Nonlinear optimal feedback control for lunar module soft landing

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Abstract In this paper, the task of achieving the soft landing of a lunar module such that the fuel consumption and the flight time are minimized is formulated as an optimal control problem. The motion of the lunar module is described in a three dimensional coordinate system. We obtain the form of the optimal closed loop control law, where a feedback gain matrix is involved. It is then shown that this feedback gain matrix satisfies a Riccati-like matrix differential equation. The optimal control problem is first solved as an open loop optimal control problem by using a time scaling transform and the control parameterization method. Then, by virtue of the relationship between the optimal open loop control and the optimal closed loop control along the optimal trajectory, we present a practical method to calculate an approximate optimal feedback gain matrix, without having to solve an optimal control problem involving the complex Riccati-like matrix differential equation coupled with the original system dynamics. Simulation results show that the proposed approach is highly effective.

Keywords Feedback control · Optimal control · Lunar module · Soft landing

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1 Introduction

The moon is the nearest celestial body to the earth. Satellites and probes have been sent out to the moon for investigations. Among these missions, landing the lunar rover or astronauts safely on the moon surface is the most challenging one for space scientists.

The mission of the lunar module soft landing starts from a circular parking orbit of the moon which is 100km high above the moon surface. According to the preselected landing target, the lunar module is decelerated and enters into a lower energy elliptical orbit, i.e., the Hohmann transfer orbit, which is coplanar with the parking orbit. The elliptical Hohmann transfer orbit has the aposelene and the perilune which are, respectively, 100km and 15km distance away from the moon surface. When the module reaches the perilune, the power descent phase begins. Since there is negligible atmosphere surrounding the moon to be used by the lunar module for deceleration, the lunar soft landing can not be performed in the same way as landing on the earth or mars. Thus, to realize the task of soft landing, one way is to use the reverse force thruster to decelerate the velocity of the lunar module starting from the perilune. This together with the attitude control thrusters will guide the module to reach the landing target with a small and safe final velocity. However the fuel of the lunar module will be consumed substantially during this process. As the mass of the lunar module is always limited, it is extremely important that the fuel consumption is minimized. In this way, more payloads can be equipped (see, for example, [2], [6], [13], [19], [21], [23]). In [4], a feedback regulation scheme is proposed based on an off-line trajectory for a vertically controlled spacecraft to achieve the soft landing on a planet without atmosphere. In [16], a nonlinear neurocontrol method is developed based on the linearized system dynamics for the soft landing of a lunar module. In [22], an optimal control law for the soft landing of a lunar module is obtained by using the Pontryagin Maximum Principle. In [20], a suboptimal guidance law is obtained for achieving the soft landing of a lunar module under the assumption that the gravitational field on the moon surface is uniform. The guidance law is expressed as a function of time-to-go. In [11], an optimal open loop control strategy for the soft landing of a lunar module with a pre-specified terminal time is obtained by using the control parameterization technique in conjunction with a time scaling transform. In most relevant papers in the literature, including those mentioned above, the system of differential equations describing the motion of the lunar module is in a two-dimensional polar coordinate system and the effect of moon rotation is not taken into account. That is, the module is assumed to descend along a vertical plane in the Lunar Central Inertial Coordinate system. Because of the moon rotation, this assumption is not realistic. A lunar module does not necessarily descend along such a vertical plane. In [24], a three-dimensional coordinate system for the lunar module soft landing is presented, where the moon rotation is taken into consideration. An open loop optimal control law is derived by using the Pontryagin Maximum Principle. In this paper, as in [24], the system of ordinary differential equations describing the motion of a lunar module is in the three-dimensional coordinate system.

The task of achieving the soft landing of a lunar module at the minimum time with the least fuel consumption can be formulated as an optimal control problem, where the system dynamic is expressed in the form of an affine system. An optimal open loop control is first obtained by using a time scaling transform [18] and the control parameterization technique [17]. Then, we derive the form of the optimal closed loop control law, which involves a feedback gain matrix, for the optimal control problem.

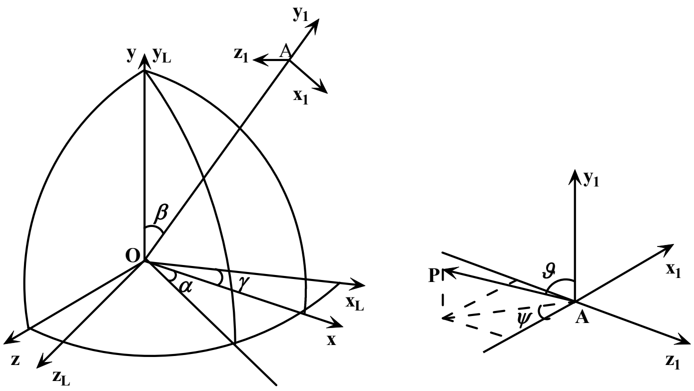

Fig. 1 Coordinate systems

The optimal feedback gain matrix is required to satisfy a Riccati-like matrix differential equation. Then, the third order B-spline function, which has been proved to be very efficient for solving optimal approximation and optimal control problems [15], is employed to construct the components of the feedback gain matrix. By virtue of the relationship between the optimal open loop control and the optimal closed loop control along the optimal trajectory, a practical computational method is presented for finding an approximate optimal feedback gain matrix, without having to solve an optimal control problem involving the complex Riccati-like matrix differential equation coupled with the original system dynamics.

2 Problem formulation

The motion of the lunar module soft landing is described in a three-dimensional coordinate system (Figure 1). Let $oxyz$ and $ox_Ly_Lz_L$ be, respectively, the Lunar Central Inertial Coordinate and Lunar Fixed Coordinate with the moon equator as the reference plane. $Ax_1y_1z_1$ is the orbit coordinate, A is the position of the lunar module. The three-dimensional coordinate forms a right handed system. α and β represent, respectively, the rotation angles between $oxyz$ and $Ax_1y_1z_1$. ϑ is the separation angle between P (the direction of the thrust force) and Ay_1. ψ is the separation angle of the projection of P onto the plane Ax_1z_1 with reference to the negative direction of Ax_1. So the direction of the thrust force P in the coordinate $Ax_1y_1z_1$ can be expressed in terms of ϑ and ψ. γ is the rotation angle between $oxyz$ and $ox_Ly_Lz_L$. Without loss of generality, we assume that $oxyz$ and $ox_Ly_Lz_L$ coincide with each other when the process of the soft landing begins. Based on Newton's second law, system dynamic equations can be derived to give [24],

$$\begin{cases} \dot{x}_L &= V_{xL}, \\ \dot{y}_L &= V_{yL}, \\ \dot{z}_L &= V_{zL}, \\ \dot{V}_{xL} &= C_1QV_r/m + g_{xL} - 2\omega_L V_{zL}, \\ \dot{V}_{yL} &= C_2QV_r/m + g_{yL}, \\ \dot{V}_{zL} &= C_3QV_r/m + g_{zL} + 2\omega_L V_{xL}, \\ \dot{m} &= -Q, \end{cases} \tag{1}$$

where

$$\begin{aligned}C_1 &= (\cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma)\sin\vartheta\cos\psi \\ &\quad -(\sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma)\sin\vartheta\sin\psi + \sin\beta\cos\gamma\cos\vartheta, \\ C_2 &= -\cos\alpha\sin\beta\sin\vartheta\cos\psi + \cos\beta\cos\vartheta + \sin\alpha\sin\beta\sin\vartheta\sin\psi, \\ C_3 &= (\cos\alpha\cos\beta\sin\gamma + \sin\alpha\cos\gamma)\sin\vartheta\cos\psi \\ &\quad -(\sin\alpha\cos\beta\sin\gamma - \cos\alpha\cos\gamma)\sin\vartheta\sin\psi + \sin\beta\sin\gamma\cos\vartheta,\end{aligned}$$

and x_L, y_L, z_L and V_{xL}, V_{yL}, V_{zL} are the positions and velocities in the Lunar Fixed Coordinate system, m is the mass of the lunar module, Q and V_r represent, respectively, the fuel consumption rate and the specific impulse of the thruster, g_{xL}, g_{yL} and g_{zL} denote the respective components of the lunar gravity in $ox_Ly_Lz_L$, and ω_L is the angular velocity of the moon rotation.

Introduce two new state equations

$$\dot{\vartheta} = v \tag{2}$$

$$\dot{\psi} = w \tag{3}$$

and let

$$\begin{aligned}\boldsymbol{x} &= [x_L, y_L, z_L, V_{xL}, V_{yL}, V_{zL}, \vartheta, \psi, m]^T \\ &= [x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9]^T\end{aligned} \tag{4}$$

$$\boldsymbol{u} = [Q, v, w]^T = [u_1, u_2, u_3]^T \tag{5}$$

The original system dynamics (1) can be rewritten in the form of a nonlinear affine system given below:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t)) + \boldsymbol{B}(\boldsymbol{x}(t), t)\boldsymbol{u}(t), \tag{6}$$

where $\boldsymbol{x} \in \mathbb{R}^9$, $\boldsymbol{u} \in \mathbb{R}^3$ and

$$\boldsymbol{f}(\boldsymbol{x}) = [x_4, x_5, x_6, g_{xL} - 2\omega_L x_6, g_{yL}, g_{zL} + 2\omega_L x_4, 0, 0, 0]^T, \tag{7}$$

$$\boldsymbol{B}(\boldsymbol{x}, t) = \begin{bmatrix} 0 & 0 & 0 & M_1 & M_2 & M_3 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T \tag{8}$$

where

$$\begin{aligned}M_1 = [&(\cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma)\sin x_7\cos x_8 \\ &- (\sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma)\sin x_7\sin x_8 \\ &+ \sin\beta\cos\gamma\cos x_7]V_r/x_9,\end{aligned} \tag{9}$$

$$\begin{aligned}M_2 = (&-\cos\alpha\sin\beta\sin x_7\cos x_8 + \sin\alpha\sin\beta\sin x_7\sin x_8 \\ &+ \cos\beta\cos x_7)V_r/x_9,\end{aligned} \tag{10}$$

and

$$\begin{aligned}M_3 = [&(\cos\alpha\cos\beta\sin\gamma + \sin\alpha\cos\gamma)\sin x_7\cos x_8 \\ &- (\sin\alpha\cos\beta\sin\gamma - \cos\alpha\cos\gamma)\sin x_7\sin x_8 \\ &+ \sin\beta\sin\gamma\cos x_7]V_r/x_9\end{aligned} \tag{11}$$

For u_1, the first component of the control $\boldsymbol{u}$, it is required to satisfy the boundedness conditions given below

$$\alpha_1 \leq u_1(t) \leq \beta_1, \quad \forall\, t \in [0, T] \tag{12}$$

We do not impose any bound on the other two components of the control $\boldsymbol{u}$. Let $\mathcal{U}$ be the set of all such controls $\boldsymbol{u} = [u_1, u_2, u_3]^T$. Elements from $\mathcal{U}$ are called admissible controls and $\mathcal{U}$ is referred to as the class of admissible controls.

The initial conditions of the soft landing are determined by the state of the lunar module in the perilune at the initial time $t_0 = 0$ and are given by

$$\boldsymbol{x}(t_0) = [x_{L0}, y_{L0}, z_{L0}, V_{xL_0}, V_{yL_0}, V_{zL0}, \vartheta_0, \psi_0, m_0]^T \tag{13}$$

Our aim is to design an optimal closed loop control law to achieve the soft landing of the lunar module such that a linear combination of the fuel consumption and the terminal time are minimized, while the terminal velocity should be approximately zero at the terminal time. The optimal control problem can be formulated as follows.

Given system (6) with the initial condition (13), find a closed loop control $\boldsymbol{u} \in \mathcal{U}$ such that the cost function

$$J = a_1 \Phi_0(\boldsymbol{x}(T)) + a_2 \int_0^T \boldsymbol{u}^T \boldsymbol{R}\, \boldsymbol{u}\, dt, \tag{14}$$

is minimized, where $\Phi_0(\boldsymbol{x}(T)) = (\boldsymbol{x}(T) - \boldsymbol{x}_d)^T \boldsymbol{S}(\boldsymbol{x}(T) - \boldsymbol{x}_d) + T$, T is the free terminal time of the soft landing, $\boldsymbol{x}_d$ is the desired terminal state vector, a_1 and a_2 are the weighting parameters which can be chosen according to the magnitudes of their corresponding terms in the cost function, $\boldsymbol{S} \in \mathbb{R}^{9\times 9}$ and $\boldsymbol{R} \in \mathbb{R}^{3\times 3}$ are, respectively, symmetric positive semidefinite and symmetric positive definite weighting matrices.

This problem is referred to as Problem (P).

3 Optimal computation control

We first proceed to solve Problem (P) as an optimal open loop control problem by using a time scaling transform and the control parameterization technique. This will provide us with an optimal open loop control and the corresponding optimal trajectory.

Let the time horizon $[0, T]$ be partitioned into p subintervals as follows:

$$0 = t_0 \leq t_1 \leq \cdots \leq t_p = T. \tag{15}$$

The switching times t_k , $1 \leq k \leq p$, are regarded as decision variables. We shall employ the time scaling transform introduced in [18] to map these switching times into a set of fixed time points $\eta_k = k/p$, $k = 1, \ldots, p$, on a new time horizon $[0, 1]$. This is easily achieved by the following differential equation

$$\frac{dt(s)}{ds} = \upsilon^p(s), \quad s \in [0, 1], \tag{16a}$$

with initial condition

$$t(0) = 0, \tag{16b}$$

where

$$\upsilon^p(s) = \sum_{k=1}^{p} \zeta_k \chi_{[\eta_{k-1},\, \eta_k]}(s). \tag{17}$$

Here, $\chi_I(s)$ denotes the indicator function of I defined by

$$\chi_I(s) = \begin{cases} 1, \ s \in I \\ 0, \ \text{elsewhere} \end{cases} \tag{18}$$

and $\zeta_k \geq 0$, $k = 1, \ldots, p$,

$$\sum_{k=1}^{p} \zeta_k = T. \tag{19}$$

Let $\boldsymbol{\zeta} = [\zeta_1, \ldots, \zeta_p]^T$ and let Θ be the set containing all such $\boldsymbol{\zeta}$.

Taking integration of (16a) with initial condition (16b), it is easy to see that, for $s \in [\eta_{l-1}, \eta_l)$,

$$t(s) = \sum_{k=1}^{l-1} \zeta_k + \zeta_l (s - \eta_{l-1}) p, \tag{20a}$$

where $l = 1, \ldots, p$. Clearly,

$$t(1) = \sum_{k=1}^{p} \zeta_k = T. \tag{20b}$$

Thus, after the time scaling transform (16a) and (16b), it follows from (6), (16a) and (16b) that

$$\dot{\hat{\boldsymbol{x}}}(s) = \upsilon^p(s) \left\{ \boldsymbol{f}(\hat{\boldsymbol{x}}(s)) + \boldsymbol{B}(\hat{\boldsymbol{x}}(s), s) \tilde{\boldsymbol{u}}(s) \right\} \tag{21a}$$

with the initial condition

$$\hat{\boldsymbol{x}}(0) = \begin{bmatrix} \boldsymbol{x}^0 \\ 0 \end{bmatrix}, \tag{21b}$$

where $\hat{\boldsymbol{x}}(s) = [\tilde{\boldsymbol{x}}(s)^T, \, t(s)]^T$, $\tilde{\boldsymbol{x}}(s) = \boldsymbol{x}(t(s))$ and $\tilde{\boldsymbol{u}}(s) = \boldsymbol{u}(t(s))$.

We now apply the control parameterization technique to approximate the control $\tilde{\boldsymbol{u}}(s) = [\tilde{u}_1(s), \tilde{u}_2(s), \tilde{u}_3(s)]^T$ as follows.

$$\tilde{u}_i^p(s) = \sum_{k=-1}^{p+1} \sigma_k^i \Omega((\frac{1}{p})s - k), \quad i = 1, 2, 3, \tag{22}$$

where

$$\Omega(\tau) = \begin{cases} 0, & |\tau| > 2 \\ -\frac{1}{6}|\tau|^3 + \tau^2 - 2\,|\tau| + \frac{4}{3}, & 1 \leq |\tau| \leq 2 \\ \frac{1}{2}|\tau|^3 - \tau^2 + \frac{2}{3}, & |\tau| < 1 \end{cases} \tag{23}$$

is the cubic spline basis function, σ_k^i, $i = 1, 2, 3$; $k = -1, 0, 1, \ldots, p+1$, are decision constants.

From (12), we have

$$\alpha_1 \leq \sigma_k^1 \leq \beta_1, \quad k = -1, 0, 1, \ldots, p+1. \tag{24}$$

Define

$$\boldsymbol{\sigma}^i = [\sigma_{-1}^i, \cdots, \sigma_{p+1}^i]^T, \quad i = 1, 2, 3, \tag{25}$$

and

$$\boldsymbol{\sigma} = [(\boldsymbol{\sigma}^1)^T, (\boldsymbol{\sigma}^2)^T, (\boldsymbol{\sigma}^3)^T]^T \tag{26}$$

Let Ξ denote the set containing all such $\boldsymbol{\sigma}$. Then, $\tilde{\boldsymbol{u}}^p(s) = [\tilde{u}_1^p(s), \tilde{u}_2^p(s), \tilde{u}_3^p(s)]^T$ is determined uniquely by the switching vector $\boldsymbol{\sigma}$ in Ξ, and vice verse. Thus, it is written as $\tilde{\boldsymbol{u}}^p(\bullet|\boldsymbol{\sigma})$. We may now state the optimal parameterization selection problem, which is an approximation of Problem (P), as follows:

Problem (Q). Given system (21a) with initial condition (21b), find a combined vector $(\boldsymbol{\sigma}, \boldsymbol{\zeta}) \in \Xi \times \Theta$, such that the cost function

$$J(\boldsymbol{\sigma}) = a_1 \hat{\Phi}_0(\hat{\boldsymbol{x}}(1|\boldsymbol{\sigma})) + a_2 \int_0^1 \upsilon^p(s|\boldsymbol{\zeta}) \tilde{\boldsymbol{u}}^p(s|\boldsymbol{\sigma})^T \boldsymbol{R} \tilde{\boldsymbol{u}}^p(s|\boldsymbol{\sigma})\, ds \tag{27}$$

is minimized, where $\hat{\Phi}_0(\hat{\boldsymbol{x}}(1|\boldsymbol{\sigma})) = (\hat{\boldsymbol{x}}(1|\boldsymbol{\sigma}) - \hat{\boldsymbol{x}}_d)^T \hat{\boldsymbol{S}} (\hat{\boldsymbol{x}}(1|\boldsymbol{\sigma}) - \hat{\boldsymbol{x}}_d)$, $\hat{\boldsymbol{x}}_d$ is the desired terminal state vector, $\hat{\boldsymbol{S}} \in \mathbb{R}^{10\times 10}$, and $\tilde{\boldsymbol{u}}^p$ is given by (22).

At this stage, we see that Problem (P) is approximated by a sequence of optimal parameter selection problems, each of which can be viewed as a mathematical programming problem and hence can be solved by existing gradient-based optimization methods (see, for example, [1], [3], [5], [8], [12], [14], [17]). A general purpose optimal control software package, called MISER 3.3 [7], was developed based on these methods, where the control is, however, approximated by piecewise constant functions (i.e., in terms of zero order spline basis functions) or piecewise linear functions (i.e., in terms of first order spline basis functions). Here, our controls are approximated in terms of cubic spline basis functions, and thus they are smooth. MISER 3.3 can be easily modified to cater for this minor modification.

Suppose that $(\tilde{\boldsymbol{u}}^{p*}, \hat{\boldsymbol{x}}^*)$ is the optimal solution of Problem (Q). Then, from (20a) and (20b), it follows that the optimal solution to Problem (P) is $(\boldsymbol{u}^*, \boldsymbol{x}^*, T^*)$, where $\boldsymbol{u}^* = [u_1^*, u_2^*, u_3^*]^T$ is the optimal open loop control, $\boldsymbol{x}^*$ is the corresponding optimal state vector, and T^* is the optimal terminal time. In view of the optimal open loop control obtained, we notice that the reverse force thruster, u_1^*, works with its maximum thrust force (i.e., at its upper bound which is a constant value) throughout the entire soft landing process. This observation is confirmed by Pontryagin's Maximum Principle (see [24]). Thus, for the computation of the optimal closed loop control problem, we set the first control variable of $\boldsymbol{u}$ to be equal to the constant value obtained through solving Problem (P) as an open loop optimal control problem.

Correspondingly, system (6) can be rewritten as

$$\dot{\boldsymbol{x}}(t) = \bar{\boldsymbol{f}}(\boldsymbol{x}(t), t) + \bar{\boldsymbol{B}} \bar{\boldsymbol{u}}(t), \tag{28}$$

where

$$\bar{\boldsymbol{f}}(\boldsymbol{x}, t) = [x_4, x_5, x_6, g_{xL} - 2\omega_L x_6 + cM_1, g_{yL} + cM_2, g_{zL} + 2\omega_L x_4 + cM_3, 0, 0, -c]^T \tag{29}$$

$$\bar{\boldsymbol{B}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T \tag{30}$$

while M_1, M_2 and M_3 remain the same as given by (9), (10) and (11), respectively. The new control vector $\bar{\boldsymbol{u}}$ is

$$\bar{\boldsymbol{u}} = [v, w]^T = [\bar{u}_1, \bar{u}_2]^T \tag{31}$$

Let $\bar{\mathcal{U}}$ be the set of all such controls. Elements from $\bar{\mathcal{U}}$ are called admissible controls and $\bar{\mathcal{U}}$ is referred to as the class of admissible controls.

The initial conditions of the soft landing remain the same as given by (13). The cost function (14) can be rewritten as

$$\bar{J} = a_1 \Phi_0(\boldsymbol{x}(T)) + a_2 \int_0^T \bar{\boldsymbol{u}}^T \bar{\boldsymbol{R}} \bar{\boldsymbol{u}} dt, \tag{32}$$

where $\bar{\boldsymbol{R}} \in \mathbb{R}^{2\times 2}$ is a symmetric positive definite matrix obtained from $\boldsymbol{R}$.

Now, the original optimal control Problem (P) is reduced to Problem ($\bar{\mathrm{P}}$) given below.

Problem ($\bar{\mathrm{P}}$). Given system (28) with the initial condition (13), find a closed loop control such that the cost function (32) is minimized.

For Problem ($\bar{\mathrm{P}}$), we have the following theorem.

Theorem 1 *The optimal closed loop control $\bar{\boldsymbol{u}}^*$ for Problem ($\bar{\mathrm{P}}$) is given by*

$$\bar{\boldsymbol{u}}^*(t) = \frac{1}{2a_2}\bar{\boldsymbol{R}}^{-1}\bar{\boldsymbol{B}}^T\boldsymbol{K}(t)\bar{\boldsymbol{f}}(\boldsymbol{x}^*(t),t), \tag{33}$$

where $\boldsymbol{x}^*$ is the optimal state, i.e. the solution of system (28) with initial condition (13) corresponding to $\bar{\boldsymbol{u}}^*$, and $\boldsymbol{K}(t)$ is the solution of the following Riccati-like differential equation

$$(\dot{\boldsymbol{K}} + \boldsymbol{K}\boldsymbol{F} + \boldsymbol{F}^T\boldsymbol{K} + \frac{1}{2}\boldsymbol{K}\boldsymbol{F}\bar{\boldsymbol{B}}\bar{\boldsymbol{R}}^{-1}\bar{\boldsymbol{B}}^T\boldsymbol{K})\bar{\boldsymbol{f}} + \boldsymbol{K}\boldsymbol{D} = 0, \tag{34a}$$

Here, $\boldsymbol{F} = \partial\,\bar{\boldsymbol{f}}/\partial\,\boldsymbol{x}$, $\boldsymbol{D} = \partial\,\bar{\boldsymbol{f}}/\partial\,t$, and

$$\boldsymbol{K}(T)\bar{\boldsymbol{f}}(\boldsymbol{x}(T),T) = a_1\frac{\partial\,\Phi_0(\boldsymbol{x}(T))}{\partial\,\boldsymbol{x}(T)} = 2a_1(\boldsymbol{x}(T)-\boldsymbol{x}_d)^T\boldsymbol{S}. \tag{34b}$$

Proof The proof is similar to that given for Theorem 3.1 in [9]. Let H be the Hamiltonian function defined by

$$H(\boldsymbol{x}(t),\bar{\boldsymbol{u}}(t),\boldsymbol{\lambda}(t)) = \boldsymbol{\lambda}^T(t)\bar{\boldsymbol{f}}(\boldsymbol{x}(t),t) + \boldsymbol{\lambda}^T(t)\bar{\boldsymbol{B}}\bar{\boldsymbol{u}}(t) - a_2\bar{\boldsymbol{u}}^T(t)\bar{\boldsymbol{R}}\bar{\boldsymbol{u}}(t), \tag{35}$$

where $\boldsymbol{\lambda}(t) \in \mathbb{R}^9$ is the costate vector.

Suppose that $(\bar{\boldsymbol{u}}^*, \boldsymbol{x}^*)$ is an optimal pair. Then, it follows from Pontryagin's Maximum Principle that

$$(i) \qquad \dot{\boldsymbol{x}}^*(t) = \left(\frac{\partial H^*}{\partial\boldsymbol{\lambda}(t)}\right)^T = \bar{\boldsymbol{f}}(\boldsymbol{x}^*(t),t) + \bar{\boldsymbol{B}}\bar{\boldsymbol{u}}^*(t) \tag{36}$$

$$(ii) \qquad \boldsymbol{x}^*(0) = \boldsymbol{x}_0 \tag{37}$$

$$(iii) \qquad \dot{\boldsymbol{\lambda}}^*(t) = -\left(\frac{\partial H^*}{\partial\boldsymbol{x}^*(t)}\right)^T \tag{38}$$

$$(iv) \qquad \boldsymbol{\lambda}^*(T) = a_1\frac{\partial\Phi_0(\boldsymbol{x}^*(T))}{\partial\boldsymbol{x}^*(T)} = 2a_1(\boldsymbol{x}^*(T)-\boldsymbol{x}_d)^T\boldsymbol{S} \tag{39}$$

$$(v) \qquad \frac{\partial H^*}{\partial\bar{\boldsymbol{u}}^*(t)} = 0 \tag{40}$$

$$(vi) \qquad H^*|_{t=T} = 0 \tag{41}$$

where $H^* = H(\boldsymbol{x}^*(t),\, \bar{\boldsymbol{u}}^*(t),\, \boldsymbol{\lambda}^*(t))$ and the terminal time T is determined by solving Problem Q.

From (40), we obtain

$$\bar{\boldsymbol{u}}^*(t) = \frac{1}{2a_2}\bar{\boldsymbol{R}}^{-1}\bar{\boldsymbol{B}}^T\boldsymbol{\lambda}^*(t) \tag{42}$$

As in [10], we postulate that the costate vector $\boldsymbol{\lambda}^*(t)$ can be expressed as

$$\boldsymbol{\lambda}^*(t) = \boldsymbol{K}(t)\bar{\boldsymbol{f}}(\boldsymbol{x}^*(t),t) \tag{43}$$

Then, it follows that

$$\bar{\boldsymbol{u}}^*(t) = \frac{1}{2a_2}\bar{\boldsymbol{R}}^{-1}\bar{\boldsymbol{B}}^T\boldsymbol{K}(t)\bar{\boldsymbol{f}}(\boldsymbol{x}^*(t),t) \tag{44}$$

Differentiating (43) with respective to t, we deduce from (36) and (44) that

$$\dot{\boldsymbol{\lambda}}^*(t) = (\dot{\boldsymbol{K}} + \boldsymbol{K}\boldsymbol{F} + \frac{1}{2a_2}\boldsymbol{K}\boldsymbol{F}\bar{\boldsymbol{B}}\bar{\boldsymbol{R}}^{-1}\bar{\boldsymbol{B}}^T\boldsymbol{K})\bar{\boldsymbol{f}}(\boldsymbol{x}^*(t), t) + \boldsymbol{K}\boldsymbol{D} \tag{45}$$

From (38) and (43), we obtain

$$\dot{\boldsymbol{\lambda}}^*(t) = -\boldsymbol{F}^T\boldsymbol{K}(t)\bar{\boldsymbol{f}}(\boldsymbol{x}^*(t), t) \tag{46}$$

Combining (45) and (46), we have

$$(\dot{\boldsymbol{K}} + \boldsymbol{K}\boldsymbol{F} + \boldsymbol{F}^T\boldsymbol{K} + \frac{1}{2a_2}\boldsymbol{K}\boldsymbol{F}\bar{\boldsymbol{B}}\bar{\boldsymbol{R}}^{-1}\bar{\boldsymbol{B}}^T\boldsymbol{K})\bar{\boldsymbol{f}} + \boldsymbol{K}\boldsymbol{D} = 0 \tag{47}$$

From (39), the terminal condition for the Riccati-like differential equation (47) is obtained as

$$\boldsymbol{\lambda}(T) = a_1\frac{\partial \Phi_0(\boldsymbol{x}(T))}{\partial \boldsymbol{x}(T)} = 2a_1(\boldsymbol{x}(T) - \boldsymbol{x}_d)^T\boldsymbol{S} \tag{48}$$

This completes the proof.

By Theorem 1, we observe that the form of the optimal closed loop control law for Problem ($\bar{\text{P}}$) is given by

$$\bar{\boldsymbol{u}}(t) = \frac{1}{2a_2}\bar{\boldsymbol{R}}^{-1}\bar{\boldsymbol{B}}^T\boldsymbol{K}(t)\bar{\boldsymbol{f}}(\boldsymbol{x}(t), t) \tag{49}$$

However, the matrix function $\boldsymbol{K}(t)$ is still required to be obtained. This task is, in fact, rather demanding. It involves solving a new optimal control problem, which we call Problem (R).

Problem (R): subject to the dynamical systems given by (28), (13), (34a) and (34b), with $\bar{\boldsymbol{u}} = \bar{\boldsymbol{R}}^{-1}\bar{\boldsymbol{B}}^T\boldsymbol{K}(t)\bar{\boldsymbol{f}}(\boldsymbol{x}(t), t)/2a_2$, find a $\boldsymbol{K}(t)$ such that the cost function (32), also with $\bar{\boldsymbol{u}} = \bar{\boldsymbol{R}}^{-1}\bar{\boldsymbol{B}}^T\boldsymbol{K}(t)\bar{\boldsymbol{f}}(\boldsymbol{x}(t), t)/2a_2$, is minimized.

For Problem (R), the dynamical system (28) is required to be solved forward in time with initial condition given by (13). On the other hand, the dynamical system (34a) should be solved backward in time with partial information on the terminal state given by (34b). This optimal control problem is, indeed, very difficult to solve.

In this paper, we propose an alternative approach to construct an approximate optimal matrix function $\boldsymbol{K}^*(t)$ without having to solve this complicated optimal control problem (R). The basic idea is explained as follows. Suppose that $\boldsymbol{u}^* = [u_1^*, u_2^*, u_3^*]^T$ is an optimal open loop control of Problem (P) and that $\boldsymbol{x}^*$ is the corresponding optimal state. As u_1^* is a constant, we fix it to the constant obtained. This gives rise to Problem ($\bar{\text{P}}$). We now consider Problem ($\bar{\text{P}}$) with $\boldsymbol{x} = \boldsymbol{x}^*$, i.e. along the optimal open loop path, and our task is to find a $\boldsymbol{K}^*(t)$ such that $\bar{\boldsymbol{u}}^{\#} = \bar{\boldsymbol{R}}^{-1}\bar{\boldsymbol{B}}^T\boldsymbol{K}^*(t)\bar{\boldsymbol{f}}(\boldsymbol{x}^*(t), t)/2a_2$ best approximates the control $\bar{\boldsymbol{u}}^*$ in the mean square sense, where $\bar{\boldsymbol{u}}^* = [u_2^*, u_3^*]^T$. Since the cost value for Problem ($\bar{\text{P}}$) with $\bar{\boldsymbol{u}}$ given by $\bar{\boldsymbol{u}}^{\#} = \bar{\boldsymbol{R}}^{-1}\bar{\boldsymbol{B}}^T\boldsymbol{K}^*(t)\bar{\boldsymbol{f}}(\boldsymbol{x}^*(t), t)/2a_1$ should be close to the cost value for Problem (P) with $\boldsymbol{u} = \boldsymbol{u}^*$, $\bar{\boldsymbol{u}}^{\#}$ can be regarded as a good approximate optimal feedback control for Problem ($\bar{\text{P}}$).

In the next section, we present a practical method to find an approximate optimal gain matrix $\boldsymbol{K}(t)$ without solving the complex optimal control problem (R).

4 A practical computational method

As the matrix function $\boldsymbol{K}(t)$ is a solution of Riccati-like differential equation, the optimal closed loop control law (49) should be smooth throughout $[0,T]$, where $T = T^*$. For this reason, $\boldsymbol{K}(t)$ is approximated in terms of cubic splines basis functions. The time horizon $[0,T^*]$ is partitioned into p equal subintervals,

$$0 = t_0 \leq t_1 \leq \cdots \leq t_p \leq t_{p+1} = T^* \tag{50}$$

Let

$$[\boldsymbol{K}(t)]_{i,j} \approx \sum_{k=-1}^{p+1} (c_{i,j,k}) \Omega((\frac{T^*}{p})t - k) \tag{51}$$

where $c_{i,j,k},\ i,j = 1,2,\ldots,9;\ k = -1,0,1,2,\ldots,p+1$, are real constant coefficients that are to be determined, p is the number of equality subintervals on $[0,T^*]$, $p+3$ is the total number of cubic spline basis functions used in the approximation of each $[\boldsymbol{K}(t)]_{i,j}$, and $\Omega(\tau)$ is defined as in (23).

Let

$$\Upsilon(\boldsymbol{K}) = \int_0^{T^*} \left\{ (u_2^*(t) - \bar{u}_1(t))^2 + (u_3^*(t) - \bar{u}_2(t))^2 \right\} dt, \tag{52}$$

where

$$\bar{\boldsymbol{u}}(t) = \begin{bmatrix} \bar{u}_1(t) \\ \bar{u}_2(t) \end{bmatrix} = \frac{1}{2a_2} \bar{\boldsymbol{R}}^{-1} \bar{\boldsymbol{B}}^T \boldsymbol{K}(t) \bar{\boldsymbol{f}}(\boldsymbol{x}^*(t), t). \tag{53}$$

Here, we see that $\bar{\boldsymbol{u}}$ is of the same form as the optimal closed loop control given by (33). Our task is to choose a $\boldsymbol{K}(t)$ such that (52) is minimized. Let $\boldsymbol{K}^*(t)$ be the optimal matrix function obtained. It is substituted into (53) to give $\bar{\boldsymbol{u}}^* = [\bar{u}_1^*, \bar{u}_2^*]^T$, which is the best approximate optimal feedback control in the mean square sense of Problem $(\bar{\mathrm{P}})$.

Our task can be posed as the following optimization problem.

Find coefficients $c_{i,j,k},\ i,j = 1,2,\ldots,9;\ k = -1,0,1,2,\ldots,p+1$, such that the cost function (52) is minimised. These optimal coefficients can be obtained by solving the following optimality conditions

$$\boldsymbol{\Gamma} = \frac{\partial \Upsilon(\boldsymbol{K})}{\partial c_{i,j,k}} = \int_0^{T^*} \frac{\partial((u_2^* - \bar{u}_1)^2 + (u_3^* - \bar{u}_2)^2)}{\partial c_{i,j,k}} dt = 0 \tag{54}$$
$$i,j = 1,2,\ldots,9,\ \ k = -1,0,1,2,\ldots,p+1$$

These are linear equations, and hence are easy to solve.

Let $\rho_i(\boldsymbol{\Lambda})$ and $\kappa_i(\boldsymbol{\Lambda})$ denote the $i-$th row and $i-$th column of the matrix $\boldsymbol{\Lambda}$. By a careful examination of (53), it is noticed that $\boldsymbol{K}(t)$ appears with $\bar{\boldsymbol{B}}^T$ multiplied from the left. If $\kappa_i(\bar{\boldsymbol{B}}^T) = 0$ for all $t \in [0,T^*]$, then $\rho_i(\boldsymbol{K}(t))$ does not affect $\bar{\boldsymbol{B}}^T\boldsymbol{K}(t)$. From (30), we see that $\kappa_i(\bar{\boldsymbol{B}}^T) = 0,\ i = 1,\ldots,6,9$, hence, there is no need to calculate those coefficients $c_{i,j,k}$ corresponding to $\rho_i(\boldsymbol{K}(t)),\ i = 1,\ldots,6,9$. From (29) and (53), we also notice that $\rho_i(\bar{\boldsymbol{f}}(\boldsymbol{x}^*(t),t)) = 0,\ i = 7,8$, and $\boldsymbol{K}(t)$ is multiplied with $\bar{\boldsymbol{f}}(\boldsymbol{x}^*(t),t)$ from the right. Thus, $\kappa_i(\boldsymbol{K}(t)),\ i = 7,8$, do not affect $\boldsymbol{K}(t)\bar{\boldsymbol{f}}(\boldsymbol{x}^*(t),t)$, and hence there is no need to calculate the corresponding components $\kappa_i(\boldsymbol{K}(t)),\ i = 7,8$. Therefore, we may set these components of $\boldsymbol{K}(t)$ to zero. In our problem, we only need to calculate 14 elements of $\boldsymbol{K}(t)$, i.e., $[\boldsymbol{K}(t)]_{i,j},\ i = 7,8;\ j = 1,2,\ldots,6,9$.

5 Numerical simulations

In this section, two examples are involved to illustrate the effectiveness of the proposed method.

5.1 Example 1

The initial conditions for the soft landing of a lunar module are given as: $x_{L0} = 819.371$km, $y_{L0} = 1428.867$km, $z_{L0} = 599.6306$km, $V_{xL0} = 1115$m/s, $V_{yL0} = -981.82$ m/s, $V_{zL0} = 816$m/s, $m_0 = 600$kg. At the initial time of the soft landing, the rotation angle $\gamma(t_0) = 0^\circ$, the specific impulse $V_r = 300 \times 9.8$m/s and the angular velocity of the moon rotation $\omega_L = 2.661699 \times 10^{-6}$rad/s. The landing target is in Mare Imbrium on the moon surface with 38.3° North latitude and 35° West longitude. When the module reaches the moon surface, the terminal velocity should be less than 3m/s. The bounds on $u_1(t)$ are: $0\text{kg/s} \le u_1(t) \le 0.51\text{kg/s}$.

In the simulation, the time horizon $[0, T]$ is partitioned into 30 subintervals. $a_1 = 10$, $a_2 = 1$, $\boldsymbol{S} = \text{diag}(1\text{e}^{-3}, 1\text{e}^{-3}, 1\text{e}^{-3}, 1\text{e}^{-3}, 1\text{e}^{-3}, 1\text{e}^{-3}, 0, 0, 0)$ and $\boldsymbol{R} = \text{diag}(1, 1, 1)$. We first use the time scaling transform (16a), (16b) and the control parameterization method (22) to construct the corresponding approximated problem (Q). Then, MISER 3.3 is utilized to solve it, giving rise to an optimal open loop control $\tilde{\boldsymbol{u}}^*(s)$ and the corresponding optimal trajectory $\tilde{\boldsymbol{x}}^*(s)$. Then, by (20a) and (20b), we obtain the optimal open loop solution, denoted by $(\boldsymbol{u}^*, \boldsymbol{x}^*, T^*)$ of Problem (P). Note that $u_1^* = 0.51$kg/s, i.e., the reverse force thruster works with its maximum thrust force $P = 1500$N. With $u_1^* = 0.51$kg/s, Problem (P) is reduced to Problem ($\bar{\text{P}}$) with $\bar{\boldsymbol{u}} = [\bar{u}_1, \bar{u}_2]^T$, where $\bar{\boldsymbol{R}}$ is chosen from $\boldsymbol{R}$ to be $\bar{\boldsymbol{R}} = \text{diag}(1, 1)$. Set $\bar{u}_1^*(t) = u_2^*(t)$ and $\bar{u}_2^*(t) = u_3^*(t)$. The corresponding optimal state of Problem ($\bar{\text{P}}$) remains the same as that of Problem (P). Substituting $(\boldsymbol{u}^*, \boldsymbol{x}^*, T^*)$ into (54), the system of linear equations can be solved by a linear equations solver within the Matlab environment. The feedback gain matrix $\boldsymbol{K}^*(t)$ obtained is substituted into (53) to give the best approximate optimal feedback control law in the mean square sense for achieving the soft landing of the lunar module.

Under the optimal feedback control, the terminal conditions of the module are $x_L(t_f) = 1117.2919$km, $y_L(t_f) = 1077.1752$km, $z_L(t_f) = 782.3021$km, $V_{xL}(t_f) = 0.6345$m/s, $V_{yL}(t_f) = -0.9852$m/s, $V_{zL}(t_f) = 0.178$m/s.

Simulation results are shown in Figure 2 to Figure 8. Figure 2 to Figure 4 are the time histories of the control outputs. It is seen that the thruster works with its maximum thrust force, the feedback angular velocity control laws coincide with the open loop ones precisely. Under the optimal feedback control, the lunar module lands on the moon surface after 542.268s, the velocities along the three directions in $ox_Ly_Lz_L$ are approaching to zero (see Figure 5 to Figure 7), the terminal velocity of the module is 1.185m/s. The distance between the lunar module and the preselected landing target is 27.98m. The terminal mass of the module is 323.443kg. The optimal descent trajectory is shown in Figure 8.

5.2 Example 2

Next, we let the lunar module soft landing start from a new perturbed initial point to testify the robustness of the optimal feedback control law against disturbances on

Table 1 Summary of simulations

simulation	distance from target	finial velocity	objective value
Case 1	33.27m	1.22m/s	5574.72
Case 2	184.1m	2.26m/s	5910.73
Case 3	0m	0m/s	5566.53

initial condition for the soft landing mission. The coordinates of the new starting point, which is 30m away from the original starting point, are $x_{L0N} = 819.375$km, $y_{L0N} = 1428.845$km and $z_{L0N} = 599.6507$km. Under the optimal feedback control, the lunar module lands on the moon surface after 542.26s, the terminal velocity is 1.22m/s, and the coordinates of the landing position are $x_{LC}(t_f) = 1117.3007$km, $y_{LC}(t_f) = 1077.1565$km and $z_{LC}(t_f) = 782.3153$km, which is 33.27m away from the preselected landing target.

For comparison, we let the lunar module soft landing start from the new initial point by using the open loop optimal control $\boldsymbol{u}^*$ obtained previously. Under the open loop optimal control, the lunar module lands on the moon surface after 543.05s with the terminal velocity which is 2.26m/s. The landing position is located at $x_{LO}(t_f) = 1117.1776$km, $y_{LO}(t_f) = 1077.1592$km and $z_{LO}(t_f) = 782.4896$km, which is 184.1m away from the desired landing target.

To exam how close to optimal the feedback control is for the soft landing when the initial position is perturbed to a new perturbed initial point, we calculate the open loop optimal control for the perturbed problem by using the control parameterization technique and the time scaling transform mentioned above. Under the new open loop optimal control, we calculate the optimal descent trajectory from which we observe that the lunar module lands on the desired landing target precisely after 542.541s. The finial velocity is 0m/s.

The simulation results are summarized in Table 1. Case 1, Case 2 and Case 3 represent the simulations from the perturbed initial point with the feedback control, the original open loop optimal control $\boldsymbol{u}^*$ and the new open loop optimal control, respectively. As we can see, the feedback control is much superior to the open loop optimal control obtained from the original initial condition when the initial position is perturbed to a new perturbed initial point. The performance of the feedback control is close to that of the optimal open loop control calculated from the perturbed initial point. The time histories of the descent trajectories are depicted in Figure 9.

6 Conclusions

The optimal control problem of lunar module soft landing is studied where a three dimensional dynamics is employed to describe the motion of the module. We first obtain an optimal open loop control by using the control parametrization method and the time scaling transform. Then, we obtained the form of the optimal closed loop control law, where the feedback gain matrix is required to satisfy a Riccati-like matrix differential equation. On this basis, a practical method was proposed to calculate the feedback gain matrix without having to solve an optimal control problem involving a complex Riccati-like differential equation coupled with the original dynamics. Simulation results showed that the proposed method is highly efficient.

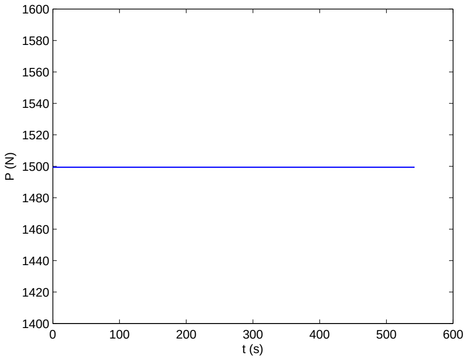

Fig. 2 Thrust force P

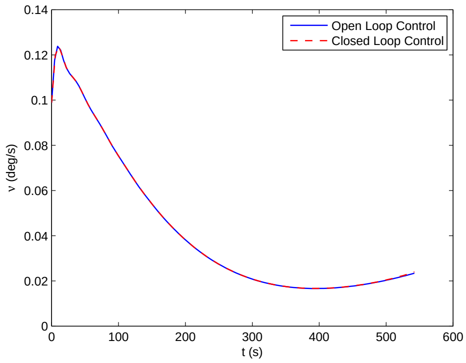

Fig. 3 Angular velocity control v

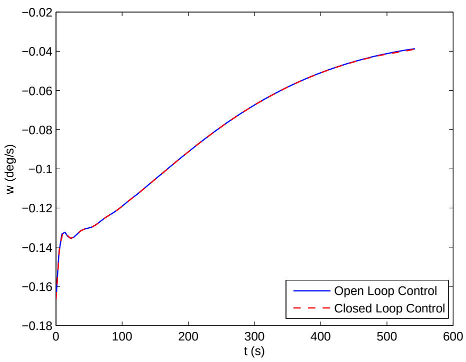

Fig. 4 Angular velocity control w

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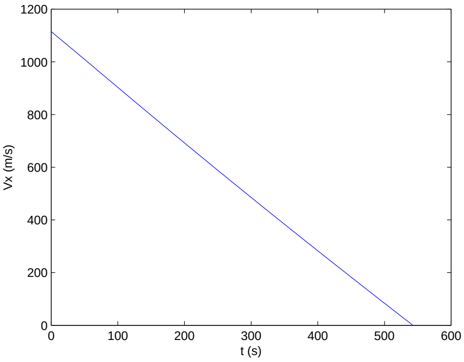

Fig. 5 Velocity along x axis

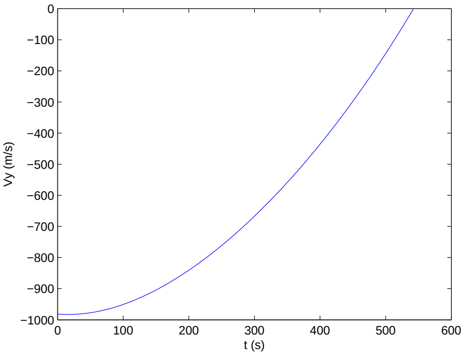

Fig. 6 Velocity along y axis

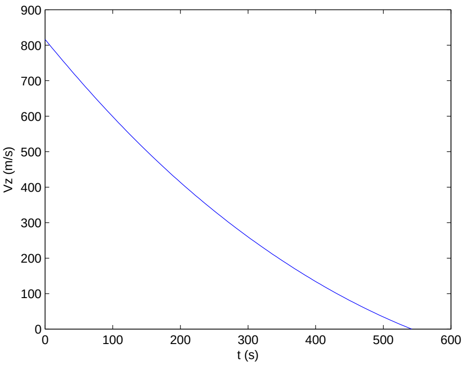

Fig. 7 Velocity along z axis

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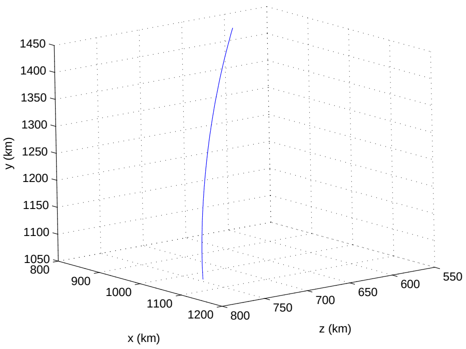

Fig. 8 Optimal descent trajectory

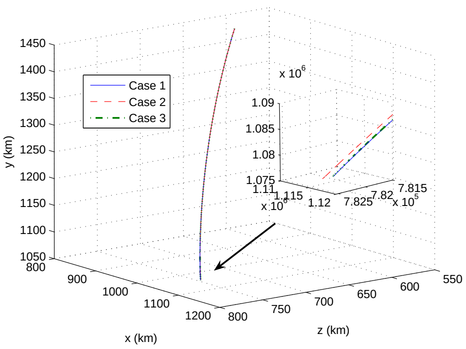

Fig. 9 Descent trajectories

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