of the graph are usually expressed in terms of bounds on certain graph parameters. Consequently, a good deal of graph theory is concerned with characterizing particular parameters. This is the subject of extremal graph theory which will form the main focus of this thesis.

Given a set of graph parameters P, a fundamental problem in extremal graph theory is finding the relationship between the parameters. This is usually accomplished by fixing some of parameters and investigating how the others vary. In the characterization problem it is often useful to study a restricted class of graphs, the so called "critical graphs". Caccetta (1992) mentioned that the term "critical" is usually used with respect to a specified graph parameter P and applies when the graph G under consideration has property P, but alteration of G (such as vertex deletion, edge deletion or edge addition) results in a graph not having property P. The resultant critical class of graphs has more structure than the general class and this structure can be utilized to yield a considerable amount of useful information. Often there is no loss of generality and quite a lot to gain, in considering such a class of graphs.

Critical graphs have been studied by many researchers; a good survey is given in Caccetta (1992). In the following we briefly mention some of this work. Under the single operation of edge-deletion the parameters that have been studied include: diameter (Glivjak and Plesník (1969a, 1969b, 1971), Glivjak et.al. (1969a, 1969b), Plesník (1975), Glivjak (1968), Gliviak (1975a, 1975b), Greenwell and Johnson (1979), Caccetta and Häggkvist (1979), Kys (1980, 1981), Fan (1987), Füredi (1992), Ananchuen and Caccetta (1993)); connectivity and edge-connectivity (Dirac (1967), Halin (1969a, 1969b, 1971), Mader (1971, 1972, 1974, 1984, 1988), Bollobás (1978), Cai (1982, 1990, 1992), Budayasa and Caccetta (1990), Budayasa et.al.(1992, 1994)); chromatic-index (Yap (1986), Gavrilov and Muzychuk (1992), Hilton and Cheng (1992)); vertex covering number (see Lovász and Plummer (1986)); and k-extendability (Ananchuen and Caccetta (1994a, 1994b, 1994c)). Under the single operation of edge addition the parameters that have been studied include: diameter (Ore (1968), Caccetta and Smyth (1987a, 1987b, 1988a, 1988b, 1989a, 1989b, 1992)) and k-extendability (Ananchuen and Caccetta (1992)). Under the single operation of vertex deletion the parameters that have been studied include: diameter (Glivjak et.al. (1969b), Glivjak and Plesník (1969a, 1969b), Plesník (1975), Gliviak (1975b, 1976), Boals and Ali (1990)); connectivity (Chartrand et.al. (1972), Entringer (1978), Hamidoune (1980), Krol and Veldman (1984)); and edge-connectivity (Cozzens and Wu (1987, 1989)).

In this thesis we study extendable graphs, that is graphs which have the property that every matching of a specified size extends to a perfect matching. In particular, we investigate graphs that are critical with respect to this extendability property under the single operation of edge addition and edge deletion. We also study graphs that are critical with respect to diameter under the operation of edge deletion. A number of the results presented in this thesis have been published in: Ananchuen and Caccetta (1992, 1993, 1994a, 1994b) or have been submitted for publication in: Ananchuen and Caccetta (1994c, 1994d, 1994e, 1994f).

Before we present our work we need to introduce some basic notation and terminology which we do in Section 1.1. In Section 1.2, we give a brief review of results on extendable graphs including those in this thesis. We conclude this chapter by reviewing results on diameter critical graphs in Section 1.3 including those in this thesis.

1.1 NOTATION AND TERMINOLOGY

Since there is considerable variation in graph theoretic notation and terminology used in the literature, we present, in this section, the basic notation and terminology that we use throughout this thesis. For the most part our notation and terminology follows that of Bondy and Murty (1976). We begin with the formal definition of a graph.

A **graph** G is an ordered triple (V(G), E(G), ψ_G) consisting of a non-empty set V(G) of **vertices** and a set E(G), disjoint from V(G), of **edges** and an **incidence function** ψ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G. If u and v are vertices of a graph G identified with an edge e, that is $\psi_G(e) = uv$, then e is said to **join** u and v, and we write e = uv. The vertices u and v are called the **ends** of e. We also say that the vertices u and v are **incident** with the edge e, and vice versa. Further, u and v are **adjacent**.

We say that a graph G is **finite** if both V(G) and E(G) are finite sets. We denote the **order** of a graph G, the number of vertices of G, by $\nu(G)$ and the **size** of a graph G, the number of edges of G, by $\varepsilon(G)$. An edge with identical ends is called a **loop**. Two or more edges joining the same pair of vertices are called **multiple** edges. A graph with no loops and no multiple edges is called a **simple** graph. The **trivial** graph is a simple graph having exactly one vertex.

For a subset U of V(G), the **neighbour** set of U in G, denoted by $N_G(U)$, is the set of all vertices of G–U adjacent to vertices of U. We denote the non-neighbour set of U by $\bar{N}_G(U)$. Thus, $\bar{N}_G(U) = V(G)\setminus(N_G(U) \cup U)$. If U = {u}, then we write $N_G(u)$ and $\bar{N}_G(u)$ instead of $N_G(\{u\})$ and $\bar{N}_G(\{u\})$, respectively.

The **degree** of a vertex u of a graph G, denoted by $d_G(u)$, is the cardinality of $N_G(u)$, the number of edges of G incident to u. It is very well-known that for every graph G

$$\sum_{u \in E(G)} d_G(u) = 2\varepsilon(G).$$

The **minimum** and **maximum** degrees of a graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. G is called an **r-regular** graph if $\delta(G) = \Delta(G) = r$. In view of the above equation, an r-regular graph on n vertices exits only if r or n is even and it has $\frac{1}{2}rn$ edges.

A **complete** graph is a graph in which every pair of vertices are adjacent. We denote a complete graph of order n by K_n. Note that K_n is (n - 1)-regular with $\varepsilon(K_n) = \frac{1}{2}n(n-1)$. K_1 is, of course, the trivial graph. An **empty** graph is one with no edges. A graph G is **bipartite** if V(G) can be partitioned into two subsets X and Y such that each edge joins a vertex of X to some vertex of Y; X and Y are called **bipartitioning sets** of G. A **complete bipartite** graph is a bipartite graph with bipartitioning sets X and Y in which each vertex of X is joined to each vertex of Y; such a graph is denoted by $K_{m,n}$ when $m = |X|$ and $n = |Y|$.

Two graphs G and H are **isomorphic**, denoted by $G \cong H$, if there exists a one to one correspondence between their vertex sets that preserves adjacency. A graph $H = (V(H), E(H), \psi_H)$ is a **subgraph** of $G = (V(G), E(G), \psi_G)$ if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and ψ_H is the restriction of ψ_G to E(H). H is a **proper subgraph** of G if $H \neq G$. We say that H is a **spanning subgraph** of G if H is a subgraph of G and V(H) = V(G).

Let V′ be a non-empty subset of V(G). The subgraph of G **induced by** V′, denoted by G[V′], is a graph with vertex set V′ and $E(G[V']) = \{ uv \in E(G) : u,v \in V' \}$. Further, G-V′ is the subgraph obtained from G by deleting the vertices of V′ together with their incident edges. When V′ = {v} we write G-v instead of G-{v}. Similarly, for E′ a non-empty subset of E(G), the **edge-induced** subgraph of G, denoted by G[E′], is a graph with edge set E′ and $V(G[E']) = \{u \in V(G) : u$ is an end vertex of an edge e of $E'\}$. Further, G-E′ is a spanning subgraph of G with edge set $E(G)\setminus E'$. If E′ = {e}, then we write G-e instead of G-{e}. Suppose u and v are non-adjacent vertices of G. Then G+uv is the graph obtained by adding the edge uv to G.

The **complement** $\overline{G}$ of a graph G is the graph with vertex set V(G), two vertices being adjacent in $\overline{G}$ if and only if they are not adjacent in G. When H is a subgraph of G, the complement $\overline{H}(G)$ of H is the subgraph G-E(H).

A **k-factor** of G is a k-regular spanning subgraph of G. A **perfect matching** in G is a 1-factor of G; it is a 1-regular spanning subgraph of G. Further, we say that M is a **matching** in G if M is a 1-regular subgraph of G. The **size** of a matching is the cardinality of the matching. A matching M is said to be a **maximum matching** if $|M| \geq |M'|$ for any other matching M' in G. A vertex u is said to be **saturated** by a matching M or **M-saturated** if u is incident to some edge of M; otherwise, u is **unsaturated** or **M-unsaturated.** For simplicity, we let V(M) denote the vertex set of the subgraph G[M] induced by a matching M.

A subset S of V(G) is called an **independent set** of G if no two vertices of S are adjacent in G. An independent set S is **maximum** if $|S| \geq |S'|$ for any other independent set S' of G. The **independence number** of G, denoted by $\alpha(G)$, is the cardinality of a maximum independent set of G. A **clique** of a simple graph G is a subset S of V(G) such that G[S] is complete. Clearly, S is a clique of G if and only if S is an independent set of $\overline{G}$ and so the two concepts are complementary.

Let G_1 and G_2 be graphs. The **union** $G_1 \cup G_2$ of G_1 and G_2 is a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The graphs G_1 and G_2 are **disjoint** if $V(G_1) \cap V(G_2) = \phi$; G_1 and G_2 are **edge disjoint** if $E(G_1) \cap E(G_2) = \phi$. The **join** $G_1 \vee G_2$ of disjoint graphs G_1 and G_2 is the graph obtained from $G_1 \cup G_2$ by joining each vertex of G_1 to each vertex of G_2. The **cartesian**

product $G_1 \times G_2$ of G_1 and G_2 is a graph with vertex set $V(G_1) \times V(G_2)$ and vertices (u_1, u_2) and (v_1, v_2) of $G_1 \times G_2$ are adjacent if either $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2 or $u_2 = v_2$ and u_1 is adjacent to v_1 in G_1.

A **walk** in a graph G is a finite, non-empty alternating sequence $W = v_0e_1v_1e_2 \ldots e_nv_n$ of vertices and edges such that, for $1 \le i \le n$, the ends of edge e_i are v_{i-1} and v_i. W is said to be a walk from v_0 to v_n, or simply a (v_0, v_n)-walk. The vertices v_0 and v_n are called the **origin** and **terminus**, respectively of W. A **section** of a walk $W = v_0e_1v_1e_2 \ldots e_nv_n$ is a walk that is a subsequence $v_ie_{i+1}v_{i+1} \ldots e_jv_j$ of consecutive terms of W; we refer to this subsequence as the (v_i, v_j)-section of W. Further, the **length** of W is the number of edges of W. In the case of a simple graph, we denote the walk $W = v_0v_1 \ldots v_n$. A **closed walk** is a walk whose origin and terminus are the same. A **cycle** C is a closed walk $v_1v_2 \ldots v_nv_1$, where $n \ge 3$ and $v_1, v_2, \ldots, v_n$ are distinct vertices. A cycle of length n, denoted by C_n, is called **n-cycle.** A **path** is a walk with distinct vertices. A **hamiltonian path** of a graph G is a path containing every vertex of G. Two paths are said to be **edge (vertex) - disjoint** if they have no edges (internal vertices) in common.

Let u and v be vertices of a graph G. The **distance** $d_G(u,v)$ between u and v is defined as the length of a shortest (u,v)-path in G; if there is no path connecting u and v we define $d_G(u,v)$ to be infinite. The **eccentricity** of u, denoted by $ec_G(u)$, is the maximum distance from u in G, that is

$$ec_G(u) = \max_{v \in V(G)} \{ d_G(u,v) \}.$$

The **diameter** of G, denoted by d(G), is the maximum eccentricity among the vertices of G. It follows that

$$d(G) = \max_{u,v \in V(G)} \{d_G(u,v)\}.$$

The **girth** of G is the length of a shortest cycle in G; if G has no cycles we define the girth of G to be infinite. A **hamiltonian cycle** of a graph G is a cycle containing every vertex of G. We say that G is **hamiltonian** if G contains a hamiltonian cycle.

Two vertices u and v of G are **connected** if there is a (u,v)-path in G. A graph G is connected if every pair of vertices of G are connected; otherwise, G is **disconnected**. A maximal connected subgraph of G is called a **component** of G; it is **odd** or **even** depending on its order. We let $o(G)$ denote the number of odd components of the graph G.

The **vertex-connectivity** (or simply **connectivity**) of G, denoted by $\kappa(G)$, is the smallest number of vertices whose removal from G results in a disconnected or trivial graph. We say that G is **k-connected** if $\kappa(G) \geq k$. Similarly, the **edge-connectivity** of G, denoted by $\kappa'(G)$, is the minimum number of edges whose removal from G results in a disconnected or trivial graph. If $\kappa'(G) \geq k$, then we say that G is **k-edge-connected**. The fundamental relationship connecting minimum degree, connectivity and edge-connectivity of a graph G is :

$$\kappa(G) \leq \kappa'(G) \leq \delta(G)$$

(see Bondy and Murty (1976) p.43).

We conclude this section by noting that all graphs considered in this thesis are finite and simple.

1.2 EXTENDABLE GRAPHS

Matching Theory has played an important role in the development of Graph Theory and Combinatorial Optimization. It has contributed significantly to both theory and application, a comprehensive account of which is given in Lovász and Plummer (1986). A fundamental theorem in Matching Theory is the well-known theorem of Tutte which gives a necessary and sufficient condition of the existence of a perfect matching in a graph. The result is:

Theorem 1.2.1: Tutte's Theorem (see Bondy and Murty (1976) p. 76)

A graph G has a perfect matching if and only if

$$o(G-S) \le |S| \qquad \text{for all } S \subset V(G). \qquad \square$$

Recall that a perfect matching in a graph G is a 1-regular spanning subgraph of G and $o(H)$ is the number of odd components in a graph H. The graph G in Figure 1.2.1 illustrates Tutte's Theorem; G does not have a perfect matching since $o(G-u) = 3 > |\{u\}|$.

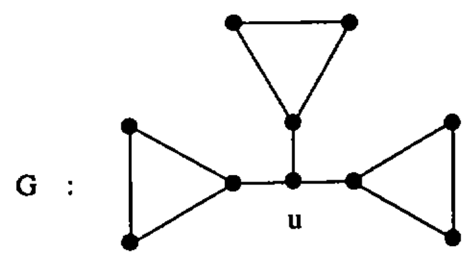


Figure 1.2.1

An interesting problem is to investigate graphs with the property that given any matching M, there exists a perfect matching containing all the edges of M. A graph with this property is called a **randomly matchable** graph. Sumner (1979) studied this concept and proved that $K_{n,n}$ and K_{2n} are the only randomly matchable graphs on 2n vertices. In view of this result, it is reasonable to relax the condition of randomly matchable by specifying the size of a matching. This leads us to the concept of extendable graphs.

Let G be a simple connected graph on 2n vertices with a perfect matching. For a given positive integer k, $1 \le k \le n - 1$, G is said to be **k-extendable** if for every matching M of size k in G, there exists a perfect matching in G containing all the edges of M. Note that a randomly matchable graph on 2n vertices is k-extendable for each value of k, $1 \le k \le n - 1$. Thus, $K_{n,n}$ and K_{2n} are k-extendable, $1 \le k \le n - 1$. The graph G_1 in Figure 1.2.2 is 1-extendable whilst the graph G_2 in Figure 1.2.2 is not 1-extendable since the edge e does not extend to a perfect matching. Notice that both G_1 and G_2 have a perfect matching.

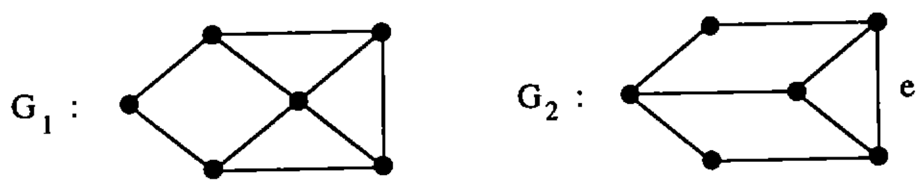


Figure 1.2.2

For convenience, a graph with a perfect matching is said to be 0-extendable.

The problem that arises is that of characterizing k-extendable graphs. This problem has been the focus of considerable attention with many authors contributing results. An excellent survey is the paper of Plummer (1991a). Most of the results are concerned with the relationship between k-extendable graphs and other graph parameters such as: connectivity (Plummer (1988a), Holton and Plummer (1991)); planarity (Holton and Plummer (1987), Plummer (1989a, 1992), Holton et.al. (1991)); genus (Plummer (1988b)); toughness (Plummer (1988c)); degree sum and neighborhood union (Plummer (1990)); and cyclic edge connectivity (Plummer (1991b), Lou and Holton (1993), Holton and Plummer (1987, 1991), Holton et.al. (1991), Aldred et.al. (1993)). In addition, the extendability of special classes of graphs such as: bipartite graphs (Plummer (1986)); claw-free graphs (Plummer (1989b)); generalized Petersen graphs (Schrag and Cammack (1989), Yu (1992b)); Cayley graphs (Chen (1992)); strongly regular graphs (Holton and Lou (1992)); have been investigated. Recently, Dean (1992) studied the matching extendability of surfaces.

Now we will mention some important results. The first result, due to Plummer, can be regarded as a fundamental result in the study of extendable graphs.

Theorem 1.2.2 (Plummer (1980))

Let G be a k-extendable graph on 2n vertices, $1 \le k \le n - 1$. Then G is $(k - 1)$-extendable. □

For $1 \le k \le n - 1$, let $\mathcal{G}_{(2n,k)}$ denote the class of k-extendable graphs on 2n vertices. Then Theorem 1.2.2 implies that

$$\mathcal{G}_{(2n,1)} \supset \mathcal{G}_{(2n,2)} \supset \mathcal{G}_{(2n,3)} \supset \dots .$$

Further, Theorem 1.2.2 together with the definitions of k-extendable graphs and randomly matchable graphs implies that an $(n - 1)$-extendable graph on $2n$ vertices is randomly matchable. Thus, $K_{n,n}$ and K_{2n} are the only $(n - 1)$-extendable graphs on $2n$ vertices.

The next result concerns the connectivity of extendable graphs.

Theorem 1.2.3: (Plummer(1980))

Let G be a k-extendable graph on $2n$ vertices, $1 \le k \le n - 1$. Then G is $(k + 1)$-connected. □

Theorem 1.2.3 together with the fact that the minimum degree of a graph is at least its connectivity implies that a k-extendable graph has minimum degree at least $k + 1$. In Section 2.3, we prove that a k-extendable graph on $2n$ vertices has minimum degree at most n or at least $2k + 1$. Further, we establish the existence of a k-extendable graph G on $2n$ vertices with $\delta(G) = j$ for every integer $j \in [k + 1, n] \cup [2k + 1, 2n - 1]$.

The next theorem gives a sufficient condition for a graph to be k-extendable.

Theorem 1.2.4: (Plummer (1980))

Let G be a graph on $2n$ vertices and $1 \le k \le n - 1$. If $\delta(G) \ge n + k$, then G is k-extendable. □

In Section 2.2, we establish that a sufficient condition for a bipartite graph G on $2n$ vertices to be k-extendable is that $\delta(G) \ge \frac{1}{2}(n + k)$. Also, in Section 2.2, we prove that a necessary and sufficient condition for a graph G on 2n vertices with $\delta(G) \geq n + k - 1$, $n - k$ even and $\frac{n}{2} \leq k \leq n - 2$ to be k-extendable is that its independence number is at most $n - k$.

The following theorem is a characterization of 1-extendable graphs.

Theorem 1.2.5: (Little et.al. (1975)).

A graph G of even order is 1-extendable if and only if

(i) $o(G-S) \leq |S|$ for all $S \subseteq V(G)$,

and

(ii) $o(G-S) = |S|$ only if S is an independent set of vertices of G. □

For any $S \subseteq V(G)$, let $M(S)$ denote a maximum matching in $G[S]$. The next few results give a characterization of k-extendable graphs. We begin with a generalization of Theorem 1.2.5.

Theorem 1.2.6: (Yu (1993))

Let G be a graph on 2n vertices and $1 \leq k \leq n - 1$. G is k-extendable if and only if for any $S \subseteq V(G)$

(i) $o(G-S) \leq |S|$,

and

(ii) $o(G-S) = |S| - 2t$, $0 \leq t \leq k - 1$, implies that $|M(S)| \leq t$. □

Theorem 1.2.7: (Lou (1994))

Let G be a graph on 2n vertices and $1 \leq k \leq n - 1$. G is k-extendable if and only if for any $S \subseteq V(G)$, $o(G-S) \leq |S| - 2d$, where $d = \min\{|M(S)|, k\}$. □

The next three theorems provide a characterization of k-extendable bipartite graphs.

Theorem 1.2.8: (Little et.al.(1975))

Let G be a bipartite graph with bipartitioning sets X and Y where $|X| = |Y|$. Then G is 1-extendable if and only if for all non-empty proper subsets S of X, $|N_G(S)| > |S|$. □

Theorem 1.2.9: (Plummer (1986))

Let G be a connected bipartite graph of order 2n with bipartitioning sets X and Y and suppose k is a positive integer such that $k \le n - 1$. Then the following are equivalent:

(a) G is k-extendable;

(b) $|X| = |Y|$ and for all non-empty subsets U of X with $|U| \le |X| - k$, $N_G(U) \ge |U| + k$;

(c) For all $x_1,\dots,x_k \in X$ and $y_1,\dots,y_k \in Y$, $G' = G-\{x_1,\dots,x_k,y_1,\dots,y_k\}$ has a perfect matching. □

Theorem 1.2.10: (Brualdi and Csima (1986))

An r-regular bipartite graph of order 2n is k-extendable, $2 \le k \le n - 1$, if and only if $r = 1$ or $r \ge \frac{1}{2}(n + k)$. □

The above six theorems give necessary and sufficient conditions for a graph to be k-extendable. They do not describe the class $\mathcal{G}_{(2n,k)}$. In section 2.4, we completely characterize the classes $\mathcal{G}_{(2n,n-1)}$ and $\mathcal{G}_{(2n,n-2)}$.

The next two results provide a method for obtaining an extendable graph from smaller extendable graphs. We use the operation of cartesian product. Recall that for graphs G_1 and G_2 the cartesian product $G_1 \times G_2$ of G_1 and G_2 is a graph with vertex set $V(G_1) \times V(G_2)$ and vertices (u_1, u_2) and (v_1, v_2) of the cartesian product graph are adjacent if either $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2 or $u_2 = v_2$ and u_1 is adjacent to v_1 in G_1.

Theorem 1.2.11: (Györi and Plummer (1992))

If G_1 and G_2 are k-extendable and ℓ-extendable graphs on $2n_1$ and $2n_2$ vertices, respectively where $1 \leq k \leq n_1 - 1$ and $1 \leq \ell \leq n_2 - 1$, then $G_1 \times G_2$ is $(k + \ell + 1)$-extendable. □

Theorem 1.2.12: (Yu (1991))

Let G_1 be a 1-extendable graph with $\nu(G_1) \geq 4$ and let $G_2,\ldots,G_k$ be connected graphs of order at least two. Then $G_1 \times G_2 \times\ldots\times G_k$ is k-extendable. □

Now we turn our attention to graphs that are critical with respect to extendability. Let G be a k-extendable graph on 2n vertices, $1 \leq k \leq n - 1$. We say that G is **critically k-extendable** or simply **k-critical**, if for every non-adjacent pair of vertices u and v of G, G+uv is not k-extendable.

Consider the graphs G_1 and G_2 in Figure 1.2.3. G_1 is a 2-critical graph whilst G_2 is not 2-critical since G_2+uv is 2-extendable

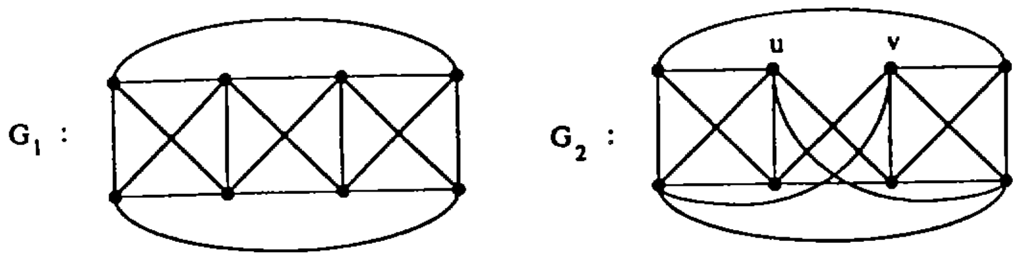


Figure 1.2.3

Plummer (1986) and Yu (1992a) independently studied the effect of adding an edge to k-extendable bipartite graphs. The result is:

Theorem 1.2.13: (Plummer (1986) and Yu (1992a))

Suppose G is a k-extendable bipartite graph. Let e be an edge of $\bar{G}$ such that G+e is still bipartite. Then G+e is also k-extendable. □

A consequence of this theorem is that the only k-critical bipartite graph on 2n vertices, $1 \le k \le n - 1$, is $K_{n,n}$.

For the non-bipartite graphs, Yu (1992a) proved that:

Theorem 1.2.14: (Yu (1992a))

Suppose G is a connected k-extendable non-bipartite graph. Then for any edge e = xy of $\bar{G}$, G+e is (k - 1)-extendable. □

The problem of characterizing k-critical graphs was posed by Saito (1989/90) and he stated without proof that the only 1-critical graphs on 2n vertices are $K_{n,n}$ and K_{2n}. Recently, Yu (1991) and Ananchuen and Caccetta (1992) independently provided the proof of this. Ananchuen and Caccetta (1992) also proved that the only k-critical graphs on 2n vertices are $K_{n,n}$ and K_{2n} for $k = n - 1$ and for $k = n - 2$ when $n \geq 5$. Further, in the same paper, we established sufficient conditions for k-extendable graphs to be k-critical. More specifically, we proved that for a k-extendable graph $G \neq K_{2n}$ on 2n vertices, $2 \leq k \leq n - 1$, if for every pair of non-adjacent vertices u and v of G there exists a dependent set S (a subset S of V(G) is dependent if G[S] has at least one edge) of G-u-v such that $o(G-(S \cup \{u,v\})) = |S|$, then G is k-critical. Moreover, for $k = 2$ this sufficient condition is also a necessary condition for non-bipartite graphs.

We also establish an upper bound on the minimum degree of k-critical graphs. We proved that if $G \neq K_{2n}$ is a k-critical graph on 2n vertices, $1 \leq k \leq n - 1$, then

$$\delta(G) \leq \begin{cases} n & \text{, for } n < 2k \\ n + 2\left\lfloor \frac{k-1}{2} \right\rfloor & \text{, for } n \geq 2k. \end{cases}$$

These results are presented in Chapter 3.

In addition to studying graphs which are critical under the operation of edge addition with respect to extendability, we also study graphs which are critical under the operation of edge deletion with respect to extendability . We refer to graphs satisfying the latter property as **minimally extendable graphs**. More precisely, let G be a k-extendable graph on 2n vertices, $1 \leq k \leq n - 1$. G is **minimally k-extendable** or simply **k-minimal** if, for every edge uv of G, G-uv is not k-extendable.

Consider the graph G_1 and G_2 in Figure 1.2.4. G_1 is 1-minimal whilst G_2 is not 1-minimal since G_2-e is 1-extendable.

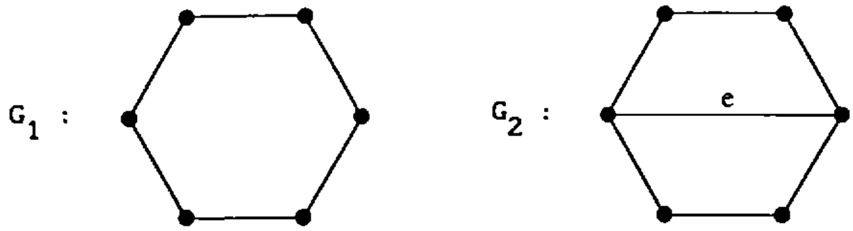


Figure 1.2.4

The concepts of k-critical and k-minimal are independent. For consider the graphs in Figure 1.2.5. It is not difficult to check that the graph G_1 is both 2-critical and 2-minimal. The graph G_2 is 2-critical but not 2-minimal. The graph G_3 is 2-minimal but not 2-critical and the graph G_4 is 2-extendable but not 2-critical or 2-minimal.

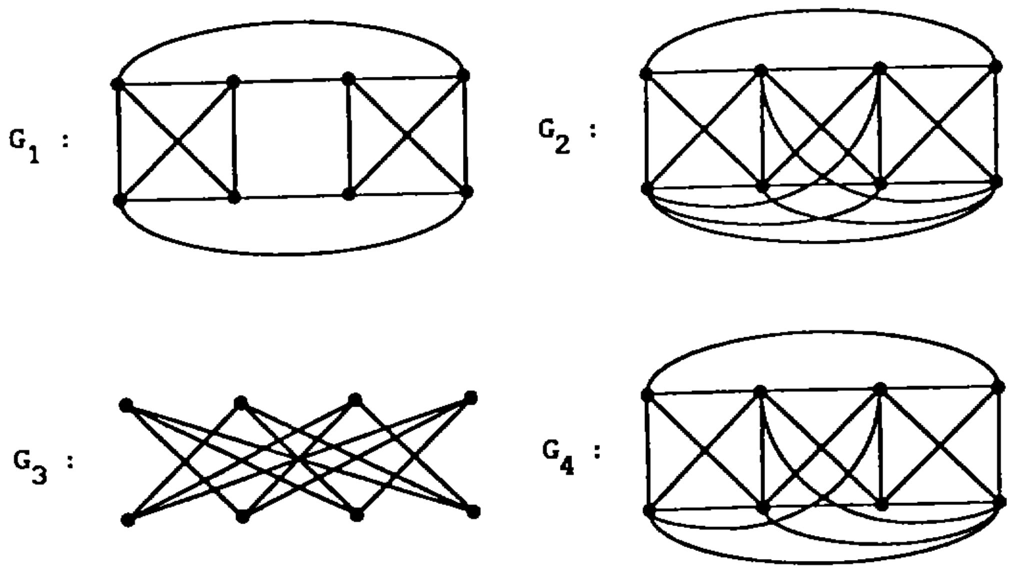


Figure 1.2.5

A natural problem that arises is that of characterizing k-minimal graphs. This problem was suggested to us by M.D.Plummer (private communication). k-extendable graphs have been studied by many authors whilst k-minimal graphs have only been studied by us (Ananchuen and Caccetta (1994a, 1994b, 1994c)). However, Plummer (1988a) studied the effect of deleting an edge of a k-extendable graph. He proved:

Theorem 1.2.15: (Plummer (1988a))

Let G be a k-extendable graph on 2n vertices, $1 \le k \le n - 1$. Then for every edge e of G, G-e is $(k - 1)$-extendable. □

In Chapter 4, we establish a number of results concerning k-minimal graphs. We prove that a k-minimal graph $G \neq K_{2n}$ on 2n vertices has $\delta(G) \le n + k - 1$. In the case of a bipartite graph G $\neq K_{n,n}$, G is k-minimal, $1 \le k \le n - 3$, only if $\delta(G) < \frac{1}{2}(n + k)$. Further, we establish necessary and sufficient conditions for a k-extendable graph to be minimal. More specifically, we prove that for a k-extendable graph G on 2n vertices, $1 \le k \le n - 1$, the following are equivalent:

- G is minimal.
- for every edge e = uv of G there exists a matching M of size k in G-e such that $V(M) \cap \{u,v\} = \phi$ and for every perfect matching F in G containing M, $e \in F$.
- for every edge e = uv of G there exists a vertex set S of G-u-v such that: $|M(S)| \ge k$; $o(G-e-S) = |S| - 2k + 2$; and u and v belong to different odd components of G-e-S.

Recall that M(S) denotes a maximum matching in G[S]. The above results together with the characterization of $\mathcal{G}_{(2n,n-1)}$ and $\mathcal{G}_{(2n,n-2)}$ allow us to completely characterize k-minimal graphs on 2n vertices for k = n - 1 and n - 2. As these results are somewhat complicated, we do not state them here.

We conclude this section by mentioning a variation of extendability to odd order graphs which we consider in Section 2.5. Yu (1993) introduced the concept of k-extendable to odd order graphs. Let G be a graph on 2n + 1 vertices having a matching of size n. For $1 \leq k \leq n - 1$, G is said to be $k\frac{1}{2}$ - **extendable** if for every vertex v of G, G-v is k-extendable. Yu obtained a relationship between k-extendable graphs and $k\frac{1}{2}$ - extendable graphs as follow:

Theorem 1.2.16: (Yu (1993))

A graph G of order 2n + 1 is $k\frac{1}{2}$ - extendable, $1 \leq k \leq n - 1$, if and only if $G \vee K_1$ is (k + 1)-extendable. □

He also established a sufficient condition for an odd order graph to be $k\frac{1}{2}$ - extendable. The result is:

Theorem 1.2.17: (Yu (1993))

Let G be a graph of order 2n + 1 with $\delta(G) \geq n + k + 1$, $1 \leq k \leq n - 1$. Then G is $k\frac{1}{2}$ - extendable. □

By applying these two results together with our characterization of $\mathcal{G}_{(2n,n-1)}$ and $\mathcal{G}_{(2n,n-2)}$ we obtain a characterization of $k\frac{1}{2}$ - extendable graphs on 2n + 1 vertices for k = n - 1 and n - 2. In fact, we prove that the only $(n - 1)\frac{1}{2}$ - extendable graph on $2n + 1 \geq 3$ vertices is K_{2n+1} and for $2n + 1 \geq 9$ vertices, the class of $(n - 2)\frac{1}{2}$ - extendable graphs consists of:

- K_{2n+1}
- graphs with minimum degree 2n - 2 and independence number at most 2
- graphs with minimum degree 2n - 1.

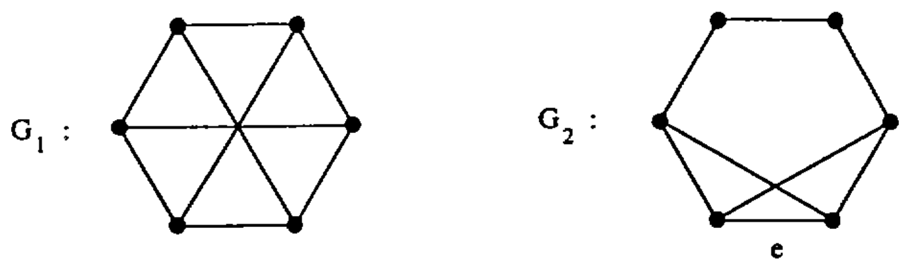
1.3 DIAMETER CRITICAL GRAPHS

In this section, we focus our attention on graphs whose diameter increases if we delete a certain number of edges. Recall that d(G), the diameter of a graph G, is the maximum distance in G; that is

$$d(G) = \max_{x,y \in V(G)} \{ d_G(x,y) \}.$$

Note that $d(G-E') \geq d(G)$ for all $E' \subseteq E(G)$. Let G be a graph of diameter k. G is said to be **strongly t-edge-critical** or simply **(k,t)-critical** if for every $E' \subseteq E(G)$, $G-E'$ has diameter greater than k if and only if $|E'| \geq t$. Denote the class of (k,t)-critical graphs by $\mathcal{G}(k,t)$.

(k,1)-critical graphs do exist. For example, the graph G_1 in Figure 1.3.1(a) is (2,1)-critical whilst the graph G_2 in Figure 1.3.1(b) having diameter 2 is not (2,1)-critical since $d(G_2-e) = 2$. In fact, every graph of diameter $k \geq 2$ having girth at least $k + 2$ is in $\mathcal{G}(k,1)$. Thus, the well-known Petersen graph and the complete bipartite graph are members of the class $\mathcal{G}(2,1)$.



(a) (b)

Figure 1.3.1

A number of authors (Glivlak (1975a, 1975b), Glivjak (1968), Glivjak and Plesnik (1969a, 1969b, 1971), Glivjak et.al. (1969a, 1969b) and Plesnik (1975)) have studied graphs in the class $\mathcal{G}(k,1)$. It is not our intention here to discuss all the results contained in these papers. However, we will state some of the important results.

Theorem 1.3.1: (Glivjak and Plesnik (1969a) and Greenwell and Johnson (1979))

Let G be a graph and let $k \geq 2$ be a natural number. Then there is a $(k,1)$-critical graph G' containing G as an induced subgraph. □

Theorem 1.3.2: (Plesnik (1975))

For any integer $k \geq 1$ and $r \geq 2$ there is a $(k,1)$-critical graph which is r-regular and r-connected. □

In view of Theorems 1.3.1 and 1.3.2 we can imagine that there are many graphs in $\mathcal{G}(k,1)$. Plesnik (1975) observed that all known examples of (k,1)-critical graphs on n vertices have at most $\frac{n^2}{4}$ edges. He proved that:

Theorem 1.3.3: (Plesnik (1975))

If G is a (k,1)-critical graph of order n for $k \geq 2$, then $\varepsilon(G) < \frac{3}{8}n(n - 1)$. □

Plesnik's observation is the same as the following conjecture posed by Simon and Murty (see Caccetta and Häggkvist (1979)) when $k = 2$.

Conjecture 1.3.1: If G is a (2,1)-critical graph of order n, then $\varepsilon(G) \leq \lfloor \frac{1}{4}n^2 \rfloor$ with equality holding if and only if $G \cong K_{\lfloor \frac{1}{2}n \rfloor, \lceil \frac{1}{2}n \rceil}$.

This conjecture has been studied by Caccetta and Häggkvist (1979) who proved the bound $0.27n^2$; Fan (1987) who improved the bound to $0.2532n^2$ and established the conjecture for $n \leq 24$ and $n = 26$. Recently, Füredi (1992) established the conjecture for extremely large n (a tower of 2's of height about 10^{14}).

Now we turn our attention to the class $\mathcal{G}(k,t)$ when $t \geq 2$. Kys (1981) conjectured that:

Conjecture 1.3.2: $\mathcal{G}(k,t) = \phi$ for $k \geq 2$ and $t \geq 2$.

In the same paper, he proved the following result:

Theorem 1.3.4: (Kys (1981))

(a) $\mathcal{G}(2,t) = \mathcal{G}(3,t) = \phi$ for $t \geq 2$.

(b) $\mathcal{G}(4,t) = \phi$ for $t \geq 3$.

(c) $\mathcal{G}(k,t) = \phi$ for $t \geq k \geq 2$. □

In Chapter 5 we will establish that:

- $\mathcal{G}(k,t) = \phi$ for $k \geq 2$, $t \geq 3$.
- $\mathcal{G}(4,t) = \mathcal{G}(5,t) = \phi$ for $t \geq 2$.

This leaves the only unresolved cases as: $k \geq 6$, $t = 2$.

CHAPTER 2

k-EXTENDABLE GRAPHS

In this chapter, we address the problem of characterizing k-extendable graphs. Throughout the chapter G always denotes a simple graph on 2n vertices having a perfect matching. Recall that G is k-extendable, if for every matching M of size k there exists a perfect matching in G containing all the edges of M.

In the previous chapter we stated a number of results relating k-extendability and minimum degree. In particular, we noted that if $\delta(G) \geq n + k$, then G is k-extendable, $1 \leq k \leq n - 1$. In Section 2.2, we establish that $\delta(G) \geq \frac{1}{2}(n + k)$ is a sufficient condition for a bipartite graph G to be k-extendable. Further, we prove a necessary and sufficient condition for a graph G with $\delta(G) \geq n + k - 1$, $n - k$ even and $\frac{n}{2} \leq k \leq n - 2$ to be k-extendable is that its independence number is at most $n - k$.

In Chapter 1, we observed that a necessary condition for k-extendability is that G is $(k + 1)$ - connected and thus $\delta(G) \geq k + 1$. In Section 2.3, we improve this result by establishing that a k-extendable graph G has $k + 1 \leq \delta(G) \leq n$ or $\delta(G) \geq 2k + 1$, $1 \leq k \leq n - 1$. Furthermore, we establish the existence of a k-extendable graph G on 2n vertices with $\delta(G) = j$ for each integer $j \in [k + 1, n] \cup [2k + 1, 2n - 1]$. Consequently, the class of k-extendable graphs becomes more complex with decreasing k.

In Section 2.4, we completely characterize k-extendable graphs for $k = n - 1$ and $n - 2$. We prove that $K_{n,n}$ and K_{2n}

are the only (n - 1)-extendable graphs. However, the structure of (n - 2)-extendable graphs is not so simple. For $2n \geq 10$, we prove that the class of (n - 2)-extendable graphs consists of :

- $K_{n,n}$
- K_{2n}
- all bipartite graphs with a perfect matching and minimum degree n - 1
- all graphs with minimum degree 2n - 3 and independence number at most 2
- all graphs with minimum degree 2n - 2.

Complete characterizations of (n - 2)-extendable graphs on 2n = 6 and 8 are also given. We conclude the chapter with a variation of the concept of extendability to odd order graphs.

2.1 PRELIMINARIES

In this section, we state two results which we make use of in our work. The first result, due to Dirac (see Bondy and Murty (1976) p.54), is a sufficient condition for a graph to contain a hamiltonian cycle.

Theorem 2.1.1: If G is a simple graph with $\nu(G) \geq 3$ and $\delta(G) \geq \frac{1}{2}\nu(G)$, then G is hamiltonian. □

The second result, due to Katerinis (1990), is a sufficient condition for bipartite graphs to have a k-factor.

Theorem 2.1.2: Let G be a bipartite graph with bipartitioning sets X and Y and k a positive integer. If

(1) $|X| = |Y|$,

(2) $\delta(G) \geq \left\lceil \frac{\lfloor X \rfloor}{2} \right\rceil \geq k$, and

(3) $|X| \geq 4k - 4\sqrt{k} + 1$ when $|X|$ is odd and $|X| \geq 4k - 2$ when $|X|$ is even,

then G has a k-factor. □

2.2. PROPERTIES OF k-EXTENDABLE GRAPHS

Our first result concerns the maximum number of independent edges in the induced subgraph of $N_G(u)$, where u is a vertex of minimum degree of a k-extendable graph G.

Theorem 2.2.1: Let G be a k-extendable graph on 2n vertices with $\delta(G) = k + t$, $1 \leq t \leq k \leq n - 1$. If $d_G(u) = \delta(G)$, then the subgraph $G[N_G(u)]$ has at most $t - 1$ independent edges.

Proof: Suppose that $d_G(u) = \delta(G)$ and $G[N_G(u)]$ has a maximum matching M of size $s \geq t$. Since G is k-extendable we must have $2s < |N(u)| = k + t \leq 2k$. So $s \leq k - 1$. Let v be an M-unsaturated vertex of $N_G(u)$. Then $M_1 = M \cup \{uv\}$ is a matching of size $s + 1 \leq k$ in G. So, by Theorem 1.2.2, M_1 can be extended to a perfect matching F in G. So $uv \in F$. Let

$$F_1 = \{xy \in F : x \in N_G(u)-\{v\},\ y \notin N_G(u)\},$$

$$A = V(F_1)\backslash N_G(u)\ ,\ \text{and}\ B = V(G)\backslash(N_G(u) \cup A \cup \{u\}).$$

Figure 2.2.1 depicts the situation with the edges of $M \cup F_1$ drawn in solid lines. Then

$$|A| = k + t - 2s - 1 \le k,$$

and hence

$$|B| = 2n - 2k - 2t + 2s \ge 2.$$

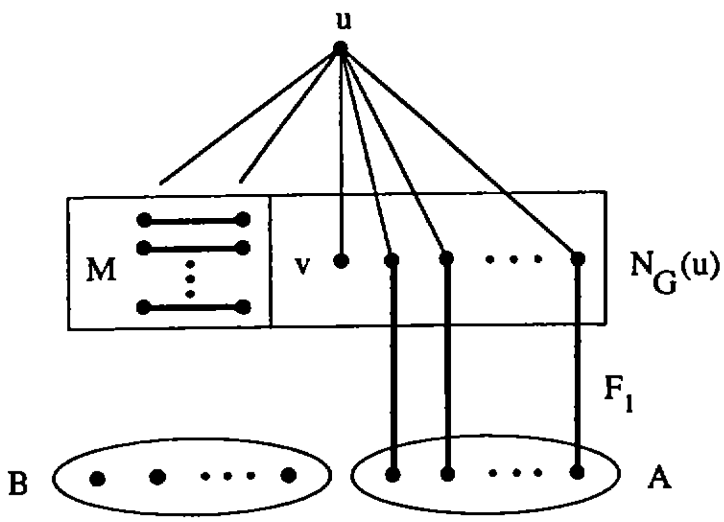


Figure 2.2.1

If v is adjacent to a vertex b of B, then $M_2 = M \cup F_1 \cup \{vb\}$ is a matching in G of size $s + (k + t - 2s - 1) + 1 = k + t - s \le k$. But then u is an isolated vertex of $G-V(M_2)$ contradicting the fact that G is k-extendable. Hence, $N_G(v) \cap B = \phi$. Now, since $d_G(v) \ge k + t$, the only possibility is for v to be adjacent to every vertex of $V(M) \cup A \cup \{u\}$ in which case $d_G(v) = k + t$.

If no vertex of B is adjacent to any vertex of $N_G(u)$, then G-A is disconnected and hence $\kappa(G) \le |A| \le k$. This contradicts Theorem 1.2.3. Let $xy \in E(G)$ with $x \in B$ and $y \in N_G(u)$. Since $y \ne v$, $y \in V(M) \cup V(F_1)$. Let $yz \in F$. Then z is in V(M) or A and so is adjacent to v. Consequently, the path x, y, z, v is an F-augmenting path in G with xy and zv not in F. But then $M_3 = (M \cup F_1 \cup \{xy, zv\}) \setminus \{yz\}$ is a matching of size $k + t - s \le k$ that saturates the vertices of $N_G(u)$, implying that G is not k-extendable. This contradiction completes the proof of the theorem. □

As a corollary we have :

Corollary 2.2.2: Let G be a k-extendable, (k + t)-regular graph on 2n vertices, $1 \le t \le k \le n - 1$. Then $G[N_G(u)]$ contains at most $t - 1$ independent edges for every vertex u of G. □

Recall that $\alpha(G)$ denotes the independence number of a graph G. Our next lemma provides an upper bound on $\alpha(G)$ for a k-extendable graph G on 2n vertices whose minimum degree is at least $2k + 1$ for $\frac{n}{2} \le k \le n - 1$.

Lemma 2.2.3: If G is a k-extendable graph on 2n vertices with $\frac{n}{2} \le k \le n - 1$ and $\delta(G) \ge 2k + 1$, then $\alpha(G) \le n - k$.

Proof: Suppose to the contrary that G contains an independent set $S = \{u_1, u_2, \ldots, u_{n-k+1}\}$ of order $n - k + 1$. Let F be a perfect matching containing the edges $u_i v_i$, $1 \le i \le n - k + 1$. Consider the graph

$$G' = G[V(G) \setminus \{u_1, u_2, \ldots, u_{n-k+1}, v_1, v_2, \ldots, v_{n-k+1}\}].$$

Clearly,

$$M = F \setminus \{u_1, u_2, \ldots, u_{n-k+1}, v_1, v_2, \ldots, v_{n-k+1}\}$$

is a perfect matching in G′ and $|M| = k - 1$.

If $e = v_i v_j \in E(G)$ for some $i \neq j$, $1 \leq i, j \leq n - k + 1$, then $M \cup \{e\}$ is a matching of size k in G. Further, as $G-V(M \cup \{e\})$ is a graph on $2n - 2k$ vertices containing the independent set S of order $n - k + 1$, this matching does not extend to a perfect matching in G, a contradiction. Consequently, $\{v_1, v_2, \ldots, v_{n-k+1}\}$ is an independent set.

Now consider the graph

$$G'' = G[V(M) \cup \{v_1, v_2\}].$$

Observe that $G-V(G'')$ is a graph on $2n - 2k$ vertices having S as an independent set of order $n - k + 1$ and thus cannot contain a perfect matching. Consequently, G'' cannot have a matching of size k. As $\nu(G'') = 2k$, Theorem 1.2.1 implies that $o(G''-S'') > |S''|$ for some $S'' \subset V(G'')$. In fact, as $|S''|$ and $o(G''-S'')$ have the same parity we have:

$$o(G''-S'') \geq |S''| + 2. \tag{2.2.1}$$

Now since G' contains a perfect matching, we have $o(G'-S') \leq |S'|$ for every $S' \subset V(G')$. If $v_1 \in S''$, then

$$\begin{aligned} o(G''-S'') &\leq o(G'-(S''\setminus\{v_1\}) + 1 \\ &\leq |S''| - 1 + 1 \\ &= |S''|, \end{aligned}$$

a contradiction. Hence, $v_1 \notin S''$ and similarly $v_2 \notin S''$. Thus, $S'' \subset V(G')$ and hence

$$o(G'-S'') \leq |S''|. \tag{2.2.2}$$

Now

$$\begin{aligned} o(G''-S'') &\leq o(G'-S'') + 2 \\ &\leq |S''| + 2 \end{aligned}$$

and so, by (2.2.1),

$$o(G''-S'') = |S''| + 2. \tag{2.2.3}$$

Further,

$$|S''| + 2 = o(G''-S'') \le o(G'-S'') + 2$$

and so by (2.2.2)

$$|S''| = o(G'-S''). \qquad (2.2.4)$$

Let w be a vertex of an odd component of $G'-S''$. Then, by (2.2.3), $v_i w \notin E(G)$, for $i = 1, 2$. Moreover, v_1 and v_2 are in different components of $G''-S''$. This together with the fact that $\{v_1, v_2, \ldots, v_{n-k+1}\}$ is an independent set gives

$$\begin{aligned} d_G(v_1) + d_G(v_2) &\le 2(n - k + 1) + 2|S''| + (2(k - 1) - |S''| - \\ &\qquad o(G'-S'')) \\ &= 2n + |S''| - o(G'-S'') \\ &= 2n. \qquad \text{(by (2.2.4))} \end{aligned}$$

But since $\delta(G) \ge 2k + 1$, we have $4k + 2 \le 2n$ and so $k < \frac{n}{2}$, a contradiction to the hypothesis of the lemma. This completes the proof. □

Remark 2.2.1: For $n \le 2k$, the graph $K_{2k} \vee (n - k)K_2$ is a k-extendable graph with minimum degree $2k + 1$ and independence number $n - k$. Thus, the upper bound in Lemma 2.2.3 is best possible. Further, the graph $K_{2k+1,2k+1}$ is k-extendable with minimum degree $n = 2k + 1$ containing an independent set of order $n = 2k + 1 > k + 1 = n - k$ for all $k \ge 1$. Thus, the lower bound on k is also best possible.

The following lemma establishes a sufficient condition for a graph G with $\delta(G) \ge n + k - 1$, $1 \le k \le n - 2$, and $n - k$ even to be k-extendable.

Lemma 2.2.4: Let G be a graph on 2n vertices with $\delta(G) \geq n + k - 1$, $1 \leq k \leq n - 2$, and $n - k$ even. If $\alpha(G) \leq n - k$, then G is k-extendable.

Proof: By Theorem 2.1.1, G contains a perfect matching. Suppose to the contrary that M is a matching of size k in G that does not extend to a perfect matching. Thus, the graph $G' = G-V(M)$ has no perfect matching. Hence, by Theorem 1.2.1, $o(G'-S') > |S'|$ for some $S' \subset V(G')$. Further, since $|S'|$ and $o(G'-S')$ have the same parity, $o(G'-S') \geq |S'| + 2$. Now

$$|S'| + o(G'-S') \leq \nu(G') = 2n - 2k,$$

and thus

$$|S'| \leq n - k - 1.$$

If $|S'| = n - k - 1$, then $o(G'-S') \geq n - k + 1$. Choosing one vertex from each component of $G'-S'$ results in an independent set of order at least $n - k + 1$, contradicting the hypothesis of the lemma. Hence

$$|S'| \leq n - k - 2. \tag{2.2.5}$$

Let H be a minimum order odd component of $G'-S'$. Noting that $\delta(G') \geq n - k - 1$ we have for $u \in V(H)$

$$n - k - 1 \leq d_{G'}(u) \leq \nu(H) - 1 + |S'|,$$

and hence

$$\nu(H) \geq n - k - |S'|.$$

Further, by the choice of H,

$$|S'| + \nu(H)\,(o(G'-S')) \leq |V(G')| = 2(n - k).$$

Consequently,

$$|S'| + (n - k - |S'|)(|S'| + 2) \le 2(n - k).$$

Now if $S' \ne \phi$, then it follows from the above inequality that $|S'| \ge n - k - 1$. But this contradicts (2.2.5). Therefore, $S' = \phi$. Now since $\delta(G') \ge n - k - 1$, G' consists of two odd components each a K_{n-k}. This contradicts the fact that $n - k$ is even, completing the proof. □

Remark 2.2.2: The bound on the minimum degree in Lemma 2.2.4 is best possible since there exists a graph with minimum degree $n + k - 2$ that has independence number at most $n - k$ which is not k-extendable. For $1 \le k \le n - 2$, such a graph is $G = \overline{K}_{n-k-1} \vee K_{n+k-2} \vee K_3$ which is drawn in Figure 2.2.2. Throughout the thesis we adopt the convention that a "double line" in our diagram denotes the join between the corresponding graphs.

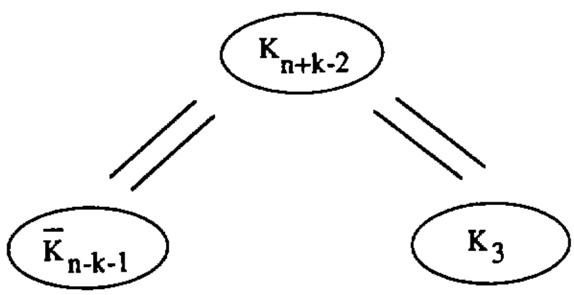


Figure 2.2.2

Clearly, G is not k-extendable, since any set of k independent edges of K_{n+k-2} does not extend to a perfect matching in G. Further, the condition " $n - k$ even " cannot be dropped since the graph $2K_{n-k} \vee K_{2k}$ has minimum degree $n + k - 1$, independence number 2 and is not k-extendable when $n - k$ is odd.

Lemmas 2.2.3 and 2.2.4 yield the following theorem :

Theorem 2.2.5: Let G be a graph on $2n$ vertices with $\delta(G) \geq n + k - 1$ and k any positive integer such that $\frac{n}{2} \leq k \leq n - 2$ and $n - k$ even. Then G is k-extendable if and only if $\alpha(G) \leq n - k$. □

Theorem 1.2.4 provides a sufficient condition on the minimum degree for a general graph to be k-extendable. Our next result establishes a minimum degree condition for a bipartite graph to be k-extendable.

Theorem 2.2.6: If G is a bipartite graph with bipartitioning sets X and Y where $|X| = |Y| = n$ and $\delta(G) \geq \frac{1}{2}(n + k)$, $1 \leq k \leq n - 1$, then G is k-extendable.

Proof: By Theorem 2.1.2, G has a perfect matching. Let M be a matching of size k in G. Consider the graph $G' = G-V(M)$ with bipartitioning sets $X' = X\backslash V(M)$ and $Y' = Y\backslash V(M)$. Clearly, $|X'| = |Y'| = n - k$. If $k = n - 1$, then $\delta(G) = n$ and thus $G \cong K_{n,n}$ which is $(n - 1)$-extendable as required. So suppose that $1 \leq k \leq n - 2$. Since $\delta(G') \geq \frac{1}{2}(n + k) - k = \frac{1}{2}(n - k) \geq 2$, Theorem 2.1.2 implies that G' has a perfect matching, as required. This completes the proof. □

Remark 2.2.3: The following construction shows that the bound on the minimum degree given in Theorem 2.2.6 is best possible. Consider the graph G displayed in Figure 2.2.3 below :

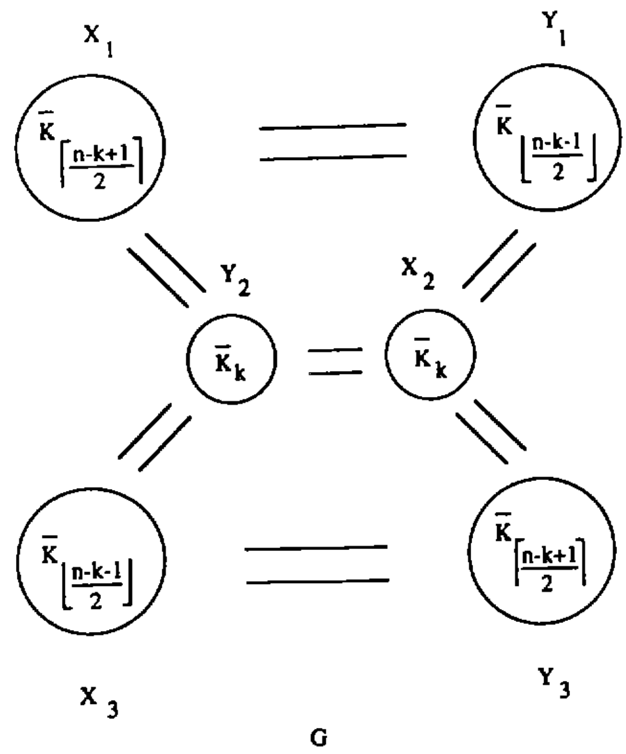


Figure 2.2.3

Clearly, G is a bipartite graph with bipartitioning sets $X = X_1 \cup X_2 \cup X_3$ and $Y = Y_1 \cup Y_2 \cup Y_3$. Further, $|X| = |Y| = n$ and $\delta(G) = \lfloor \frac{n + k - 1}{2} \rfloor$. Now since $G-X_2-Y_2$ is a bipartite graph without a perfect matching and the subgraph $G[X_2 \cup Y_2]$ is a $K_{k,k}$ and thus contains a matching of size k, G is not k-extendable. As our definition of k-extendability is for graphs with a "perfect matching" we need to show that G has such a matching. Note that $1 \le \lceil \frac{n - k + 1}{2} \rceil - \lfloor \frac{n - k - 1}{2} \rfloor \le 2$ and thus G has a perfect matching for every $k \ge 2$. In fact, only for $k = 1$ and $n - k$ even does G not have a perfect matching. In this case G has a maximum matching of size $n - 1$ with 2 unsaturated vertices, say x and y with $x \in X_1$ and $y \in Y_3$. But then the graph $G' = G+xy$ has a perfect matching, is bipartite and has $\delta(G') = \lfloor \frac{n + k - 1}{2} \rfloor$.

Further, G′ is not k-extendable.

Remark 2.2.4: We established in Theorem 2.2.6 that for sufficiency the " regularity " condition in Theorem 1.2.10 can be replaced by " minimum degree ". However, this is not the case for necessity as illustrated by the graph displayed in Figure 2.2.4 below. Note that this non-regular graph is just the bipartite graph H_2 defined in the proof of Lemma 2.3.3 (proved in the next section) which is 1-extendable.

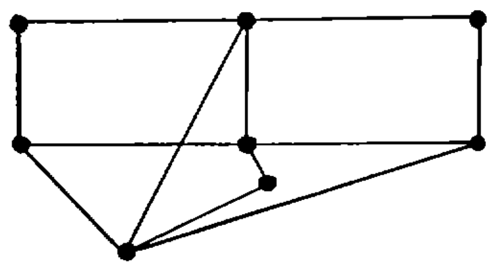


Figure 2.2.4

2.3 MINIMUM DEGREE OF k-EXTENDABLE GRAPHS

In this section, we establish a necessary condition, in terms of the minimum degree, for k-extendability. This condition, as we shall see in the next section, plays a crucial role in the characterization of k-extendable graphs for $k = n - 1$ and $n - 2$.

Theorem 2.3.1: If G is a k-extendable graph on 2n vertices, $1 \le k \le n - 1$, then $k + 1 \le \delta(G) \le n$ or $\delta(G) \ge 2k + 1$.

Proof: Suppose the assertion is false. Then, by Theorem 1.2.3, there exists a k-extendable graph G, $1 \le k \le n - 1$, with $n + 1 \le$

$\delta(G) \leq 2k$. Thus $\frac{1}{2}(n + 1) \leq k \leq n - 1$ and $G \neq K_{2n}$. Let $d_G(u) = r = \delta(G)$. By Theorem 2.2.1, the subgraph $H = G[N_G(u)]$ contains a maximum matching M with

$$|M| \leq r - k - 1 \leq k - 1.$$

Further, $|M| \geq 1$ as otherwise vertices of H have degree at most $2n - r < r$ in G.

Let F be a perfect matching in G containing M. Consider $M' = \{ vw \in F \mid v, w \in \overline{N}_G(u) \}$. Clearly,

$$\begin{aligned} |M'| &= \frac{1}{2} \{(2n - r - 1) - (r - 2|M| - 1)\} \\ &= n - r + |M| \\ &\leq |M| - 1 \\ &\leq k - 2. \end{aligned}$$

If $G[\overline{N}_G(u)\backslash V(M')]$ contains an edge e, then $M' \cup \{e\}$ is a matching of size at most $k - 1$ which does not extend to a perfect matching in G, since H-V(M) is an independent set of order $r - 2|M|$ and

$$\begin{aligned} |\overline{N}_G(u)\backslash V(M' \cup \{e\})| &= 2n - r - 1 - 2|M'| - 2 \\ &= 2n - r - 2(n - r + |M|) - 3 \\ &= r - 2|M| - 3. \end{aligned}$$

Hence, $G[\overline{N}_G(u)\backslash V(M')]$ has no edges.

Now, since $|M| \leq r - k - 1$,

$$|\overline{N}_G(u)\backslash V(M')| = r - 2|M| - 1 \geq 2k - r + 1 \geq 1.$$

Thus, there exists a vertex $x \in \overline{N}_G(u)\backslash V(M')$. If $xy \notin E(G)$ for all $y \in V(M)$, then

$$\begin{aligned} d_G(x) &\leq 2|M'| + r - 2|M| \\ &\leq 2(|M| - 1) + r - 2|M| \\ &= r - 2 \end{aligned}$$

$< r,$

a contradiction. So $xy \in E(G)$ for some $y \in V(M)$. Let $yy' \in M$ and consider $M'' = M' \cup \{xy, y'u\}$. M'' is a matching of size at most k that does not extend to a perfect matching in G, since

$$|N_G(u)\backslash\{y, y'\}| - |\bar{N}_G(u)\backslash(V(M') \cup \{x\})|$$
$$= (r - 2) - (r - 2|M| - 2)$$
$$= 2|M|$$

and $H-y-y'$ has at most $|M| - 1$ independent edges. This contradicts the k-extendability of G and completes the proof of our theorem. □

We now consider the realizability problem associated with the above result. Let $\mathcal{G}(2n, k, j)$ denote the class of k-extendable graphs on 2n vertices with minimum degree j. We establish that $\mathcal{G}(2n, k, j) \neq \phi$ for every integer $j \in [k + 1, n] \cup [2k + 1, 2n - 1]$. We begin with the case $2k + 1 \le j \le 2n - 1$.

Lemma 2.3.2: For every integer j, $2k + 1 \le j \le 2n - 1$, $\mathcal{G}(2n, k, j) \neq \phi$.

Proof: We distinguish two cases according to the parity of j.

Case 1: j odd.

Let $j = 2t + 1$, $k \le t \le n - 1$, and consider the graph $H_j = (n - t)K_2 \vee K_{2t}$. Clearly, $\nu(H_j) = 2n$, $\delta(H_j) = j$ and H_j has a perfect matching. We now prove that H_j is k-extendable. Let M be a matching of size k in H_j. It is convenient to write H'_j =

$(n - t)K_2$ and $H''_j = K_{2t}$ so that $H_j = H'_j \vee H''_j$. Further, let $M = X \cup Y \cup Z$ where

$$X = \{\ ab \in M \mid a,\ b \in V(H'_j)\},$$

$$Y = \{\ ab \in M \mid a \in V(H'_j) \text{ and } b \in V(H''_j)\}, \text{ and}$$

$$Z = \{\ ab \in M \mid a,\ b \in V(H''_j)\}.$$

We denote the sets of M - unsaturated vertices of H'_j and H''_j by A and B respectively. Let A_i, $i = 0, 1$, denote the set of vertices of A having degree i in $H'_j - V(M)$. Figure 2.3.1 illustrates our notation.

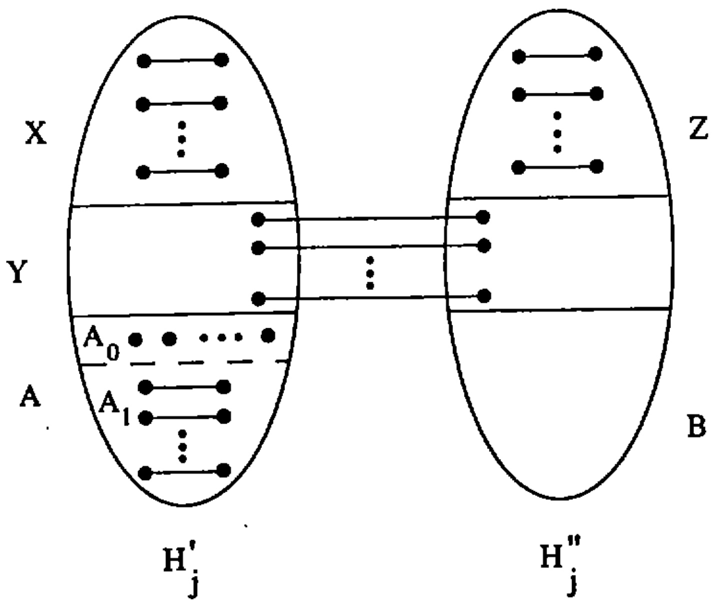


Figure 2.3.1

Clearly, $|A_0|$ and $|B|$ have the same parity. Further, $|A_0| \leq |Y|$ and

$$|B| = 2t - 2|Z| - |Y|$$

$$= 2t - 2(|Z| + |Y|) + |Y|$$

$$= 2t - 2(k - |X|) + |Y|$$

$$\text{(since } k = |X| + |Y| + |Z|)$$

$$= 2(t - k) + 2|X| + |Y|$$

$$\geq 2|X| + |Y| \qquad \text{(since } t \geq k)$$

$$\geq |A_0|.$$

The matching M can be extended to a perfect matching in G as follows. The edges joining vertices of A_1 form a matching M'. Let M'' be any set of $|A_0|$ independent edges joining vertices of A_0 and B. Note that, since $|A_0|$ and $|B|$ have the same parity, the graph $H_j - V(M \cup M' \cup M'')$ is a complete graph of even order and hence has a perfect matching M'''. Now $M \cup M' \cup M'' \cup M'''$ is the required perfect matching.

Case 2: j even.

Let $j = 2t$, $k + 1 \leq t \leq n - 1$. Consider the graph H_j obtained from the graph $(n - t)K_2 \vee K_{2t-1}$ by adding a vertex u and joining u to every vertex of K_{2t-1} and exactly one vertex of $(n - t)K_2$. Thus, $H_j = ((n - t - 1)K_2 \cup P_3) \vee K_{2t-1}$ where P_3 is the path on 3 vertices. Clearly, $\nu(H_j) = 2n$, $\delta(H_j) = j$ and H_j has a perfect matching. For convenience, we write $H'_j = (n - t - 1)K_2 \cup P_3$ and $H''_j = K_{2t-1}$ so that $H_j = H'_j \vee H''_j$. We now prove that H_j is k-extendable.

Let M be a matching of size k in H_j. Define X, Y, Z, A and B as in Case 1. Further, let A_i, $i = 0, 1, 2$, denote the set of vertices of A having degree i in $H'_j - V(M)$. Figure 2.3.2 depicts the situation.

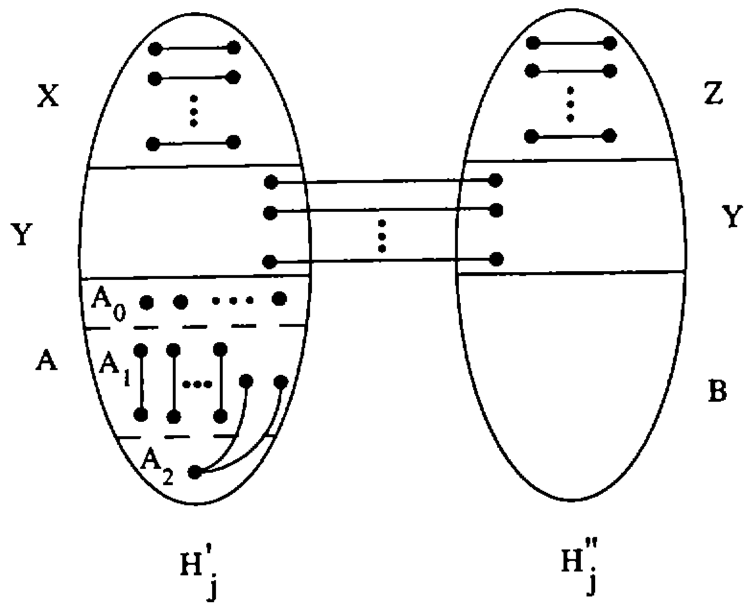


Figure 2.3.2

Consider B. We have

$$|B| = (2t - 1) - 2|Z| - |Y|$$

$$= (2t - 1) - 2(|Z| + |Y|) + |Y|$$

$$= (2t - 1) - 2(k - |X|) + |Y|$$

$$\text{(since } k = |X| + |Y| + |Z|)$$

$$= 2(t - k + |X|) + |Y| - 1.$$

Thus, $|B|$ and $|Y|$ have different parity. Since $|A_1|$ is even and

$$|Y| = 2(n - t - |X|) - |A_1| - (|A_0| + |A_2|) + 1,$$

$|A_0| + |A_2|$ and $|Y|$ have different parity. Hence, $|B|$ and $|A_0| + |A_2|$ have the same parity. Now it is not too difficult to verify that $|A_0| \le |Y| + 1$ and if $A_2 \ne \phi$, then $|A_0| \le |Y|$. Since $t \ge k + 1$ and $|X| \ge 0$,

$$|B| = 2(t - k + |X|) + |Y| - 1$$

$\geq |Y| + 1.$

Now the matching M can be extended to a perfect matching in H_j as follows.

If $A_2 = \phi$, then we take M′, M″ and M‴ as defined in Case 1. So we need to consider only the case $A_2 \neq \phi$. Let $P_3 = u,v,w$. Clearly, $A_2 = \{v\}$ and $u, w \in A_1$. The edges joining vertices of $A_1 \setminus \{u,w\}$ together with the edge uv form a matching M′ that saturates every vertex of $(A_1 \cup A_2)\setminus\{w\}$. Let M″ be any set of $|A_0| + 1$ independent edges joining vertices of $A_0 \cup \{w\}$ and B. The matching M″ is possible since $|A_0| \leq |Y|$ when $A_2 \neq \phi$ and $|B| \geq |Y| + 1 \geq |A_0| + 1$. Now the graph $H_j - V(M \cup M' \cup M'')$ is a complete graph of even order (as $|B|$ and $|A_0| + |A_2| = |A_0| + 1$ have the same parity) and hence has a perfect matching M‴. The matching $M \cup M' \cup M'' \cup M'''$ is the required matching. This completes the proof of our lemma. □

We now consider the range $k + 1 \leq j \leq n$.

Lemma 2.3.3: For every integer j, $k + 1 \leq j \leq n$, $\mathcal{G}(2n, k, j) \neq \phi$.

Proof: For $k + 1 \leq j \leq n$, define $G_0 = \overline{K}_{n-j} \vee \overline{K}_{j-1} \vee \overline{K}_j \vee \overline{K}_{n-j}$. Form the graph $\hat{G}$ from G_0 by adding a perfect matching between the vertices of the two $\overline{K}_{n-j}$'s. Observe that $\hat{G}$ has 2n - 1 vertices and minimum degree j. We form the graph H_j from $\hat{G}$ by adding the vertex u and joining u to every vertex of $\overline{K}_j$. Figure 2.3.3 illustrates the construction; for later reference we identify subgraphs G_1, G_2, G_3, and G_4 as indicated.

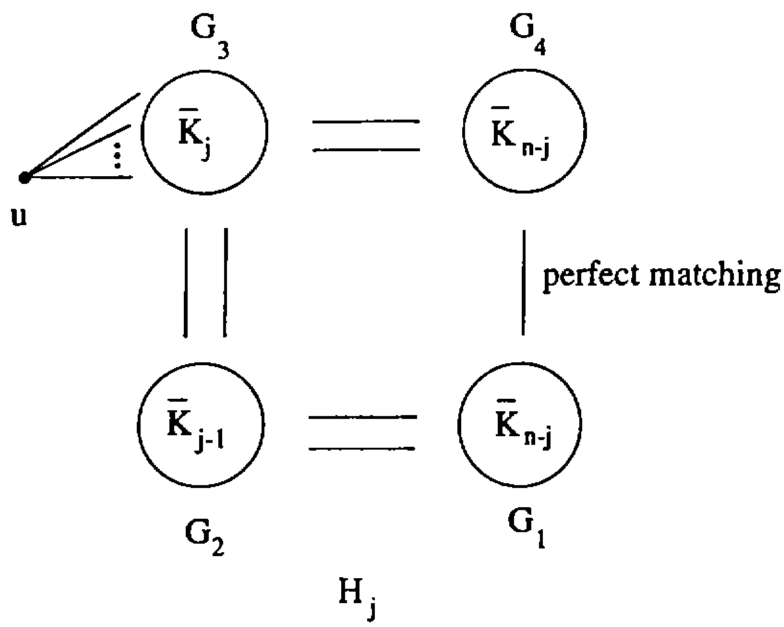


Figure 2.3.3

We will establish that $H_j \in \mathcal{G}(2n, k, j)$. As $\delta(H_j) = j$ we need only show that H_j is k-extendable. Observe that if H_j-u-v is k-extendable for every $v \in V(G_3)$, then H_j is also k-extendable(Theorem 1.2.2). Hence, it is sufficient to show that H_j-u-v is k-extendable for every $v \in V(G_3)$.

Let $v \in V(G_3)$ and consider the graph $H'_j = H_j-u-v$. Let M' be a matching in H'_j of size k. Further, let

$$M' = M_{12} \cup M_{23} \cup M_{34} \cup M_{41},$$

where M_{rs} denotes the edges of M' joining vertices of G_r to vertices of G_s, $1 \le r \ne s \le 4$.

Consider the subgraph $H'_j[(G_1 \cup G_4)-V(M')]$. Clearly, vertices of $G_1 \cup G_4$ have degree 0 or 1 in $H'_j[(G_1 \cup G_4)-V(M')]$. Let A_i and B_i, $i = 0, 1$, denote the sets of vertices of $G_1-V(M')$ and

$|A_1| = |B_1|$. Since, in H'_j, each vertex of A_0 (B_0) is joined to exactly one vertex of G_4 (G_1) incident to an edge of M_{34} (M_{12}), we have $|A_0| \le |M_{34}|$ and $|B_0| \le |M_{12}|$. Further,

$$|A_0| = (n - j) - |M_{12}| - |M_{41}| - |A_1|$$

and

$$|B_0| = (n - j) - |M_{34}| - |M_{41}| - |B_1|.$$

Thus

$$|A_0| - |B_0| = |M_{34}| - |M_{12}|. \qquad (2.3.1)$$

Since $|M_{12}| + |M_{23}| + |M_{34}| + |M_{41}| = k$ and $j \ge k + 1$,

$$\begin{aligned}\nu(G_3-(V(M') \cup \{v\})) &= (j - 1) - |M_{23}| - |M_{34}| \\ &= (j - 1) - k + |M_{12}| + |M_{41}| \\ &\ge |M_{12}| + |M_{41}| \\ &\ge |B_0| \qquad \text{(since } |B_0| \le |M_{12}|).\end{aligned}$$

Similarly,

$$\begin{aligned}\nu(G_2-V(M')) &= (j - 1) - |M_{12}| - |M_{23}| \\ &= (j - 1) - k + |M_{34}| + |M_{41}| \\ &\ge |M_{34}| + |M_{41}| \\ &\ge |A_0| \qquad \text{(since } |A_0| \le |M_{34}|).\end{aligned}$$

Now the matching M′ can be extended to a perfect matching in G as follows. Let M'_{41} be the set of edges joining vertices of A_1 and B_1. Further, let M'_{12} be the set of $|A_0|$ independent edges joining vertices of A_0 and $G_2-V(M')$ and M'_{34} the set of $|B_0|$ independent edges joining vertices of B_0 and $G_3-(V(M') \cup \{v\})$.

independent edges joining vertices of B_0 and $G_3-(V(M') \cup \{v\})$. M'_{12} and M'_{34} exist since $\nu(G_2-V(M')) \geq |A_0|$ and $\nu(G_3-(V(M') \cup \{v\})) \geq |B_0|$.

Now the graph $H'_j-V(M' \cup M'_{12} \cup M'_{34} \cup M'_{41})$ is a complete bipartite graph with bipartitioning sets $V(G'_2) = V(G_2) \setminus V(M' \cup M'_{12})$ and $V(G'_3) = V(G_3) \setminus (V(M' \cup M'_{34}) \cup \{v\})$. Consider $V(G'_2)$ and $V(G'_3)$. We have

$$\begin{aligned} |V(G'_2)| &= (j - 1) - |M_{12}| - |M_{23}| - |A_0| \\ &= (j - 1) - |M_{12}| - |M_{23}| - (|M_{34}| - |M_{12}| + |B_0|), \quad \text{by (2.3.1)} \\ &= (j - 1) - |M_{23}| - |M_{34}| - |B_0| \\ &= |V(G'_3)|. \end{aligned}$$

Thus, $H'_j-V(M' \cup M'_{12} \cup M'_{34} \cup M'_{41})$ has a perfect matching M'_{23}. Hence, $M' \cup M'_{12} \cup M'_{23} \cup M'_{34} \cup M'_{41}$ is the required matching. This completes the proof of our lemma. □

Lemmas 2.3.2 and 2.3.3 yield the following theorem:

Theorem 2.3.4: For every integer $j \in [k + 1, n] \cup [2k + 1, 2n - 1]$, $\mathcal{G}(2n, k, j) \neq \phi$. □

Remark 2.3.1: For $n \geq 2k$, every integer in the interval $[k + 1, 2n - 1]$ is realizable as a minimum degree of a k-extendable graph on 2n vertices, $1 \leq k \leq n - 1$. For $n \leq 2k - 1$, this is not the case as no integer in the interval $[n + 1, 2k]$ can be the minimum degree of a k-extendable graph on 2n vertices, $1 \leq k \leq n - 1$, by Theorem 2.3.1.

2.4 CHARACTERIZATION OF k-EXTENDABLE GRAPHS ON 2n VERTICES FOR $k = n - 1$ AND $n - 2$

Theorem 2.3.1 provides the possible values of the minimum degree of a k-extendable graph G on 2n vertices for $1 \le k \le n - 1$. For the case $k = n - 1$, we have G is $(n - 1)$-extendable only if $\delta(G) = n$ or $2n - 1$. We observed, in Chapter 1, that $(n - 1)$-extendable graphs on 2n vertices are randomly matchable and $\mathcal{G}_{(2n,n-1)} = \{K_{n,n}, K_{2n}\}$. Sumner (1979) and Ananchuen and Caccetta (1994a) independently and using different methods proved this result. Sumner's method relied on establishing the regularity of randomly matchable graphs whilst our proof takes advantage of Theorem 2.3.1. Our proof is presented below.

Theorem 2.4.1: G is an $(n - 1)$-extendable graph on $2n \ge 4$ vertices if and only if $G \cong K_{n,n}$ or K_{2n}.

Proof: We need only prove the necessity condition as $K_{n,n}$ and K_{2n} are clearly $(n - 1)$ - extendable. So suppose that G is $(n - 1)$-extendable and $G \ncong K_{n,n}$ and K_{2n}. Then $\delta(G) = n$.

Let $d_G(u) = n$. By Theorem 2.2.1, $N_G(u)$ is independent. Consequently, every vertex of $N_G(u)$ is adjacent to every vertex of $\bar{N}_G(u)$. Consider any vertex v of $N_G(u)$, $d_G(v) = n$ and so $N_G(v)$ is independent. Hence, $\bar{N}_G(u)$ is an independent set and therefore $G \cong K_{n,n}$, a contradiction. This completes the proof of the theorem. □

The simplicity of the structure of $(n - 1)$-extendable graphs does not carry over to the case of $(n - 2)$-extendable graphs. Theorem 2.3.1 establishes that the minimum degree of an $(n-2)$-extendable graph on $2n$ vertices is: $n-1$; n; $2n-3$; $2n-2$; or $2n-1$. Further, by Theorem 2.3.4, each of these values is realizable. We now proceed to characterize $(n-2)$-extendable graphs on $2n$ vertices according to minimum degree.

Lemma 2.4.2: Let G be a graph on $2n \geq 8$ vertices with a perfect matching and $\delta(G) = n - 1$. Then G is $(n-2)$-extendable if and only if G is bipartite.

Proof: The sufficiency follows from Theorem 2.2.6. We need only prove the necessity. So let G be an $(n-2)$-extendable graph with $\delta(G) = n - 1$.

Let u be a vertex of degree $n - 1$. By Theorem 2.2.1, $N_G(u)$ is an independent set of vertices. The subgraph $H = G[\overline{N}_G(u)]$ has at least one edge, since otherwise $\overline{N}_G(u) \cup \{u\}$ is an independent set of $n + 1$ vertices implying that G has no perfect matching. If xy and $x'y'$ are independent edges of H, then the graph

$$G' = G-\{x,y,x',y'\}$$

has $2n - 4$ vertices and contains $N_G(u)$ as an independent set of $n - 1$ vertices. Thus, G' cannot have a perfect matching, contradicting the fact that G is 2-extendable. Hence, H contains only one independent edge, xy say.

Now since G is $(n-1)$-connected (Theorem 1.2.3) and $|\overline{N}_G(u)| = n \geq 4$, at least one of x or y is adjacent to a vertex of $N_G(u)$. Suppose that $xz \in E(G)$ with $z \in N_G(u)$. If $yw \in E(G)$, $w \in N_G(u)$, $w \neq z$, then the graph $G-\{x,y,z,w\}$ contains two disjoint independent sets $\{u\} \cup (\bar{N}_G(u)\backslash\{x,y\})$ and $N_G(u)\backslash\{z,w\}$ of order $n - 1$ and $n - 3$, respectively, and so cannot have a perfect matching. This contradicts the fact that G is 2-extendable, Hence, $|N_G(y) \cap N_G(u)| \le 1$.

Suppose that $N_G(y) \cap N_G(u) \ne \phi$. Then $yz \in E(G)$. The above argument implies that $N_G(x) \cap N_G(u) = \{z\}$. Now, by Theorem 2.2.1, each of x, y and z has degree at least n in G. Consequently, x and y are joined to every other vertex of $\bar{N}_G(u)$. But then since $n \ge 4$, H contains at least two independent edges. This contradiction establishes that $N_G(y) \cap N_G(u) = \phi$. Hence, $d_G(y) = n - 1$ and the set of vertices $N_G(y) = \bar{N}_G(u)\backslash\{y\}$ must be (by Theorem 2.2.1) an independent set. Consequently, $N_G(u) \cup \{y\}$ and $\{u\} \cup (\bar{N}_G(u)\backslash\{y\})$ are independent sets of n vertices of G, proving that G is bipartite. □

Remark 2.4.1: Lemma 2.4.2 is best possible in the sense that there exists an $(n - 2)$-extendable graph on $2n = 6$ vertices with minimum degree $n - 1$ that is not bipartite. Figure 2.4.1 displays such a graph.

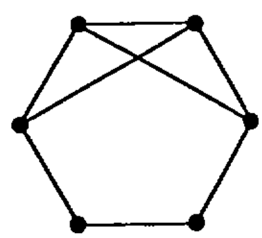


Figure 2.4.1

Lemma 2.4.3: Let G be a graph on $2n \ge 10$ vertices with $\delta(G) = n$. Then G is $(n - 2)$-extendable if and only if $G \cong K_{n,n}$.

Proof: The sufficiency is obvious as $K_{n,n}$ is k-extendable for $1 \le k \le n - 1$. So we need to prove only the necessity. We do this by following a similar strategy to that used in the proof of the previous lemma.

Let G be an (n - 2)-extendable graph with $\delta(G) = n$. If G contains an independent set X of n vertices, then $V(G)\setminus X$ is also an independent set, since otherwise G cannot be 3-extendable. But then, since $\delta(G) = n$, $G \cong K_{n,n}$. Hence, we may suppose that G contains at most n - 1 independent vertices.

Let $d_G(u) = n$. Theorem 2.2.1 together with the above assumption implies that the subgraph $G[N_G(u)]$ contains only one independent edge, vw say. Now for the edge vw to be extendable to a perfect matching in G, the subgraph $H = G[\overline{N}_G(u)]$ must have edges. If H contains two independent edges xy and x'y', then the graph $G' = G-\{x,y,x',y',v,w\}$ has 2n - 6 vertices and contains an independent set of order n - 2 and hence cannot contain a perfect matching. This contradicts the fact that G is 3-extendable.Hence H contains only one independent edge, say xy. Consequently, either $d_H(x) = d_H(y) = 2$ (and x and y lie on a triangle) or at least one of x or y, say x, has degree 1 in H. So x must be joined to at least $n - 2 \ge 3$ vertices of $N_G(u)$. Hence, $xz \in E(G)$, for some $z \in N_G(u)\setminus\{v,w\}$. Further, since $n \ge 5$, y must be joined to a vertex $z' \in V(G)\setminus\{x,v,w,z\}$.

If $z' \notin \overline{N}_G(u)$, then $G-\{v,w,x,y,z,z'\}$ is a bipartite graph with bipartitioning sets $N_G(u)\setminus\{v,w,z,z'\}$ and $\{u\} \cup (\overline{N}_G(u)\setminus\{x,y\})$ of order n - 4 and n - 2, respectively. Hence, the matching {vw, xz, yz'} does not extend to a perfect matching, a contradiction. Therefore, $z' \in \overline{N}_G(u)$ and hence, since $d_G(y) \ge n$, $t = |N_G(y) \cap$

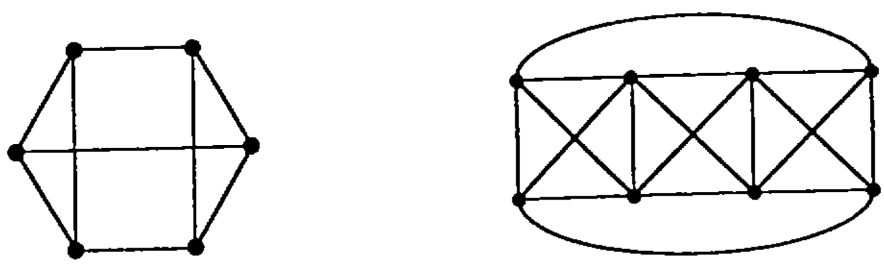
$N_G(u)|$ is 2 or 3. We claim that $\bar{N}_G(u)\setminus\{y\} \subseteq N_G(y)$. This is clearly so when $t = 2$. If $t = 3$, then vw and xz are independent edges of $G[N_G(y)]$ and hence, by Theorem 2.2.1, $d_G(y) \geq n + 1$ and so $\bar{N}_G(u)\setminus\{y\} \subseteq N_G(y)$. Now, since $n \geq 5$, $\{u\} \cup (\bar{N}_G(u)\setminus\{y\})$ is an independent set of $n - 1$ vertices of $G-\{v,w,y,z\}$. Hence, since G is 1-extendable $yz \notin E(G)$. Consequently,

$$N_G(y) = \{v,w\} \cup (\bar{N}_G(u)\setminus\{y\}).$$

Further, $d_H(x) = 1$ and so x must be joined to at least $n - 1$ vertices of $N_G(u)$. Hence, xv or $xw \in E(G)$.

Without any loss of generality suppose that $xv \in E(G)$. Then $\{uw, xv\}$ does not extend to a perfect matching in G, since the subgraph $G-\{u,v,w,x\}$ has an independent set $\{y\} \cup (N_G(u)\setminus\{v,w\})$ of order $n - 1$ and so cannot have a perfect matching. This proves that $N_G(u)$ is an independent set of size $> n - 1$ and completes the proof of the lemma. □

Remark 2.4.2: Lemma 2.4.3 is best possible in the sense that there are $(n - 2)$-extendable graphs on $6 \leq 2n \leq 8$ vertices with $\delta(G) = n$ which are not $K_{n,n}$. Two of these graphs are shown in Figure 2.4.2.



(a) (b)

Figure 2.4.2

Our next lemma concerns the case when $\delta(G) = 2n - 3$. By substituting $k = n - 2$ in Theorem 2.2.5 we have:

Lemma 2.4.4: Let G be a graph on $2n \geq 8$ vertices with $\delta(G) = 2n - 3$. Then G is $(n - 2)$-extendable if and only if $\alpha(G) \leq 2$. □

In view of theorems 1.2.4 and 2.3.1 and lemmas 2.4.2, 2.4.3 and 2.4.4 we can now state a characterization of (n-2)-extendable graphs.

Theorem 2.4.5: Let G be a graph on $2n \geq 10$ vertices. Then G is $(n - 2)$-extendable if and only if G:

(i) is $K_{n,n}$ or K_{2n}, or

(ii) is a bipartite graph with a perfect matching and minimum degree $n - 1$, or

(iii) has minimum degree $2n - 3$ and $\alpha(G) \leq 2$, or

(iv) has minimum degree $2n - 2$. □

Remark 2.4.3: There exist $(n - 2)$-extendable graphs for each type specified in Theorem 2.4.5. Clearly, $K_{n,n}$-{a perfect matching} satisfies type (ii). $2K_2 \vee K_{2n-4}$ satisfies type (iii) and $K_{2n}-e$, for some edge e of K_{2n}, is of type (iv).

Remark 2.4.4: An $(n - 2)$-extendable graph has order at least 6. Theorem 2.4.5 provides a characterization for $2n \geq 10$. The graphs displayed in Figures 2.4.1 and 2.4.2 (b) indicate that this bound is best possible.

We conclude this section by completing the characterization of (n - 2)-extendable graphs of order $2n \geq 6$. In view of Theorem 2.4.5 we need only consider the cases n = 3 and 4. We start with the case n = 4. We have already seen, in Lemma 2.4.2, that the only members of $\mathcal{G}(8, 2, 3)$ are bipartite graphs with a perfect matching and minimum degree 3. In fact,

$$\mathcal{G}(8, 2, 3) = \{K_{4,4}-M_t \mid M_t \text{ is a matching of size } t, \; 1 \leq t \leq 4\}.$$

Also, by Lemma 2.4.4, all members of $\mathcal{G}(8, 2, 5)$ have independence number at most 2. There are 30 non-isomorphic graphs in $\mathcal{G}(8, 2, 5)$ as listed in Table 2.4.1. We obtained this list by considering the degree sequence of $G \in \mathcal{G}(8, 2, 5)$; it is convenient to consider the complement $\overline{G}$ which, by lemma 2.4.4, is triangle free. Note that P_t, C_t and W_t in the Table 2.4.1 denote the path, cycle and wheel of order t, respectively.

degree sequence of G	$\bar{G}$	G
5,5,5,5,5,5,5,5	C_8 $2C_4$	K_8-{a hamiltonian cycle} $2K_2 \vee 2K_2$
5,5,6,6,6,6,6,6	$2P_3 \cup K_2$ $P_4 \cup 2K_2$	$[(K_1 \cup K_2) \vee (K_1 \cup K_2)] \vee 2K_1$ $P_4 \vee C_4$
5,5,5,5,6,6,6,6	$P_3 \cup P_5$ $2P_4$ $C_4 \cup 2K_2$ $P_6 \cup K_2$	$(K_1 \cup K_2) \vee (K_5$- {a hamiltonian path}) $P_4 \vee P_4$ $2K_2 \vee C_4$ (K_6-{a hamiltonian path}) $\vee 2K_1$
5,5,5,5,5,5,6,6	$P_3 \cup C_5$ $P_4 \cup C_4$ $C_6 \cup K_2$ P_8	$(K_1 \cup K_2) \vee C_5$ $P_4 \vee 2K_2$ (K_6-{a hamiltonian cycle})$\vee 2K_1$ K_8-{a hamiltonian path}
5,6,6,6,6,6,6,7	$P_3 \cup 2K_2 \cup K_1$	$(K_1 \cup K_2) \vee W_5$
5,6,6,6,6,7,7,7	$P_3 \cup K_2 \cup 3K_1$	$(K_1 \cup K_2) \vee (K_5$-an edge e)
5,5,6,6,6,6,7,7	$2P_3 \cup 2K_1$ $P_4 \cup K_2 \cup 2K_1$	$[(K_1 \cup K_2) \vee (K_1 \cup K_2)] \vee K_2$ $P_4 \vee (K_4$-an edge e)

degree sequence of G	$\overline{G}$	G
5,5,5,6,6,6,6,7	$P_3 \cup P_4 \cup K_1$ $P_5 \cup K_2 \cup K_1$	$(K_1 \cup K_2) \vee (P_4 \vee K_1)$ $(K_5$-{a hamiltonian path}$) \vee P_3$
5,6,6,7,7,7,7,7	$P_3 \cup 5K_1$	$(K_1 \cup K_2) \vee K_5$
5,5,6,6,7,7,7,7	$P_4 \cup 4K_1$	$P_4 \vee K_4$
5,5,5,6,6,7,7,7	$P_5 \cup 3K_1$	$(K_5$-{a hamiltonian path}$) \vee K_3$
5,5,5,5,6,6,7,7	$C_4 \cup K_2 \cup 2K_1$ $P_6 \cup 2K_1$	$2K_2 \vee (K_4$-an edge e$)$ $(K_6$-{a hamiltonian path}$) \vee K_2$
5,5,5,5,5,6,6,7	$P_3 \cup C_4 \cup K_1$ $C_5 \cup K_2 \cup K_1$ $P_7 \cup K_1$	$(K_1 \cup K_2) \vee (2K_2 \vee K_1)$ $C_5 \vee P_3$ $(K_7$-{a hamiltonian path}$) \vee K_1$
5,5,5,5,7,7,7,7	$C_4 \cup 4K_1$	$2K_2 \vee K_4$
5,5,5,5,5,7,7,7	$C_5 \cup 3K_1$	$C_5 \vee K_3$
5,5,5,5,5,5,7,7	$C_6 \cup 2K_1$	$(K_6$-{a hamiltonian cycle}$) \vee K_2$
5,5,5,5,5,5,5,7	$C_7 \cup K_1$	$(K_7$-{a hamiltonian cycle}$) \vee K_1$

Table 2.4.1

As (Theorem 1.2.4) every graph G with $\delta(G) \geq 6$ on 8 vertices is 2-extendable, we need only consider the class $\mathcal{G}(8, 2, 4)$. Figure 2.4.2(b) shows a non-bipartite graph in $\mathcal{G}(8, 2, 4)$. We now establish that $\mathcal{G}(8, 2, 4)$ contains exactly 7 non-isomorphic graphs. We begin with the following lemma.

Lemma 2.4.6: Let $G \in \mathcal{G}(8, 2, 4)\backslash K_{4,4}$ and let u be a vertex of G with degree 4. Then $G[N_G(u)] \cong K_1 \cup K_3$.

Proof: Let $H = G[N_G(u)]$. By Theorem 2.2.1, H contains at most one independent edge. First we suppose that $E(H) = \phi$. If $v_1v_2 \in G[\bar{N}_G(u)]$, then $G-v_1-v_2$ is a graph on 6 vertices containing an independent set of order 4 and thus G cannot have a perfect matching containing the edge v_1v_2. This contradicts the fact that G is 2-extendable. Hence, $G[\bar{N}_G(u)]$ has no edges. But then $G \cong K_{4,4}$, a contradiction. Consequently, $E(H) \neq \phi$.

Let $V(H) = N_G(u) = \{x,y,z,v\}$, $\bar{N}_G(u) = \{a,b,c\}$ and suppose without any loss of generality that $xy \in E(H)$. Then, since H cannot have two independent edges, $zv \notin E(G)$. Since G is 2-extendable, the edge xy is contained in a perfect matching F in G. Clearly, F must contain an edge of $G[\bar{N}_G(u)]$, ab say. Now if $\{x,z,v\}$ is an independent set of vertices of G, then G cannot have a perfect matching containing the edges uy and ab, contradicting the extendability of G. Therefore, x must be joined to at least one of z or v. Similarly, $\{y,z,v\}$ cannot be an independent set of vertices of G and thus y must be joined to at least one of z or v. Since H contains at most one independent edge, the only possibility is for $H \cong K_1 \cup K_3$. This completes the proof of the lemma. □

Remark 2.4.5: Consider the proof of Lemma 2.4.6 above. It follows that if $G[\{x,y,z\}] \cong K_3$, then $d_G(v) \le 4$. Since $\delta(G) = 4$, $d_G(v) = 4$. Further, $N_G(v) = \{u\} \cup \bar{N}_G(u)$. Thus, G contains the graph G* displayed in Figure 2.4.3 as a spanning subgraph. Moreover, if $xa \in E(G)$ with $d_G(x) = 4$, then $d_G(a) = 4$.

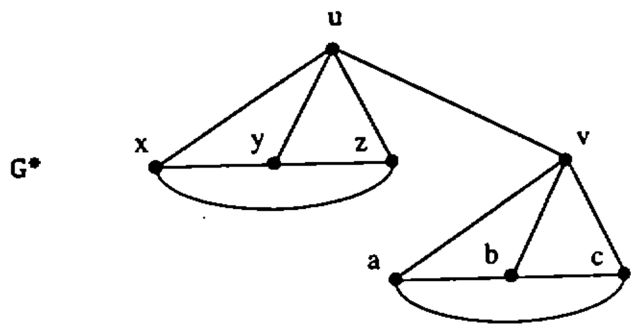


Figure 2.4.3

In the following we find it convenient to refer to the graph G* in our proof.

Corollary 2.4.7: Let $G \in \mathcal{G}(8, 2, 4)\backslash K_{4,4}$ be a 4-regular graph. Then G is the graph displayed in Figure 2.4.4.

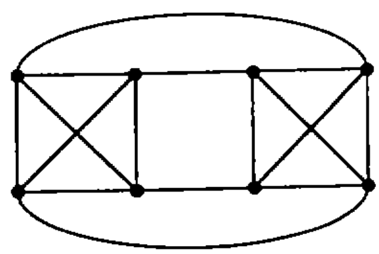


Figure 2.4.4

Proof: Since G^* is a spanning subgraph of G and G is 4-regular, the only possibility is that G is obtained from G^* by joining the vertices of the $\{x,y,z\}$ and $\{a,b,c\}$ with a perfect matching. Hence, G is the graph of Figure 2.4.4 as required. □

Corollary 2.4.8: Let $G \in \mathcal{G}(8, 2, 4)\backslash K_{4,4}$. Then $\Delta(G) \le 6$. Further, if $\Delta(G) = 6$, then there are exactly two vertices of degree 4.

Proof: Since G^* is a spanning subgraph of G and $d_G(u) = d_G(v) = 4$, $\Delta(G) \le 6$.

Suppose G contains at least three vertices of degree 4. Without any loss of generality we may suppose that $d_G(x) = 4$ and $xa \in E(G)$. Then, by Remark 2.4.5, $d_G(a) = 4$ and thus G cannot contain a vertex of degree 6. Hence, if $\Delta(G) = 6$, then G has exactly two vertices of degree 4, as required. □

Lemma 2.4.9: (i) $G \in \mathcal{G}(8, 2, 4)$ with $\Delta(G) = 5$ if and only if G is one of the graphs (up to isomorphism) displayed in Figure 2.4.5.

(ii) $G \in \mathcal{G}(8, 2, 4)$ with $\Delta(G) = 6$ if and only if G is one of the graphs (up to isomorphism) displayed in Figure 2.4.6.

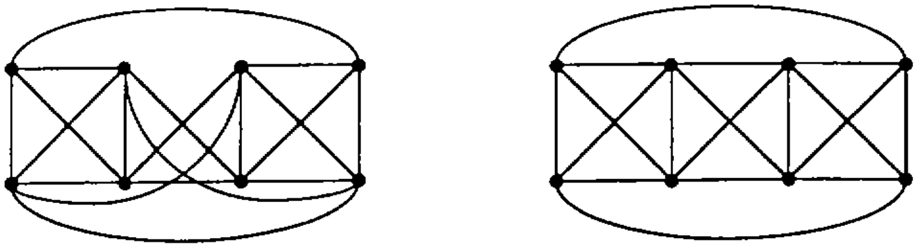
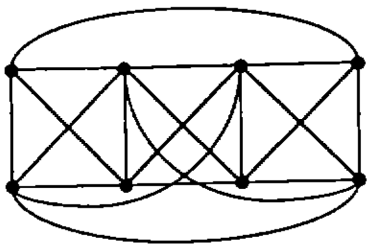
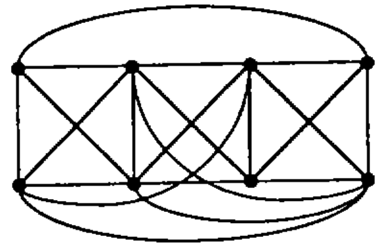


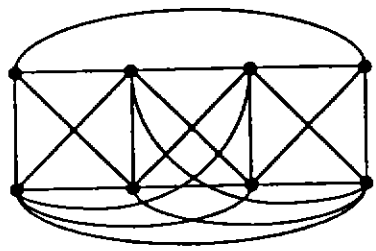
Figure 2.4.5



(a)



(b)



(c)

Figure 2.4.6

Proof: It is not too difficult to verify that the graphs in figures 2.4.5 and 2.4.6 are 2-extendable. Now let $G \in \mathcal{G}(8, 2, 4)$. It is sufficient to consider the bipartite subgraph G^{**} of G^{*} with bipartitioning sets $\{x,y,z\}$ and $\{a,b,c\}$. Using Lemma 2.4.6, Remark 2.4.5 and the minimum degree of G, it is not too difficult to show that $\{x,y,z\}$ and $\{a,b,c\}$ must have the same degree sequence in G^{**}.

Suppose $\Delta(G) = 6$. By Corollary 2.4.8, u and v are the only two vertices of degree 4. Hence, each vertex of $\{x,y,z,a,b,c\}$ must have degree at least 2 in G^{**}. It easily follows that there are three non-isomorphic graphs in $\mathcal{G}(8, 2, 4)$ with $\Delta(G) = 6$; these graphs are displayed in Figure 2.4.6.

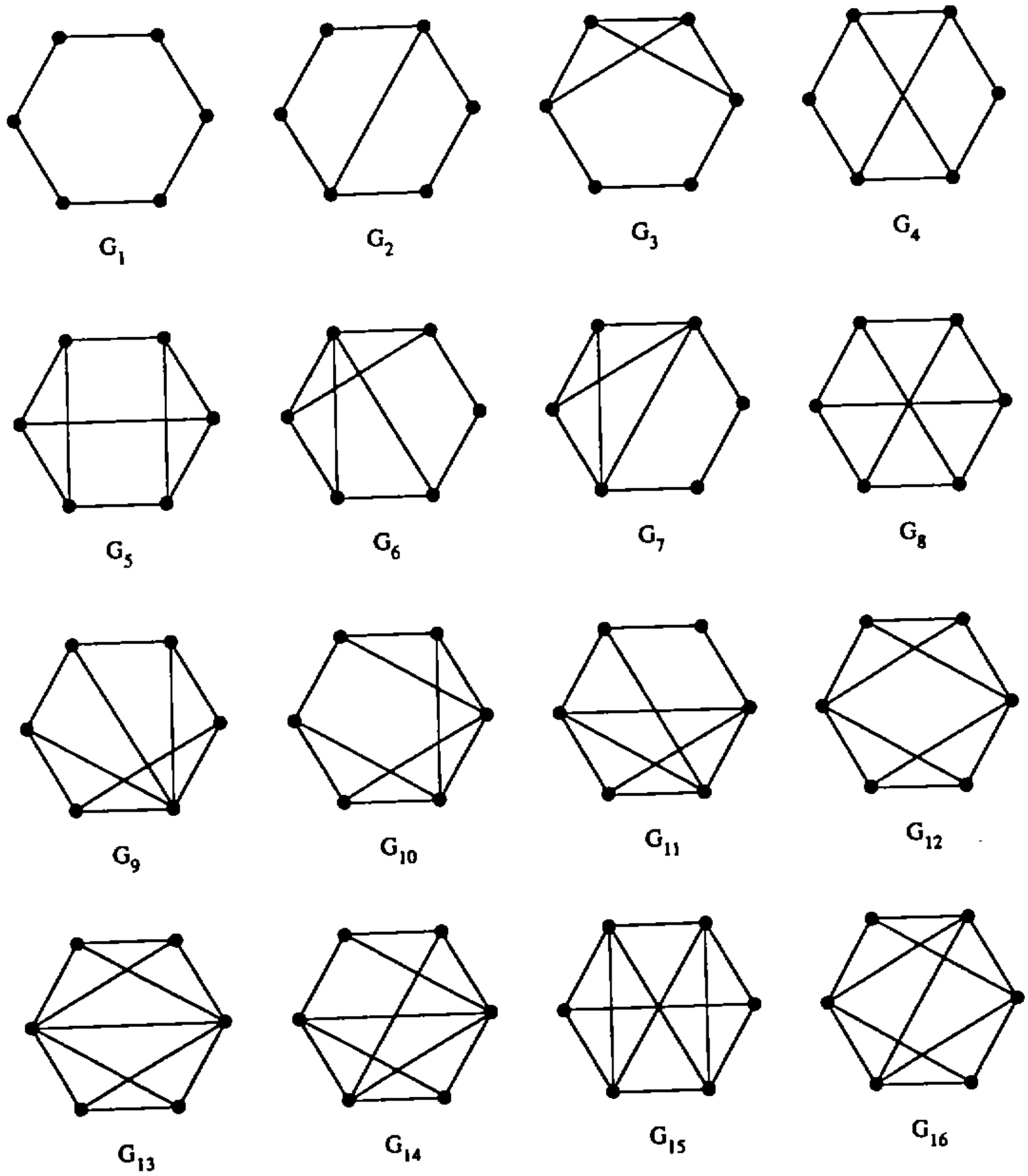
Next, we suppose that $\Delta(G) = 5$. Then each vertex of $\{x,y,z,a,b,c\}$ must have degree at least 1 in G^{**}. Without any loss of generality, we may assume that $d_G(x) = 5$ and $\{a,b\} \subseteq N_G(x)$. By Remark 2.4.6, $d_G(a) \geq 5$ and $d_G(b) \geq 5$. Since $\Delta(G) = 5$, $d_G(a) = d_G(b) = 5$. Hence, G must have at least 4 vertices of degree 5. Therefore, G must be one of the graphs displayed in Figure 2.4.5, as required. □

Lemma 2.4.9 and corollaries 2.4.7 and 2.4.8 together yield the following theorem:

Theorem 2.4.10: $|\ \mathcal{G}(8, 2, 4)\ | = 7$. The class $\mathcal{G}(8, 2, 4)$ consists of $K_{4,4}$ and the six graphs in figures 2.4.4, 2.4.5 and 2.4.6. □

Now we turn our attention to a characterization of (n - 2)-extendable graphs on 2n = 6 vertices. Theorem 2.3.1 ensures that a 1-extendable graph G on 6 vertices has minimum degree 2, 3, 4, or 5. It turns out that the class $\mathcal{G}_{(6,1)} = \bigcup_{\delta=2}^{5} \mathcal{G}(6,1,\delta)$ has 24 members. This can be established directly through a tedious and detail case analysis. A simpler alternative is to take advantage of the complete catalogue of graphs on 6 vertices (see Harary (1972) pp 218-224). Of the 60 graphs that satisfy the degree requirement, only 24 of them are 1-extendable; this can be established by routine checking. We summarize the result in the following theorem.

Theorem 2.4.11: There are exactly 24 non-isomorphic 1-extendable graphs on 6 vertices, namely the graphs displayed in Figure 2.4.7.



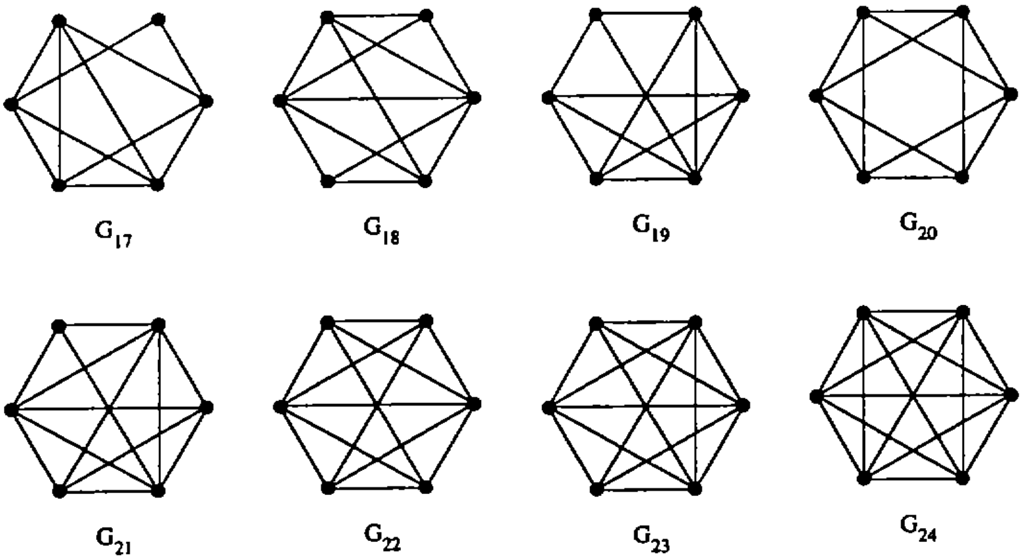


Figure 2.4.7 □

2.5 A VARIATION OF EXTENDABILITY TO ODD ORDER GRAPHS

As mentioned in Section 1.2, Yu (1993) introduced a variation of extendability to odd order graphs. Let G be a graph on $2n + 1$ vertices having matching of size n. Recall that, for $1 \le k \le n - 1$, G is $k\frac{1}{2}$ - extendable if for every vertex v of G, G-v is k-extendable.

Observe that K_{2n+1} is $k\frac{1}{2}$ - extendable for every k, $1 \le k \le n - 1$. On the other hand, $K_{n,n+1}$ is not $k\frac{1}{2}$ - extendable for any k, $1 \le k \le n - 1$, as deleting a vertex v from the smaller of the bipartitioning sets results in a $K_{n-1,n+1}$ which is clearly not k-extendable.

Theorems 1.2.16 and 1.2.17 together with the characterization of k-extendable graphs on 2n vertices, $k = n - 1$ and $n - 2$, given in theorems 2.4.1 and 2.4.5, respectively provide a characterization of $k\frac{1}{2}$ - extendable graphs on $2n + 1$ vertices for $k = n - 1$ and $n - 2$. More specifically, we have the following two results.

Theorem 2.5.1: Let G be a graph on $2n + 1 \geq 3$ vertices. Then G is $(n - 1)\frac{1}{2}$ - extendable if and only if $G \cong K_{2n+1}$.

Proof: The sufficiency is immediate and so we need to prove only the necessity. Let G be an $(n - 1)\frac{1}{2}$ - extendable graph on $2n + 1$ vertices. Then, by Theorem 1.2.16, the graph $G \vee K_1$ is an n-extendable graph of order $2n + 2$. Hence, by Theorem 2.4.1, $G \vee K_1 \cong K_{n+1,n+1}$ or K_{2n+2}. Observe that $E(G) \neq \phi$ since G-v has a perfect matching for every $v \in V(G)$. Consequently, $G \vee K_1$ cannot be a bipartite graph and thus $G \vee K_1 \cong K_{2n+2}$. Hence, $G \cong K_{2n+1}$, as required. This completes the proof. □

Our next result concerns the case $k = n - 2$.

Theorem 2.5.2: Let G be a graph on $2n + 1 \geq 9$ vertices. Then G is $(n - 2)\frac{1}{2}$ - extendable if and only if G is :

(i) $K_{2n + 1}$, or

(ii) $\delta(G) = 2n - 2$ and $\alpha(G) \leq 2$, or

(iii) $\delta(G) = 2n - 1$.

Proof: Let G be an $(n - 2)\frac{1}{2}$ - extendable graph on $2n + 1$ vertices. Then, by Theorem 1.2.16, $G \vee K_1$ is an $(n - 1)$-extendable graph on $2n + 2$ vertices. So ,by Theorem 2.4.5, $G \vee K_1$:

- is $K_{n+1, n+1}$ or K_{2n+2}, or
- is a bipartite graph with a perfect matching and minimum degree n, or
- has minimum degree 2n - 1 and has independence number at most 2, or
- has minimum degree 2n.

Now since $E(G) \neq \phi$, $G \vee K_1$ cannot be a bipartite graph. Further, since $\delta(G) = \delta(G \vee K_1) - 1$ and G is an induced subgraph of $G \vee K_1$, we conclude that G satisfies one of (i) to (iii) in our theorem, as required.

We now establish the sufficiency part. Clearly, K_{2n+1} is $(n - 2)\frac{1}{2}$ - extendable. If $\delta(G) = 2n - 1$, then, by Theorem 1.2.17, G is $(n - 2)\frac{1}{2}$ - extendable, as required. So the only remaining case is that when $\delta(G) = 2n - 2$ and $\alpha(G) \leq 2$.

In this case $\delta(G \vee K_1) = 2n - 1$ and $\alpha(G \vee K_1) \leq 2$; note that G is an induced subgraph of $G \vee K_1$. Hence, by Lemma 2.4.4, $G \vee K_1$ is an (n - 1)-extendable graph on 2n + 2 vertices. Consequently, G is $(n - 2)\frac{1}{2}$ - extendable, as required. This completes the proof of the theorem. □

CHAPTER 3

CRITICALLY k-EXTENDABLE GRAPHS

In this chapter, we consider the problem of characterizing k-critical graphs. Recall that a graph G is k-critical if G is k-extendable and G+uv is not k-extendable for every non-adjacent pair of vertices u and v of G. Observe that $K_{n,n}$ and K_{2n} are k-critical for all k, $1 \le k \le n - 1$. On the other hand, the cycle C_{2n} of order $2n \ge 6$ is 1-extendable but not 1-critical.

In Chapter 2, we established the existence of a k-extendable graph G on 2n vertices with $\delta(G) = j$ for each integer $j \in [k + 1, n] \cup [2k + 1, 2n - 1]$. A consequence of this result is that the class of k-extendable graphs becomes more complex with decreasing k and thus the characterization problem for k-extendable graphs becomes more difficult as k decreases. As we mentioned in Chapter 1, the critical subclass of graphs has more structure than the general class and this structure can be utilized to yield a considerable amount of useful information.

In this chapter we establish a number of properties of k-critical graphs. In Section 3.2, we present sufficient conditions for k-extendable graphs to be k-critical. More specifically, we prove that for a k-extendable graph $G \ne K_{2n}$ on 2n vertices, $2 \le k \le n - 1$, if for every pair of non-adjacent vertices u and v of G there exists a dependent set S (a subset S of V(G) is dependent if G[S] has at least one edge) of G-u-v such that $o(G-(S \cup \{u,v\}) = |S|$, then G is k-critical. Moreover, for k = 2 this sufficient condition is also a necessary condition for non-bipartite graphs. Thus, a 2-extendable graph G on $2n \ge 6$ vertices is 2-critical if and only if G is K_{2n} or $K_{n,n}$ or for every pair of non-adjacent vertices u and v of G there exists a dependent set of G-u-v such that $o(G-(S \cup \{u,v\})) = |S|$. We also show that there exist 2-critical non-bipartite graphs which are not complete.

In Section 3.3, we establish a necessary condition, in terms of the minimum degree, for k-critical graphs. In fact, we prove that if $G \ne K_{2n}$ is a k-critical graph on 2n vertices, $1 \le k \le n - 1$, then

$$\delta(G) \le \begin{cases} n & , \text{ for } n < 2k \\ n + 2 \left\lfloor \frac{k-1}{2} \right\rfloor, & \text{ for } n \ge 2k. \end{cases}$$

This result and the characterization of k-extendable graphs in Section 2.4 play a crucial role in the characterization of k-critical graphs.

In Section 3.4, we establish that $K_{n,n}$ and K_{2n} are the only k-critical graphs on 2n vertices for $k = 1$, $n - 1$ and for $k = n - 2$ and $n \ge 5$. The bound on the number of vertices for the case $k = n - 2$ is best possible in the sense that there exist $(n - 2)$-critical graphs on 8 vertices which are not complete or complete bipartite graphs. Further, we characterize $(n - 2)$-critical graphs on $2n = 6$ and 8 vertices. We show that there are exactly two 1-critical graphs on 6 vertices and five 2-critical graphs on 8 vertices.

3.1 PRELIMINARIES

In this section, we state a number of results from the literature which we make use of in our work.

Let M be a maximum matching in a graph G. The **deficiency** def(G) of G is defined as the number of M-unsaturated vertices of G. Recall that we denote the number of odd components in a graph H by $o(H)$. We begin by stating Berge's Formula (see Lovász and Plummer (1986) p.90):

Theorem 3.1.1: For every graph G

$$\text{def}(G) = \max\,\{o(G-X) - |X| : X \subseteq V(G)\}. \qquad \square$$

Before stating a necessary condition for 2-extendable graphs we need the following definitions. A graph G is **bicritical** if G-u-v has a perfect matching for every pair of vertices u and v. A graph G is **elementary** if the graph G′ induced by the edges

$$E' = \{e : e \in E(G) \text{ and } e \text{ is in some perfect matching in } G\}$$

is connected. Plummer (1980) proved the following result.

Theorem 3.1.2: Let G be a 2-extendable graph with $2n \geq 6$ vertices. Then G is either bicritical or elementary bipartite. □

3.2 SOME SUFFICIENT CONDITIONS

In this section, we establish some sufficient conditions for k-extendable graphs to be critical. Our first result concerns regular graphs of diameter 2.

Theorem 3.2.1: Let G be a k-extendable, (k + t)-regular graph, $1 \le t \le k \le n - 1$, on 2n vertices having diameter 2. Let w be any vertex of G and u and v any pair of non-adjacent vertices of $N_G(w)$. If $G[N_G(w)\backslash\{u,v\}]$ has exactly t - 1 independent edges, then G is k-critical.

Proof: Let M be a matching of size t - 1 in $G[N_G(w)\backslash\{u,v\}]$. Then $M_1 = M \cup \{uw\}$ is a matching of size $t \le k$ in G and so can be extended to a perfect matching F in G. Let

$$F_1 = \{xy \in F : x \in N_G(w)\backslash\{u,v\} \ , \ y \notin N_G(w)\}.$$

Since, by Theorem 2.2.1, $G[N_G(w)]$ has at most t - 1 independent edges, $|F_1| = k - t$. But then $M_2 = M \cup F_1 \cup \{uv\}$ is a matching in G+uv of size k and G+uv-V(M_2) has w as an isolated vertex. Hence, G is k-critical, proving the theorem. □

Remark 3.2.1: The graph G(2k,2k) obtained by joining two disjoint K_{2k}'s by a perfect matching satisfies the conditions in Theorem 3.2.1. Hence, as G(2k,2k) is k-extendable, it is k-critical.

In establishing the remaining results in this chapter we make frequent use of the following fact which follows immediately from the definition of k-critical graphs.

Remark 3.2.2: Let G be a k-extendable graph on 2n vertices, $1 \le k \le n - 1$. G is k-critical if and only if for every non-adjacent pair of vertices u and v of G, G-u-v is not $(k - 1)$-extendable.

Our next result provides a sufficient condition for any k-extendable graph to be k-critical. We make use of the following terminology. We call a subset S of V(G) **dependent** if G[S] has at least one edge.

Theorem 3.2.2: Let $G \ne K_{2n}$ be a k-extendable graph on 2n vertices, $2 \le k \le n - 1$. If for every pair of non-adjacent vertices u and v of G there exists a dependent set S of G-u-v such that $o(G-(S \cup \{u,v\})) = |S|$, then G is k-critical. Moreover, the converse is true for a non-bipartite graph G and $k = 2$.

Proof: Let u and v be non-adjacent vertices of G satisfying the hypothesis of the theorem. Then $G' = G-u-v$ contains a dependent set S such that

$$\begin{aligned} |S| &= o(G-(S \cup \{u,v\})) \\ &= o(G'-S). \end{aligned}$$

Hence, by Theorem 1.2.5, G' is not 1-extendable. Consequently, G' is not $(k - 1)$-extendable and thus G is k-critical.

Suppose that G is a 2-critical non-bipartite graph. Consider the graph $G'' = G-x-y$, where x and y are non-adjacent vertices of G. G'' has a perfect matching by Theorem 3.1.2 but is not

1-extendable. Hence, by Theorem 1.2.5, there exists a dependent set S such that $o(G''-S) = |S|$. Therefore, $o(G-(S \cup \{x,y\})) = |S|$, as required. This completes the proof of the theorem. □

In view of Theorem 1.2.13 we have the following corollary.

Corollary 3.2.3: Let G be a 2-extendable graph on $2n \geq 6$ vertices. Then G is 2-critical if and only if G is K_{2n} or $K_{n,n}$ or for every pair of non-adjacent vertices u and v of G there exists a dependent set S of G-u-v such that $o(G-(S \cup \{u,v\})) = |S|$. □

Remark 3.2.3: There exist 2-critical non-bipartite graphs which are not complete. For example, the graphs drawn in Figure 3.2.1.

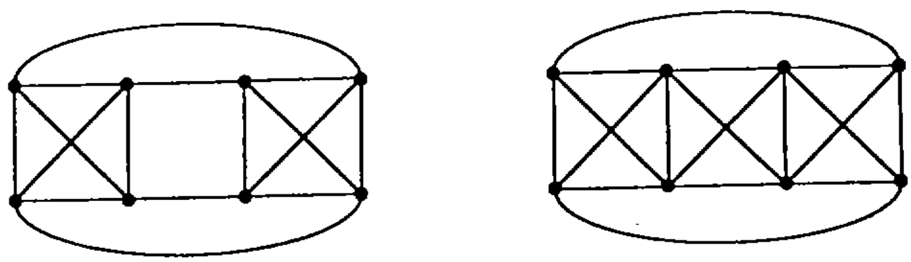


Figure 3.2.1

Remark 3.2.4: None of the graphs in Figure 3.2.1 are 1-critical since, in each case, the deletion of any pair of non-adjacent vertices results in a graph having a perfect matching. Thus, a k-critical graph need not be (k - 1)-critical.

3.3 MINIMUM DEGREE OF k-CRITICAL GRAPHS

Theorem 1.2.3 implies that a k-extendable graph G has minimum degree at least $k + 1$. Further, Theorem 1.2.4 gives us a clue that the minimum degree of a k-critical graph on 2n vertices should be no greater than $n + k$. In fact, as we shall establish in this section, this is indeed the case. We start with the following two lemmas.

Lemma 3.3.1: Let $G \neq K_{2n}$ be a k-critical graph on 2n vertices, $1 \leq k \leq n - 1$, and u and v any pair of non-adjacent vertices of G. Let M be a matching of size $k - 1$ in G-u-v. Then the graph $G' = G-(V(M) \cup \{u,v\})$ has a matching of size at least $n - k - 1$.

Proof: Suppose G' has a maximum matching M' of size at most $n - k - 2$. Then

$$\begin{aligned} \operatorname{def}(G') &= |V(G')| - 2|M'| \\ &= 2(n - k) - 2|M'| \\ &\geq 4. \end{aligned}$$

By Theorem 3.1.1, there exists a subset S' of $V(G')$ such that

$$o(G'-S') - |S'| = \operatorname{def} G' \geq 4.$$

Put $S = S' \cup \{u,v\}$ and $G_1 = G-V(M)$. Then

$$o(G_1-S) - |S| = o(G'-S') - |S'| - 2 \geq 2.$$

Then $\operatorname{def}(G_1) \geq 2$, implying that G is not k-extendable. This contradiction completes the proof of the lemma. □

Lemma 3.3.2: Let G be a connected graph on $2n$ vertices with $\delta(G) \geq n - 1$ having a maximum matching M of size $n - 1$. Then for M-unsaturated vertices u and v of G, $N_G(u) = N_G(v)$. Furthermore, no two vertices of $N_G(u)$ are joined by an edge of M, and the vertices of $V(G)\backslash N_G(u)$ form an independent set.

Proof: Let $M = \{x_i y_i : 1 \leq i \leq n - 1\}$. Observe that if $x_i u \in E(G)$ then $y_i v \notin E(G)$ as otherwise M is not maximum. Let

$$M_1 = \{x_i y_i \in M : ux_i,\ uy_i \in E(G)\},$$
$$M_2 = \{x_i y_i \in M : vx_i,\ vy_i \in E(G)\}, \text{ and}$$
$$M_3 = M\backslash(M_1 \cup M_2).$$

From our earlier observation it follows that $M_1 \cap M_2 = \phi$. By definition, if $x_i y_i \in M_3$, then u and v can each be joined to at most one of x_i and y_i. Consequently,

$$\begin{aligned} 2(n - 1) \leq d_G(u) + d_G(v) &\leq 2(|M_1 \cup M_2 \cup M_3|) \\ &= 2|M| \\ &= 2(n - 1), \end{aligned}$$

and hence each of u and v must be joined to exactly one end of each edge of M_3. In fact, $N_G(u) \cap V(M_3) = N_G(v) \cap V(M_3)$.

If $M_3 = \phi$, then, since G is connected, we have an M-augmenting path between u and v, contradicting the maximality of M. Hence, $M_3 \neq \phi$. We next establish that $M_1 = \phi$.

Suppose $M_1 \neq \phi$. Let X and Y respectively denote the vertices of $V(M_3)$ adjacent and non-adjacent to u. If $ab \in E(G)$ with $a \in Y$

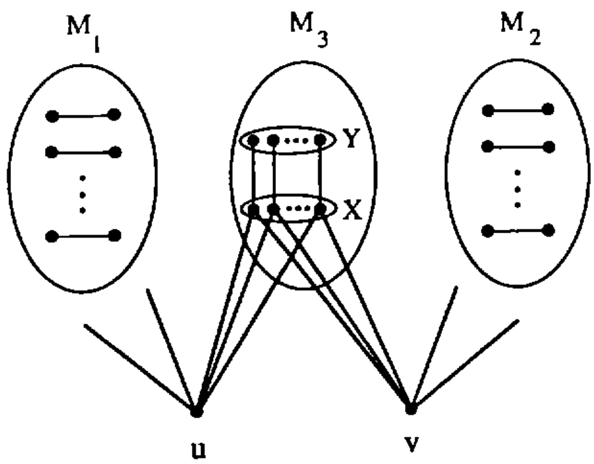


Figure 3.3.1

and $b \notin X$, then G contains an M-augmenting u,v path, contradicting the maximality of M. Hence, Y is an independent set of vertices of G and no vertex of Y is joined to any vertex of $V(M_1) \cup V(M_2)$. Consequently, for $w \in Y$ we have $d_G(w) \le |X| \le n - 2$, a contradiction. Therefore, $M_1 = \phi$ and similarly $M_2 = \phi$. This proves the lemma. □

Theorem 3.3.3: If $G \ne K_{2n}$ is a k-critical graph on 2n vertices, $1 \le k \le n - 1$, then

$$\delta(G) \le \begin{cases} n & , \text{ for } n < 2k \\ n + 2\lfloor \frac{k-1}{2} \rfloor, & \text{for } n \ge 2k. \end{cases} \qquad (3.3.1)$$

Proof: Let u and v be any pair of non-adjacent vertices of G and M a matching of size $k - 1$ in G-u-v such that the graph $G' = G-(V(M) \cup \{u,v\})$ has no perfect matching. Notice that the subgraph $G[V(M) \cup \{u,v\}]$ has a maximum matching of size at most $k - 1$, for otherwise G is not k-extendable. We distinguish two cases according to the value of k.

Case 1: $n < 2k$.

Suppose that $\delta(G) \geq n + 1$. Let M' be a maximum matching in the graph G' defined above. By Lemma 3.3.1, $|M'| = n - k - 1$ (note that $\nu(G') = 2n - 2k$). Let x and y be the two M'-unsaturated vertices of G'. Clearly, x and y are not adjacent. Since $\delta(G) \geq n + 1$ and M' is a maximum matching in G', there must be an edge e of M such that x and y are adjacent to different end vertices of e, say a and b, respectively. Then $M' \cup \{xa, yb\}$ is a matching of size $n - k + 1 \leq k$. But

$$G-(V(M') \cup \{x,a,y,b\}) = G[(V(M)\setminus\{a,b\}) \cup \{u,v\}]$$

has a matching of size at most $k - 2$. This contradiction proves that $\delta(G) \leq n$ for $n < 2k$.

Case 2: $n \geq 2k$.

Suppose that $\delta(G) \geq n + k$. Let $G_0 = G-u-v$. Then

$$|V(G_0)| = 2(n - 1)$$

and

$$\delta(G_0) \geq \delta(G) - 2 \geq (n - 1) + (k - 1).$$

By Theorem 1.2.4, G_0 is $(k - 1)$-extendable contradicting the fact that G is k-critical. Hence, $\delta(G) \leq n + k - 1$. Thus, if k is odd we are done. So suppose k is even. For this case we will prove that $\delta(G) \leq n + k - 2$.

Suppose that $\delta(G) = n + k - 1$. Now by the choice of G',

$$\delta(G') \geq \delta(G) - 2k = n - k - 1.$$

We now prove that G' is connected. Suppose that G' is disconnected. Then G' contains exactly two components as

$$\nu(G') = 2(n - k) \le 2(\delta(G') + 1).$$

In fact, G' consists of two disjoint K_{n-k}'s. Since G' has no perfect matching, $n - k$ and hence n must be odd.

Since $\delta(G) = n + k - 1$, every vertex of G' must be adjacent, in G, to every vertex of $V(M) \cup \{u,v\}$. Let x and y be any two non-adjacent vertices of G'. Now consider the graph $\hat{G} = G+xy$. We will establish that G' is connected by showing that $\hat{G}$ is k-extendable.

Suppose $\hat{G}$ is not k-extendable. Then since G is k-extendable, there exists a set $\hat{M}$ of k independent edges, with $xy \in \hat{M}$, that does not extend to a perfect matching in $\hat{G}$. If $ab \in \hat{M}$ and $a, b \notin V(G')$, then $\hat{M}' = (\hat{M}\setminus\{xy,ab\}) \cup \{xa,yb\}$ is a matching in G of size k with $V(\hat{M}) = V(\hat{M}')$. But then G cannot be k-extendable, a contradiction. We get a similar contradiction when $ab \in \hat{M}$ with $a \in V(G')$ and $b \notin V(G')$. We conclude therefore that $V(\hat{M}) \subseteq V(G')$. If $V(\hat{M}) \ne V(G')$ then the graph $G'' = G-V(M \cup \hat{M})$ consists of $\bar{K}_2 \vee (K_{2p} \cup K_{2q})$ for some p and q. Note that $V(\bar{K}_2) = \{u,v\}$. But G'' has a perfect matching implying that $\hat{M}$ is k-extendable. Hence, $V(\hat{M}) = V(G')$ and so $n - k = k$ implying that n is even, a contradiction. Therefore, $\hat{G}$ is k-extendable, contradicting the criticality of G. Hence, G' is connected.

Now Lemma 3.3.1 together with the fact that G' has no perfect matching implies that G' has a maximum matching M' of size $n - k - 1$. Let u' and v' be the two M'-unsaturated vertices of G'. By Lemma 3.3.2, $N_{G'}(u') = N_{G'}(v')$. Let $N_{G'}(u') =$

$\{x_1, x_2, \ldots, x_{n-k-1}\}$. Lemma 3.3.2 implies that no two x_i's are joined by an edge of M' and the set $V(G')\setminus N_{G'}(u')$ is an independent set of vertices. Since $\delta(G) = n + k - 1$ and $G[V(M) \cup \{u,v\}]$ has a maximum matching of size at most $k - 1$, at least one of u or v, say u, is joined to a vertex, w say, of $N_{G'}(u')$. (See Figure 3.3.2).

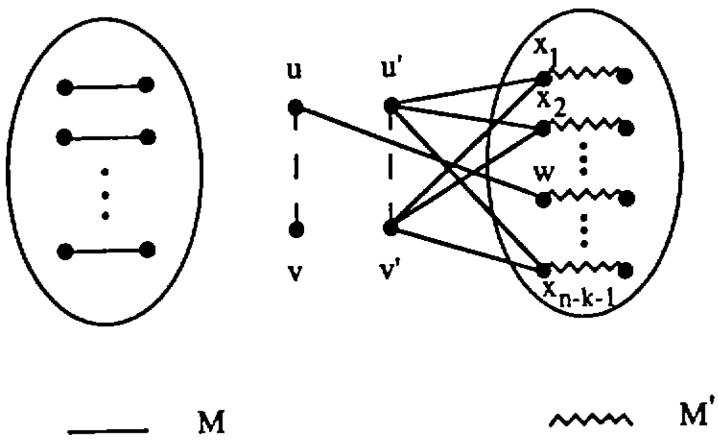


Figure 3.3.2

Consider the matching $M'' = M \cup \{uw\}$. The subgraph $G''' = G-V(M'')$ contains a set $S = \{v\} \cup (N_{G'}(u')\setminus\{w\})$ such that $o(G'''-S) > |S|$. Hence, G''' does not contain a perfect matching and so G is not k-extendable, a contradiction. This completes the proof of the theorem. □

Theorems 1.2.4 and 3.3.3 together yield the following corollary:

Corollary 3.3.4: Let $G \neq K_{2n}$ be a graph on 2n vertices, $1 \leq k \leq n - 1$. If $\delta(G) \geq n + k$, then G is k-extendable but not k-critical. □

Remark 3.3.1: For $n < 2k$ the graph $K_{n,n}$ achieves the bound (3.3.1). For $n = 2k$ the graphs H_1 and H_2 drawn in Figure 3.3.3 achieve the bound given in (3.3.1) for k odd and even, respectively. Recall that in our diagram a "double line" denotes the join. That H_1 and H_2 are k-critical is easily established. For example, in the case of H_1 if $uv \notin E(H_1)$, then u and v are in diagonally opposite K_k's and so for odd k it is easy to find a matching M of size k, with $uv \in M$, such that $H_1-V(M)$ consists of two odd components.

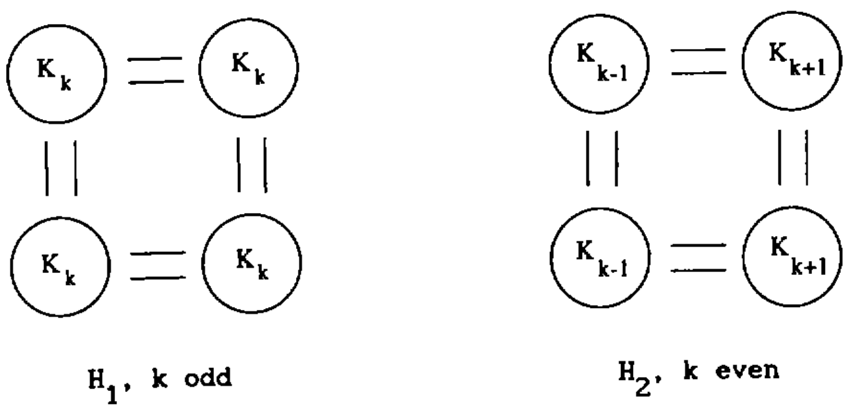


Figure 3.3.3

3.4 CHARACTERIZATION OF k-CRITICAL GRAPHS ON 2n VERTICES FOR k = 1, n - 1 and n - 2

In this section, we characterize k-critical graphs on 2n vertices for k = 1, n - 1 and n - 2. By applying theorems 2.3.1, 2.4.1, 2.4.5 and 3.3.3, it is not too difficult to establish the characterization of k-critical graphs on 2n vertices for k = n - 1 and k = n - 2. However, the case k = 1 is not so obvious. We have observed, in Section 2.3, that for every n, k and j with $k + 1 \le j \le n$ or $2k + 1 \le j \le 2n - 1$, there exists a k-extendable graph G on 2n vertices with $\delta(G) = j$. This means that the class of 1-extendable graphs is large. In fact, every graph on 2n vertices with minimum degree at least n + 1 is 1-extendable (Theorem 1.2.4). Although there exists a characterization of 1-extendable graphs (Theorem 1.2.5), it does not provide us with a list of 1-extendable graphs. By imposing the criticality condition upon 1-extendable graphs we are able to specify 1-critical graphs.

As mentioned in Chapter 1, Yu (1991) and Ananchuen and Caccetta (1992) independently proved that the only 1-critical graphs on 2n vertices are $K_{n,n}$ or K_{2n}. The methods in these two papers are totally different; we present ours below. In our work we make frequent use of the following fact:

> A graph G is 1-critical if and only if G is 1-extendable and G-u-v has no perfect matching for every pair of non-adjacent vertices u and v of G.

Our first task is to establish that 1-critical graphs are regular.

Lemma 3.4.1: If G is a 1-critical graph on 2n vertices, then G is regular.

Proof: Suppose to the contrary that G is not regular. Let $\delta(G) = r$. Since G is connected, there exist adjacent vertices u and v with $d_G(u) = r$ and $d_G(v) > r$.

Let F be a perfect matching in G containing edge uv. Let

$$A = \{xy \in F | x \in N_G(u)\backslash\{v\},\ y \notin N_G(u)\}, \text{ and}$$

$$B = \{xy \in F | x,y \in N_G(u)\}.$$

If v is adjacent to $x \in N_G(u)\backslash\{v\}$ and $xy \in A$, then G-u-y has a perfect matching, namely $(F\backslash\{uv,xy\}) \cup \{vx\}$. But this contradicts the fact that G is 1-critical. Hence, v is not adjacent to any vertex of $N_G(u) \cap V(A)$. Consequently, since $|A| + 2|B| = r - 1$, v is joined to a vertex, w say, different from u that does not belong to $V(A) \cup V(B)$. Let wz be the edge of G belonging to F. The choice of w implies that $wz \notin A \cup B$. Now $(F\backslash\{uv,wz\}) \cup \{vw\}$ is a perfect matching in G-u-z, contradicting the criticality of G. This proves the lemma. □

The following theorem provides a characterization of 1-critical graphs.

Theorem 3.4.2: A graph G on 2n vertices is 1-critical if and only if $G \cong K_{n,n}$ or K_{2n}.

Proof: The sufficiency is obvious as $K_{n,n}$ and K_{2n} are k-critical for $1 \le k \le n - 1$. So we need to prove the necessity.

Let G be 1-critical. Then, by Lemma 3.4.1, G is r-regular for some $r \geq 2$. Take u, v, F, A and B as in the proof of Lemma 3.4.1. Then $r = |A| + 2|B| + 1$ and v is not adjacent to any vertex of $N_G(u) \cap V(A)$. We now prove that $G \cong K_{n,n}$ when $B = \phi$.

Suppose $B = \phi$. If $vw \in E(G)$, with $w \in \bar{N}_G(u)\backslash V(A)$, then

$$F' = (F\backslash\{uv,ww'\}) \cup \{vw\},$$

where $ww' \in F$, is a perfect matching in $G-u-w'$. But then G is not 1-critical. Hence, v is not adjacent to any vertex of $\bar{N}_G(u)\backslash V(A)$. Now since v has degree r it must be joined to every vertex of $V(A) \cap \bar{N}_G(u)$. Let x be any vertex of $N_G(u)\backslash\{v\}$. Suppose that $xy \in E(G)$ with $y \neq u$ and $y \notin \bar{N}_G(u) \cap V(A)$. Let xx' and yy' belong to F. Then v is adjacent to at least one of x' or y', say x'. Since $B = \phi$, u is not adjacent to y'. Now

$$(F\backslash\{uv,xx',yy'\}) \cup \{vx',xy\}$$

is a perfect matching in $G-u-y'$, contradicting the criticality of G. Hence, $N_G(u)$ is an independent set, each vertex of which is adjacent to every vertex of $\bar{N}_G(u) \cap V(A)$. Consequently, $\bar{N}_G(u)\backslash V(A) = \phi$. Hence, $r = n$ and $G \cong K_{n,n}$.

We next prove that $G \cong K_{2n}$ when $B \neq \phi$. Suppose $B \neq \phi$. Consider the edge $bb' \in B$. If $vb \notin E(G)$, then $(F\backslash\{uv,bb'\}) \cup \{ub'\}$ is a perfect matching in $G-v-b$, contradicting the criticality of G. Hence, $V(B) \subseteq N_G(v)$. A similar argument establishes that any two vertices of $V(B)$ are adjacent. Therefore, the vertices u,v and $V(B)$ form a complete subgraph in G. Now let $aa' \in A$ with $a \notin N_G(u)$. If $va \notin E(G)$, then $(F\backslash\{uv,aa'\}) \cup \{ua'\}$ is a perfect matching in $G-v-a$, contradicting the criticality of G. Hence, v is joined to every vertex of $V(A)$

$\cap \bar{N}_G(u)$. Consider any edge $bb' \in B$. If $ab \notin E(G)$, then $(F\setminus\{aa',bb',uv\}) \cup \{ua',vb'\}$ is a perfect matching in $G-a-b$, a contradiction. Consequently, each vertex of $\bar{N}_G(u) \cap V(A)$ is adjacent to every vertex of $V(B) \cup \{v\}$.

Suppose s,t are non-adjacent vertices with $s \in V(A) \cap N_G(u)$ and $t \in V(A) \cap \bar{N}_G(u)$. Let $tt', ss' \in A$. Now

$$(F\setminus\{ss',tt',uv\}) \cup \{ut',vs'\}$$

is a perfect matching in $G-s-t$, a contradiction. Hence, each vertex of $V(A) \cap \bar{N}_G(u)$ is adjacent to every vertex of $V(A) \cap N_G(u)$. Consequently, $N_G(u) \subseteq N_G(a)$ for every $a \in V(A) \cap \bar{N}_G(u)$. Further, since G is r-regular, $N_G(u) = N_G(a)$.

Now suppose that $\bar{N}_G(u)\setminus V(A) \neq \phi$ and let $p \in \bar{N}_G(u)\setminus V(A)$. Since G is r-regular, p is not adjacent to any vertex of $(V(A) \cap \bar{N}_G(u))$ or $(\{v\} \cup V(B))$. Since G is connected, we may choose p such that $pq \in E(G)$ for some $q \in V(A) \cap N_G(u)$. Let $pp', qq' \in F$. Now

$$(F\setminus\{pp',qq',uv\}) \cup \{pq,vq'\}$$

is a perfect matching in $G-u-p'$, a contradiction. Hence, $\bar{N}_G(u)\setminus V(A) = \phi$. We complete the proof by showing that $A = \phi$.

Suppose $A \neq \phi$ and let $a_1 \in V(A) \cap N_G(u)$. Since a_1 is not joined to v or any vertex of $V(B)$, we have

$$r = |A| + 2|B| + 1 \leq 2|A|$$

and hence $|A| \geq 2|B| + 1 \geq 3$. Let $a_2 \in V(A) \cap N_G(u)$ and a_1a_1', $a_2a_2' \in A$. If $a_1a_2 \in E(G)$, then $(F\setminus\{a_1a_1',a_2a_2'\}) \cup \{a_1a_2\}$ is a perfect matching in $G-a_1'-a_2'$. Since $a_1'a_2' \notin E(G)$, this contradicts the criticality of G. Hence, the vertices of $N_G(u) \cap V(A)$ form an independent set. But then $d_G(a_1) \leq |A| + 1 < r$, a contradiction.

This proves that $A = \phi$ and hence $G \cong K_{2n}$. This completes the proof of the theorem. □

By Theorem 2.4.1 and the fact that $K_{n,n}$ and K_{2n} are k-critical for all k, $1 \le k \le n - 1$, we have a characterization of (n - 1)-critical as follows :

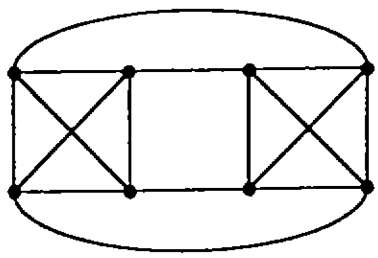
Theorem 3.4.3: Let G be a graph on $2n \ge 4$ vertices. Then G is (n - 1)-critical if and only if $G \cong K_{n,n}$ or K_{2n} □

We now turn our attention to (n - 2) - critical graphs. From theorems 2.3.1 and 3.3.3 we conclude that an (n - 2)-critical graph G has $\delta(G) = n - 1$; n; or $2n - 1$ for $2n \ge 6$. This fact together with Theorem 2.4.5 and a consequence of Theorem 1.2.13 allows us to state the following characterization of (n - 2)-critical graphs.

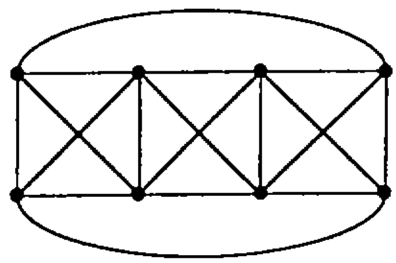
Theorem 3.4.4: A graph G on $2n \ge 10$ vertices is (n - 2)-critical if and only if $G \cong K_{n,n}$ or K_{2n}. □

Remark 3.4.1: Theorem 3.4.4 is best possible in the sense that there are (n - 2)-critical graphs on $2n = 8$ vertices which are not complete or complete bipartite graphs. For example, the graphs drawn in Figure 3.2.1.

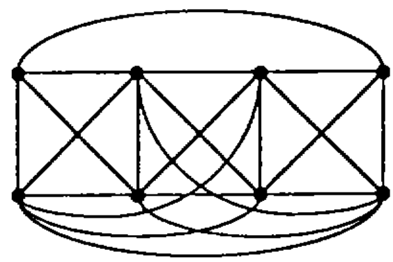
We have observed that an (n - 2)-extendable graph has order at least 6. Theorem 3.4.4 characterizes (n - 2)-critical graphs of order $2n \ge 10$. Theorem 3.4.2 ensures that the only (n - 2)-critical graphs on 2n = 6 vertices are $K_{n,n}$ and K_{2n}. The remaining case is when 2n = 8. Consider the graphs displayed in Figure 3.4.1. It is not too difficult to show that each G_i is 2-critical.



G_1



G_2



G_3

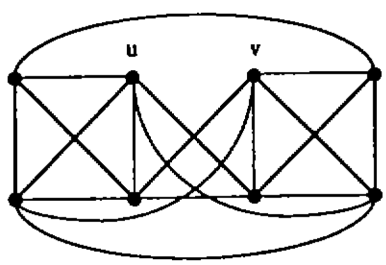
Figure 3.4.1

We will show that besides $K_{4,4}$ and K_8 the above are the only 2-critical graphs on 8 vertices.

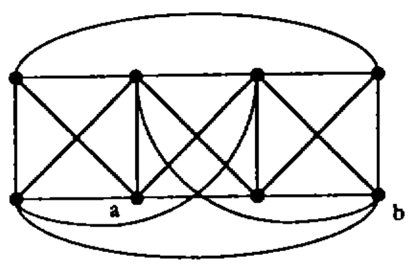
Theorem 3.4.5: G is a 2-critical graph on 8 vertices if and only if G is $K_{4,4}$ or K_8 or one of the graphs (up to isomorphism) displayed in Figure 3.4.1.

Proof: The sufficiency is obvious. Now we will prove the necessity.

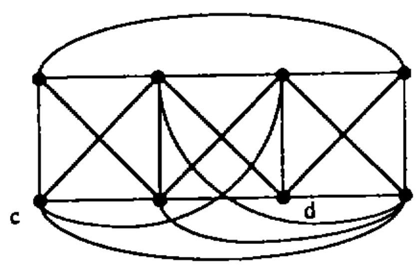
Suppose G is a 2-critical graph on 8 vertices. As noted above, $\delta(G) = 3$, 4 or 7. Clearly, if $\delta(G) = 7$, then $G \cong K_8$. Suppose $\delta(G) = 3$. Then, by Lemma 2.4.2, G is bipartite and hence, by Theorem 1.2.13, G is not critical. So the only remaining case is $\delta(G) = 4$. By Theorem 2.4.10, there are exactly seven 2-extendable graphs on 8 vertices with $\delta(G) = 4$. One of them is $K_{4,4}$ and the rest are the graphs displayed in figures 3.4.1 and 3.4.2. Notice that $H_1 + uv \cong H_2$, $H_2 + ab \cong H_3$ and $H_3 + cd \cong$



H_1



H_2



H_3

Figure 3.4.2

G_3. Since H_2, H_3, and G_3 are 2-extendable, H_1, H_2 and H_3 are not 2-critical. This completes the proof of our theorem. □

CHAPTER 4

MINIMALLY k-EXTENDABLE GRAPHS

In Chapter 3, we considered the problem of characterizing k-critical graphs which are k-extendable, but adding any edge that does not belong to their edge sets destroys the property of extendability. In this chapter, we consider the problem of characterizing graphs which are k-extendable but the deletion of any edge results in a graph that is not k-extendable. We have established that $K_{n,n}$ and K_{2n} are k-extendable and also k-critical for all k, $1 \le k \le n - 1$. We will show (Theorem 4.1.6) that these graphs are k-minimal if and only if $k = n - 1$. The cycle C_{2n} of order $2n \ge 6$ is an example of a 1-minimal graph which is not 1-critical. However, there exist k-extendable graphs which are both critical and minimal; an example is the graph G(2k, 2k) obtained by joining two disjoint K_{2k}'s by a perfect matching.

In Section 4.1, we establish necessary and sufficient conditions for k-extendable graphs to be k-minimal. More specifically, we prove that for a k-extendable graph G on 2n vertices, $1 \le k \le n - 1$, the following are equivalent:

- G is minimal.
- for every edge $e = uv$ of G there exists a matching M of size k in G-e such that $V(M) \cap \{u,v\} = \phi$ and for every perfect matching F in G containing M, $e \in F$.
- for every edge $e = uv$ of G there exists a vertex set S of G-u-v such that : $|M(S)| \ge k$; $o(G-e-S) = |S| - 2k + 2$; and u and v belong to different odd components of G-e-S (recall

that $M(S)$ denotes a maximum matching in $G[S]$).

Further, we prove that a sufficient condition for a k-extendable $(k+t)$-regular graph G on $2n$ vertices, $1 \le t \le k \le n - 1$, to be k-minimal is that for every edge uv of G, $G[N_G(u)\setminus\{v\}]$ or $G[N_G(v)\setminus\{u\}]$ contains exactly $t - 1$ independent edges.

In Section 4.2, we establish that a k-minimal graph $G \ne K_{2n}$ on $2n$ vertices, $1 \le k \le n - 1$, has minimum degree at most $n + k - 1$. This result also plays a crucial role in the characterization of k-minimal graphs. For a bipartite graph G on $2n$ vertices, we prove that if $G \ne K_{n,n}$ is k-minimal, $1 \le k \le n - 3$, then $\delta(G) < \frac{1}{2}(n + k)$.

Section 4.3 focuses on the characterization of k-minimal graphs on $2n$ vertices for $k = n - 1$ and $n - 2$. We prove that $K_{n,n}$ and K_{2n} are the only $(n - 1)$-minimal graphs on $2n$ vertices while the class of $(n - 2)$-minimal graphs on $2n$ vertices, $n \ge 5$, consists of :

- $(n - 1)$-regular bipartite graphs
- $(2n - 3)$-regular graphs
- graphs with $2n - 1$ vertices of degree $2n - 3$ and one vertex of degree $2n - 1$
- graphs with $2n - 2$ vertices of degree $2n - 3$ and two vertices of degree $2n - 2$, u and v say, such that $N_G(u)\setminus\{v\} = N_G(v)\setminus\{u\}$.

We also characterize $(n - 2)$-minimal graphs on $2n$ vertices for $2n = 6$ and 8. We show that there are exactly four 1-minimal graphs on 6 vertices and nine 2-minimal graphs on 8 vertices.

4.1 PROPERTIES OF MINIMALLY k-EXTENDABLE GRAPHS

Consider a k-minimal graph G. Since for every edge e of G, G-e is not k-extendable, there exists a matching M in G-e of size k that does not extend to a perfect matching in G-e. Our first result concerns the size of a maximum matching in G-e-V(M).

Lemma 4.1.1: Let G be a k-minimal graph on 2n vertices, $1 \le k \le n - 1$ and e any edge of G. If M is a matching of size k in G-e that does not extend to a perfect matching in G-e, then G-e-V(M) has a maximum matching of size $n - k - 1$.

Proof: Let M′ be a maximum matching in G′ = G-e-V(M). Since G is k-minimal, $|M'| \le n - k - 1$. Suppose that $|M'| \le n - k - 2$. Then

$$\begin{aligned} def(G') &= |V(G')| - 2|M'| \\ &= 2(n - k) - 2|M'| \\ &\ge 4. \end{aligned}$$

By Theorem 3.1.1, there exists a subset S′ of V(G′) such that

$$o(G'-S') - |S'| = def(G') \ge 4.$$

Let xy be an edge of M. Put $S'' = S' \cup \{x,y\}$ and $G'' = G' \cup \{x,y\}$. Then $o(G''-S'') = o(G'-S')$ and hence

$$o(G''-S'') - |S''| = o(G'-S') - |S'| - 2 \ge 2.$$

Then $def(G'') \ge 2$, implying that G-e is not (k - 1)-extendable, contradicting Theorem 1.2.15. This completes the proof of the lemma. □

The following result concerns a sufficient condition for k-extendable (k + t)-regular graphs to be k-minimal.

Theorem 4.1.2: Let G be a k-extendable (k + t)-regular graph on 2n vertices, $1 \le t \le k \le n - 1$. If for every edge uv of G, $G[N_G(u)\backslash\{v\}]$ or $G[N_G(v)\backslash\{u\}]$ contains exactly t - 1 independent edges, then G is minimal.

Proof: Let $uv \in E(G)$. Without any loss of generality, assume that $G[N_G(u)\backslash\{v\}]$ contains exactly t - 1 independent edges. Let M be a set of these edges. Then $|M| = t - 1$. Thus, $M \cup \{uv\}$ is a matching of size $t \le k$ which can be extended to a perfect matching F in G. Let

$$F' = \{ xy \in F \mid x \in N_G(u)\backslash(V(M) \cup \{v\})\}$$

Since, by Theorem 2.2.1, $G[N_G(u)]$ has at most t - 1 independent edges, $|F'| = k - t + 1$. But then $M' = M \cup F'$ is a matching in G-uv of size $(t - 1) + (k - t + 1) = k$ and G-uv-V(M') contains u as an isolated vertex. Hence, G is k-minimal as required. □

We remark that the graph G(2k, 2k) obtained by joining two disjoint K_{2k}'s by a perfect matching satisfies the conditions in Theorem 4.1.2. Since G(2k, 2k) is k-extendable, it is also k-minimal.

Our next two lemmas yield necessary and sufficient conditions for k-extendable graphs to be k-minimal.

Lemma 4.1.3: Let G be a k-extendable graph on 2n vertices, $1 \le k \le n - 1$. Then G is minimal if and only if for every edge e = uv of G there exists a matching M of size k in G-e such that $V(M) \cap \{u,v\} = \phi$ and for every perfect matching F in G containing M, $e \in F$.

Proof: The sufficiency is obvious, so we need only prove the necessity. Let $e = uv$ be an edge of G and M a matching of size k in G-e that does not extend to a perfect matching in G-e. We first show that $V(M) \cap \{u,v\} = \phi$.

Suppose to the contrary that $V(M) \cap \{u,v\} \neq \phi$. First we assume that $\{u,v\} \subseteq V(M)$. Then $G-V(M) = G-e-V(M)$. Hence, M is extendable in G only if it is extendable in G-e, a contradiction. Hence, $\{u,v\} \subseteq V(M)$. So we need only consider the case when exactly one of u or v belongs to V(M).

Without any loss of generality, assume

$$V(M) \cap \{u,v\} = \{u\}.$$

Since M is a matching in G-e, there exists a vertex $u' \in V(G)\setminus\{v\}$ such that $uu' \in M$. If F is a perfect matching in G containing M, then $uv \notin F$ since $uu' \in F$. Consequently, F is a perfect matching in G-e containing M which contradicts the choice of M. Hence, $u \notin V(M)$. This proves that $V(M) \cap \{u,v\} = \phi$.

Since M is a matching in G-e, M is also a matching in G. If there exists a perfect matching F′ in G containing M such that $uv \notin F'$, then F′ is a perfect matching in G-e containing M, a contradiction. Hence, every perfect matching in G containing M must contain edge uv. This proves our result. □

Recall that M(S) denotes a maximum matching in G[S]. We now establish another necessary and sufficient condition for k-extendable graphs to be k-minimal.

Lemma 4.1.4: Let G be a k-extendable graph on 2n vertices, $1 \le k \le n-1$. Then G is minimal if and only if for every edge $e = uv$ of G there exists a vertex set S of G-u-v such that :

(i) $|M(S)| \ge k$;

(ii) $o(G-e-S) = |S| - 2k + 2$;

and (iii) u and v belong to different odd components of G-e-S.

Proof: The sufficiency follows directly from Theorem 1.2.7. We need only consider the necessity. Let $e = uv$ be an edge of G. Since G is minimal, G-e is not k-extendable. Thus, by Theorem 1.2.7, there exists a set $S_0 \subseteq V(G-e)$ such that $o(G-e-S_0) > |S_0| - 2d_0$, where $d_0 = \min\{|M(S_0)|, k\}$. But,by Theorem 1.2.15 G-e is $(k-1)$-extendable and so we have for every $S_1 \subseteq V(G-e)$, $o(G-e-S_1) \le |S_1| - 2d_1$, where $d_1 = \min\{|M(S_1)|, k-1\}$. Now if $|M(S_0)| \le k-1$, then

$$o(G-e-S_0) > |S_0| - 2d_0 = |S_0| - 2|M(S_0)|$$

and

$$o(G-e-S_0) \le |S_0| - 2d_1 = |S_0| - 2|M(S_0)|,$$

a contradiction. Hence, $|M(S_0)| \ge k$, proving (i). Thus, we have $d_0 = k$ and $d_1 = k - 1$. Consequently,

$$o(G-e-S_0) > |S_0| - 2d_0 = |S_0| - 2k$$

and

$$o(G-e-S_0) \le |S_0| - 2d_1 = |S_0| - 2(k-1).$$

Since $\nu(G)$ is even, S_0 and $o(G-e-S_0)$ have the same parity. Hence,

$$o(G-e-S_0) = |S_0| - 2k + 2,$$

proving (ii).

Now we establish (iii). Since G is k-extendable, by Theorem 1.2.7 and the fact that $|M(S_0)| \geq k$, we have

$$o(G-S_0) \leq |S_0| - 2k.$$

Now making use of the fact that

$$o(G-e-S_0) \leq o(G-S_0) + 2,$$

we conclude that

$$|S_0| - 2k + 2 = o(G-e-S_0) \leq o(G-S_0) + 2 \leq |S_0| - 2k + 2.$$

Hence,

$$o(G-e-S_0) = o(G-S_0) + 2.$$

This implies that e must be an edge joining two different odd components of $G-e-S_0$. Consequently, u and v belong to different odd components of $G-e-S_0$ and clearly $S_0 \cap \{u,v\} = \phi$. This proves (iii) and thus completes the proof of our lemma. □

Lemmas 4.1.3 and 4.1.4 together yield the following theorem :

Theorem 4.1.5: Let G be a k-extendable graph on 2n vertices, $1 \leq k \leq n - 1$. Then the following are equivalent:

(a) G is minimal.

(b) For every edge e = uv of G there exists a matching M of size k in G-e such that $V(M) \cap \{u,v\} = \phi$ and for every perfect matching F in G containing M, $e \in F$.

(c) For every edge e = uv of G there exists a vertex set S of G-u-v such that : $|M(S)| \geq k$; $o(G-e-S) = |S| - 2k + 2$;

and u and v belong to different odd components of G-e-S.

□

Clearly, the graphs $K_{n,n}$ and K_{2n} are k-extendable for each k, $1 \le k \le n - 1$. However, it is not so obvious that $K_{n,n}$ and K_{2n} are k-minimal if and only if $k = n - 1$. We prove this in our next result.

Theorem 4.1.6: (a) K_{2n} is k-minimal, $1 \le k \le n - 1$, if and only if $k = n - 1$.

(b) $K_{n,n}$ is k-minimal, $1 \le k \le n - 1$, if and only if $k = n - 1$.

Proof: (a) First we will prove the sufficiency. Let $e = uv \in K_{2n}$. By Theorem 4.1.5 (b), there exists a matching M of size k in $K_{2n}-uv$ such that $V(M) \cap \{u,v\} = \phi$ and for every perfect matching F in K_{2n} containing M, $e \in F$.

If $\nu(K_{2n}-(V(M) \cup \{u,v\})) \ge 2$, then there exists a perfect matching F_1 in K_{2n} containing M such that $e \notin F_1$, since $K_{2n}-V(M)$ is a 1-factorable graph on $2n - 2k$ vertices, a contradiction. Consequently,

$$\nu(K_{2n}-(V(M) \cup \{u,v\})) = 0.$$

Hence, $n = k + 1$ as required.

Now we show that K_{2n} is $(n - 1)$-minimal. Clearly, K_{2n} is $(n - 1)$-extendable. Let $e = xy$ be an edge of K_{2n}. Then $K_{2n}-x-y \cong K_{2n-2}$. Clearly, K_{2n-2} contains a matching M_1 of size $n - 1$ and M_1 does not extend to a perfect matching in $K_{2n}-xy$, since M_1 saturates the neighbour set of x and y in $K_{2n}-xy$. Therefore, K_{2n} is minimal. This completes the proof of (a).

The proof of (b) is similar. □

We conclude this section by making the following observations.

Remark 4.1.1: Theorem 1.2.2 asserts that a k-extendable graph is (k - 1)-extendable. However, a k-minimal graph G need not be (k - 1)-minimal. For example, the graph in Figure 4.4.1 is 2-minimal but not 1-minimal.

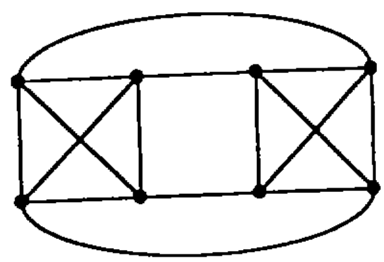


Figure 4.1.1

Remark 4.1.2: Consider any k-extendable graph G on 2n vertices, $1 \le k \le n - 1$. If $d_G(u) = k + 1$ or $d_G(v) = k + 1$ for every edge e = uv of G, then G is k-minimal. This implies that a (k + 1)-regular k-extendable graph on 2n vertices, $1 \le k \le n - 1$, is k-minimal. Thus, the k-cube Q_k which is a k-regular (k - 1)-extendable graph (see Györi and Plummer (1992)) is (k - 1)-minimal.

4.2 MINIMUM DEGREE OF k-MINIMAL GRAPHS

Theorem 1.2.3 implies that a k-extendable graph G has minimum degree at least k + 1. A useful result in our work on k-critical graphs was an upper bound on the minimum degree. Our next theorem establishes a similar upper bound on the minimum degree of k-minimal graphs.

Theorem 4.2.1: If $G \neq K_{2n}$ is a k-minimal graph on 2n vertices, $1 \leq k \leq n - 1$, then $\delta(G) \leq n + k - 1$.

Proof: If $k = n - 1$, then, since $G \neq K_{2n}$, we have $\delta(G) \leq 2n - 2 = n + k - 1$ and we are done. So we may assume that $1 \leq k \leq n - 2$. Suppose to the contrary that $\delta(G) \geq n + k$. Let e be an edge of G. Since G-e is not k-extendable, G-e has a matching M of size k such that M does not extend to a perfect matching in G-e. On the other hand, we have $\delta(G-V(M)) \geq n + k - 2k = n - k = \frac{1}{2}\,\nu(G-V(M))$ and $\nu(G-V(M)) = 2(n - k) \geq 4$, since $k \leq n - 2$. Hence, by Theorem 2.1.1, G-V(M) is hamiltonian and hence G-V(M) has a hamiltonian cycle of even order 2(n - k). Since every even cycle has two disjoint perfect matchings, G-V(M) has at least two disjoint perfect matchings M_1 and M_2. Clearly, either $e \notin M_1$ or $e \notin M_2$, since $M_1 \cap M_2 = \phi$. Without any loss of generality, assume that $e \notin M_1$. But then $F = M_1 \cup M$ is a perfect matching in G-e with $M \subseteq F$, contradicting the assumption on M. Thus, $\delta(G) \leq n + k - 1$. □

Theorems 1.2.4 and 4.2.1 together yield the following corollary:

Corollary 4.2.2: Let $G \neq K_{2n}$ be a graph on $2n$ vertices, $1 \leq k \leq n - 1$. If $\delta(G) \geq n + k$, then G is k-extendable but not k-minimal. □

Remark 4.2.1: An alternative proof of Theorem 4.2.1 can be obtained by applying the property of minimality together with Theorem 2.1.1 and Lemma 3.3.2.

Remark 4.2.2: The upper bound of $n + k - 1$ given in Theorem 4.2.1 is not always achievable. The characterization of $(n - 1)$-minimal graphs given in the next section (Lemma 4.3.1) shows that the bound is not achievable for the case $k = n - 1$. On the other hand, our characterization of $(n - 2)$-minimal graphs given in Theorem 4.3.11 shows that the bound is achievable for the case $k = n - 2$; an example is the graph $\bar{K}_2 \vee (K_{2n-2}-\{\text{a hamiltonian cycle}\})$. It would be interesting to determine when the bound is achievable.

The following result concerns the upper bound on the minimum degree of k-minimal bipartite graphs.

Theorem 4.2.3: If $G \neq K_{n,n}$ is a k-minimal bipartite graph on $2n$ vertices, $1 \leq k \leq n - 3$, then $\delta(G) < \frac{1}{2}(n + k)$.

Proof: Suppose to the contrary that G is a k-minimal bipartite graph on $2n$ vertices, $1 \leq k \leq n - 3$ with $\delta(G) \geq \frac{1}{2}(n + k)$. Let X and Y be bipartitioning sets of G and $e = uv \in E(G)$ where $u \in X$, $v \in Y$. Clearly, $\delta(G-e) \geq \delta(G) - 1 \geq \frac{1}{2}(n + k) - 1$ and G-e has the same bipartitioning sets as G.

If $\delta(G-e) > \frac{1}{2}(n + k) - 1$, then, by Theorem 2.2.6, G-e is k-extendable, contradicting the minimality of G. Hence, $\delta(G-e) = \frac{1}{2}(n + k) - 1$. This implies that $n + k$ is even.

Since G is k-minimal, by Lemma 4.1.3, there exists a matching M of size k in G-e such that $V(M) \cap \{u,v\} = \phi$ and e belongs to every perfect matching in G containing M. Let F be a perfect matching in G containing M. Consider the graph

$$G' = G-(V(M) \cup \{u,v\}).$$

Clearly,

$$F' = F\backslash(V(M) \cup \{u,v\})$$

is a perfect matching in G' and $\delta(G') \geq \frac{1}{2}(n - k) - 1$. Let

$$N_{G'}(u) = \{u_1, u_2, \ldots, u_r\}, \text{ and}$$
$$N_{G'}(v) = \{v_1, v_2, \ldots, v_s\}.$$

Clearly, $r,s \geq \frac{1}{2}(n - k) - 1$. If $u_iv_j \in F'$ for some $1 \leq i \leq r$ and $1 \leq j \leq s$, then

$$F'' = (F\backslash\{uv, u_iv_j\} \cup \{uu_i, vv_j\}$$

is a perfect matching in G-e containing M, contradicting the minimality of G. Hence, $u_iv_j \notin F'$ for all $1 \leq i \leq r$ and $1 \leq j \leq s$. Now let

$$A = \{u_i' : u_iu_i' \in F', 1 \leq i \leq r\},$$

$$B = \{v_j' : v_jv_j' \in F', 1 \leq j \leq s\},$$

$$C = X\backslash(V(M) \cup N_{G'}(v) \cup A \cup \{u\}), \text{ and}$$

$$D = Y\backslash(V(M) \cup N_{G'}(u) \cup B \cup \{v\}).$$

Note that $A \cap N_{G'}(v) = B \cap N_{G'}(u) = \phi$ and $|C| = |D|$. Figure 4.2.1 illustrates our notation; note that the edges of $M \cup \{e\}$ are indicated in bold whilst those of F' are indicated in wavy.

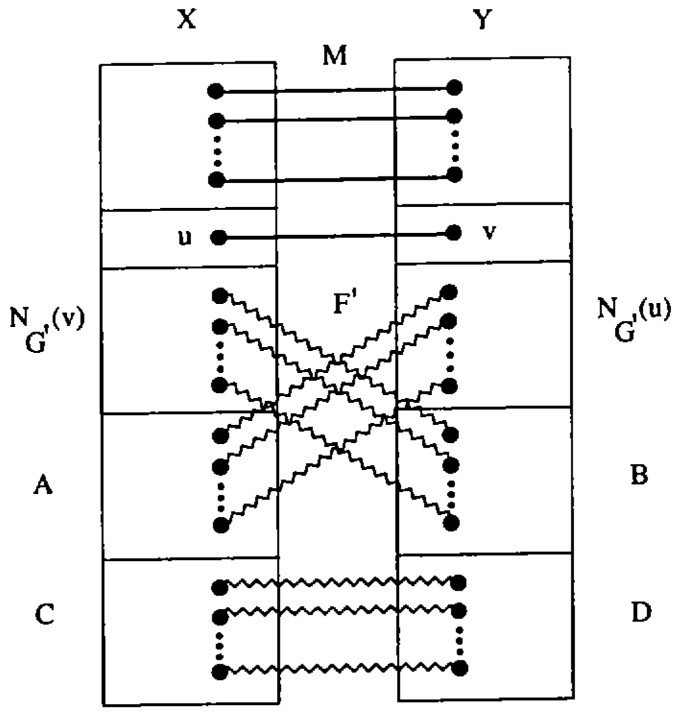


Figure 4.2.1

We claim that $u_i'w \notin E(G)$ for every $u_i' \in A$ and $w \in B \cup \{v\}$. Suppose this is not the case and $u_i'w \in E(G)$ for some i, $1 \le i \le r$, and $w \in B \cup \{v\}$. Then G contains the F-alternating path:

$$uu_iu_i'v, \qquad \text{if } w = v$$

or

$$uu_iu_i'v_j'v_jv, \quad \text{if } w = v_j',$$

implying the existence of a perfect matching in G-e containing M, a contradiction. Hence, $u_i'w \notin E(G)$ for every $u_i' \in A$ and $w \in B \cup \{v\}$. Similarly, $v_j'w \notin E(G)$ for every $v_j' \in B$ and $w \in A \cup \{u\}$. It follows that

$$\tfrac{1}{2}(n + k) \le d_G(u_i') \le n - s - 1, \qquad \text{for } 1 \le i \le r,$$

and

$$\tfrac{1}{2}(n + k) \le d_G(v_j') \le n - r - 1, \qquad \text{for } 1 \le j \le s.$$

Consequently, $r,s \le \frac{1}{2}(n - k) - 1$. Hence, since $r, s \ge \frac{1}{2}(n - k) - 1$, the only possibility is for $r = s = \frac{1}{2}(n - k) - 1 \ge 1$ (note that as $n + k$ is even, $n - k \ge 3$ is even and thus $n - k$ is at least 4).

Consider the vertex $u'_i \in A$. We have $N_G(u'_1) \subseteq Y\backslash(B \cup \{v\})$. Now, since

$$|Y| - |B \cup \{v\}| = n - 1 - \frac{1}{2}(n - k) + 1 = \frac{1}{2}(n + k),$$

we have $N_G(u'_i) = Y\backslash(B \cup \{v\})$ for every $u'_i \in A$. Similarly, $N_G(v'_j) = X\backslash(A \cup \{u\})$ for every $v'_j \in B$. Now

$$\begin{aligned} |D| = |C| &= n - k - 1 - |N_{G'}(u)| - |B| \\ &= n - k - 1 - 2s \\ &= 1. \end{aligned}$$

Let $C = \{c\}$ and $D = \{d\}$. Then G contains the edges cv'_j, $1 \le j \le s$ and du'_i, $1 \le i \le r$. But then

$$uu_1u'_1dcv'_1v_1v$$

is an F-alternating path in G and hence

$$F''' = (F\backslash\{uv,\ u_1u'_1,\ cd,\ v_1v'_1\}) \cup \{uu_1,\ u'_1d,\ cv'_1,\ v_1v\}$$

is a perfect matching in G-e containing M, a contradiction. This completes the proof of the theorem. □

Remark 4.2.3: Theorem 4.2.3 is best possible in the sense that for $k = n - 2$, there exists an $(n - 2)$-minimal bipartite graph with minimum degree $n - 1$; for example the $(n - 1)$-regular bipartite graph.

4.3 CHARACTERIZATION OF k-MINIMAL GRAPHS ON 2n VERTICES FOR k = n - 1 AND n - 2

In this section, we concentrate on characterizing k-minimal graphs on 2n vertices for k = n - 1 and n - 2. The characterization of k-extendable graphs on 2n vertices for k = n - 1 and k = n - 2 (Theorem 2.4.1 and Theorem 2.4.5) and theorems 4.1.6 and 4.2.1 are very essential for establishing the results in this section. We begin with the case k = n - 1.

Lemma 4.3.1: G is an (n - 1)-minimal graph on $2n \geq 4$ vertices if and only if $G \cong K_{n,n}$ or K_{2n}.

Proof: It follows immediately from theorems 2.4.1 and 4.1.6. □

We can now turn our attention to (n - 2)-minimal graphs. From theorems 2.3.1, 4.1.6 and 4.2.1, we conclude that an (n - 2)-minimal graph G has $\delta(G) = n - 1$, n or 2n - 3 for $2n \geq 6$. Further, from Lemma 2.4.3 and Theorem 4.1.6, $\delta(G) \neq n$ for $2n \geq 10$. We thus have :

Lemma 4.3.2: If G is an (n - 2)-minimal graph on $2n \geq 6$ vertices, then $\delta(G) = n - 1$, n or 2n - 3. Furthermore, for $2n \geq 10$, $\delta(G) \neq n$. □

We establish our characterization of (n - 2)-minimal graphs by considering two cases according to the values of the minimum degree.

Theorem 4.3.3: G is an (n - 2)-minimal graph on $2n \geq 8$ vertices with $\delta(G) = n - 1$ if and only if G is an (n - 1)-regular bipartite graph.

Proof: It follows from Lemma 2.4.2 and Theorem 1.2.3 that an (n - 1)-regular bipartite graph G on $2n \geq 8$ vertices is (n - 2)-minimal and so the sufficiency is immediate. We need to consider the necessity part.

Let G be an (n - 2)-minimal graph with $\delta(G) = n - 1$. Then, by Lemma 2.4.2, G is bipartite with bipartitioning sets, A and B say, of order n. We need to establish that G is (n - 1)-regular. Suppose that this is not the case. Then, since $\delta(G) = n - 1$ and $|A| = |B| = n$, G contains vertices $x \in A$ and $y \in B$ that have degree n. So $xy \in E(G)$. But then G-xy is a bipartite graph with $\delta(G) = n - 1$ and hence, by Theorem 2.2.6, is (n - 2)-extendable. This contradicts the minimality of G and completes the proof of our theorem. □

Remark 4.3.1: The bound on n in Theorem 4.3.3 is best possible as an (n - 2)-minimal graph G on 6 vertices with $\delta(G) = n - 1$ exists which is neither bipartite nor regular; for example, the graph of Figure 2.4.1.

Characterizing the (n - 2)-minimal graphs having minimum degree 2n - 3 is a more complicated exercise. We begin by establishing some sufficient conditions for (n - 2)-extendable graphs with $\delta(G) = 2n - 3$ to be minimal.

Lemma 4.3.4: If G is a $(2n - 3)$-regular $(n - 2)$-extendable graph on $2n \geq 8$ vertices, then G is minimal.

Proof: Let $e = uv \in E(G)$ and consider $G' = G[N_G(u)\setminus\{v\}]$. Clearly, $\nu(G') = 2n - 4$ and $\delta(G') \geq 2n - 7$. We claim that G' has a perfect matching M. For $2n \geq 10$, this follows from Theorem 2.1.1 as $\delta(G') \geq 2n - 7 \geq \frac{1}{2}\nu(G')$. For $2n = 8$, $\delta(G') \geq 1$ and so either G' has a perfect matching or an independent set of size 3. Now, by Lemma 2.4.4, G' must have a perfect matching as required. Now M is a matching of size $n - 2$ which clearly does not extend to a perfect matching in G-uv. This completes the proof of the lemma. □

Remark 4.3.2: Lemma 4.3.4 is best possible in the sense that $K_{3,3}$ is a 3-regular 1-extendable graph on 6 vertices but is not minimal, by Theorem 4.1.6.

Lemma 4.3.5: Let G be an $(n - 2)$-extendable graph on $2n \geq 8$ vertices. If G has only one vertex of degree $2n - 1$ and $2n - 1$ vertices of degree $2n - 3$, then G is minimal.

Proof: Follows from the proof of Lemma 4.3.4, since every edge of G is incident to at least one vertex of degree $2n - 3$. □

Remark 4.3.3: Lemma 4.3.5 is best possible in the sense that the wheel W_6 (drawn in Figure 4.3.1) on 6 vertices satisfies our hypothesis but is not minimal since W_6-e is still 1-extendable.

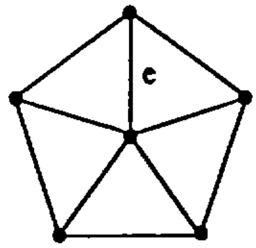


Figure 4.3.1

Lemma 4.3.6: Let G be an (n - 2)-extendable graph on $2n \geq 8$ vertices. If G has 2n - 2 vertices of degree 2n - 3 and two vertices of degree 2n - 2, u and v say, such that $N_G(u)\backslash\{v\} = N_G(v)\backslash\{u\}$, then G is minimal.

Proof: Let e = xy be an edge of G. If $d_G(x) = 2n - 3$, then the proof of Lemma 4.3.4 is valid and establishes that G is minimal. So the only case we need to consider is $d_G(x) = d_G(y) = 2n - 2$. That is x = u and y = v. Now since $N_G(u)\backslash\{v\} = N_G(v)\backslash\{u\}$, G contains a vertex w that is not joined to u or v. Clearly, $N_G(w) = N_G(u)\backslash\{v\}$. Consider a vertex $z \in N_G(u)\backslash\{v\}$ and the subgraph $G' = G[N_G(u)\backslash\{v,z\}]$. Observe that $\nu(G') = 2n - 4$ and $\delta(G') \geq 2n - 7$ and so, as in the proof of Lemma 4.3.4, G′ contains a perfect matching M. Now M does not extend to a perfect matching in G-uv since the induced subgraph on {u,v,z,w} in G-uv is $K_{1,3}$. This completes the proof of the lemma. □

Remark 4.3.4: The condition $N_G(u)\backslash\{v\} = N_G(v)\backslash\{u\}$ in Lemma 4.3.6 is essential, since there exists an (n - 2)-extendable graph which violates this condition and is not minimal. Let P_4 be a path on 4 vertices and $H = P_4 \vee (K_{2n-4}$-{a hamiltonian cycle}), $n \geq 4$. It is easy to show that H is an (n - 2)-extendable graph with two vertices, u and v say, of degree 2n - 2 and 2n - 2 vertices of degree 2n - 3. Clearly, u and v are internal vertices of P_4. Further, $N_G(u)\setminus\{v\} \neq N_G(v)\setminus\{u\}$. It is not difficult to show that H-uv is (n - 2)-extendable. Hence, H is not minimal.

Now we can establish a characterization of (n - 2)-minimal graphs on 2n vertices with minimum degree 2n - 3. We begin with the following lemma.

Lemma 4.3.7: Let G be a k-minimal graph on 2n vertices, $1 \leq k \leq n - 2$ with $\Delta(G) = 2n - 1$. If $d_G(u) = 2n - 1$, then $d_G(v) \leq 2k + 1$ for every $v \in V(G)\setminus\{u\}$.

Proof: Suppose to the contrary that there exists a vertex $v \neq u$ of G with $d_G(v) \geq 2k + 2$. Since G is minimal and $uv \in E(G)$, it follows from Lemma 4.1.3 that there exists a matching M of size k in G-uv with $V(M) \cap \{u,v\} = \phi$. Let F be a perfect matching in G containing M. Thus, $uv \in F$ (Lemma 4.1.3). Since $k \leq n - 2$, $|V(G)\setminus(V(M) \cup \{u,v\})| \geq 2n - 2k - 2 \geq 2$. Because $d_G(v) \geq 2k + 2$, there exists a vertex $x \in V(G)\setminus(V(M) \cup \{u,v\})$ with $vx \in E(G)$. Let $xy \in F$. Clearly, $y \notin V(M) \cup \{u,v\}$. Further, $uy \in E(G)$ since $d_G(u) = 2n - 1$. Thus,

$$F_0 = (F\setminus\{uv, xy\}) \cup \{vx, uy\}$$

is a perfect matching containing M and $uv \notin F_0$. But this contradicts Lemma 4.1.3 and hence proves our lemma. □

Lemmas 4.3.5 and 4.3.7 together yield the following theorem.

Theorem 4.3.8: Let G be an $(n-2)$-extendable graph on $2n \geq 8$ vertices with $\delta(G) = 2n - 3$ and $\Delta(G) = 2n - 1$. Then G is minimal if and only if G has only one vertex of degree $2n - 1$ and $2n - 1$ vertices of degree $2n - 3$. □

Theorem 4.3.9: Let G be an $(n-2)$-extendable graph on $2n \geq 8$ vertices with $\delta(G) = 2n - 3$ and $\Delta(G) = 2n - 2$. Then G is minimal if and only if G has $2n - 2$ vertices of degree $2n - 3$ and two vertices of degree $2n - 2$, u and v say, such that $N_G(u)\setminus\{v\} = N_G(v)\setminus\{u\}$.

Proof: The sufficiency follows from Lemma 4.3.6. So we need only prove the necessity. Let G be an $(n-2)$-minimal graph with $\delta(G) = 2n - 3$ and $\Delta(G) = 2n - 2$. Then the number of vertices of degree $2n - 3$ must be even and hence the number of vertices of degree $2n - 2$ is also even. Thus, G contains at least two vertices of degree $2n - 2$. We need to prove that there are exactly 2 such vertices of G. Suppose to the contrary that u, v and w are three vertices of degree $2n - 2$. We distinguish two cases according to whether or not $uv \in E(G)$.

Case 1: $uv \notin E(G)$. Then $N_G(u) = N_G(v) = V(G)\setminus\{u,v\}$ and $w \in N_G(u)$. Let M be any matching in G of size $n - 2$ with $V(M) \cap \{u,w\} = \phi$; such an M exists by Lemma 4.1.3. Consider the subgraph $G' = G-V(M)$. Let $V(G') = \{u,w,x,y\}$. Note that v could be x or y.

If $v = x$, then $F = M \cup \{xw, uy\}$ is a perfect matching in G-uw, contradicting the minimality of G (Lemma 4.1.3). Hence,

$v \notin \{x,y\}$. This implies that $x, y \in N_G(u)$. Since $d_G(w) = 2n - 2$, $wy \in E(G)$ or $wx \in E(G)$. Without any loss of generality, assume $wy \in E(G)$. Then $F' = M \cup \{wy, ux\}$ is a perfect matching in $G-uw$, again contradicting the minimality of G (Lemma 4.1.3). This proves Case 1.

Case 2: $uv \in E(G)$. Let $a \in \bar{N}_G(u)$ and M' a matching in G with $V(M') \cap \{u,v\} = \phi$. If $av \in E(G)$, then an argument similar to that used in Case 1 establishes the existence of a perfect matching F'', in $G-uv$, containing M' such that $uv \notin F''$. Hence, by Lemma 4.1.3, $av \notin E(G)$. This implies that $N_G(u)\setminus\{v\} = N_G(v)\setminus\{u\}$. Since $\delta(G) = 2n - 3$, $N_G(a) = N_G(u)\setminus\{v\}$ and $w \neq a$. Thus, $w \in N_G(u)\setminus\{v\}$. Again a similar argument to that used in Case 1 establishes the existence of a perfect matching in $G-uw$ containing a matching M'' of size $n - 2$ with $V(M'') \cap \{u,w\} = \phi$. This contradicts the minimality of G. Hence, u and v are the only two vertices of degree $2n - 2$ of G. Moreover, $N_G(u)\setminus\{v\} = N_G(v)\setminus\{u\}$ follows directly from the proof and completes the proof of the theorem.

□

The following result which follows from Lemma 2.4.4 and theorems 4.3.8 and 4.3.9 gives us information on the induced subgraph of a neighbour set of a vertex having maximum degree in $(n - 2)$-minimal graphs with $\delta(G) = 2n - 3$.

Lemma 4.3.10: Let G be a non-regular $(n - 2)$-minimal graph on $2n \geq 8$ vertices with $\delta(G) = 2n - 3$. If $d_G(u) = \Delta(G)$ and $H = G[N_G(u)]$, then $\bar{H}$, the complement of H, is a 2-regular triangle-free graph or a 2-regular triangle free graph plus an isolated vertex. □

Lemmas 4.3.2 and 4.3.4 and theorems 4.3.3, 4.3.8 and 4.3.9 together allow us to state the following characterization of $(n - 2)$-minimal graphs on $2n$ vertices.

Theorem 4.3.11: Let G be an $(n - 2)$-extendable graph on $2n \geq 10$ vertices. Then G is minimal if and only if G :

(i) is an $(n - 1)$-regular bipartite graph, or

(ii) is a $(2n - 3)$-regular graph, or

(iii) contains one vertex of degree $2n - 1$ and $2n - 1$ vertices of degree $2n - 3$, or

(iv) contains $2n - 2$ vertices of degree $2n - 3$ and two vertices of degree $2n - 2$, u and v say, such that $N_G(u)\setminus\{v\} = N_G(v)\setminus\{u\}$. □

Remark 4.3.5: There exist $(n - 2)$-minimal graphs for each type specified in Theorem 4.3.11. Examples are : $K_{n,n}$-{a perfect matching}; K_{2n}-{a hamiltonian cycle}; $K_1 \vee (K_{2n-1}$-{a hamiltonian cycle}); and $\bar{K}_2 \vee (K_{2n-2}$-{a hamiltonian cycle}), respectively.

We have observed that an $(n - 2)$-extendable graph has order at least 6. Theorem 4.3.11 characterizes $(n - 2)$-minimal graphs of order $2n \geq 10$. We conclude this section by completely characterizing all $(n - 2)$-minimal graphs on 6 and 8 vertices. We begin with the case $2n = 8$. Consider the graphs displayed in Figure 4.3.2.

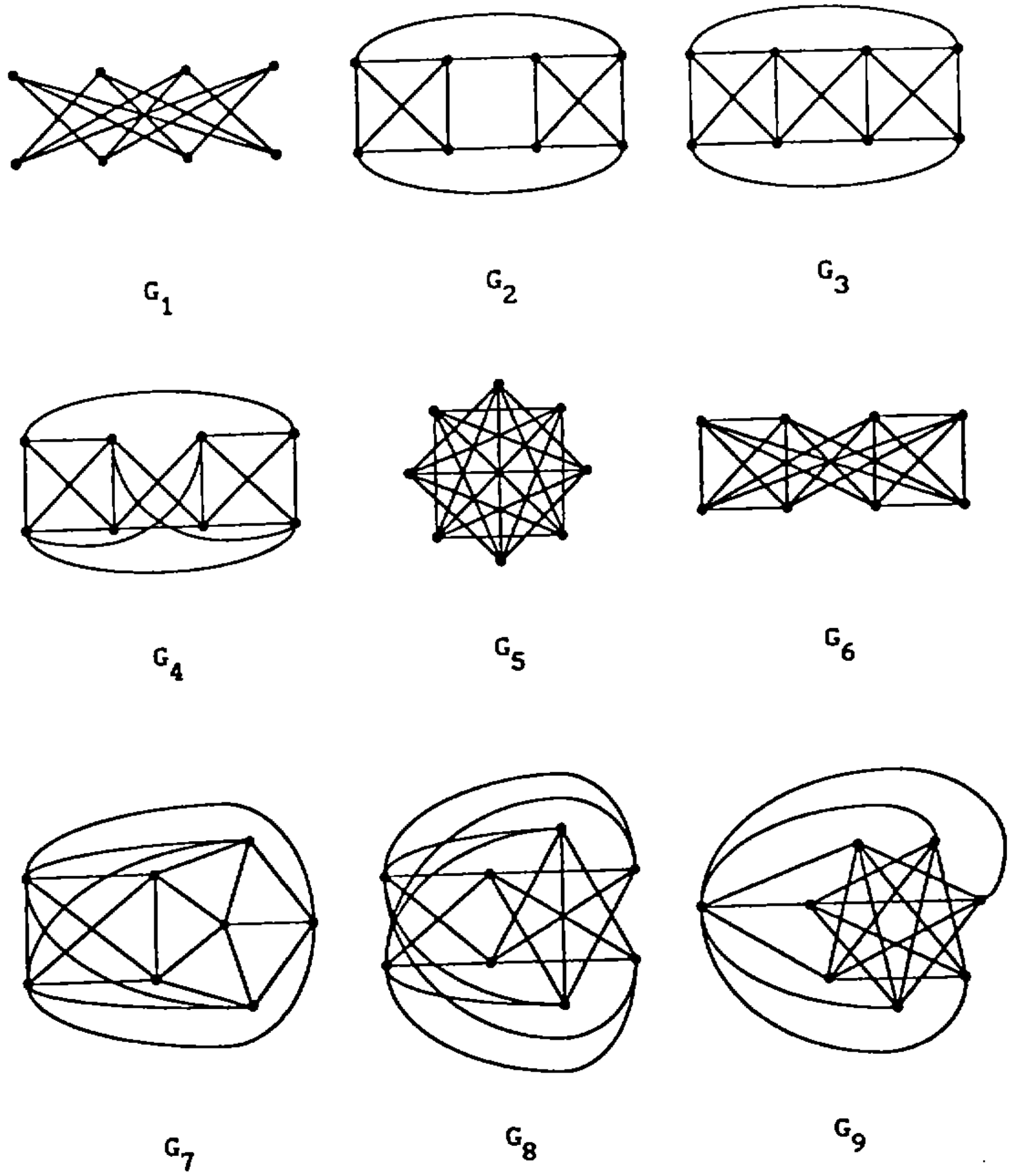


Figure 4.3.2

It is not too difficult to verify that each G_i is 2-minimal. We will first prove that these graphs are the only 2-minimal graphs on 8 vertices.

Theorem 4.3.12: Let G be a 2-minimal graph on 8 vertices. Then G is one of the graphs (up to isomorphism) displayed in Figure 4.3.2.

Proof: By Lemma 4.3.2, $\delta(G) = 3$, 4 or 5. If $\delta(G) = 3$, then, by Theorem 4.3.3, G is 3-regular bipartite graph. Hence G is the graph G_1. Suppose $\delta(G) = 4$. Then, by Theorem 2.4.10, there are exactly seven members in $\mathfrak{G}(8, 2, 4)$. Three of them are G_2, G_3 and G_4. The rest are $K_{4,4}$ and the graphs H_1, H_2 and H_3 in Figure 4.3.3.

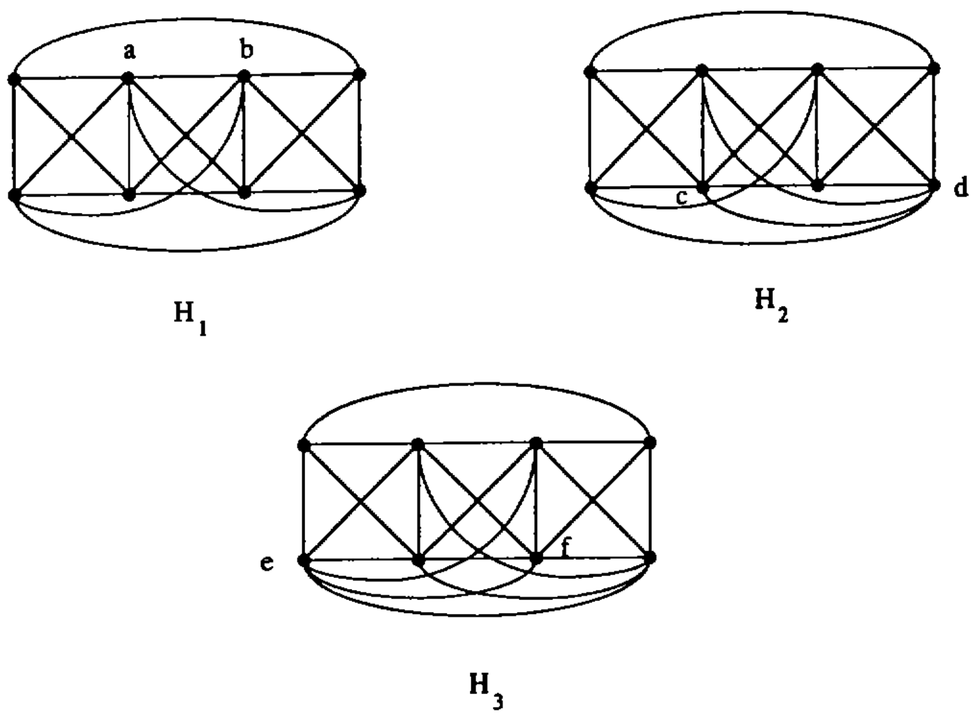


Figure 4.3.3

By Theorem 4.1.6(b), $K_{4,4}$ is not 2-minimal. Consider the graphs H_1, H_2 and H_3. Observe that $H_1 - ab \cong G_4$, $H_2 - cd \cong H_1$ and $H_3 - ef \cong H_2$. Hence, H_1, H_2 and H_3 are not 2-minimal. This proves the theorem for the case $\delta(G) = 4$.

The only remaining case is $\delta(G) = 5$. If G is a 5-regular 2-extendable graph, then, by Lemma 4.3.4, G is 2-minimal. According to Table 2.4.1, there are exactly two 5-regular 2-extendable graphs on 8 vertices, namely G_5 and G_6. So we need to consider the case when G is non-regular. We proceed according to $\Delta(G)$ which is 6 or 7.

Suppose $\Delta(G) = 6$. Then, by Theorem 4.3.9, G contains exactly two vertices of degree 6, u and v say, such that $N_G(u)\setminus\{v\} = N_G(v)\setminus\{u\}$. Table 2.4.1 showed that there are four 2-extendable graphs with degree sequence 5, 5, 5, 5, 5, 5, 6, 6. But only two of them, G_7 $((K_1 \cup K_2) \vee C_5)$ and G_8 $((K_6-\{\text{a hamiltonian cycle}\}) \vee 2K_1)$ satisfy the above mentioned condition concerning neighbour sets. Finally, consider the case $\Delta(G) = 7$. By Theorem 4.3.8 and Table 2.4.1, there is exactly one such graph, namely G_9. This completes the proof of our theorem. □

Now we turn our attention to the case $2n = 6$. Consider the graphs displayed in Figure 4.3.4.

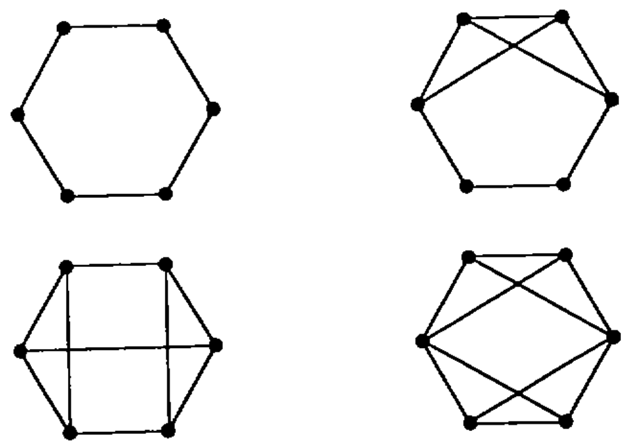


Figure 4.3.4

It is easy to check that each of these graphs is 1-minimal. We now prove that these are the only 1-minimal graphs on 6 vertices.

Theorem 4.3.13: There are exactly four non-isomorphic 1-minimal graphs on 6 vertices, namely the graphs displayed in Figure 4.3.4.

Proof: By Theorem 2.4.11, there are exactly 24 non-isomorphic 1-extendable graphs on 6 vertices. Four of them are the graphs in Figure 4.3.4, the other twenty are the graphs $H_1, \ldots, H_{20}$ in Figure 4.3.5.

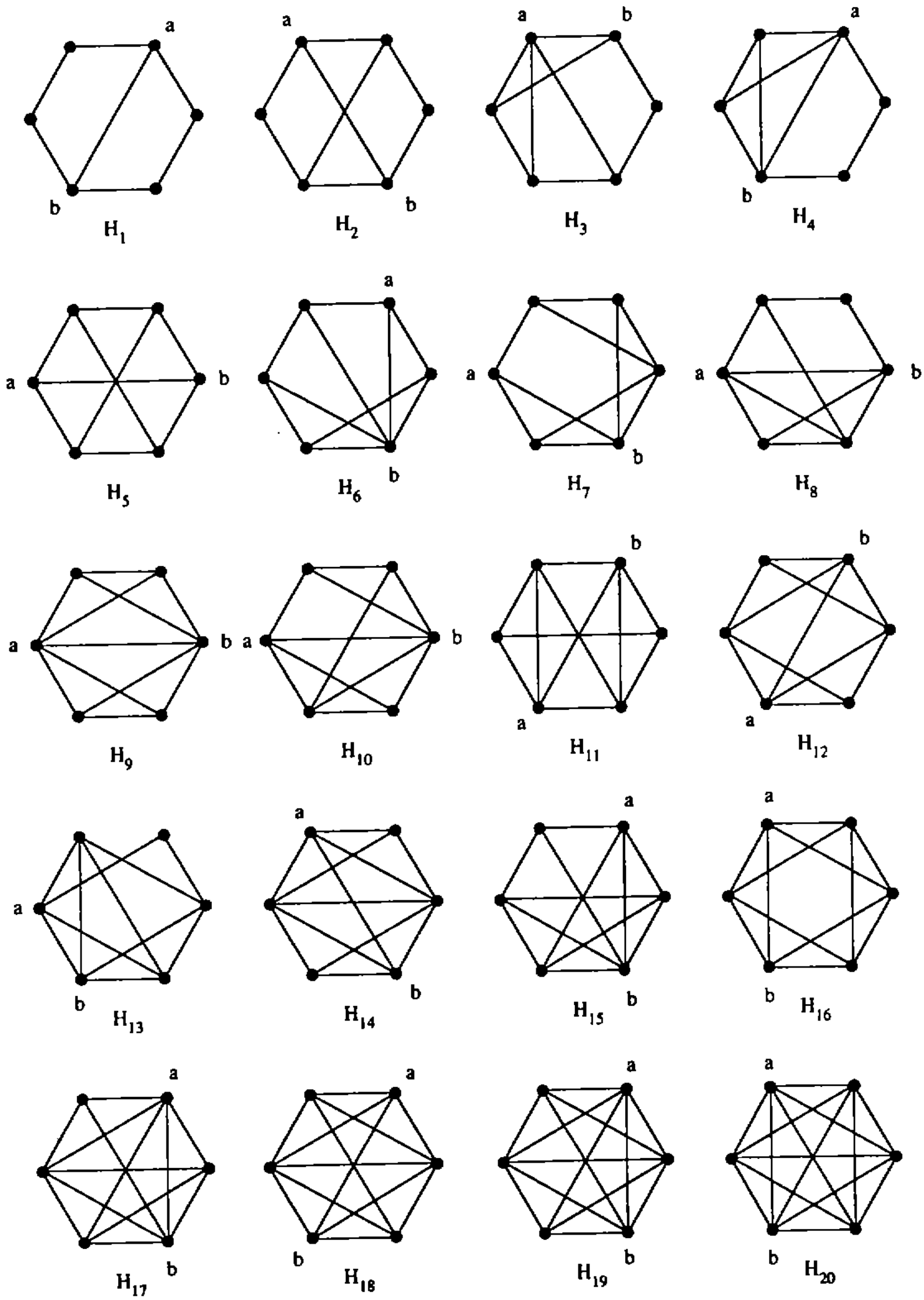


Figure 4.3.5

Notice that for each i, i = 1,...,20, H_i-ab is 1-extendable. Hence, none of the graphs displayed in Figure 4.3.5 are 1-minimal. This completes the proof of our theorem. □

Remark 4.3.6: By Theorem 4.2.1, H_{16}, H_{18}, H_{19} and H_{20} are not 1-minimal since each of them has minimum degree at least 4.

CHAPTER 5

STRONGLY EDGE-CRITICAL GRAPHS OF GIVEN DIAMETER

In chapters 3 and 4, we investigated critical and minimal graphs with respect to the extendability property. In this chapter, we consider graphs that are edge-critical with respect to diameter. Recall that a simple undirected graph G of diameter k is said to be strongly t-edge-critical or simply (k,t)-critical if for every $E' \subseteq E(G)$, $G-E'$ has diameter greater than k if and only if $|E'| \geq t$ and we denote the class of (k,t)-critical graphs by $\mathcal{G}(k,t)$.

For $t \geq 2$, Kys (1981) conjectured that $\mathcal{G}(k,t) = \phi$ for $k \geq 2$. He proved the conjecture for about half the cases: for $k = 2$; $k = 3$; $k = 4$ and $t \geq 3$; and for $t \geq k \geq 2$. In this chapter, we establish that for $t \geq 2$, $\mathcal{G}(4,t) = \mathcal{G}(5,t) = \phi$ and for $t \geq 3$, $\mathcal{G}(k,t) = \phi$ for every $k \geq 2$. This leaves the only unresolved cases as: $k \geq 6$, $t = 2$.

Section 5.1 contains some preliminary results proved by Kys (1981). One interesting result is that for $t \geq 2$, $\mathcal{G}(k,t) = \phi$ implies that $\mathcal{G}(k,t + 1) = \phi$. Hence, to prove that $\mathcal{G}(k,t) = \phi$; $k \geq 2$, $t \geq 2$ it is sufficient to prove that $\mathcal{G}(k, 2) = \phi$; $k \geq 2$.

In Section 5.2, we present some important properties of (k,t)-critical graphs which are crucial in establishing our main results. In particular, we prove that if u is a vertex of $G \in \mathcal{G}(k,t)$; $k \geq 2$, $t \geq 2$ and the eccentricity of u is k, then $d_G(u) \geq 2t - 2$. Moreover, if $k = 4$ or 5 and $t = 2$, then $d_G(u) \geq 3$.

In Section 5.3, we show that $\mathcal{G}(k,t) = \phi$ for $k \geq 2$ and $t \geq 3$. We conclude this chapter by proving, in Section 5.4, that

$\mathcal{G}(k,t) = \phi$ for $k = 4$ or 5 and $t \geq 2$.

5.1. PRELIMINARIES

Let G be a graph of diameter k and u any vertex of G. Recall that the eccentricity of a vertex u is

$$ec_G(u) = \max_{v \in V(G)} \left\{ d_G(u,v) \right\}.$$

Let $L_i(u)$ denote the vertices of G that are at a distance i from u, $i = 0,1,2,\ldots,ec_G(u)$. We call $\{L_i(u)\colon i = 0,1,\ldots,ec_G(u)\}$ the **distance decomposition** of V(G) from the vertex u.

We denote the length of a path P in G by $|P|$. Further, for $E' \subseteq E(G)$, $P \cap E'$ denotes the set of edges of G which belong to P and E'. We now state a number of results of Kys (1981) which we make use of in our work.

Lemma 5.1.1: For $t \geq 2$ if $G \in \mathcal{G}(k,t)$, then $\delta(G) \geq t$. □

Lemma 5.1.2: For $t \geq 2$ if $\mathcal{G}(k,t) = \phi$, then $\mathcal{G}(k,t+1) = \phi$. □

Lemma 5.1.3: Let $G \in \mathcal{G}(k,t)$, $k \geq 2$, $t \geq 2$, and let $E' = \{e_1,e_2,\ldots,e_t\}$ be any set of t edges of G. Then for every two vertices m and n of G with $d_{G-E'}(m,n) > k$ there are t (m,n)-paths $P_1, P_2,\ldots, P_t$ in G such that $|P_i| \leq k$ and $P_i \cap E' = \{e_i\}$, $i = 1,2,\ldots,t$. □

Lemma 5.1.4: Let $G \in \mathcal{G}(k,t)$, $k \geq 2$, $t \geq 2$, and u a vertex of G having $ec_G(u) = k$. Then no two vertices of $L_k(u)$ are joined in G. □

Lemma 5.1.5: Let $G \in \mathcal{G}(k,t)$, $k \geq 2$, $t \geq 2$, and let u, x be vertices of G with $d_G(u,x) = k$. Let E′ be a set of t edges of G containing the edges uv and xy with $v \in L_1(u)$ and $y \in L_{k-1}(u)$. If for $m \in L_r(u)$ and $n \in L_s(u)$, $d_{G-E'}(m,n) > k$, then $r + s = k$. Furthermore, if every edge of $E' \setminus \{uv, xy\}$ is incident to u or x, then $d_G(m,n) = k$. □

Note that the vertices m and n in the above lemma exist for some r and s since G is (k,t)-critical.

5.2 PROPERTIES OF (k,t)-CRITICAL GRAPHS OF GIVEN DIAMETER

To establish our main results we need, in addition to the lemmas mentioned in Section 5.1, a number of further properties concerning the class $\mathcal{G}(k,t)$. Before presenting these new results we need to introduce some further terminology.

Let P be an (a,b)-path in a graph G. We say that vertex x **preceeds** vertex y on P if the (a,y)-section of P, denoted by P(a,y), contains the vertex x.

Our first lemma is essentially an extension of Lemma 5.1.5.

Lemma 5.2.1: Let $G \in \mathcal{G}(k,t)$, $k \geq 2$, $t \geq 2$, and let u, x be vertices of G with $d_G(u,x) = k$. Let E′ be a set of t edges of G containing the edges uv and xy with $v \in L_1(u)$ and $y \in L_{k-1}(u)$. If for $m \in L_r(u)$ and $n \in L_s(u)$, $d_{G-E'}(m,n) > k$, then there exists an (m,n)-path P_1 in G of length at most k containing the edge uv such that either

(i) $|P_1(m,v)| = r - 1$ and $|P_1(u,n)| = s$

or

$$\text{(ii)} \qquad |P_1(m,u)| = r \text{ and } |P_1(v,n)| = s - 1.$$

Proof: Lemma 5.1.3 implies the existence of an (m,n)-path P_1 of length at most k containing the edge uv. So we need only establish that P_1 satisfies condition (i) or (ii). Suppose that v preceeds u on P_1. Then clearly $|P_1(m,v)| \geq r - 1$ and $|P_1(u,n)| \geq s$. Further

$$|P_1(m,v)| = |P_1| - |P_1(u,n)| - 1 \leq k - s - 1,$$

and hence, since by Lemma 5.1.5, $r + s = k$,

$$|P_1(m,v)| \leq r - 1.$$

This proves (i). When u preceeds v on P_1 the same argument yields (ii). This completes the proof of the lemma. □

As a corollary we have:

Corollary 5.2.2: Assume the hypothesis of Lemma 5.2.1 and let uw be an edge of $E'\backslash\{uv,xy\}$. If P_2 is an (m,n)-path of length at most k in G containing the edge uw, then w preceeds u on P_2 if condition (i) of Lemma 5.2.1 holds.

Proof: Suppose that condition (i) of Lemma 5.2.1 holds and u preceeds w on P_2. Then condition (ii) of Lemma 5.2.1 holds for P_2. But then, by Lemma 5.1.3

$$P_2(m,u) \cup P_1(u,n)$$

contains an (m,n)-path in G–E′ of length at most $r + s = k$, a contradiction. This completes the proof. □

Remark 5.2.1: If the length of P_i, $i = 1, 2$ is exactly k, then at most two edges of P_i join vertices of $L_j(u)$ to vertices of $L_{j+1}(u)$, $0 \le j \le k - 1$. Furthermore, there is exactly one edge of P_i between $L_j(u)$ and $L_{j+1}(u)$ for $r \le j \le s - 1$.

In the proofs that follow we make frequent use of the following simple fact which follows from Lemma 5.1.4.

Lemma 5.2.3: Let $G \in \mathcal{G}(k,t)$, $k \ge 2$, $t \ge 2$. If $d_G(u,x) = k$, then $d_G(v,x) = k - 1$ for every $v \in N_G(u)$. □

Our next two lemmas are important in establishing a lower bound on the degree of vertices of $G \in \mathcal{G}(k,t)$ having eccentricity k.

Lemma 5.2.4: Let $G \in \mathcal{G}(k,t)$, $k \ge 2$, $t \ge 2$, and let u, x be vertices of G with $d_G(u,x) = k$. Let P_1 be a (v,x)-path, $v \in L_1(u)$, in G of length $k - 1$ and E' a set of t edges of $E(G)\setminus(\{uv\} \cup E(P_1))$ containing the edges uw and xy. If for $m \in L_r(u)$ and $n \in L_s(u)$, $d_{G-E'}(m,n) > k$ and $r + s = k$, then $r \ge 1$ and $s \ge 1$. Moreover, if $t \ge 3$ and there are at least two edges of E' incident to u, then $r \ge 2$ and $s \ge 2$.

Proof: Without any loss of generality suppose that $r \le s$. We need to prove that $r \ne 0$. By Lemma 5.1.3 there exist (m,n)-paths Q_1 and Q_2 in G of length at most k such that

$$Q_1 \cap E' = \{uw\}$$

and

$$Q_2 \cap E' = \{xy\}.$$

If $r = 0$, then $s = k$ and thus $m = u$ and $n \in L_k(u)$. Since P_1 is a (v,x)-path in $G-E'$ of length $k - 1$ and $uv \notin E'$, $n \neq x$. But then the path Q_2 which contains the edge xy cannot be of length at most k, a contradiction. Hence, $r \neq 0$, proving the first part of the lemma.

Now suppose that $t \geq 3$ and $uz \in E'$, $z \neq w$. Let Q_3 be the (m,n)-path in G of length at most k such that

$$Q_3 \cap E' = \{uz\}.$$

Suppose that $r = 1$. Then $s = k - 1$ and so $m \in L_1(u)$ and $n \in L_{k-1}(u)$. If $m = w$, then Q_3 has length greater than k, since $uw \notin Q_3$. Hence, $m \neq w$ and similarly $m \neq z$. Furthermore, every (m,n)-path in G containing uw or uz of length at most k must contain the edge mu. But then

$$d_{G-E''}(m,n) \geq d_{G-E'}(m,n) > k$$

where

$$E'' = \{mu\} \cup (E' \setminus \{uw,uz\}) ,$$

contradicting the fact that G is (k,t)-critical. This proves that $r \neq 1$ thus completing the proof of the lemma. □

Lemma 5.2.5: Let $G \in \mathcal{G}(k,t)$, $k \geq 2$, $t \geq 2$, and let $E' = \{e_1, e_2, \ldots, e_t\}$ be any set of t edges of G. If for every two vertices m and n of G with $d_G(m,n) = k$ and $d_{G-E'}(m,n) > k$ there are t (m,n)-paths $P_1, P_2, \ldots, P_t$ such that $P_i \cap E' = \{e_i\}$, $i = 1,2,\ldots,t$, then the paths $P_1, P_2, \ldots, P_t$ are pairwise edge-disjoint.

Proof: Clearly, if $e' \in P_i \cap P_j$, $i \neq j$, then $d_{G-E''}(m,n) \geq d_{G-E'}(m,n) > k$, where $E'' = \{e'\} \cup (E' \setminus \{e_i, e_j\})$, contradicting the fact that $G \in \mathcal{G}(k,t)$. This proves the lemma. □

We are now ready to prove the main result of this section.

Theorem 5.2.6: Let $G \in \mathcal{G}(k,t)$, $k \geq 2$, $t \geq 2$. If $ec_G(u) = k$, then $d_G(u) \geq 2t - 2$.

Proof: Let $L_1(u) = \{u_1, u_2, \ldots, u_\ell\}$ and $x \in L_k(u)$. Then by Lemma 5.1.1, $\ell \geq t$ and hence we only need to consider the case $t \geq 3$. Since $G \in \mathcal{G}(k,t)$, there are, in G, at least t edge-disjoint (u,x)-paths of length k. Let $P_1, P_2, \ldots, P_t$ be any t such paths and without any loss of generality suppose that $uu_i \in P_i$, $i = 1,2,\ldots,t$.

Now consider the t edges

$$E' = \{uu_1, uu_2, \ldots, uu_{t-1}, xy\}$$

where $y \notin P_t$. Then, by lemmas 5.1.5 and 5.2.4, there exist vertices $m \in L_r(u)$ and $n \in L_s(u)$ with $d_{G-E'}(m,n) > k$, $r + s = k$ and $s \geq r \geq 2$. Further, $d_G(m,n) = k$. Lemma 5.1.3 implies the existence of (m,n)-paths $Q_1, Q_2, \ldots, Q_t$, in G, of length k with $Q_i \cap E' = \{uu_i\}$ for $i = 1,2,\ldots,t-1$ and $Q_t \cap E' = \{xy\}$. These t paths are, by Lemma 5.2.5, pairwise edge-disjoint. Now, since each Q_i, $i = 1,2,\ldots,t-1$, contains 2 edges incident to u, $d_G(u) \geq 2(t-1)$, as required. □

For the case when $G \in \mathcal{G}(k,2)$, $k = 4$ or 5 we have the following lower bound on the degree of a vertex of G having eccentricity k.

Theorem 5.2.7: Let $G \in \mathcal{G}(k,2)$, $k = 4$ or 5. If $ec_G(u) = k$, then $d_G(u) \geq 3$.

Proof: Suppose to the contrary that $d_G(u) \leq 2$. Then, by Lemma 5.1.1, $d_G(u) = 2$. Let $L_1(u) = \{v,w\}$, $x \in L_k(u)$ and let P_1 and P_2 be the two edge-disjoint (u,x)-paths in G. Without any loss of generality let $uv \in P_1$ and $uw \in P_2$. Now consider the edges $E' = \{uv,xy\}$, where $y \notin P_2$. Then, by lemmas 5.1.5 and 5.2.4, there exist vertices $m \in L_r(u)$ and $n \in L_s(u)$ with $d_{G-E'}(m,n) > k$, $r + s = k$ and $s \geq r \geq 1$.

As in the proof of Theorem 5.2.6 there exist (m,n)-paths Q_1 and Q_2 in G of length k with $Q_1 \cap E' = \{uv\}$ and $Q_2 \cap E' = \{xy\}$.

If v preceeds u on Q_1, then, since $d_{G-uw}(u,n) \leq k$, we have $d_{G-uw}(v,n) \leq k - 1$. Let R denote a (v,n)-path of length at most $k - 1$ in G-uw. Now, since $k = 4$ or 5 and $s \geq r \geq 1$, we have $r = 1$ or 2. If $r = 1$, then $m = v$ and hence $d_{G-E'}(m,n) = d_{G-uw}(m,n) \leq k - 1$, a contradiction. If $r = 2$, then $mv \in E(G)$ and hence

$$R \cup \{mv\}$$

is an (m,n)-path of length at most k in G-E′, a contradiction. Hence, v does not preceed u on Q_1. A similar argument will establish that u cannot preceed v on Q_1. Hence the lemma. □

5.3 THE CLASS OF (k,t)-CRITICAL GRAPHS FOR k ≥ 2 AND t ≥ 3

In this section, we prove that $\mathcal{G}(k,t) = \phi$ for $k \geq 2$ and $t \geq 3$.

Theorem 5.3.1: $\mathcal{G}(k,t) = \phi$ for $k \geq 2$ and $t \geq 3$.

Proof: In view of Lemma 5.1.2 we need only prove that $\mathcal{G}(k,3) = \phi$ for $k \geq 2$. Assume to the contrary that $\mathcal{G}(k,3) \neq \phi$, $k \geq 2$, and let $G \in \mathcal{G}(k,3)$.

Let u be a vertex of G with $ec_G(u) = k$. Let $L_1(u) = \{u_1, u_2, \ldots, u_\ell\}$ and $x \in L_k(u)$. Theorem 5.2.6 implies that $\ell \geq 4$. Since $G \in \mathcal{G}(k,3)$, there are at least three edge-disjoint (u,x)-paths of length k. Let P_1, P_2 and P_3 be three such paths and assume without any loss of generality that $uu_i \in P_i$, $i = 1,2,3$. Now consider the edges

$$E' = \{uu_1, uu_2, xy\},$$

where $y \notin P_3$. As in the proof of Theorem 5.2.6, there exist vertices $m \in L_r(u)$ and $n \in L_s(u)$ with $d_{G-E'}(m,n) > k$, $r + s = k$, $s \geq r \geq 2$, $d_G(m,n) = k$ and pairwise edge-disjoint (m,n)-paths Q_1, Q_2 and Q_3, in G, of length k with $Q_i \cap E' = \{uu_i\}$, for $i = 1,2$, and $Q_3 \cap E' = \{xy\}$.

Since $s \geq r \geq 2$, $k = r + s \geq 4$, thus we have nothing to prove for $k \leq 3$. For $k \geq 4$ we establish our contradiction by considering the distance decomposition of vertex m. Clearly, $u \in L_r(m)$ and $x \in L_s(m)$. Lemma 5.2.1 and Corollary 5.2.2 imply that either $u_1, u_2 \in L_{r-1}(m)$ (when u_1 preceeds u on Q_1) or $u_1, u_2 \in L_{r+1}(m)$ (when u preceeds u_1 on Q_1). Further, y is in $L_{s-1}(m)$ or $L_{s+1}(m)$.

Choose vertices m_1, $m_2 \in L_1(m)$ and $n_1 \in L_{k-1}(m)$ such that $m_1, n_1 \notin Q_1 \cup Q_2 \cup Q_3$ and $m_2 \in Q_2$. Such vertices exist since, by Theorem 5.2.6, both m and n have degree at least four. Let

$$E'' = \{mm_1, mm_2, nn_1\}.$$

We will establish that $d(G-E'') = k$, contradicting the criticality of G. Suppose to the contrary that $d(G-E'') > k$. Then there exist vertices $a \in L_{r^*}(m)$ and $b \in L_{s^*}(m)$ with $d_{G-E''}(a,b) > k$ and $r^* + s^* = k$. Further, by lemmas 5.1.3, 5.1.5, 5.2.4 and 5.2.5

we have : $r^* \geq 2$, $s^* \geq 2$; and pairwise edge-disjoint (a,b)-paths R_1, R_2 and R_3, in G, of length k with $R_i \cap E'' = \{mm_i\}$ for $i = 1,2$ and $R_3 \cap E'' = \{nn_1\}$.

Let H be the subgraph of G formed by taking the union of the three paths R_1, R_2 and R_3. Observe that H is a connected graph of diameter k containing m and n. We will establish the required contradiction by showing that H contains an (m,n)-path $\hat{Q}$ of length at most k such that $\hat{Q} \cap E' = \phi$. Note that such a $\hat{Q}$ would also be an (m,n)-path of length at most k in G–E′, a contradiction.

We assume without any loss of generality that $s^* \geq r^*$. Now we distinguish three cases according to the value of r^*.

Case 1: $2 \leq r^* \leq r - 1$

In this case $s^* \geq s + 1$ since $k = r^* + s^* = r + s$. Since $r \leq s$ we have $r^* \leq r - 1 < r \leq s < s^* \leq k - 2$. The situation is depicted in Figure 5.3.1 below. Note that in all our figures we write L_i for $L_i(m)$.

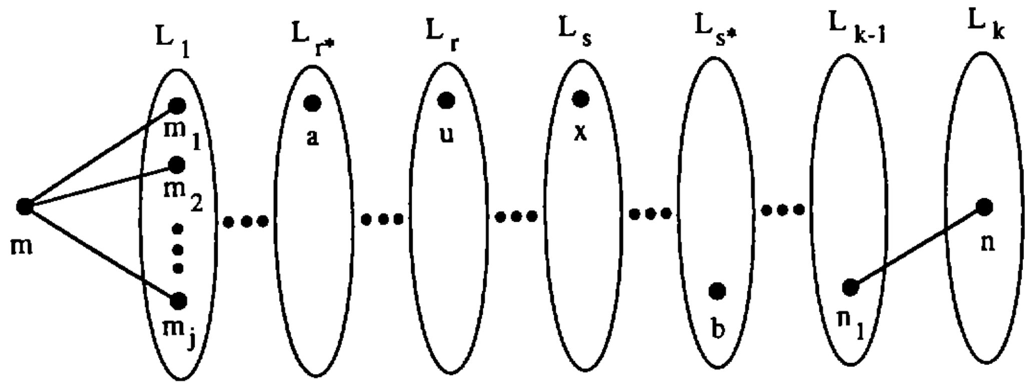


Figure 5.3.1

Consequently, $xy \notin R_3(b,n)$ since y is in $L_{s-1}(m)$ or $L_{s+1}(m)$, $x \in L_s(m)$ and $|R_3| = k$. Further, by Remark 5.2.1, the section $R_3(b,n)$ contains neither uu_1 nor uu_2. Now if $R_2 \cap E' = \phi$, then

$$R_2(m,b) \cup R_3(b,n)$$

is an (m,n)-path of length

$$|R_2(m,b)| + |R_3(b,n)| = s^* + k - s^* = k$$

in G-E′, a contradiction. Hence, $R_2 \cap E' \neq \phi$.

Suppose $uu_1 \in R_2$. If m_2 preceeds m on R_2, then the subgraph

$$R_2(a,m_2) \cup Q_2(m_2,u) \cup R_2(u,b)$$

contains (see Figure 5.3.2) an (a,b)-path of length at most

$$(r^* - 1) + (r - 1) + (s^* - r) = r^* + s^* - 2 < k$$

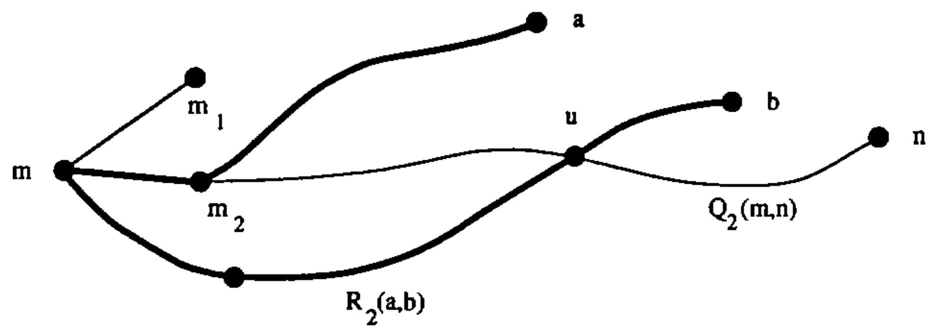


Figure 5.3.2

having no edges of E″, a contradiction. Hence, m preceeds m_2 on R_2. But then the subgraph

$$R_2(a,m) \cup Q_1(m,u) \cup R_2(u,b)$$

contains (see Figure 5.3.3) an (a,b)-path of length at most

$$r^* + r + s^* - r = k,$$

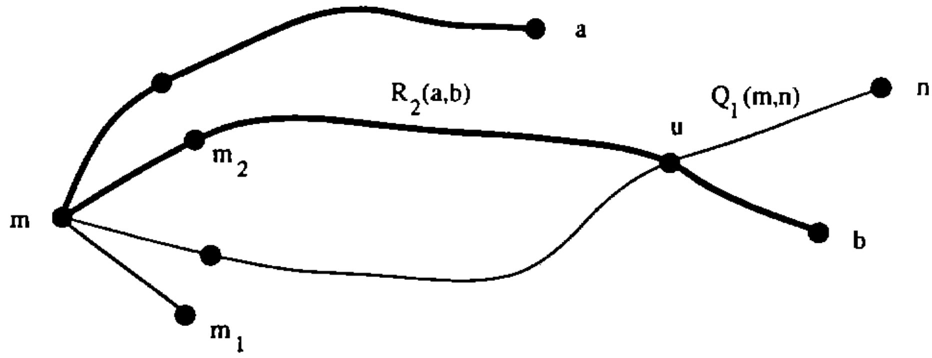


Figure 5.3.3

having no edges of E″, again a contradiction. So $uu_1 \notin R_2$. Similarly, $uu_2 \notin R_2$.

The only possibility then is for $xy \in R_2$. In this case $xy \notin R_1 \cup R_3$. Consequently, if $R_1 \cap \{uu_1, uu_2\} = \phi$, then the subgraph

$$R_1(m,b) \cup R_3(b,n)$$

contains an (m,n)-path of length at most $s^* + k - s^* = k$ in G-E′, a contradiction. Hence, $R_1 \cap \{uu_1, uu_2\} \neq \phi$.

Suppose $uu_1 \in R_1$. If m preceeds m_1 on R_1, then the subgraph

$$R_1(a,m) \cup Q_1(m,u) \cup R_1(u,b)$$

contains (see Figure 5.3.4) an (a,b)-path of length at most

$$r^* + r + s^* - r = k$$

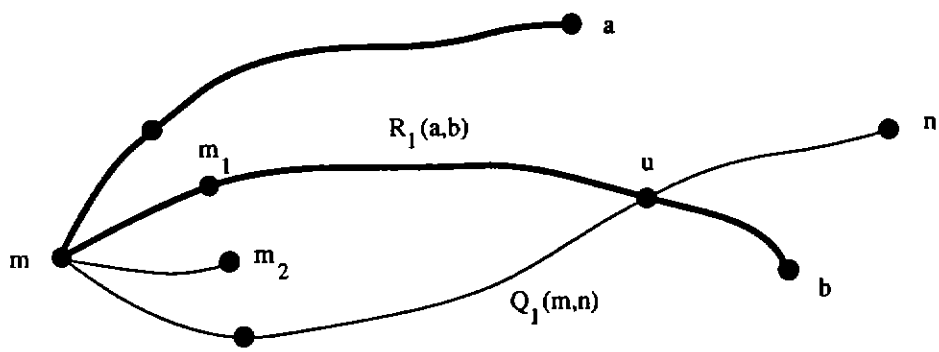


Figure 5.3.4

having no edges of E″, a contradiction. Therefore, m_1 preceeds m on R_1. But then, by Lemma 5.2.1 and Corollary 5.2.2, m_2 preceeds m on R_2. Consequently, the subgraph

$$R_2(a,m_2) \cup Q_2(m_2,u) \cup R_1(u,b)$$

contains (see Figure 5.3.5) an (a,b)-path of length at most

$$r^* - 1 + r - 1 + s^* - r = k - 2$$

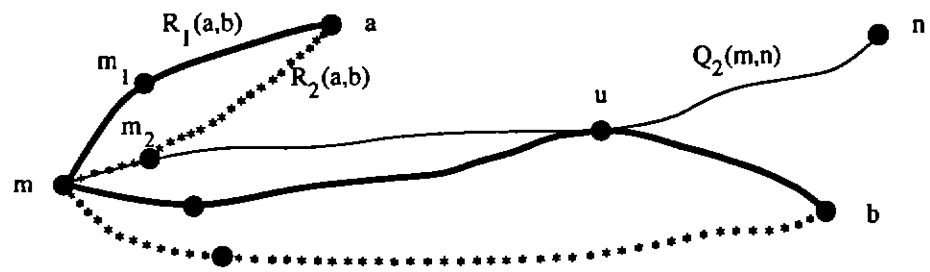


Figure 5.3.5

having no edges of E″, again a contradiction. Hence, $uu_1 \notin R_1$. Similarly, $uu_2 \notin R_1$. Therefore, $R_1 \cap \{uu_1,uu_2\} = \phi$ and hence $xy \notin R_2$. This completes the proof for Case 1.

Case 2: $r^* = r$

In this case $s^* = s$ and so $a,u \in L_r(m)$ and $b,x \in L_s(m)$. Note that a could be u and b could be x. Suppose first that $a = u$. Then $b \neq x$, since otherwise $Q_1(u,m) \cup Q_3(m,x)$ would be a (u,x)-path in G–E″ of length k. Consequently, $R_3(b,n) \cap E' = \phi$, since $r \le s$, $|R_3| = k$, $b \in L_s(m)$ and Remark 5.2.1. Therefore, if $R_2 \cap E' = \phi$, then as in Case 1

$$R_2(m,b) \cup R_3(b,n)$$

is an (m,n)-path of length k in G–E′. Hence, $R_2 \cap E' \neq \phi$.

Suppose that $uu_1 \in R_2$. If m_2 preceeds m on R_2, then the subgraph

$$Q_1(a,m) \cup R_2(m,b)$$

contains an (a,b)-path of length k having no edges of E″, a contradiction. Hence, m preceeds m_2 on R_2. But then the subgraph

$$Q_2(u,m_2) \cup R_2(m_2,b)$$

contains an (a,b)-path of length $(r^* - 1) + (s^* - 1) < k$ having no edges of E″, again a contradiction. Hence, $uu_1 \notin R_2$. Similarly, $uu_2 \notin R_2$. So the only possibility is for $xy \in R_2$ and hence $y \in L_{s-1}(m)$. But then, noting Remark 5.2.1, we must have $b = x$, a contradiction. This proves that $a \neq u$.

Again we will prove that $R_2 \cap E' = \phi$. Suppose this is not the case. Since $a, u \in L_r(m)$ and $a \neq u$, if $uu_1 \in R_2$ or $uu_2 \in R_2$, then m preceeds u on R_2. As in the proof of Case 1, we have $R_2 \cap \{uu_1,uu_2\} = \phi$ and $R_1 \cap \{uu_1,uu_2\} = \phi$. The only possibility is for $xy \in R_2$ and hence $xy \notin R_1 \cup R_3$. Recall that $b \in L_s(m)$. Consequently, if $r < s$, then clearly $b \neq u$ and if $r = s$, then, by a similar argument that used in case $a = u$, we can establish that $b \neq u$. Consequently, $R_3(b,n) \cap \{uu_1,uu_2\} = \phi$ and thus $R_3(b,n) \cap E' = \phi$. But then

$$R_1(m,b) \cup R_3(b,n)$$

is an (m,n)-path of length k in G−E′, a contradiction. Thus, $xy \notin R_2$ and hence $R_2 \cap E' = \phi$. Now if $xy \notin R_3(b,n)$, then

$$R_2(m,b) \cup R_3(b,n)$$

is an (m,n)-path of length k in G−E′, a contradiction. Hence, $xy \in R_3(b,n)$. Consequently, since $a \neq u$, $R_3(a,n) \cap E' = \phi$. But then

$$R_2(m,a) \cup R_3(a,n)$$

is an (m,n)-path of length k in G−E′, a contradiction. This completes the proof of the Case 2.

Case 3: $r^* \geq r + 1$

In this case, since $r^* + s^* = r + s$, we have $r + 1 \leq r^* \leq s^* \leq s - 1$. Hence, $r \leq s - 2$. Now, since $x \in L_s(m)$ and $|R_1| = |R_2| = k$, we have $xy \notin R_1 \cup R_2$. Further, $R_3 \cap \{uu_1, uu_2\} = \phi$ since $|R_3| = k$, $u \in L_r(m)$, $r \leq s - 2$ and $s^* \geq r + 1$. As in the previous cases we show that $R_2 \cap E' = \phi$.

Suppose that $uu_1 \in R_2$. If m_2 preceeds m on R_2, then one of the following subgraphs occurs :

$$R_2(a,u) \cup Q_1(u,m) \cup R_2(m,b)$$

or

$$R_2(a,m_2) \cup Q_2(m_2,u) \cup R_2(u,b).$$

As each of these contains an (a,b)-path of length at most k having no edges of E'', we have a contradiction. Hence, m preceeds m_2 on R_2. But then one of the following subgraphs occurs :

$$R_2(a,u) \cup Q_2(u,m_2) \cup R_2(m_2,b)$$

or

$$R_2(a,m) \cup Q_1(m,u) \cup R_2(u,b).$$

As each of these contains an (a,b)-path of length at most k having no edge of E'', we again have a contradiction. Hence, $uu_1 \notin R_2$. Similarly, $uu_2 \notin R_2$ and so $R_2 \cap E' = \phi$. Now if $xy \in R_3(a,n)$, then $R_2(m,b) \cup R_3(b,n)$ is an (m,n)-path of length k in $G-E'$, a contradiction. Hence, $xy \notin R_3(a,n)$. If $xy \in R_3(b,n)$, then

$$R_2(m,a) \cup R_3(a,n)$$

is an (m,n)-path of length k in $G-E'$, again a contradiction. Consequently, $R_3 \cap E' = \phi$. Hence, $R_2 \cup R_3$ contains an (m,n)-path of length k having no edges of E', a contradiction. This completes the proof of the theorem. □

5.4 THE CLASS OF (k,t)-CRITICAL GRAPHS FOR $k = 4$ OR 5 AND $t \geq 2$

We now consider the case $(k,t) = (4,2)$. We begin with the following lemma.

Lemma 5.4.1: Let $G \in \mathcal{G}(4,2)$ and let $E' = \{uv,xy\}$ be a set of edges of G such that $d_G(u,x) = d_{G-E'}(u,x) = 4$. If $m \in L_r(u)$ and $n \in L_s(u)$, $d_{G-E'}(m,n) > 4$, then either $r = 1$ or $s = 1$.

Proof: The situation here is very similar to that in the proof of Theorem 5.3.1. Thus, there exist (m,n)-paths Q_1 and Q_2, in G, of length 4 with $Q_1 \cap E' = \{uv\}$ and $Q_2 \cap E' = \{xy\}$. Further, there are edges $E'' = \{mm_1, nn_1\}$ with $m_1, n_1 \notin Q_1 \cup Q_2$. There exist vertices $a \in L_{r^*}(m)$ and $b \in L_{s^*}(m)$ with $d_{G-E''}(a,b) > 4$, $r^* + s^* = 4$ and (a,b)-paths R_1 and R_2, in G, of length 4 with $R_1 \cap E'' = \{mm_1\}$ and $R_2 \cap E'' = \{nn_1\}$.

The subgraph $R_1 \cup R_2$ is a cycle of length 8 containing the vertices m and n. Consequently, $E' \subseteq R_1 \cup R_2$, since otherwise there would exist an (m,n)-path of length 4 not containing edges of E'.

Now assume that $r \neq 1$ and $s \neq 1$. Then, by lemmas 5.1.5 and 5.2.4, $r \geq 2$, $s \geq 2$, and $r + s = 4$. Thus, $r = s = 2$. We can without any loss of generality assume that v preceeds u on Q_1 and $r^* \leq s^*$. We now distinguish two cases according to the location of x and y on Q_2.

Case 1: x preceeds y on Q_2

The situation is depicted in Figure 5.4.1.

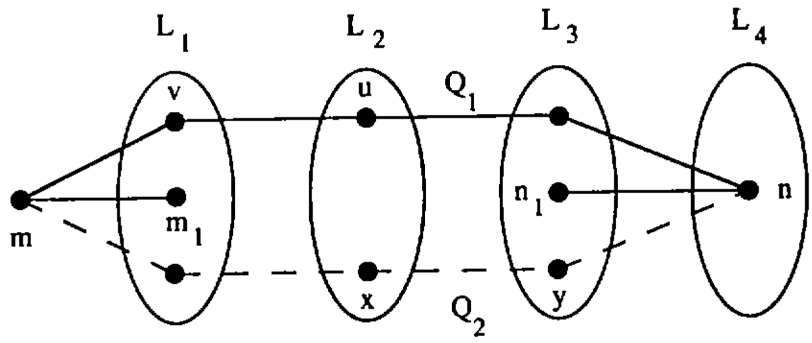


Figure 5.4.1

Suppose first that $r^* = 1$. Then $a \in L_1(m)$ and $b \in L_3(m)$. Since $nn_1 \in R_2$ and $|R_2| = 4$, $bn \in E(G)$. If $a = v$, then $R_1 \cap E' = \{xy\}$ since $E' \subseteq R_1 \cup R_2$ and R_1 has length 4 and passes through m_1. But then $y = b$ and hence $Q_1(v,n) \cup \{ny\}$ is an (a,b)-path of length 4 in G-E″, a contradiction. Hence, $a \neq v$. Since $E' \subseteq R_1 \cup R_2$, $a = m_1$ as otherwise $vu \notin R_1 \cup R_2$. Further, $R_1 \cap E' = \{uv\}$ and $R_2 \cap E' = \{xy\}$. By Theorem 5.2.7, $d_G(a) \geq 3$. Thus, there exists a vertex $a_1 \in N_G(a) \backslash (R_1 \cup R_2)$. Hence, by Lemma 5.2.3, $d_G(a_1,b) = 3$. Since $d_{G-E''}(a,b) > 4$, the (a_1,b)-path $\hat{R}$ of length 3 must contain one of the edges of E″. The only possibility is for $a_1 \in L_2(m)$, $nn_1 \in \hat{R}$ and thus $xy \notin \hat{R}$. But then $\{ma, aa_1\} \cup \hat{R}(a_1,n)$ is an (m,n)-path of length 4 in G-E′, a contradiction. This proves that $r^* \neq 1$.

Next we suppose that $r^* = 2$. Then $a,b \in L_2(m)$. Suppose $a = u$. Since $Q_1(u,m) \cup Q_2(m,x)$ is a (u,x)-path of length 4 in G-E″, $b \neq x$. But then $xy \notin R_1 \cup R_2$ as otherwise $|R_1|$ or $|R_2|$ is greater than 4, a contradiction. Hence, $a \neq u$. By the same argument we establish that a, b, u and x are distinct vertices. Since $|R_1| = |R_2| = 4$, neither R_1 nor R_2 contains xy. But then $E' \nsubseteq R_1 \cup R_2$, a contradiction. This completes the proof of Case 1.

Case 2: y preceeds x on Q_2

The situation is depicted in Figure 5.4.2.

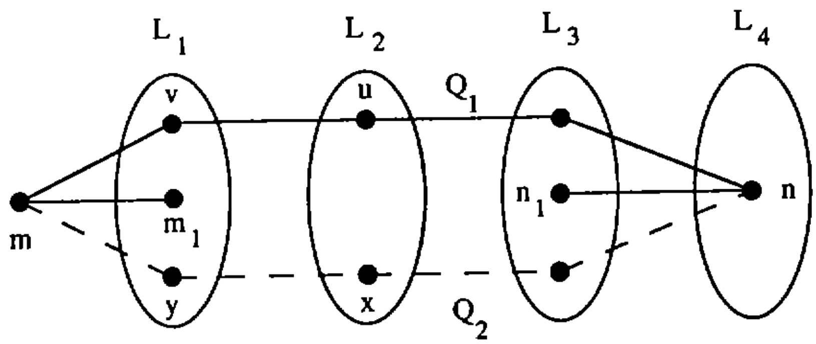


Figure 5.4.2

Clearly, if a = u (a = x), then b ≠ x (b ≠ u). Since $mm_1 \in R_1$, $|R_1| = 4$, $nn_1 \in R_2$ and $|R_2| = 4$ we must have $|R_1 \cap E'| \le 1$ and $|R_2 \cap E'| \le 1$. Further, if $|R_1 \cap E'| = 1$, then $R_2 \cap E' = \phi$. Hence, $E' \nsubseteq R_1 \cup R_2$, a contradiction. This completes the proof of the lemma. □

Theorem 5.4.2: $\mathfrak{F}(4,t) = \phi$ for $t \ge 2$.

Proof: In view of Lemma 5.1.2 we need only prove that $\mathfrak{F}(4,2) = \phi$. Assume to the contrary that $\mathfrak{F}(4,2) \ne \phi$ and let $G \in \mathfrak{F}(4,2)$.

Letting u and x be vertices of G with $d_G(u,x) = 4$ and following the same line of argument as in the proof of Theorem 5.3.1 we define edge-disjoint (u,x)-paths P_1 and P_2 of length 4 with $uv \in P_1$ and $uw \in P_2$. Further, we define

$$E' = \{uv, xy\}$$

where $y \notin P_2$. Observe that $d_{G-E'}(u,x) = 4$. Hence, by Lemma

5.4.1, there exist vertices $m \in L_1(u)$ and $n \in L_3(u)$ with $d_{G-E'}(m,n) > 4$. We take E″, a, b, Q_1, Q_2, R_1 and R_2 as in the proof of Lemma 5.4.1. Then $E' \subseteq R_1 \cup R_2$. Clearly, $d_{G-E''}(m,n) = 4$. By Lemma 5.4.1, $r^* = 1$ or $s^* = 1$. Without any loss of generality, assume that $r^* = 1$. Then, by Lemma 5.1.5, $s^* = 3$. We distinguish three cases according to the location of v and u on Q_1 and x and y on Q_2.

Case 1: v preceeds u on Q_1 and x preceeds y on Q_2

Then m = v and n = y. Figure 5.4.3 depicts the situation.

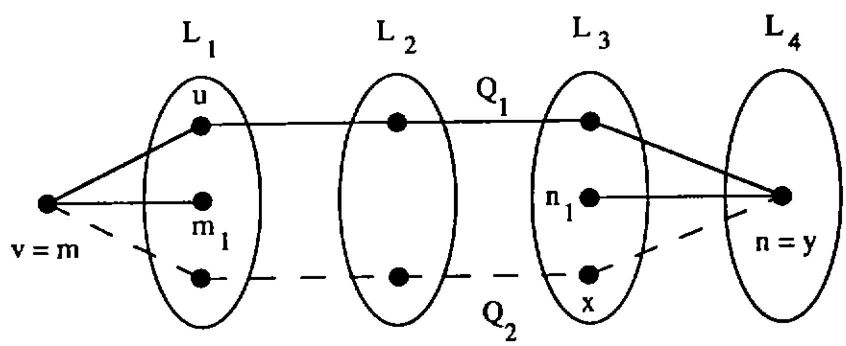


Figure 5.4.3

Observe that $xy \notin R_1$, since $mm_1 \in R_1$ and $|R_1| = 4$. Similarly, $uv \notin R_2$. As in the proof of Lemma 5.4.1, $E' \subseteq R_1 \cup R_2$. Consequently, $uv \in R_1$ and $xy \in R_2$.

Since $a \in L_1(m)$, $uv \in R_1$ and $|R_1| = 4$, we have a = u or m_1. Further, $bn \in E(G)$ since $nn_1 \in R_2$ and $|R_2| = 4$. If a = u, then b $\neq$ x and a must preceed m on R_1. But then $R_1(m,b) \cup \{bn\}$ is an (m,n)-path of length 4 in G–E′, a contradiction. Therefore, a $\neq$ u. Hence, $a = m_1$. Similarly, $b = n_1$. Now every (a,b)-path T of length 4, in G, must contain exactly one edge of E″. Further, if $m_1m \in T$ ($n_1n \in T$), then $mu \in T$ ($xy \in T$), for otherwise $T(m,b) \cup \{bn\}$ ($\{mm_1\} \cup T(m_1,n)$) is an (m,n)-path of length 4 in G–E′, a contradiction. Now $d_{G-E'}(a,b) > 4$, $a \in L_2(u)$ and $b \in L_2(u)$, contradicting Lemma 5.4.1. This completes the proof of Case 1.

Case 2: v preceeds u on Q_1 and y preceeds x on Q_2

Then $v = m$ and $y \in L_2(m)$. Figure 5.4.4 depicts the situation.

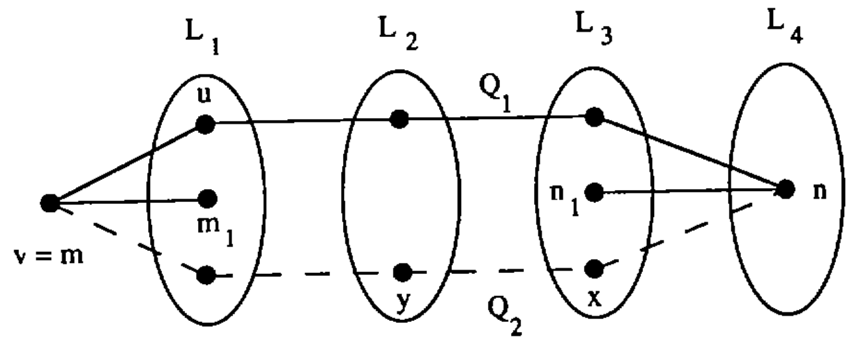


Figure 5.4.4

Observe that $uv \notin R_2$, since $nn_1 \in R_2$ and $|R_2| = 4$. Hence, since $E' \subseteq R_1 \cup R_2$, $uv \in R_1$.

Since $a \in L_1(m)$, $uv \in R_1$ and $|R_1| = 4$, we have $a = u$ or m_1. As in Case 1 above $a \neq u$. Consequently, $a = m_1$. Since $d_G(a) \geq 3$, there is a vertex $\beta \notin R_1 \cup R_2$ that is adjacent to a. By Lemma 5.2.3, $d_G(\beta,b) = 3$. Let S be a (β,b)-path of length 3. Since $d_{G-E''}(a,b) > 4$, S must contain mm_1 or nn_1. Therefore, since $b \in L_3(m)$, $\beta \in L_2(m)$. Now, since $S = (\beta, n_1, n, b)$, we have (m, m_1, β, n_1, n) is an (m,n)-path of length 4 in G-E', a contradiction. This completes the proof of Case 2.

Case 3: u preceeds v on Q_1 and y preceeds x on Q_2

The situation is depicted in Figure 5.4.5.

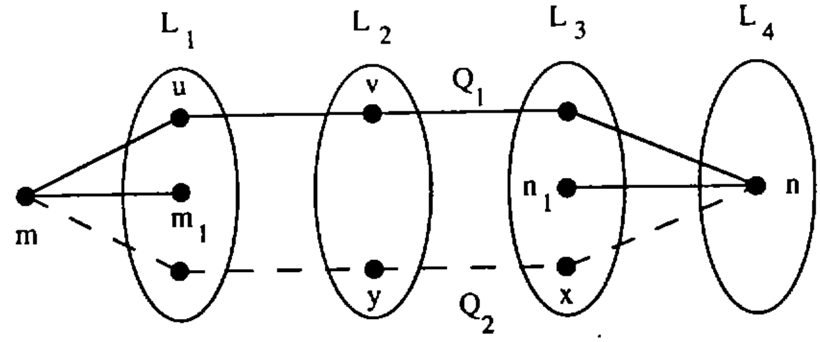


Figure 5.4.5

Recall that $r^* = 1$, then $a \in L_1(m)$ and $b \in L_3(m)$. Since $nn_1 \in R_2$ and $|R_2| = 4$, $bn \in E(G)$. If $a \neq u$ and m_1, then $uv \notin R_1 \cup R_2$ since $|R_1| = |R_2| = 4$. Consequently, $E' \not\subseteq R_1 \cup R_2$, a contradiction. Hence, $a = u$ or m_1. Similar argument as in Case 1 above establishes a contradiction to Lemma 5.4.1. This completes the proof of Case 3 and the proof of our theorem. □

The method of proof used in Lemma 5.4.1 and Theorem 5.4.2 can be applied to the case $k = 5$ with very little change. We give the detail of these proofs in Lemma 5.4.3 and Theorem 5.4.4. However, the methods do not extend beyond $k = 5$ and so the case $k \geq 6$, $t = 2$ remains unresolved.

Lemma 5.4.3: Let $G \in \mathcal{G}(5,2)$ and let $E' = \{uv, xy\}$ be edges of G such that $d_G(u,x) = d_{G-E'}(u,x) = 5$. If $m \in L_r(u)$, $n \in L_s(u)$ and $d_{G-E'}(m,n) > 5$, then either $r = 1$ or $s = 1$.

Proof: The situation here is very similar to that in the proof of Lemma 5.4.1 except for the diameter. Thus, there exist (m,n)-paths Q_1 and Q_2, in G, of length 5 with $Q_1 \cap E' = \{uv\}$ and $Q_2 \cap E' = \{xy\}$. Further, there are edges $E'' = \{mm_1, nn_1\}$ with m_1,

$n_1 \notin Q_1 \cup Q_2$. There exist vertices $a \in L_{r^*}(m)$ and $b \in L_{s^*}(m)$ with $d_{G-E''}(a,b) > 5$ and $r^* + s^* = 5$ and (a,b)-paths R_1 and R_2, in G, of length 5 with $R_1 \cap E'' = \{mm_1\}$ and $R_2 \cap E'' = \{nn_1\}$.

The subgraph $R_1 \cup R_2$ is a cycle of length 10 containing the vertices m and n. Consequently, $E' \subseteq R_1 \cup R_2$, since otherwise there would exist an (m,n)-path of length 5 not containing edges of E'.

Now assume that $r \neq 1$ and $s \neq 1$. Then, by Lemmas 5.1.5 and 5.2.4, $r \geq 2$, $s \geq 2$ and $r + s = 5$. Thus, $r = 2$ and $s = 3$ or $r = 3$ and $s = 2$. We can without any loss of generality assume that v preceeds u on Q_1, and $r^* \leq s^*$. We now distinguish two cases according to the value of r and s.

Case 1 : $r = 3$ and $s = 2$.

Subcase 1.1 x preceeds y on Q_2.

The situation is depicted in Figure 5.4.6.

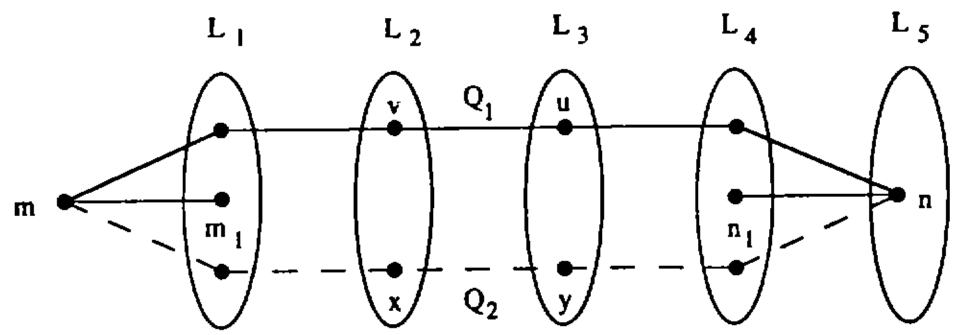


Figure 5.4.6

Suppose $r^* = 1$. Then $a \in L_1(m)$, $b \in L_4(m)$ and $|R_1 \cap E'| = |R_2 \cap E'| = 1$. Without any loss of generality, assume $vu \in R_1$ and $xy \in R_2$. Hence, $ax \in E(G)$ and $av \notin E(G)$ since $|R_2| = 5 = |R_1|$ and $d_G(a,b) = 5$. By Theorem 5.2.7, $d_G(a) \geq 3$. Thus, there exists a vertex $a_1 \in N_G(a) \setminus (R_1 \cup R_2)$. Hence, by Lemma 5.2.3, $d_G(a_1,b) = 4$.

Since $d_{G-E''}(a,b) > 5$, the (a_1,b)-path $\hat{R}$ of length 4 must contain one of the edges of E''. The only possibility is for $a_1 \in L_2(m)$, $nn_1 \in \hat{R}$ and thus $a_1 \neq x$ and $a_1 \neq v$. But then $\{ma, aa_1\} \cup \hat{R}(a_1,n)$ is an (m,n)-path of length 5 in $G-E'$, a contradiction. This proves that $r^* \neq 1$.

Next we suppose that $r^* = 2$. Then $a \in L_2(m)$ and $b \in L_3(m)$. Clearly, if $R_1 \cap E' = \phi$ or $R_2 \cap E' = \phi$, then $E' \nsubseteq R_1 \cup R_2$, since $E' \nsubseteq R_1$ and $E' \nsubseteq R_2$, a contradiction. Hence, $R_1 \cap E' \neq \phi$ and $R_2 \cap E' \neq \phi$. Suppose $xy \in R_1$ and $vu \in R_2$. Since $|R_1| = |R_2| = 5$, $mm_1 \in R_1$ and $nn_1 \in R_2$, we have $a = v$ and $b = y$, contradicting the fact that $Q_1(v,m) \cup Q_2(m,y)$ is a (v,y)-path in $G-E''$. Again we get a similar contradiction when $uv \in R_1$ and $xy \in R_2$. This completes the proof of subcase 1.1.

Subcase 1.2 y preceeds x on Q_2

Figure 5.4.7 depicts the situation.

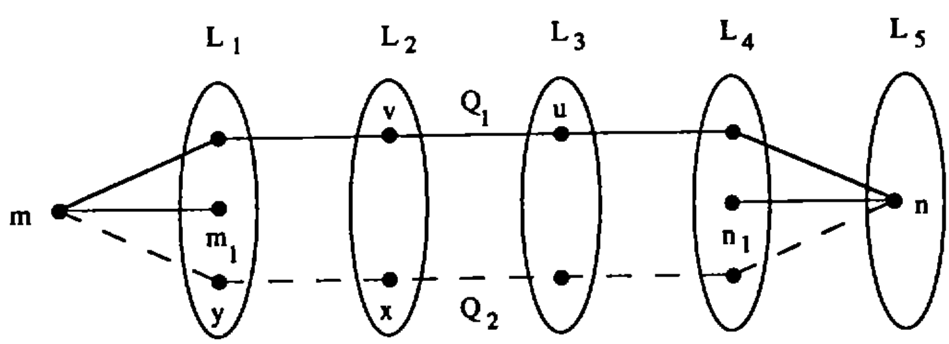


Figure 5.4.7

Suppose $r^* = 1$. Thus, $a \in L_1(m)$, $b \in L_4(m)$ and $|R_1 \cap E'| = |R_2 \cap E'| = 1$. If $a \neq m_1$ and $a \neq y$, then $yx \notin R_1 \cup R_2$, a contradiction. Hence, $a = m_1$ or $a = y$. Using the same argument as that used in Subcase 1.1 to prove $r^* \neq 1$, we can establish that $r^* \neq 1$. Hence, $r^* = 2$. Thus, $a \in L_2(m)$, $b \in L_3(m)$. Clearly, xy

$\notin R_2$. If $E' \subseteq R_1$, then $a = x$ and $b = u$, which contradicts to the fact that $Q_2(x,m) \cup Q_1(m,u)$ is a (x,u)-path in $G-E''$. Hence, $E' \nsubseteq R_1$. Therefore, $xy \in R_1$ and $uv \in R_2$ since $E' \subseteq R_1 \cup R_2$ and $xy \notin R_2$. Thus, $a = v$. By applying a similar argument to that used in Subcase 1.1 to prove $r^* \neq 1$, we can establish that $r^* \neq 2$. This completes the proof of Case 1.

Case 2 $r = 2$ **and** $s = 3$.

Subcase 2.1 x **preceeds** y **on** Q_2.

The situation is depicted in Figure 5.4.8.

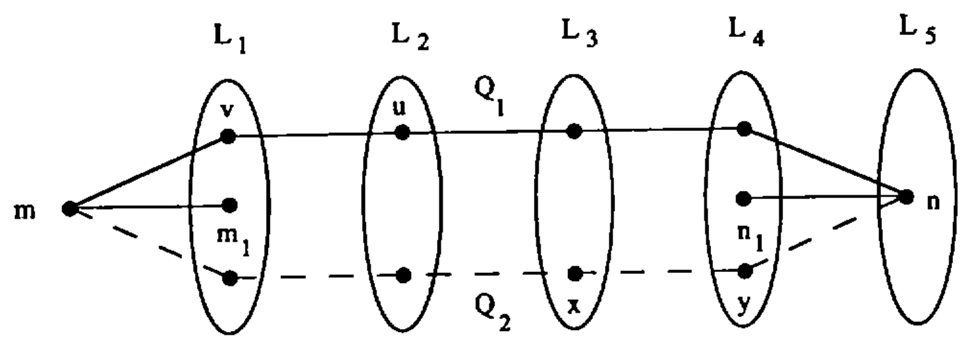


Figure 5.4.8

Suppose first that $r^* = 1$. Then $a \in L_1(m)$ and $b \in L_4(m)$. Since $nn_1 \in R_2$ and $|R_2| = 5$, $bn \in E(G)$. If $a = v$, then $R_1 \cap E' = \{xy\}$ since R_1 has length 5 and $R_1 \cap E'' = \{mm_1\}$. But then $y = b$ and hence $Q_1(v,n) \cup \{ny\}$ is a (v,y)-path of length 5 in $G-E''$, a contradiction. Hence, $a \neq v$. If $a \neq m_1$, then $uv \notin R_1 \cup R_2$ and thus $E' \nsubseteq R_1 \cup R_2$, a contradiction. Hence, $a = m_1$.

Similarly, $b = n_1$. Now every (a,b)-path T of length 5 must contain exactly one edge of E''. Further, if $m_1m \in T$ ($n_1n \in T$), then $vu \in T$ ($xy \in T$) for otherwise $T(m,b) \cup \{bn\}$ ($\{mm_1\} \cup T(m_1,n)$) is an (m,n)-path of length 5 in $G-E'$, a contradiction. Now $d_{G-E'}(a,b) > 5$, $a \in L_3(u)$ and $b \in L_2(u)$, contradicting case 1.

Hence, $r^* \neq 1$.

The only possibility is $r^* = 2$ and $s^* = 3$. Clearly, $xy \notin R_1$ and $vu \notin R_2$. Since $E' \subseteq R_1 \cup R_2$, $vu \in R_1$ and $xy \in R_2$. It is not difficult to verify that $a \neq u$. By applying an argument similar to that used in Subcase 1.1 to prove $r^* \neq 1$, we can establish that $r^* \neq 2$. This completes the proof of Subcase 2.1.

Subcase 2.2 y preceeds x on Q_2.

Figure 5.4.9 depicts the situation.

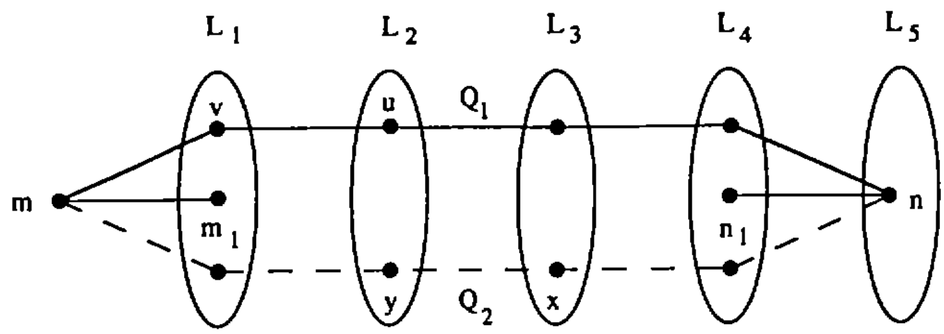


Figure 5.4.9

This situation is the same as Subcase 1.2. This completes the proof of the lemma. □

Theorem 5.4.4: $\mathcal{G}(5,t) = \phi$ for $t \geq 2$.

Proof: In view of Lemma 5.1.2 we need only prove that $\mathcal{G}(5,2) = \phi$. Assume to the contrary that $\mathcal{G}(5,2) \neq \phi$ and let $G \in \mathcal{G}(5,2)$. Letting u and x be vertices of G with $d_G(u,x) = 5$ and following the same line of argument as in the proof of Theorem 5.3.1 we define edge-disjoint (u,x)-paths P_1 and P_2 of length 5 with $uv \in P_1$ and $uw \in P_2$. Further, we define

$$E' = \{uv, xy\}$$

where $y \notin P_2$. Observe that $d_{G-E'}(u,x) = 5$. Hence, by Lemma 5.4.3, there exist vertices $m \in L_1(u)$ and $n \in L_4(u)$ with $d_{G-E'}(m,n) > 5$. We take E'', a, b, Q_1, Q_2, R_1 and R_2 as in the proof of Lemma 5.4.3. Then $E' \subseteq R_1 \cup R_2$. Clearly, $d_{G-E''}(m,n) = 5$. By Lemma 5.4.3, $r^* = 1$ or $s^* = 1$. Without any loss of generality, assume $r^* = 1$. Then, by Lemma 5.1.5, $s^* = 4$. We distinguish three cases according to the location of v and u on Q_1 and x and y on Q_2.

Case 1 v preceeds u on Q_1 and x preceeds y on Q_2

Then v = m and n = y. Figure 5.4.10 depicts the situation.

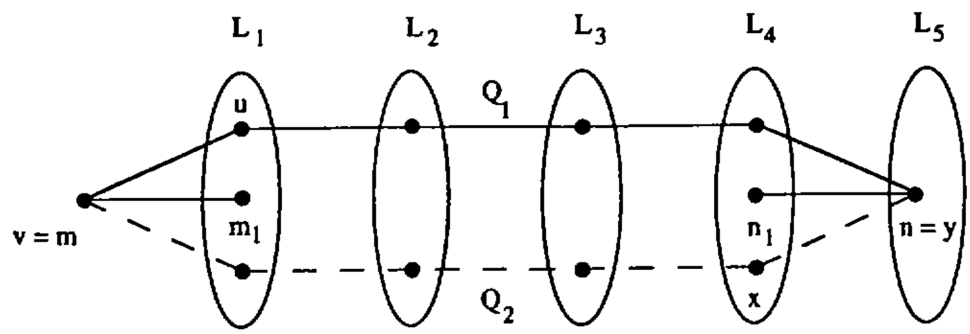


Figure 5.4.10

Observe that $xy \notin R_1$, since $mm_1 \in R_1$ and $|R_1| = 5$. Similarly, $uv \notin R_2$. Because $E' \subseteq R_1 \cup R_2$, $uv \in R_1$ and $xy \in R_2$.

Since $a \in L_1(m)$ and $b \in L_4(m)$, $bn \in E(G)$. Further, since $uv \in R_1$, $a = u$ or $a = m_1$. Similarly, $b = n_1$ or $b = x$. If $a = u$, then $b \neq x$ and a must preceed m on R_1. But then $R_1(m,b) \cup \{bn\}$ is an (m,n)-path of length 5 in G-E', a contradiction. Therefore, $a \neq u$. Hence, $a = m_1$. Similarly, $b = n_1$. Now every (a,b)-path T of length 5 in G must contain exactly one edge of E''. Further, if $m_1m \in T$ ($n_1n \in T$), then $mu \in T$ ($xy \in T$) for otherwise $T(m,b) \cup$

$\{bn\}$ $(\{mm_1\} \cup T(m_1,n))$ is an (m,n)-path of length 5 in G-E', a contradiction. Now $d_{G-E'}(a,b) > 5$, $a \in L_2(u)$ and $b \in L_3(u)$, contradicting Lemma 5.4.3. This completes the proof of Case 1.

Case 2 v preceeds u on Q_1 and y preceeds x on Q_2

Then v = m. The situation is depicted in Figure 5.4.11.

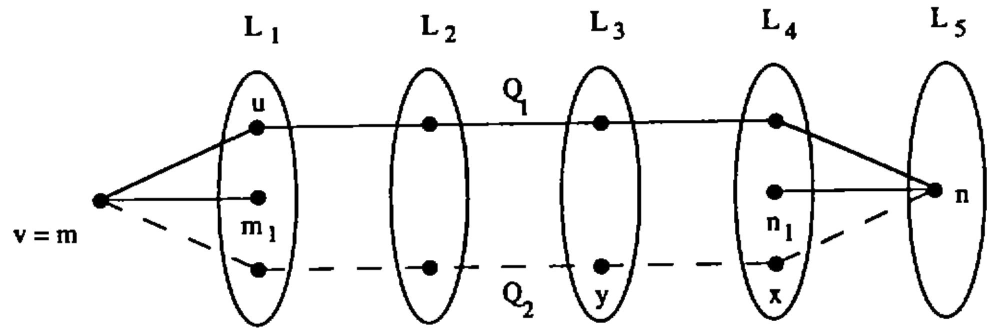


Figure 5.4.11

Observe that $uv \notin R_2$, since $nn_1 \in R_2$ and $|R_2| = 5$. Because $E' \subseteq R_1 \cup R_2$, we have $uv \in R_1$.

Since $a \in L_1(m)$, $uv \in R_1$ and $|R_1| = 5$, we have $a = u$ or $a = m_1$. As in Case 1 above, $a \neq u$. Consequently, $a = m_1$. If $xy \notin R_2$, then $\{mm_1\} \cup R_2(m_1,n)$ is an (m,n)-path in G-E' of length 5, a contradiction. Hence, $xy \in R_2$. This implies that $b = n_1$, since otherwise $|R_2| > 5$. Using the same argument as that used in Case 1, we can establish that, in this case, $r^* \neq 1$ which contradicts to Lemma 5.4.3. This completes the proof of Case 2.

Case 3 u preceeds v on Q_1 and y preceeds x on Q_2

The situation is depicted in Figure 5.4.12.

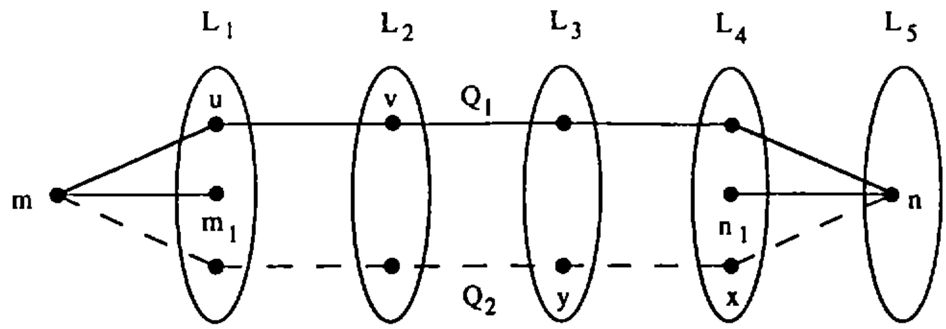


Figure 5.4.12

Since $nn_1 \in R_2$ and $|R_2| = 5$, $bn \in E(G)$. If $a \neq u$ and m_1, then $uv \notin R_1 \cup R_2$, since $|R_1| = |R_2| = 5$. Consequently, $E' \not\subseteq R_1 \cup R_2$, a contradiction. Hence, $a = u$ or m_1. Now using a similar argument as in Case 1 above establishes $r^* \neq 1$, contradicting Lemma 5.4.3. This completes the proof of our theorem. □

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