Department of Mathematics and Statistics

OPTIMAL CONTROL PROBLEMS WITH CONSTRAINTS ON THE STATE AND CONTROL AND THEIR APPLICATIONS

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Declaration

I affirm that the material in this thesis is the result of my own original research and has not been submitted for any other degree, diploma, or award.

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Abstract

In this thesis, we consider several types of optimal control problems with constraints on the state and control variables. These problems have many engineering applications. Our aim is to develop efficient numerical methods for solving these optimal control problems.

In the first problem, we consider a class of discrete time nonlinear optimal control problems with time delay and subject to constraints on states and controls at each time point. These constraints are called all-time-step constraints. A constraint transcription technique in conjunction with a local smoothing method is used to construct a sequence of approximate discrete time optimal control problems involving time delay in states and controls and subject to nonlinear inequality constraints in canonical form. These approximate optimal control problems are special cases of a general discrete time optimal control problems with time delay appearing in the state and control and subject to nonlinear inequality constraints in canonical form. Thus, we devise an efficient gradient-based computational method for solving this general optimal control problem. The gradient formulas needed for the cost and the canonical constraint functions are derived. With these gradient formulas, the discrete time optimal control problem with time delay appearing in states and controls and subject to nonlinear inequality constraints in canonical form is solvable as an optimization problem with inequality constraints by the Sequential Quadratic Programming (SQP) method. With this computational method, each of the approximate problems constructed from the original optimal control problem can be solved. A practical problem arising from the study of a tactical logistic decision analysis problem is considered and solved by using the computational method that we have developed.

In the second problem, we consider a general class of maximin optimal control problems, where the violation avoidance of the continuous state inequality constraints is to be maximized. An efficient computational method is developed for solving this general maximin optimal control problem. In this computational method, the constraint transcription method is used to construct a smooth approximate function for each of the continuous state inequality constraints, where the accuracy of the approximation is controlled by an accuracy parameter. We then obtain a sequence of smooth approximate optimal control problems, where the integral of the summation of these smooth approximate functions is taken as its cost function. A necessary condition and a sufficient condition are derived

showing the relationship between the original maximin problem and the sequence of the smooth approximate problems. We then construct a violation avoidance function from the solution of each of the smooth approximate optimal control problems and the original continuous state inequality constraints in such a way that the problem of finding an optimal control of the maximin optimal control problem is equivalent to the problem of finding the largest root of the violation avoidance function. The control parameterization technique and a time scaling transform are applied to these smooth approximate optimal control problems. Two practical problems are considered as applications. The first one is an obstacle avoidance problem of an autonomous mobile robot, while the second one is the abort landing of an aircraft in a windshear downburst. The proposed computational method is then applied to solve these problems.

In the third problem, we consider a class of optimal PID control problems subject to continuous inequality constraints and terminal equality constraint. By applying the constraint transcription method and a local smoothing technique to these continuous inequality constraint functions, we construct the corresponding smooth approximate functions. We use the concept of the penalty function to append these smooth approximate functions to the cost function, forming a new cost function. Then, the constrained optimal PID control problem is approximated by a sequence of optimal parameter selection problems subject to only terminal equality constraint. Each of these optimal parameter selection problems can be viewed and hence solved as a nonlinear optimization problem. The gradient formulas of the new appended cost function and the terminal equality constraint function are derived, and a reliable computation algorithm is given. The method proposed is used to solve a ship steering control problem.

In the fourth problem, we consider a class of optimal control problems subject to equality terminal state constraints and continuous inequality constraints on the state and/or control variables. After the control parameterization together with a time scaling transformation, the problem is approximated by a sequence of optimal parameter selection problems with equality terminal state constraints and continuous inequality constraints on the state and/or control. An exact penalty function is constructed for these terminal equality constraints and continuous inequality constraints. It is appended to the cost function to form a new cost function, giving rise to an unconstrained optimal parameter selection problem. The convergence analysis shows that, for a sufficiently large penalty parameter, a local minimizer of the unconstrained optimization problem is a local minimizer of the optimal parameter selection problem with terminal equality constraints and continuous inequality constraints. The relationships between the approximate optimal parameter selection problems and the original optimal control problem are also discussed. Finally, the method proposed is applied to solve three nontrivial optimal control problems.

List of publications

The following papers (which have been published or accepted for publication) were completed during PhD candidature:

- B. Li, K. L. Teo, G. H. Zhao and G. R. Duan, "An efficient computational approach to a class of minimax optimal control problems with applications," *Australian and New Zealand Industrial and Applied Mathematics Journal*, vol. 51, pp. 162-177 , 2009.

- B. Li, K. L. Teo and G. R. Duan, "Optimal control for a class of discrete time time-delayed problem with canonical constraints," *The 1st World Congress on Global Optimization in Engineering and Science*, China, 1-5, June, 2009.

- B. Li, K. L. Teo and G. R. Duan, "Maximum obstacle avoidance problem," *The 4th International Conference on Optimization and Control with Applications*, Harbin, China, 6-11, June, 2009.

- G. R. Duan, D. K. Gu and B. Li, "Optimal control for final approach of rendezvous with non-cooperative target," *Pacific Journal of Optimization*, vol. 6, no. 3, pp. 521-532, 2010.

- B. Li, K.L Teo and G. R. Duan, "Optimal control computation for discrete time time-delayed optimal control problem with all-time-step inequality constraints," *International Journal of Innovative Computing, Information and Control*, vol. 6, no. 7, pp. 3157-3175, 2010.

- K. L. Teo, B. Li, C. C. Lim and G. R. Duan, "An optimal PID controller design for nonlinear optimal control problems with continuous inequality constraints," *The International Conference on Optimization and Control*, Hunan University, Guiyang, China, 18-23, July, 2010.

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Contents

CHAPTER 1

Introduction

1.1 Motivation and Background

An optimal control problem is to find a function such that a performance measure (also called objective function or cost function) is minimized subject to a dynamical system and a set of algebraic constraints. It has a wide range of applications in many areas such as space technology, defence, environmental science, operations research, economics, biology, mechanical engineering, electrical engineering, chemical engineering, civil engineering and even social sciences.

1.2 Numerical Methods

Numerical Methods In many practical real-world optimal control problems, there are rigid requirements to be satisfied at every time point in the planning horizon. Such requirements are often expressed as continuous inequality constraints on the state and/or control. These algebraic constraints are the most difficult constraints in optimal control problems. Optimal control problems subject to such continuous inequality constraints have been studied intensively in the literature. See, for example, [101–103], and the references cited therein. In [102], necessary conditions for optimality have been derived for various types of constrained optimal control problems. However, many real-world practical problems are much too complex to admit analytical solutions by using these necessary conditions for optimality, and they can only be solved numerically. There are already some numerical methods, such as the shooting method [31], [32], the discretization method [109–111], the non-smooth Newton method [107, 108], and the control parameterization method [96–98, 100, 101, 103], available in the literature.

1.2.1 Shooting and Multiple Shooting Methods

The shooting method is developed by Bryson [31] and Breakwell [32] to solve a two-point boundary-value-problem (TPBVP) obtained from applying the maximum principle. It is based on a guess of the solution of the co-state equations at the initial time point. Then, both the state and co-state equations are integrated forward in time. The solution of the co-state equations at the terminal time is compared with the terminal condition of the co-state equations obtained from the maximum principle. The error is then used to update the initial guess for the co-state equations. This process is continued until the conditions of the maximum principle are satisfied. However, the TPBVP is very sensitive to the initial guess of the co-state equations. The failure of achieving convergence is common for this method. Thus, the multiple shooting method is proposed in [33] to overcome the shortcomings of the shooting method. The multiple shooting method divides the time horizon into many subintervals, and the shooting method is used in every subinterval until the conditions of the maximum principle and the continuities of all the junction points are satisfied. However, this method remains sensitive to the initial guess of the co-state equations, although to a lesser extent.

1.2.2 The Direct Collocation Method

The discretization method, which is also called the direct optimization method, is an effective method algorithm for small scale optimal control problems. By the discritization of the state and control variables, the direct collocation method transforms a constrained optimal control problem into a finite dimensional nonlinear optimization problem which can be solved by standard nonlinear programming methods, such as the SQP method. Initially, a coarse discritization is carried out and a solution is obtained. The optimality is checked by applying second-order sufficient conditions (SSC). Then, a sequence of refinement steps are taken, yielding a sequence of increasingly complicated approximate problems with more accurate solutions. Clearly, for large scale optimal control problems, the computational burden can become too enormous for effective implementation. In [109], two discretization methods, that transcribe optimal control problems into nonlinear programming problems, are discussed. These nonlinear programming problems are solvable by using the SQP method. It is also shown that the SQP method can be used to check the SSC and to calculate the co-state variables.

1.2.3 Control Parameterization Methods

Consider optimal control problems subject to canonical equality as well as inequality constraints on the state and/or the control variables. The idea of the control parameterization method ([34–36] and many others) is to partition the time horizon into a number of subintervals. Then the control is approximated by a piecewise constant or piecewise linear function with possible discontinuities at the partition points. The heights of the piecewise constant functions are regarded as decision variables. In this way, the optimal control problem is approximated by a sequence of finite dimensional optimal parameter selection problems, each of which can be regarded as a mathematical programming problem and solved by existing standard optimization techniques. The convergence of the sequence of the approximate optimal parameter selection problems to the original optimal control problem has also been established in [36]. It has been shown [36] that many types of constraints encountered in practice can be transformed into respective canonical constraints. The most difficult constraints are the ones which are required to be satisfied at each time point. These constraints are often expressed as continuous inequality constraints on the state and/or control variables. They are handled by the constraint transcription method reported in [103], where the continuous inequality constraints are approximated by functions in integral form involving a smoothing parameter. These integral functions are either regarded as conventional inequality constraints or appended into the cost function by using the concept of the penalty function method to form a new cost function. For the first case, the original optimal control problem is approximated by a sequence of nonlinear optimization problems with inequality constraints in integral form, and each of which can be solved by standard constrained optimization methods, such as the sequential quadratic programming (SQP) method. For the second case, we obtain a sequence of unconstrained nonlinear optimization problems and each of which is solvable by standard unconstrained nonlinear optimization techniques, such as conjugate gradient method or any quasi-Newton method [36].

In practice, the partition time points in the control parameterization method should also be regarded as decision variables for better efficiency. For this, a time scaling transform (originally called the control parameterization enhancing transform) is introduced in [87]. It maps the varying partition time points into fixed time points in a new time horizon via introducing an additional control variable, called the time scaling control. After the time scaling transform, the resulting optimal control problem is in the same form as that obtained by the standard control parameterization approach. This time scaling transform has been applied with great success to convert many highly complex optimal control problems, such as the optimal impulsive optimal control problems [39,41], optimal control problems involving switched systems [40], and the optimal discrete valued control problems [42], into respective standard optimal control problems solvable by the standard control parameterization approach. For the continuous inequality constraints on the state and/or control variables, they are handled by the constraint transcription technique which is originally developed in [103] to handle continuous inequality constraints on the state variables only. It is extended in [97] to the case where both the state and control are allowed to appear explicitly in the continuous inequality constraints. The nonlinear optimization problems, either constrained or unconstrained, are controlled by two parameters, where one controls the accuracy and the other controls the feasibility. The convergence for the method developed in [87, 97, 103] can be slow. This is because there are two parameters to be adjusted. More specifically, it is required to initiate a value for the accuracy parameter. Then, the feasibility parameter is required to be adjusted until the continuous inequality constraints are satisfied. The accuracy parameter is then reduced and followed by the adjustment of the feasibility parameter until the continuous inequality constraints are satisfied. The process is repeated until the required accuracy is achieved. Furthermore, there is no theoretical result showing that a local optimal solution of the nonlinear inequality constrained optimization problem is a local optimal solution of the optimal parameter selection problem with continuous inequality constraints on the state and/or control.

The optimal control software package, MISER 3.3 [37], is implemented based on the control parameterization technique, the time scaling transform and the constraint transcription technique. Many practically important problems have been successfully solved by using the optimal control software package, MISER 3.3. See, for example, [36, 37] and the relevant references cited therein.

1.2.4 Non-smooth Newton Methods

In [107] and [108], necessary conditions are stated in terms of a local minimum principle and it is transformed into an equivalent non-smooth equation in appropriate Banach spaces by using the Fischer-Burmeister function. The nonlinear and non-smooth functions are then solved by a non-smooth Newton's method. The global convergence and locally superlinear convergence under certain regularity conditions are proved in [107], while the local quadratic convergence is proved and a globalization strategy based on the minimization of the squared residual norm is suggested in [108].

1.3 Discrete Time Optimal Control Problems

Discrete time optimal control problems can be considered as finite dimensional optimization problems. There are numerous methods that are available in the literature for solving discrete time optimal control problems.

In [38], a class of discrete time optimal control problems is studied under the framework of nonlinear programming. More specifically, the Kuhn-Tucker theorem is applied to this problem and a discrete maximum principle similar to Pontryagin maximum principle is derived. However, it is not widely used in practice. The main reason is that the resulting two-point boundary-value problem obtained from the application of the discrete maximum principle is difficult to solve.

In principle, a discrete time optimal control problem can be solved as a mathematical programming problem by regrading all the states and the control variables as decision variables and the difference equations as algebraic constraints. However, the computational burden can become much too enormous for efficient implementation when there are many states and control variables. It is particularly so when the number of the time steps is large. Also, the algebraic constraints arising from the difference equations are not easy to be satisfied, if they are nonlinear. Furthermore, when the problem is subject to all-time-step constraints, additional nonlinear inequality constraints are required to be imposed at every time step. Thus, even a small scale discrete time optimal control problem with all-time-step constraints can become a large sale mathematical programming problem involving a large number of decision variables and many nonlinear equality and inequality constraints.

In [61], a discrete time optimal control problem is solved as a mathematical programming problem, where only the control variables are considered as decision variables. The state variables are obtained through solving the difference equations. In this way, there are fewer decision variables and the optimization problem does not involve the algebraic constraints arising from the difference equations. However, the number of decision variables can still be very large if the number of the time steps is large. For the all-time-step inequality constraints, they are handled by the constraint transcription technique developed in [103], where each of the all-time-step constraints is approximated by a sequence of canonical constraints in the form of the cost function.

The dynamical programming method provides a nonlinear feedback control law, while the discrete maximum principle or the mathematical programming approach gives rise to only open loop control law. However, only small scale discrete time optimal control problems without all-time-step inequality constraints can be solved effectively by the dynamic programming method.

For many natural and man-made systems, inherent delays exist during the transmission of information between different parts of the systems. As a consequence, it gives rise to time delayed systems for which the evolution of current states depends on the past and present values of states and controls. Optimal control of time delayed systems has been an active research area since 1960s. For problems involving continuous time systems with time delay, many papers are now available. See, for example, [44], [45–59]. Amongst these references, several computational methods (see [46–49, 51–62]) are suggested. For problems involving discrete time systems with time delay, there are much fewer papers available in the literature. In [63], Kuhn-Tucker theorem of nonlinear programming (see [64]) is used to derive a discrete maximum principle similar to Pontryagin maximum principle for an optimal discrete time system with a pure delay. However, no efficient computational algorithm is proposed using this discrete maximum principle. In [65], optimal tracking control for discrete time systems with time delay and a quadratic cost function, which is affected by persistent disturbances, is considered. However, no constraints on states or controls are involved. In [36], computational methods are proposed for several classes of optimal control problems with constraints on states and/or controls. These include a discrete time optimal control problem subject to constraints on states and controls at each time point. However, no time delay is involved in the problem considered.

For discrete time optimal control problems with time delays, it has many applications in areas such as logistics. To be more specific, it is known that logistics is playing an essential role in modern warfare. Contemporary military thoughts suggests that the logistic support for the modern warfare could be better described in terms of network structure. Furthermore, the management of the dynamic system behavior will be needed so as to maximize performance. This network nature of the modern warfare is constructed through a system of linked support hubs or nodes, which describe the dynamic distribution of the stock among the distribution hubs, support bases, exchange points and costumer locations through the supply routes. This type of network is preferred by the armed forces, because it enables the transfer of resources throughout and across the system, enhancing the responsiveness and robustness of support and reducing layering, linearity and the need for excess redundancy. In [60], motivated by the tactical logistic decision analysis problem, a class of discrete time optimal control problems subject to constraints at each time point with time delay appearing in the control is considered. First, the discrete time optimal control problem with the constraints ignored is solved by using the method suggested in [67], yielding the unconstrained optimal control. Then the control and the state are saturated when they violate their respective constraints. Clearly, such a control is, in general, not an optimal control for the discrete time optimal control problems with time delay and subject to constraints at each time point. More importantly, it is, in reality, impossible to saturate the states when their constraints are violated. It is thus clear that the problem considered in [60] has not yet been solved satisfactorily.

1.4 Minimax Optimal Control Problems

A minimax optimal control problem (also known as a Chebyshev optimal control problem) is an optimal control problem, where its performance measure cannot be represented by functions in the form of Bolza. It is described as minimizing the maximal value of a given function of the state and control variables over a prescribed time interval. It is well known that a maxmin problem is equivalent to a minimax problem. Minimax optimal control problems have been widely studied in the literature. See, for example, Warga [69, 70], Johnson [71], Powers [72], Holmaker [73–78]. Several sets of necessary conditions have been derived, and they have been used in the construction of numerical methods. For the method reported in Warga [69], a transformation is used to convert the minimax optimal control problem into a standard optimal control problem in the form of Bolza with additional inequality constraints on the state and/or control variables. Then, this standard optimal control problem can be dealt with by the existing methods and theories. This idea has been widely adopted, see, for example, Miele [75–77], Bock [79] and Oberle [68]. Minimax optimal control problems have many real world applications. Let us look at two applications below, where the first one is the robot navigation while the second one is the abort landing of an aircraft in the presence of windshear.

Robot navigation problems have been extensively studied in the literature, see, for example, [1–4]. One approach, which is known as the reactive approach (see [80]), is to design a specific control law for each behavior within a collection of behaviors, dedicating to perform a specific task. The robot switches between different behaviors when different circumstances are encountered in the environment. The advantage for this method is that the design task is quite simple because the controller is designed with only a limited set of objectives under consideration. The second one is called the deliberative approach. The motion of the robot is carefully designed in advance and some optimal performance, like minimizing the energy consumption, is specified. This approach is often used in structured environments, e.g. in industrial settings. In [81], the problem of a single robot moving towards a goal while avoiding obstacles is considered, where the optimal sequence of switches between the go-to-goal mode and the avoid-obstacle mode is to be found. This optimization problem is solved as an optimal control problem. The radius of the obstacle, called the safety distance, is regarded as the control parameter. However, when the robot crosses the guard, it changes its mode. But then, it may be steered back towards the guard, and hence resulting in traversing the guard many times in a short time interval. To avoid this chattering situation, the single guard is replaced in [82] by two circles with a common center at the obstacle. When the robot is in the goal-approach mode and is outside the inner circle, it will change its mode when the inner circle is crossed. On the other hand, when the robot is in the avoid-obstacle mode and is inside the outer circle, it will change its mode when it traverses towards the outer circle. The radii are regarded as control parameters. The optimal radii are obtained through solving an optimal control problem by using a gradient-based method. However, none of these two approaches solves this obstacle avoidance problem optimally.

The presence of low-altitude windshear is a microscale meteorological phenomenon usually taking place in subtropical regions. Windshear is a difference in wind speed and direction over a short distance in the atmosphere. It is observed that microburst and downbursts are often caused by thunderstorms. The microburst and downburst are the movement of high speed descending air in a short time and then spreading out from its center horizontally with high velocity. It is highly dangerous when an airplane is encountered with such a windshear when it is taking off or landing, even for a highly skilled pilot. This is because the aircraft may encounter a headwind followed by a tailwind, both coupled with a downdraft. The transition from the headwind to the tailwind leads to an acceleration and the resulting windshear inertia force can be as large as the drag of the aircraft, and sometimes as large as the thrust of the engines. Several fatal accidents are caused by the windshear. Two such incidents are: (i) The crash of a Boeing B-727 from PANAM Flight 759 on July 9, 1982 at New Orleans International Airport; and (ii) the crash of a Lockheed L-1011 from Delta Airlines Flight 191on August 2, 1985.

When the pilot of an aircraft detects a low-altitude windshear, he/she has two choices: (i) penetration landing; or (ii) abort landing. If the initial altitude is low, penetration landing is always chosen. However, if the initial altitude is high enough, abort landing is a much safer procedure than penetration landing. Thus, abort landing problem, penetration landing problem and other control and guidance problems involving windshear have been extensively studied in the literature. See, for example, [5–24, 83–85].

If the abort landing problem is studied as an optimization problem, the global information on the wind flow field is assumed available. Then the task is to determine the optimal trajectory for the maximization of the ground clearance so as to transfer the aircraft from a descending path to an ascending path safely. For the guidance studies, it is assumed that only local information on the wind flow is available. In this situation, a near-optimal trajectory is to be determined to approximate the behavior of the optimal trajectory by using only the local information. Although the optimal trajectory is not implementable due to the limitations of the on-board computer capacity, the study can provide the benchmark for the guidance schemes and piloting strategies for evaluating the existing guidance schemes and piloting strategies.

The optimal abort landing problem is to determine the optimal trajectory for the maximization of the ground clearance while transferring an aircraft from the descending path to an ascending path. In other words, it is to minimize the peak value of the altitude drop. This problem can be formulated as a minimax optimal control problem. In [21], the abort landing problem is studied in a vertical plane. It assumes that, when a plane encounters a windshear, the pilot increases the power setting at a constant time rate until the maximum power setting is reached. After that, the power setting is held constant. As a result, the only controls are the angle of attack and its time derivative. The problem is formulated as a minimax optimal control problem. This minimax optimal control problem is then converted into a Bolza problem through suitable transformations. The Bolza problem is solved by using the dual sequential gradient-restoration algorithm

(DSGRA) in [29] and [30]. The simulation is carried out under different combinations of windshear intensities, initial altitudes and power setting rates. It is found that, for strong-to-severe windshear, the optimal trajectory includes three branches: a descending flight branch, followed by a nearly horizontal flight branch, followed by an ascending flight branch after the aircraft has passed through the shear region. Along the optimal trajectory, the point of minimum velocity is reached at about the time when the shear ends. The peak altitude drop depends on the windshear intensity, the initial altitude, and the power setting rate. It increases as the windshear intensity increases and the initial altitude increases. It decreases as the power setting rate increases. In [21], a benchmark optimal trajectory is obtained for the optimal abort landing problem. From our studies in Chapter 3, we find that the optimal trajectory obtained is in the same shape. In [83], the problem is solved by using the gradient-restoration method. In [84] and [85], the problems is also converted into a Bolza problem through suitable transformations. However, the Bolza problem is solved by using the multiple shooting method, which requires a good guess of the initial condition. This is not an easy task to achieve.

1.5 Optimal Closed Loop Control Problems

The numerical methods reviewed in Section 1.2 are for finding open loop optimal controls. However, feedback controls are preferred in engineering applications. It is well known that an analytic linear feedback control law can be obtained for linear quadratic optimal control problems. For nonlinear optimal control problems, an analytic feedback control law is, however, not available except for some very special cases [43]. For nonlinear optimal control problems, even a numerical solution of optimal feedback control is very difficult to obtain. It involves solving a nonlinear HJB partial differential equation. The neighboring extremal method, originating in the 1960s, is considered as an effective method in dealing with feedback control of nonlinear optimal control problems. However, it is a local feedback control effective only in a small region around the open loop optimal trajectory. Furthermore, HJB partial differential equation approach is not effective when the optimal control problems are subject to constraints on the state.

For nonlinear optimal control problems, a simple feedback control is in the form of a PID controller, which consists of three parallel control terms for proportional, integral and derivative control actions. The PID controller design technique is widely accepted in industry. It is one of the most reliable methods used in practice. Since 1940s, many methods have been developed for tuning the PID controllers. Some methods use information on the open loop step response, such as the Coon-Cohen reaction curve method, Ziegler-Nichols frequency response method. Some commercial autotuners, which are developed based on these simple tuning methods, are available in the market. However, these methods often do not provide good tuning because they do not make use of full information of the dynamic behaviors. The Ziegler-Nichols method, which has been widely used in the process industry for the tuning of the PID parameters, usually gives rise to oscillatory set point responses. From 1980s, some automatic tuning methods have been suggested, such as the phase margin method [27] and the refined Ziegler-Nichols method [28]. However, the parameters for the proportional, integral and derivative control terms are still not easy to tune. For optimal control problems with hard constraints on the state and/or PID controller, the tuning of the PID parameters is much harder. It is particularly so for the tuning of the parameter for the integral term of the PID controller. The main reason is that the integral term of the PID controller performs the integral action over a period of time. Because of the accumulation effect, a large value of the parameter for the integral control will cause huge overshoot. On the other hand, if the parameter for the integral control is chosen to be very small, while the overshoot can become small, the steady state error will take a long time to reduce in the presence of constant disturbances.

Steering control problems have been well studied since 1920s. One of its applications is in ship steering. It is required to design an autopilot such that the ship is steered to the desired course automatically. Technically, the measured heading angle is compared with the desired course, and the difference is used as the input to the autopilot. Then autopilot generates an output signal by the integrated algorithm to the rudder servomechanism interface leading to suitable control signals to drive the ship rudder. The functions of the autopilot can be divided into the course changing and course keeping. Course change demands a fast response when a ship is maneuvering in a confined waterway. Course keeping demand a control to maintain the ship in a desired course with minimum rudder activity even under certain disturbances. Tight control is not recommended because it makes the rudder so active that additional rudder drag is always resulted. Thus, the rudder should be controlled in such a way that the propulsion losses caused by the resultant yawing and the motion of the rudder are minimized. The steering for the course keeping is more difficult than that for course changing.

For the autopilot design of the ship steering problem, PID control is considered to be one of the most reliable methods and has been widely used. (see, for example, [88, 89, 92]). Until now almost all commercially available ship autopilots were designed based on the PID controller. There are numerous advantages, resulting in the popularity of using PID controllers. For example, the structure of a PID controller is simple, involving only three parameters for the proportional, integral and derivative terms of the PID controller. It is easily implemented in practice, and convenient to adjust or reset by engineers. Furthermore, a PID controller is highly reliable. Consequently, it is widely used by engineers. However, there are also some shortcomings associated with a PID controller. For example, the parameters of a PID controller are difficult to tune when the output is required to move within a highly confined region. This is so even for an experienced technician.

Adaptive control is another popular control strategy for the ship steering problems (see, for example, [25, 26, 90, 93, 95]), though it is not as widely used as PID control. The advantage for the adaptive control is that it doesn't need the a priori knowledge of the dynamics of the plant. It works adaptively when there is some change of the steering characteristics. In [25], the sensitive model approach and the Lyapunov approach are studied, and simulations based on these two methods are carried out. Results show that there is no significant difference between these two methods. In [26], the autopilot is designed based on the model reference adaptive control. Different optimal behaviors in terms of ship steering are studied. It shows that the rate of turn or the turning circle should be adjusted during the course changing mode, and the accuracy against the economy should be adjusted during the course keeping mode. In [95], a self tuning autopilot is designed based on the recursive least-squared estimation method to reconcile the conflicting demands of course keeping and of course changing. It is achieved by formulating the course keeping performance criterion to minimize the heading error, propulsion losses and rudder activity, while the course changing performance criterion to minimize the heading error. However, adaptive controllers are harder to design. In particular, they may not be applicable when the output is required to move within a highly confined region.

1.6 Overview of the Thesis

In previous sections, we gave a brief introduction of the computational methods available in the literature for solving optimal control problems. The backgrounds of the problems that we will study in the following chapters and their applications are also given. The purpose of this thesis is to present new computational methods for several constrained optimal control problems. We briefly describe these problems below.

In Chapter 2, we consider a class of discrete time nonlinear optimal control problems with time delay and subject to inequality constraints on states and controls at each time point. In other words, for each time point, there is an inequality nonlinear constraint. These inequality constraints are called all-time-step inequality constraints. A constraint transcription technique is used in conjunction with a local smoothing method to construct a sequence of approximate discrete time optimal control problems involving time delay in states and controls and subject to nonlinear inequality constraints in canonical form. These approximate optimal control problems are special cases of a general discrete time optimal control problems with time delay appearing in the state and control and subject to nonlinear inequality constraints in canonical form. Thus, an efficient gradient-based computational method is devised for solving this general optimal control problem.

The gradient formulas needed for the cost and the canonical constraint functions are derived. With these gradient formulas, it is shown that the discrete time optimal control problem with time delay appearing in states and controls and subject to nonlinear inequality constraints in canonical form is solvable as an optimization problem with inequality constraints by the Sequential Quadratic Programming (SQP) method. With this computational method, each of the approximate problems constructed from the original optimal control problem can be solved. A tactical logistic problem is considered and studied. It is a problem of decision making for the distribution of resources within a network of support, where the network seeks to mimic how logistic support might be delivered in a military area of operations. The problem is formulated as a discrete time optimal control problem with a time delay appearing in the control, where the physical limitations in capacity at locations and the requirements for stock are formulated as all-time-step inequality constraints. The objective is to minimize the combat power cost function.

In Chapter 3, we consider a general class of maxmin optimal control problems, where the violation avoidance of the continuous state constraints is to be maximized. An efficient computational method for solving this general maxmin optimal control problem is devised. In this computational method, the constraint transcription method is used to construct a smooth approximate function for each of the continuous state inequality constraints, where the accuracy of the approximation is controlled by an accuracy parameter. This yields a sequence of smooth approximate optimal control problems, where the integral of the summation of these smooth approximate functions is taken as its cost function. A necessary condition and a sufficient condition are derived showing the relationship between the original maxmin optimal control problem and the sequence of the smooth approximate optimal control problems. A violation avoidance function is introduced from the solution of each of the smooth approximate optimal control problems and the original continuous state inequality constraints in such a way that the problem of finding an optimal control of the maxmin optimal control problem is equivalent to the problem of finding the largest root of the violation avoidance function. The control parameterization technique and a time scaling transform are applied to these smooth approximate optimal control problems. Two practical problems are considered and studied. The first one is an obstacle avoidance problem of an autonomous mobile robot. The objective is to control the robot such that it will avoid the obstacles as far as possible. The second one is the well known abort landing of an aircraft in a windshear downburst. It is required to control the aircraft such that the minimum altitude is maximized. They are both solved by the computational method developed in this chapter.

In Chapter 4, we consider a class of optimal PID control problems subject to continuous inequality constraints and terminal equality constraints. In other words, we are required to find the optimal PID parameters such that a cost function is minimized, while the continuous inequality constraints and terminal equality constraint are satisfied. The constraint transcription method and a local smoothing technique are applied to construct smooth approximate function in integral form for each of these continuous inequality constraints. The concept of the penalty function is then used to append these smooth approximate functions in integral form to the cost function, forming a new cost function. In this way, the constrained optimal PID control problem is approximated by a sequence of optimal parameter selection problems subject to only terminal equality constraint. Each of these optimal parameter selection problems is solvable as a nonlinear optimization problem. The gradient formulas of the new appended cost function and the terminal equality constraint function are derived, and a reliable computation algorithm is given for the tuning of the optimal PID parameters. The method proposed is used to design an optimal PID controller for a ship steering problem in its full generality without resorting to simplification or linearization. This PID controller is served as an autopilot such that the ship is steered to the desired course automatically under certain disturbances.

In Chapter 5, we focus on a class of optimal control problems subject to equality terminal state constraints and continuous inequality constraints on the state and/or control variables. It is required to find an optimal control such that the performance function is minimized, and the equality terminal state constraints and continuous inequality constraints on both the state and control variables are satisfied. It is a challenging problem and has been intensively studied, because such constraints are often encountered in practice. After the application of the control parameterization technique and a time scaling transformation, the constrained optimal control problem is approximated by a sequence of optimal parameter selection problems with equality terminal state constraints and continuous inequality constraints on the state and/or control. A new exact penalty functions are constructed for these terminal equality constraints and continuous inequality constraints. They are appended to the cost function to form a new cost function, giving rise to an unconstrained optimal parameter selection problem. The convergence analysis shows that, for a sufficiently large penalty parameter, a local minimizer of the unconstrained optimization problem is a local minimizer of the optimal parameter selection problem with terminal equality constraints and continuous inequality constraints. The relationships between the sequence of the approximate optimal parameter selection problems and the original constrained optimal control problem are discussed. Finally, the method proposed is applied to solve three nontrivial optimal control problems.

We conclude the thesis with some concluding remarks and some suggestion for future research.

The structure of this thesis is shown in the following figure.

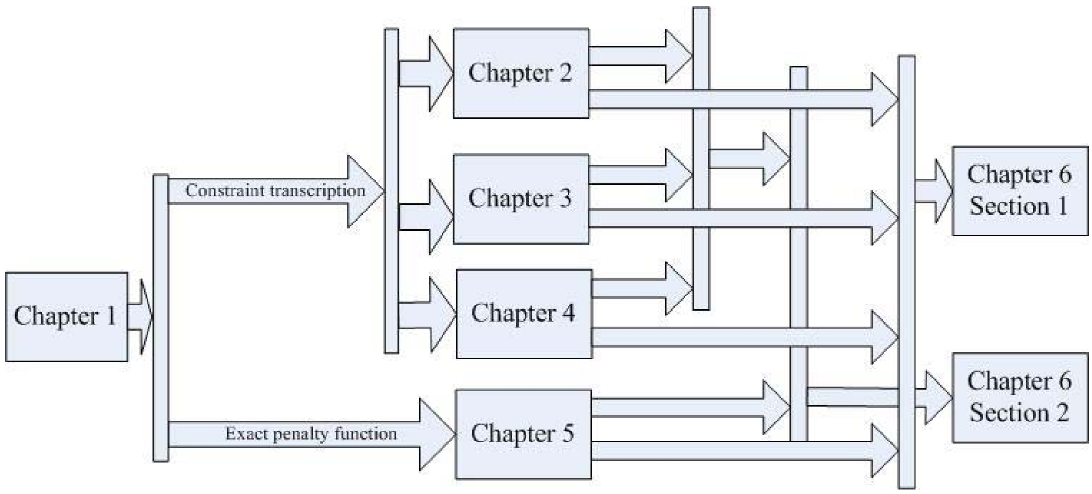

Figure 1.1: The structure of the thesis

CHAPTER 2

Discrete Time Optimal Control Problems with Time Delay and All-time-step Inequality Constraints

2.1 Introduction

In this chapter, we consider a class of discrete time nonlinear optimal control problems with time delay and subject to constraints on states and controls at each time point. These constraints are called all-time-step constraints. This problem, in principle, can be solved as a nonlinear optimization problem, where the cost function is minimized with respect to both the state and control variables subject to the all-time-step inequality constraints and the difference equations which are regarded as equality constraints. This approach will give rise to many nonlinear constraints. These include the equality constraints that arise from the difference equations. These equality constraints are nonlinear. It is acknowledged that nonlinear optimization problems with nonlinear equality constraints are difficult to solve, as the satisfaction of the nonlinear equality constraints are difficult to maintain during the optimization process. Another approach is to use the system of difference equations to calculate the state for a given control. In this way, only the control variables are decisions variables and the constraints contain only the all-time-step inequality constraints. Using this approach, the problem can also be regarded as a nonlinear optimization problem subject to nonlinear inequality constraints at each time point. To solve such a problem using a gradient-based method, such as SQP approximation scheme [36], we need the values and the gradients of the cost function and the all-time-step inequality constraint functions. If the number of the time steps is large, the number of nonlinear inequality constraints will also be large. To calculate the gradient of each of these nonlinear inequality constraints, the same number of associated co-state systems are also required to be solved, one for each of these nonlinear inequality constraints. Clearly, the computational complexity is rather high.

This chapter is the author's work in [112]. In this chapter, we use the constraint transcription technique introduced in [61] in conjunction with a local smoothing method to construct a sequence of approximate discrete time optimal control problems involving time delay in states and controls and subject to nonlinear inequality constraints in canonical form. Rigorous convergence analysis shows that the optimal solutions of the approximate problems converge to the optimal solution of the original optimal control problem. Furthermore, it is noted that these approximate optimal control problems are special cases of a general discrete time optimal control problems with time delay appearing in the state and control and subject to nonlinear inequality constraints in canonical form. Thus, we devise an efficient gradient-based computational method for solving this general optimal control problem. The gradient formulas needed for the cost and the canonical constraint functions are derived. With these gradient formulas, the discrete time optimal control problem with time delay appearing in states and controls and subject to nonlinear inequality constraints in canonical form is solvable as an optimization problem with inequality constraints by any gradient-based optimization method, such as the SQP approximation scheme (see [36]).

With this computational method, each of the approximate problems constructed from the original optimal control problem can be solved. As an application, we consider the tactical logistic decision analysis problem formulated in [60]. In [60], the optimal control problem with the constraints ignored is solved by using the method suggested in [67], yielding the unconstrained optimal control. Then the control and the state are saturated when they violate their respective constraints. Clearly, such a control is, in general, not an optimal control for the discrete time optimal control problems with time delay and subject to constraints at each time point under consideration. More importantly, it is, in reality, impossible to saturate the states when their constraints are violated. It is thus clear that the problem considered in [60] has not yet been solved satisfactorily. With our method, the constraints are considered explicitly in the development of our algorithm. Thus, it is not surprising that the optimal cost obtained using our algorithm is much less than that reported in [60]. More importantly, the optimal control obtained using our algorithm is such that the all-time-step constraints are satisfied at each time point. This is unlike the approach used in [60] by artificially saturating the state and control when the constraints are violated. Although saturating the control is possible, the task of saturating the state is really not realistic in practice.

2.2 Problem Statement

Consider a process described by the following system of difference equations with time delay:

$$x(k+1) = f(k, x(k), x(k-h), u(k), u(k-h)), \; k = 0, 1, \ldots, M-1, \tag{2.1a}$$

where

$$x = [x_1, x_2 \ldots, x_n]^\top \in \mathbb{R}^n, \; u = [u_1, u_2, \ldots u_r]^\top \in \mathbb{R}^r,$$

are, respectively, the state and control vectors, while $f = [f_1, f_2, \ldots, f_n]^\top \in \mathbb{R}^n$ is a given function and h is the delay time, which is an integer satisfying $0 < h < M$. Here, we consider the case where there is only one time delay. The extension to the case involving many time delays is straightforward but is more involved in terms of notation.

The initial functions for the state and control functions are:

$$\begin{aligned} x(k) &= \phi(k), \; k = -h, -h+1, \ldots, -1, \; x(0) = x^0, && (2.1\text{b}) \\ u(k) &= \gamma(k), \; k = -h, -h+1, \ldots, -1, && (2.1\text{c}) \end{aligned}$$

where

$$\phi(k) = [\phi_1, \phi_2, \ldots, \phi_n]^\top, \; \gamma(k) = [\gamma_1, \gamma_2, \ldots, \gamma_r]^\top,$$

are given functions from $k = -h, -h+1, \ldots -1$ into $\mathbb{R}^n$ and $\mathbb{R}^r$, respectively, and x^0 is a given vector in $\mathbb{R}^n$. Define

$$U = \left\{ \nu = [v_1, v_2, \ldots v_r]^\top \in \mathbb{R}^r : \alpha_i \le v_i \le \beta_i, \; i = 1, 2, \ldots, r \right\}, \tag{2.2}$$

where α_i, $i = 1, 2, \ldots, r$, and β_i, $i = 1, 2, \ldots, r$, are given real numbers. Note that U is a compact and convex subset of $\mathbb{R}^r$.

Consider the all-time-step inequality constraints on the state and control variables given below:

$$h_i\left(k, x(k), u\left(k\right)\right) \le 0, \; k = 0, 1, \ldots M-1; \; i = 1, 2, \ldots, N_2, \tag{2.3}$$

where h_i, $i = 1, 2, \cdots, N_2$, are given real-valued functions.

A control sequence $u = \{u(0), u(1), \ldots, u(M-1)\}$ is said to be an admissible control if $u(k) \in U$, $k = 0, 1, \ldots, M-1$, where U is defined by (2.2). Let $\mathcal{U}$ be the class of all such admissible controls. If a $u \in \mathcal{U}$ is such that the all-time-step inequality constraints (2.3) are satisfied, then it is called a feasible control. Let $\mathcal{F}$ be the class of all such feasible controls.

We now state our problem formally as follows:

Problem (Q2) Given system (2.1a)- (2.1c), find a control $u \in \mathcal{F}$ such that the cost function

$$g_0(u) = \Phi_0(x(M)) + \sum_{k=0}^{M-1} \mathcal{L}_0(k, x(k), x(k-h), u(k), u(k-h)) \tag{2.4}$$

is minimized over $\mathcal{F}$, where Φ_0 and $\mathcal{L}_0$ are given real-valued functions.

2.3 Approximation

In this section, we shall use the constraint transcription technique introduced in [61] to approximate each of the all-time-step inequality constraints by a sequence of inequality constraints in canonical form. In this way, we will obtain a sequence of discrete time optimal control problems with time delay and subject to canonical constraints. Therefore, to solve Problem (Q2), it is required to solve a sequence of discrete time optimal control problems with time-delay and subject to canonical constraints. In this section, we shall construct these approximate optimal control problems and then show the convergence of these approximate optimal control problems to the original optimal control problem. In section **??**, we shall develop a computational method for solving a general class of discrete time optimal control problems with time-delay and subject to canonical constraints. This general optimal control problem contains the approximate optimal control problem as special cases.

To begin, we first note that the all-time-step inequality constraints (2.3) are equivalent to the following equality constraints:

$$g_i(u) = \sum_{k=0}^{M-1} \max\{h_i(k, x(k), u(k)), 0\} = 0, \; i = 1, 2, \ldots, N_2. \tag{2.5}$$

Thus, the set $\mathcal{F}$ of feasible controls can be written as:

$$\mathcal{F} = \{u(k) \in U, k = 0, 1, \ldots, M-1 : g_i(u) = 0, \; i = 1, 2, \ldots, N_2\}, \tag{2.6}$$

where U is defined by (2.2). However, the functions appeared in (2.5) are nonsmooth. Thus, for each $i = 1, 2, \ldots, N_2$, we shall approximate the nonsmooth function

$$\max\{h_i(k, x(k), u(k)), 0\}$$

by a smooth function

$$\mathcal{L}_{i,\varepsilon}(k, x(k), u(k)) \tag{2.7}$$

given by

$$\mathcal{L}_{i,\varepsilon} = \begin{cases} 0, & \text{if} \quad h_i < -\varepsilon, \\ \dfrac{(h_i+\varepsilon)^2}{4\varepsilon}, & \text{if} \ -\varepsilon \le h_i \le \varepsilon, \\ h_i, & \text{if} \quad h_i > \varepsilon, \end{cases} \tag{2.8}$$

where $\varepsilon > 0$ is an adjustable constant with small value. Then, the all-time-step inequality constraints (2.3) are approximated by the inequality constraints in canonical form defined by

$$-\frac{\varepsilon}{4} + g_\varepsilon(u) \le 0, \tag{2.9}$$

where

$$g_\varepsilon(u) = \sum_{i=1}^{N_2}\sum_{k=0}^{M-1} \mathcal{L}_{i,\varepsilon}(k, x(k), u(k)). \tag{2.10}$$

Define

$$\mathcal{F}_\varepsilon = \left\{ u(k) \in U,\ k = 0, 1, \ldots, M-1 : -\frac{\varepsilon}{4} + g_\varepsilon(u) \le 0 \right\}. \tag{2.11}$$

Now, we can define a sequence of approximate problems Q(ε), where $\varepsilon > 0$, below.
Problem(Q2(ε)) Problem (Q2) with (2.3) replaced by

$$G_\varepsilon(u) = -\frac{\varepsilon}{4} + g_\varepsilon(u) \le 0,\ i = 1, 2, \ldots, N_2. \tag{2.12}$$

In Problem (Q2(ε)), our aim is to find a control u in $\mathcal{F}_\varepsilon$ such that the cost function (2.4) is minimized over $\mathcal{F}_\varepsilon$. For each $\varepsilon > 0$, Problem (Q2(ε)) is a special case of a general discrete time optimal control problem with time-delay and subject to canonical constraints defined below.
Problem (P2) Given system (2.1a)-(2.1c), find an admissible control $u \in \mathcal{U}$ such that the cost function

$$g_0(u) = \Phi_0(x(M)) + \sum_{k=0}^{M-1} \mathcal{L}_0(k, x(k), x(k-h), u(k), u(k-h)) \tag{2.13}$$

is minimized over $\mathcal{U}$ subject to the following constraints in canonical form:

$$g_i(u) = 0,\ i = 1, 2, \ldots, N_e, \tag{2.14a}$$
$$g_i(u) \le 0,\ i = N_e + 1, N_e, \ldots, N, \tag{2.14b}$$

where

$$g_i(u) = \Phi_i(x(M)) + \sum_{k=0}^{M-1} \mathcal{L}_i(k, x(k), x(k-h), u(k), u(k-h)). \tag{2.15}$$

We shall develop an efficient computational method for solving Problem (P2) in the next section. In the rest of this section, our aim is to establish the required convergence properties of Problems (Q2(ε)) to Problem (Q2). We assume that the following assumptions are satisfied.

Assumption 2.1. For each $k = 0, 1, \dots, M-1,\ f(k, \cdot, \cdot, \cdot, \cdot)$ is continuously differentiable on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^r$.

Assumption 2.2. For each $i = 1, 2, \dots, N_2$, and for each $k = 0, 1, \dots, M-1,\ h_i(k, \cdot, \cdot)$ is continuously differentiable on $\mathbb{R}^n \times \mathbb{R}^r$.

Assumption 2.3. Φ_0 is continuously differentiable on $\mathbb{R}^n$.

Assumption 2.4. For each $k = 0, 1, \dots, M-1,\ \mathcal{L}_0(k, \cdot, \cdot, \cdot, \cdot)$ is continuously differentiable on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^r$.

Assumption 2.5. For any control u in $\mathcal{F}$, there exists a control $\bar{u} \in \mathcal{F}^0$ such that $\alpha\bar{u} + (1-\alpha)u \in \mathcal{F}^0$ for all $\alpha \in (0, 1]$, where $\mathcal{F}^0$ is the interior of $\mathcal{F}$, meaning that if $u \in \mathcal{F}^0$, then

$$h_i(k, x(k), u(k)) < 0,\ \ i = 1, \dots, N_2,$$

for all $k = 0, 1, \dots, M-1$.

Remark 2.1. Under Assumption 2.5, it can be shown that for any u in $\mathcal{F}$ and $\delta > 0$, there exists a $\bar{u} \in \mathcal{F}^0$ such that

$$\max_{0 \le k \le M-1} \|u(k) - \bar{u}(k)\| \le \delta,$$

In what follows, we shall present an algorithm for solving Problem (Q2) as a sequence of Problems (Q2(ε)).

Algorithm 2.1.

Step 1. Set $\varepsilon = \varepsilon_0$.

Step 2. Solve Problem (Q2(ε)) as a nonlinear programming problem, and obtain an optimal solution.

Step 3. Set $\varepsilon = \varepsilon/10$, and go to Step 2.

Remark 2.2. ε_0 is usually set as 1.0×10^{-2}; and the algorithm is terminated as "successful exit" when $\varepsilon < 10^{-7}$.

Remark 2.3. In Step 2, we need to solve Problem (Q2(ε)) for each $\varepsilon > 0$, which is a special case of Problem (P2). Thus, we will develop an efficient computational method for solving Problem (P2) in Section **??**.

To establish the convergence properties of Problems (Q2(ε)) to Problem (Q2), we need

Lemma 2.1. *If u_ε is a feasible control of Problem (Q2(ε)), then it is also a feasible control of Problem (Q2).*

Proof. Suppose u_ε is not a feasible control of Problem (Q2). Then, there exit some $i \in \{1, 2, \ldots, N_2\}$ and $k \in \{0, 1, \ldots, M-1\}$ such that

$$h_i\left(k, x(k\,|u_\varepsilon), u_\varepsilon(k)\right) > 0.$$

This, in turn, implies that

$$\mathcal{L}_{i,\varepsilon}\left(k, x(k\,|u_\varepsilon), u_\varepsilon(k)\right) > \frac{\varepsilon}{4},$$

and hence,

$$g_\varepsilon\left(u_\varepsilon\right) > \frac{MN_2\varepsilon}{4} > \frac{\varepsilon}{4}.$$

That is,

$$-\frac{\varepsilon}{4} + g_\varepsilon\left(u_\varepsilon\right) > 0.$$

This is a contradiction to the constraints specified in (2.12). This completes the proof. □

Theorem 2.1. *Let u^* be an optimal control of Problem (Q2) and let u_ε^* be an optimal control of Problem (Q2(ε)). Then,*

$$\lim_{\varepsilon \to 0} g_0(u_\varepsilon^*) = g_0(u^*).$$

Proof. By Assumption 2.5, there exists a $\bar{u} \in \mathcal{F}^0$ such that

$$u_\alpha \equiv \alpha\bar{u} + (1-\alpha)u^* \in \mathcal{F}^0,\ \forall \alpha \in (0,1]\,.$$

Thus, for any $\delta_1 > 0, \exists$ an $\alpha_1 \in (0,1]$ such that

$$g_0(u^*) \leq g_0(u_\alpha) \leq g_0(u^*) + \delta_1,\ \forall \alpha \in (0,\alpha_1]\,. \tag{2.16}$$

Choose $\alpha_2 = \alpha_1/2$. Then, it is clear that $u_{\alpha_2} \in \mathcal{F}^0$. Thus, there exists a $\delta_2 > 0$ such that

$$h_i(k, x(k\,|u_{\alpha_2}), u_{\alpha_2}) < -\delta_2,\ i = 1, 2, \ldots, N_2,$$

for all $k,\ 0 \leq k \leq M-1$. Let $\varepsilon = \delta_2$. Then, it follows from the definition of $\mathcal{L}_{i,\varepsilon}$ given by (2.8) that $\mathcal{L}_{i,\varepsilon} = 0$. Thus, (2.12) is satisfied and hence $u_{\alpha_2} \in \mathcal{F}_\varepsilon$. Let u_ε^* be an optimal control of Problem (Q2(ε)). Clearly, $u_\varepsilon^* \in \mathcal{F}_\varepsilon$ and

$$g_0(u_\varepsilon^*) \leq g_0(u_{\alpha_2}). \tag{2.17}$$

However,

$$g_0(u^*) \leq g_0(u_\varepsilon^*). \tag{2.18}$$

Thus, if follows from (2.16), (2.17) and (2.18) that

$$g_0(u^*) \leq g_0(u^*_\varepsilon) \leq g_0(u_{\alpha_2}) \leq g_0(u^*) + \delta_1.$$

Letting $\varepsilon \to 0$ and noting that $\delta_1 > 0$ is arbitrary, the conclusion of the theorem follows readily. This completes the proof. □

Theorem 2.2. *Let u^*_ε and u^* be optimal controls of Problems (Q2(ε)) and (Q2), respectively. Then, there exists a subsequence of $\{u^*_\varepsilon\}$, which is again denoted by the original sequence, and a control $\bar{u} \in \mathcal{F}$ such that, for each $k = 0, 1, .., M-1$,*

$$\lim_{\varepsilon \to 0} \|u^*_\varepsilon(k) - \bar{u}(k)\| = 0. \tag{2.19}$$

Furthermore, $\bar{u}$ is an optimal control of Problem (Q2).

Proof. Since U is a compact subset of $\mathbb{R}^r$, and $\{u^*_\varepsilon\}$, as a sequence in ε, is such that $u^*_\varepsilon(k) \in U$, for $k = 0, 1, .., M-1$, it is clear that there exists a subsequence, which is again denoted by the original sequence, and a control parameter vector $\bar{u} \in \mathcal{U}$ such that, for each $k = 0, 1, .., M-1$,

$$\lim_{\varepsilon \to 0} \|u^*_\varepsilon(k) - \bar{u}(k)\| = 0. \tag{2.20}$$

By induction, we can show, by using Assumption 2.1 and (2.20), that, for each $k = 0, 1, \ldots, M$,

$$\lim_{\varepsilon \to 0} \|x(k|u^*_\varepsilon) - x(k\,|\bar{u})\| = 0. \tag{2.21}$$

Thus, by Assumption 2.2, we have, for each $k = 0, 1, .., M$,

$$\lim_{\varepsilon \to 0} h_i(k, x(k|u^*_\varepsilon), u^*_\varepsilon(k)) = h_i(k, x(k\,|\bar{u}), \bar{u}(k)),\ i = 1, 2, \ldots, N_2. \tag{2.22}$$

By Lemma 2.1, $u^*_\varepsilon \in \mathcal{F}$ for all $\varepsilon > 0$. Thus, it follows from (2.22) that $\bar{u} \in \mathcal{F}$. Next, by Assumption 2.1, we deduce from (2.20) and (2.21) that

$$\lim_{\varepsilon \to 0} g_0(u^*_\varepsilon) = g_0(\bar{u}). \tag{2.23}$$

For any $\delta_1 > 0$, it follows from Remark 2.1 that there exists a $\hat{u} \in \mathcal{F}^0$ such that, for each $k = 0, 1, \ldots, M-1$,

$$\|u^*(k) - \hat{u}(k)\| \leq \delta_1. \tag{2.24}$$

By Assumption 2.1 and induction, we can show that, for any $\rho_1 > 0$, there exists a $\delta_1 > 0$ such that for each $k = 0, 1, \ldots, M$,

$$\|x(k\,|u^*) - x(k\,|\hat{u})\| \leq \rho_1, \tag{2.25}$$

whenever (2.24) is satisfied. Using (2.24), (2.25) and Assumption 2.4, it follows that, for any $\rho_2 > 0$, there exists a $\hat{u} \in \mathcal{F}^0$ such that

$$g_0(u^*) \leq g_0(\hat{u}) \leq g_0(u^*) + \rho_2. \tag{2.26}$$

Since $\hat{u} \in \mathcal{F}^0$, we have, for each $k = 0, 1, \ldots, M$,

$$h_i(k, x(k\,|\hat{u}), \hat{u}(k)) < 0, \;\; i = 1, 2, \ldots, N_2,$$

and hence there exists a $\delta > 0$ such that, for each $k = 0, 1, \ldots, M$,

$$h_i(k, x(k\,|\hat{u}), \hat{u}(k)) \leq -\delta, \;\; i = 1, 2, \ldots, N_2. \tag{2.27}$$

Thus, in view of (2.11), we see that

$$\hat{u} \in \mathcal{F}_\varepsilon,$$

for all ε, $0 \leq \varepsilon \leq \delta$. Therefore,

$$g_0(u_\varepsilon^*) \leq g_0(\hat{u}). \tag{2.28}$$

Using (2.26) and (2.28), and noting that $u_\varepsilon^* \in \mathcal{F}$, we obtain

$$g_0(u^*) \leq g_0(u_\varepsilon^*) \leq g_0(u^*) + \rho_2. \tag{2.29}$$

Since $\rho_2 > 0$ is arbitrary, it follows that

$$\lim_{\varepsilon \to 0} g_0(u_\varepsilon^*) = g_0(u^*). \tag{2.30}$$

Combining (2.23) and (2.30), we conclude that $\bar{u}$ is an optimal control of Problem (Q2). This completes the proof. □

2.4 Computational Method

In this section, we shall develop an efficient computational method for solving Problem (P2) as a nonlinear mathematical programming problem, where the SQP approximation scheme is used together with the active set strategy. There are several efficient implementations of SQP available (see, for example, the subroutines NLPQL and NLPQLP written by Schittkowski [66]). For doing this, it is required to calculate, for each control sequence $u = \{u(0), u(1), \ldots, u(M-1)\}$, the values of the cost function $g_0(u)$ and the constraints functions $g_{i,\varepsilon}(u)$, $i = 1, 2, \ldots, N$, as well as their gradients. The calculation of the values of the cost function (2.13) and the canonical constraint functions given by (2.14a) and (2.14b) corresponding to each $u \in \mathcal{U}$ can be done as follows.

For each $u = \{u(0), u(1), \ldots, u(M-1)\}$, where $u(k) \in U$, $k = 0, 1, \ldots, M-1$, with U being defined by (2.2), we solve system ((2.1a), (2.1b), (2.1c)) to obtain the corresponding solution sequence $x(k\,|u)$, $k = 0, 1, \ldots, M-1$. Then, the value of the cost function (2.13) and the values of the canonical constraint functions given by (2.14a) and (2.14b) are calculated.

To calculate the gradients of the cost and constraint functions, we will derive the required gradient formulas corresponding to each control sequence

$$u = \{u(0), u(1), \ldots, u(M-1)\}$$

as follows.

For each $i = 0, 1, \ldots, N$, let

$$H_i\left(k, x(k), y(k), z(k), u(k), v(k), w(k), \lambda^i(k+1), \bar{\lambda}^i(k)\right)$$

be the corresponding Hamiltonian sequence defined by

$$\begin{aligned}
& H_i\left(k, x(k), y(k), z(k), u(k), v(k), w(k), \lambda^i(k+1), \bar{\lambda}^i(k)\right) \\
= \; & \mathcal{L}_i(k, x(k), y(k), u(k), v(k)) \\
& + \mathcal{L}_i(k+h, z(k), x(k), w(k), u(k)) e(M-k-h) \\
& + \left(\lambda^i(k+1)\right)^\top f(k, x(k), y(k), u(k), v(k)) \\
& + \left(\bar{\lambda}^i(k)\right)^\top f(k+h, z(k), x(k), w(k), u(k)) e(M-k-h),
\end{aligned} \tag{2.31}$$

where $e(\cdot)$ denotes the Heaviside function defined by

$$e(k) = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0, \end{cases} \tag{2.32}$$

and

$$y(k) = x(k-h), \tag{2.33a}$$

$$z(k) = x(k+h), \tag{2.33b}$$

$$v(k) = u(k-h), \tag{2.33c}$$

$$w(k) = u(k+h), \tag{2.33d}$$

$$\bar{\lambda}^i(k) = \lambda^i(k+h+1). \tag{2.33e}$$

For each control u, $\lambda^i(\cdot|u)$ is the solution of the following co-state system

$$\left(\lambda^i(k)\right)^\top = \frac{\partial H_i(k)}{\partial x(k)}, \; k = M-1, M-2, \ldots, 0, \tag{2.34}$$

with boundary conditions

$$\left(\lambda^i(M)\right)^\top = \frac{\partial \Phi_i(x(M))}{\partial x(M)}, \tag{2.35a}$$

$$\lambda^i(k) = 0, \; k > M. \tag{2.35b}$$

We set

$$z(k) = 0, \; \forall k = M-h+1, M-h+2, \ldots, M, \tag{2.36}$$

and

$$w(k) = 0, \; \forall k = M-h, M-h+1, \ldots, M. \tag{2.37}$$

Then, the gradient formulas for the cost functions (for $i = 0$) and constraint functions (for $i = 1, \ldots, N$) are given in the following theorem.

Theorem 2.3. *Let $g_i(u)$, $i = 0, 1, \ldots, N$, be defined by (2.13) (the cost function for $i = 0$) and (2.15) (the constraint functions for $i = 1, 2, \ldots, N$). Then, for each $i = 0, 1, \ldots, N$, the gradient of the function $g_i(u)$ is given by*

$$\frac{\partial g_i(u)}{\partial u} = \left[\frac{\partial H_i(0)}{\partial u(0)}, \frac{\partial H_i(1)}{\partial u(1)}, \ldots, \frac{\partial H_i(M-1)}{\partial u(M-1)}\right], \tag{2.38}$$

where

$$H_i(k) = H_i\left(k, x(k|u), y(k|u), z(k|u), u(k), v(k), w(k), \lambda^i(k+1|u), \overline{\lambda}^i(k|u)\right),$$
$$k = 0, 1, \ldots, M-1.$$

Proof. Define

$$u = [(u(0))^\top, (u(1))^\top, \ldots, (u(M-1))^\top]^\top. \tag{2.39}$$

Let the control u be perturbed by $\varepsilon \hat{u}$, where $\varepsilon > 0$ is a small real number and $\hat{u}$ is an arbitrary but fixed perturbation of u given by

$$\hat{u} = [(\hat{u}(0))^\top, (\hat{u}(1))^\top, \ldots, (\hat{u}(M-1))^\top]^\top. \tag{2.40}$$

Then, we have

$$u_\varepsilon = u + \varepsilon\hat{u} = [(u(0,\varepsilon))^\top, (u(1,\varepsilon))^\top, \ldots, (u(M-1,\varepsilon))^\top]^\top, \tag{2.41}$$

where

$$u(k,\varepsilon) = u(k) + \varepsilon\hat{u}(k), \ \ k = 0, 1, \ldots, M-1. \tag{2.42}$$

For brevity, let $x(k) = x(k|u)$ be the solution of the system (2.1a)-(2.1c) corresponding to the control u. Furthermore, let the perturbed solution be denoted by

$$x(k,\varepsilon) = x(k \mid u_\varepsilon), \ \ k = 1, 2, \ldots, M. \tag{2.43}$$

Then,

$$x(k+1,\varepsilon) = f(k, x(k,\varepsilon), y(k,\varepsilon), u(k,\varepsilon), v(k,\varepsilon)). \tag{2.44}$$

The variation of the state for $k = 0, 1, \ldots, M-1$ is:

$$\begin{aligned}
\triangle x(k+1) &= \left.\frac{\mathrm{d}x(k+1,\varepsilon)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \\
&= \frac{\partial f(k, x(k), y(k), u(k), v(k))}{\partial x(k)}\triangle x(k) \\
&+ \frac{\partial f(k, x(k), y(k), u(k), v(k))}{\partial y(k)}\triangle y(k) \\
&+ \frac{\partial f(k, x(k), y(k), u(k), v(k))}{\partial u(k)}\hat{u}(k) \\
&+ \frac{\partial f(k, x(k), y(k), u(k), v(k))}{\partial v(k)}\triangle v(k),
\end{aligned} \tag{2.45a}$$

where

$$\triangle x(k) = 0, \ \ k \leq 0, \tag{2.45b}$$

$$\triangle u(k) = 0, \ \ k < 0. \tag{2.45c}$$

From (2.45b) and (2.45c), we obtain

$$\triangle y(k) = 0, \ \ k = 0, 1, \ldots, h, \tag{2.46a}$$

and

$$\triangle v(k) = 0, \ \ k = 0, 1, \ldots, h-1. \tag{2.46b}$$

Define

$$\bar{\mathcal{L}}_i = \mathcal{L}_i(k, x(k), y(k), u(k), v(k)), \tag{2.47a}$$
$$\hat{\mathcal{L}}_i = \mathcal{L}_i(k+h, z(k), x(k), w(k), u(k)), \tag{2.47b}$$
$$\bar{f} = f(k, x(k), y(k), u(k), v(k)), \tag{2.47c}$$
$$\hat{f} = f(k+h, z(k), x(k), w(k), u(k)), \tag{2.47d}$$
$$\bar{H}_i = H_i(k). \tag{2.47e}$$

By chain rule and (2.47a), it follows that

$$
\begin{aligned}
\frac{\partial g_i(u)}{\partial u}\hat{u} &= \lim_{\varepsilon \to 0} \frac{g_i(u_\varepsilon) - g_i(u)}{\varepsilon} \equiv \left.\frac{\mathrm{d}g_i(u_\varepsilon)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \\
&= \frac{\partial \Phi_i(x(M))}{\partial x(M)} \triangle x(M) \\
&\quad + \sum_{k=0}^{M-1} \left[\frac{\partial \bar{\mathcal{L}}_i}{\partial x(k)} \triangle x(k) + \frac{\partial \bar{\mathcal{L}}_i}{\partial y(k)} \triangle y(k) \right. \\
&\quad \left. + \frac{\partial \bar{\mathcal{L}}_i}{\partial u(k)} \hat{u}(k) \frac{\partial \bar{\mathcal{L}}_i}{\partial v(k)} \triangle v(k) \right]. \qquad (2.48)
\end{aligned}
$$

From (2.33a), (2.33c) and (2.47b), we have

$$
\begin{aligned}
&\sum_{k=0}^{M-1} \left\{ \left(\frac{\partial \bar{\mathcal{L}}_i}{\partial y(k)} \right) \triangle y(k) + \left(\frac{\partial \bar{\mathcal{L}}_i}{\partial v(k)} \right) \triangle v(k) \right\} \\
= &\sum_{k=0}^{M-1} e(M-k-h) \left[\left(\frac{\partial \hat{\mathcal{L}}_i}{\partial x(k)} \right) \triangle x(k) \right. \\
&\left. + \left(\frac{\partial \hat{\mathcal{L}}_i}{\partial u(k)} \right) \hat{u}(k) \right]. \qquad (2.49)
\end{aligned}
$$

Substituting (2.49) into (2.48), and then using (2.31) and (2.47a)-(2.47e), we obtain

$$
\begin{aligned}
\frac{\partial g_i(u)}{\partial u}\hat{u} &= \left(\frac{\partial \Phi_i(x(M))}{\partial x(k)} \right) \triangle x(M) \\
&\quad + \sum_{k=0}^{M-1} \left[\left(\frac{\partial \bar{H}_i}{\partial x(k)} \right) \triangle x(k) + \left(\frac{\partial \bar{H}_i}{\partial u(k)} \right) \hat{u}(k) \right. \\
&\quad - \left(\lambda^i(k+1) \right)^\top \frac{\partial \bar{f}}{\partial x} \triangle x(k) \\
&\quad - \left(\bar{\lambda}^i(k) \right)^\top \frac{\partial \hat{f}}{\partial x(k)} \triangle x(k) e(M-k-h) \\
&\quad - \left(\lambda^i(k+1) \right)^\top \frac{\partial \bar{f}}{\partial u(k)} \hat{u}(k) \\
&\quad \left. - \left(\bar{\lambda}^i(k) \right)^\top \frac{\partial \hat{f}}{\partial u(k)} \triangle u(k) e(M-k-h) \right]. \qquad (2.50)
\end{aligned}
$$

Using (2.35b) and the definition of $e(\cdot)$, it follows that

$$\begin{aligned}
&\sum_{k=0}^{M-1}\left(\bar{\lambda}^i(k)\right)^\top\left[\frac{\partial \hat{f}}{\partial x(k)}\triangle x(k)+\frac{\partial \hat{f}}{\partial u(k)}\triangle u(k)\right]e(M-k-h)\\
=&\sum_{k=0}^{M-h-1}\left(\bar{\lambda}^i(k)\right)^\top\left[\frac{\partial \hat{f}}{\partial x(k)}\triangle x(k)+\frac{\partial \hat{f}}{\partial u(k)}\triangle u(k)\right]e(M-k-h)\\
=&\sum_{k=h}^{M-1}\left(\lambda^i(k+1)\right)^\top\left[\frac{\partial \bar{f}}{\partial y(k)}\triangle y(k)+\frac{\partial \bar{f}}{\partial v(k)}\triangle v(k)\right]. \qquad (2.51)
\end{aligned}$$

As $\triangle y(k)=0$, for $0\le k\le h$, and $\triangle v(k)=0$, $0<k\le h$, we have

$$\begin{aligned}
&\sum_{k=h}^{M-1}\left(\lambda^i(k+1)\right)^\top\left[\frac{\partial \bar{f}}{\partial y(k)}\triangle y(k)+\frac{\partial \bar{f}}{\partial v(k)}\triangle v(k)\right]\\
=&\sum_{k=0}^{M-1}\left(\lambda^i(k+1)\right)^\top\left[\frac{\partial \bar{f}}{\partial y(k)}\triangle y(k)+\frac{\partial \bar{f}}{\partial v(k)}\triangle v(k)\right]. \qquad (2.52)
\end{aligned}$$

Combining (2.51) and (2.52), we obtain

$$\begin{aligned}
&\sum_{k=0}^{M-1}\left(\bar{\lambda}^i(k)\right)^\top\left[\frac{\partial \hat{f}}{\partial x(k)}\triangle x(k)+\frac{\partial \hat{f}}{\partial u(k)}\triangle u(k)\right]e(M-k-h)\\
=&\sum_{k=0}^{M-1}\left(\lambda^i(k+1)\right)^\top\left[\frac{\partial \bar{f}}{\partial y(k)}\triangle y(k)+\frac{\partial \bar{f}}{\partial v(k)}\triangle v(k)\right]. \qquad (2.53)
\end{aligned}$$

From (2.45a) and (2.53), it follows from (2.50) that

$$\begin{aligned}
\frac{\partial g_i(u)}{\partial u}\hat{u} =& \left(\frac{\partial \Phi_i(x(M))}{\partial x(k)}\right)\triangle x(M)\\
&+\sum_{k=0}^{M-1}\left\{\left(\frac{\partial \bar{H}_i}{\partial x(k)}\right)\triangle x(k)\right.\\
&\left.+\left(\frac{\partial \bar{H}_i}{\partial u(k)}\right)\hat{u}(k)-\left(\lambda^i(k+1)\right)^\top\triangle x(k+1)\right\}. \qquad (2.54)
\end{aligned}$$

Thus, by (2.34), (2.47c) and (2.54), we obtain

$$\begin{aligned}\frac{\partial g_i(u)}{\partial u}\hat{u} &= \left(\frac{\partial \Phi_i(x(M))}{\partial x(k)}\right)\triangle x(M)\\&+\sum_{k=0}^{M-2}\left[\left(\frac{\partial \bar{H}_i(k)}{\partial x(k)}\right)\triangle x(k)\right.\\&\left.-\left(\frac{\partial \bar{H}_i(k+1)}{\partial x(k)}\right)\triangle x(k+1)\right]\\&+\frac{\partial \bar{H}_i(M-1)}{\partial x(k)}\triangle x(M-1)-\left(\lambda^i(M)\right)^\top \triangle x(M)\\&+\sum_{k=0}^{M-1}\left[\left(\frac{\partial \bar{H}_i}{\partial u(k)}\right)\hat{u}(k)\right]. \end{aligned} \tag{2.55}$$

Therefore, by substituting (2.35a) and (2.45b) into (2.49), it follows that

$$\frac{\partial g_i(u)}{\partial u}\hat{u} = \left[\frac{\partial H_i(0)}{\partial u(0)}, \frac{\partial H_i(1)}{\partial u(1)}, \ldots, \frac{\partial H_i(M-1)}{\partial u(M-1)}\right]\hat{u}.$$

Since $\hat{u}$ is arbitrary, we obtain

$$\frac{\partial g_i(u)}{\partial u} = \left[\frac{\partial H_i(0)}{\partial u(0)}, \frac{\partial H_i(1)}{\partial u(1)}, \ldots, \frac{\partial H_i(M-1)}{\partial u(M-1)}\right].$$

This completes the proof. □

The values of the cost and constraint functions as well as their gradients are calculated as described in following algorithm.

Algorithm 2.2.

Step 1. For a given control sequence $u = \{u(0), u(1), \ldots, u(M-1)\}$ with $u(k) \in U$, $k = 0, 1, \ldots, M-1$, compute the solution $x(k)$, $k = 0, 1, \ldots, M-1$, of system (1) by solving time delayed difference equations (2.1a) with initial conditions (2.1b) and (2.1c) forward from $k = 0$ to $k = M$.

Step 2. Calculate the values of the cost function $g_0(u)$ and the constraint functions $g_{i,\varepsilon}(u)$, $i = 1, 2, \ldots, N$, by using the control sequence $u = \{u(0), u(1), \ldots, u(M-1)\}$, and the corresponding solution sequence $x(k)$, $k = 0, 1, \ldots, M-1$.

Step 3. For each $i = 0,\ 1, \ldots,\ N$, compute the co-state solution $\lambda^i(k)$, $k = M-1, M-2, \ldots, 0$, by solving co-state difference equations (2.34) with terminal conditions (2.35a), (2.35b), (2.36) and (2.37) backward, from $k = M, M-1, \ldots, 0$. Thus, for each $i = 1, 2, \ldots, N$, $\lambda^i(k)$, $k = M-1, M-2, \ldots, 0$, are obtained.

Step 4. Calculate the gradients of the cost function $g_0(u)$ and the constraint functions $g_i(u)$, $i = 1, 2, \ldots, N$, according to the formulas given in Theorem 2.3.

Based on Algorithm 2.2, Problem (P2) can be solved as a nonlinear mathematical programming problem by using the SQP approximating scheme. The subroutines NLPQL and NLPQLP coded in [66] are two examples of efficient implementations of SQP.

2.5 A Tactical Logistic Decision Analysis Problem

We now consider a tactical logistic decision analysis problem studied in [60]. It is a problem of decision making for the distribution of resources within a network of support, where the network seeks to mimic how logistic support might be delivered in a military area of operations. The problem is formulated as a discrete time optimal control problem with a time delay appearing in the control, where the physical limitations in capacity at locations and the requirements for stock are formulated as all-time-step constraints. The objective is to minimize the combat power cost function. The procedure for constructing the "optimal" control reported in [60] is as follows. The optimal control is first obtained, by using the method reported in [67], for the optimal control problem with the constraints ignored. Then, the control and the state are forced to be saturated when they violated their constraints. Clearly, the control so obtained is not, in general, an optimal control. Furthermore, it is, in reality, impossible to saturate the state of the system. Thus, the problem has not yet been solved successfully in [60]. Thus, in this section, both the control constraints and the all-time-step constraints are considered explicitly during the computation of our optimal control. Therefore, the solution obtained satisfies all the constraints.

We now recall the optimal control model of tactical logistic decision analysis problem formulated by [60] as follows. Let $x(t) = [x_1(t), x_2(t), \ldots, x_5(t)]^\top$ be the state vector, where $x_i(t),\ i = 1, 2, \ldots, 5$, denote the stocks of logistic resources at the five locations at time t. Let $u(t) = [u_1(t), u_2(t), \ldots, u_8(t)]^\top$ be the control vector, where $u_i(t),\ i = 1, 2, \ldots, 8$, denote the stocks dispatched for supply along eight routes during the time period t. It is assumed that there is a delay of one time period between the dispatch of material from a supply location and the receipt at a receiving location. $A(t)$ denotes the proportions of stock at respective locations that are available for the next time period; $B(t)$ denotes the proportions of stock along respective supply routes that are providing the supply. The bounds for $x(t)$ and $u(t)$ reflect, respectively, the physical limitations in capacity at locations and supply routes. Furthermore, the criteria are the footprint and the physical distribution effort, which are the average combat power spent in protecting logistic resources located in the network and the average combat power spent in maintaining and protecting distribution effort along supply routes. $Q(t)$ and $R(t)$ denote, respectively, the opportunity cost to combat power of protecting the logistic resources at all the locations and along all the supply routes. Then, the model is

$$x(t+1) = Ax(t) + B_0u(t) + B_1u(t-1), \tag{2.56a}$$

$$x(0) = x_0,\ u(-1) = 0, \tag{2.56b}$$

$$x_{\min} \le x(t) \le x_{\max}, \tag{2.57a}$$

$$u_{\min} \le u(t) \le u_{\max}, \tag{2.57b}$$

where

$$A = \begin{bmatrix} 0.95 & 0 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0.75 & 0 & 0 \\ 0 & 0 & 0 & 0.75 & 0 \\ 0 & 0 & 0 & 0 & 0.85 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.95 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.87 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.75 & 0 & 0 & 0.7 \\ 0 & 0 & 0.8 & 0 & 0 & 0.8 & 0.7 & 0 \\ 0 & 0 & 0 & 0.85 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$x_0 = \begin{bmatrix} 3500 \\ 800 \\ 400 \\ 400 \\ 200 \end{bmatrix}.$$

The cost function is

$$G = \frac{1}{2}x^\top(T)Qx(T) + \sum_{t=0}^{T-1}\frac{1}{2}\left\{x^\top(t)Qx(t) + u^\top(t)Ru(t)\right\}, \tag{2.58}$$

where

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 2.5 \end{bmatrix}, \ R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

The logistic network for the example is as shown in Figure 2.1. The constraints (2.57a)

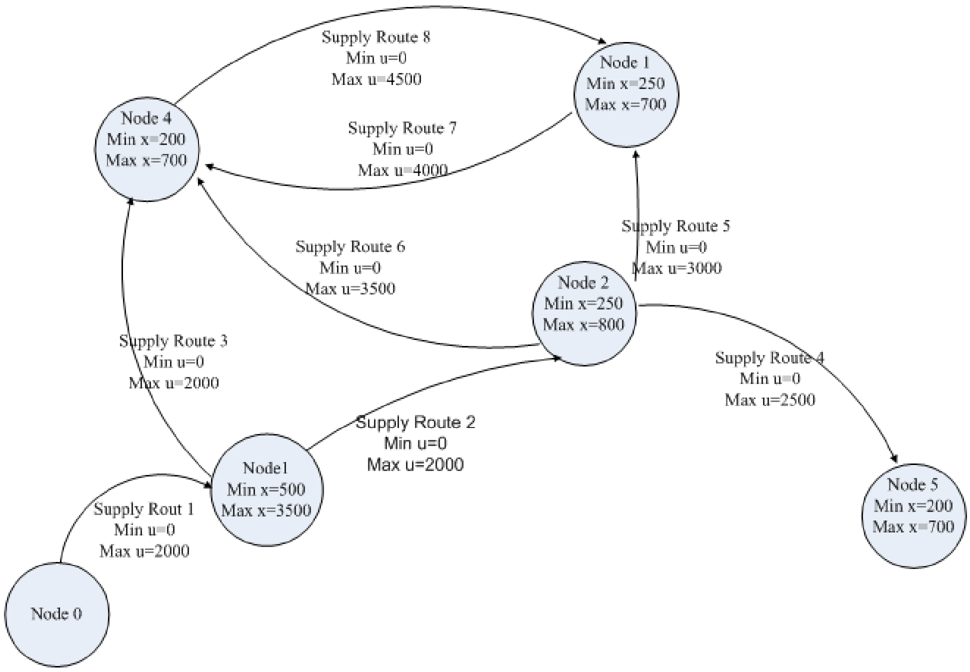

Figure 2.1: Example for the logistic network

and (2.57b) can be rewritten as:

$$\begin{aligned} g_i(u) = x_{i,\min} - x_i(k) \leq 0,\ k = 0, 1, \ldots, M-1, \\ i = 1, 2, \ldots, 5, \end{aligned} \tag{2.59}$$

$$\begin{aligned} g_i(u) = x_i(k) - x_{i,\max} \leq 0,\ k = 0, 1, \ldots, M-1, \\ i = 6, 7, \ldots, 10. \end{aligned} \tag{2.60}$$

The constraints (2.59) and (2.60) are all-time-step inequality constraints. We now use the constraint transcription method described in Section **??** to approximate these all-time-step inequality constraints by a sequence (in $\varepsilon > 0$) of the inequality constraints given below:

$$-\frac{\varepsilon}{4} + g_\varepsilon(u) \leq 0 \tag{2.61}$$

where $g_\varepsilon(u)$ is constructed from $g_i(u)$ according to (2.10) and (2.8). In this way, we obtain a sequence (in $\varepsilon > 0$) of discrete time optimal control problems with time delay and subject to canonical constraints (2.61). For each $\varepsilon > 0$, the corresponding discrete time optimal control problem with time delay and subject to canonical constraints (2.61) can be solved as an nonlinear optimization problem as explained in Section **??**. The values of the cost function and the canonical constraint functions are calculated as mentioned in Step 1 to Step 2 of Algorithm 2.2. Their gradients are calculated as explained in Algorithm 2.2 using the gradient formulas obtained in Theorem 2.3. The initial value of ε is chosen as 1.0×10^{-2}. ε is reduced to $\varepsilon/10$ in each iteration. It is found that the change in the cost function value is negligible after ε is reduced to 1.0×10^{-7}. Thus, the corresponding optimal cost function value (1.68×10^7) obtained is taken as the optimal cost function value. This value is much less than that obtained in [60], which is 3.5×10^7.

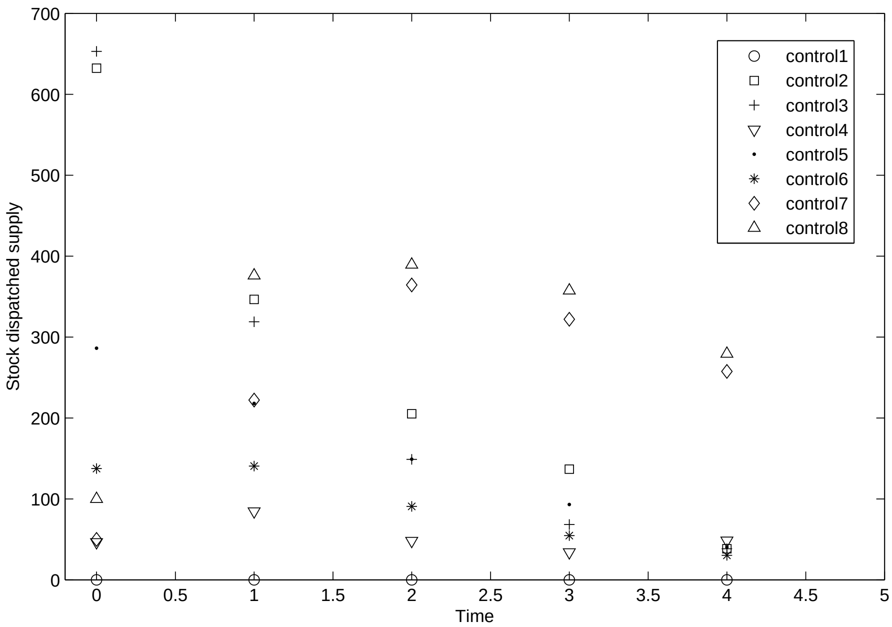

Figure 2.2: Stock dispatched supply

The optimal control and the corresponding optimal state obtained using our method are depicted in Figures 2.2 to Figure 2.4. By careful examination of these figures, we see that the constraints on the control and the all-time-step constraints are satisfied at each time point. From Figure 2.2, we see that $u_1(k) = 0$ for $k = 0, 1, \ldots, 4$, indicating no stock being dispatched along the supply route 1 to Node 1. This is because $u_1(k)$ could only contribute extra stock to Node 1 through the supply route 1 from Node 0, and the initial stock in Node 1 is large, twice as large as those in the other nodes. Thus, it is clear that

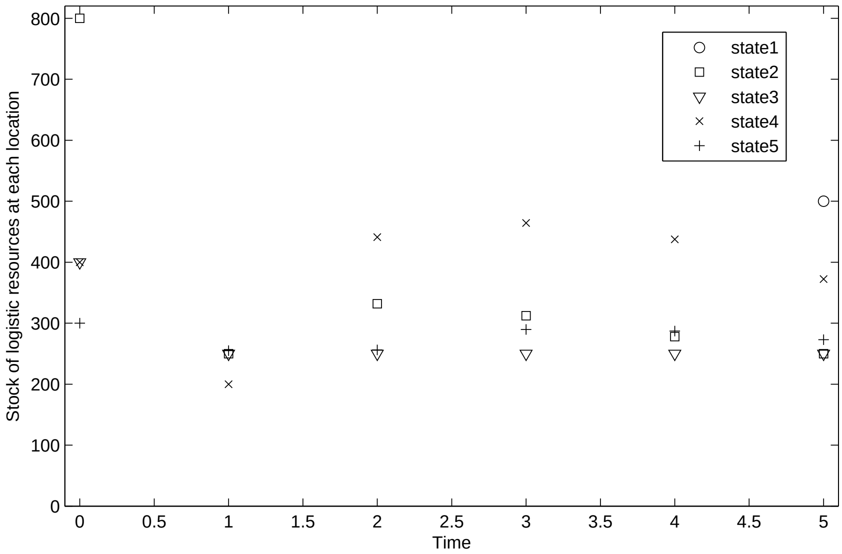

Figure 2.3: Stock at each location

stock should be moved out of Node 1 to other nodes quickly through the supply routes 2 and 3 so as to decrease the cost of holding the stock in Node 1. Also from Figure 2.2, we see that $u_2(k)$ and $u_3(k)$ are very large at $k = 0$, meaning that a large amount of stock is dispatched from Node 1 to the other nodes of the network at $k = 0$. From the structure of the network, it is clear that there is only one supply route to Node 5 with no supply route coming out of it. This means that Node 5 is a pure receiver of stock from other nodes. In view of the limits imposed on the maximum stock in various nodes, we see form Figure 2.3 that the amount of stock that is moved along the supply route 4 to Node 5 is low for $k = 0, 1, \ldots, 4$. The structure of the network depicted in Figure 2.1 clearly reveals that there are 4 supply routes (i.e., supply routes 3, 6 and 7) to Node 4 with only one supply route (i.e., supply route 8) coming out of it. For Node 3, there are 2 supply routes (i.e., supply routs 2 and 4) in and only 1 (i.e., supply route 7) out. By virtue of these observations, the amounts of stock along the supply routes for which the stocks are moved out should be large. This is confirmed in Figure 2.2 that $u_k(k)$ and $u_8(k)$, which denote, respectively, the amounts of stock being moved out from Node 3 along the supply route 7 and Node 4 along the supply route 8 are large for $k = 1, 2, 3, 4$. Their values are quite low at $k = 0$. This is due to the appearance of time delay along the supply routes, which indicates that that nodes cannot receive stock instantaneously. The stocks arrive with delay. For Node 2 , there are 3 supply routes (i.e., supply routes 4, 5 and 6) out but only 1 supply route (i.e., supply route 2) in. As shown in Figure 2.2, we see that the amounts of the stock being moved out along the supply routes 4, 5 and 6 are relatively

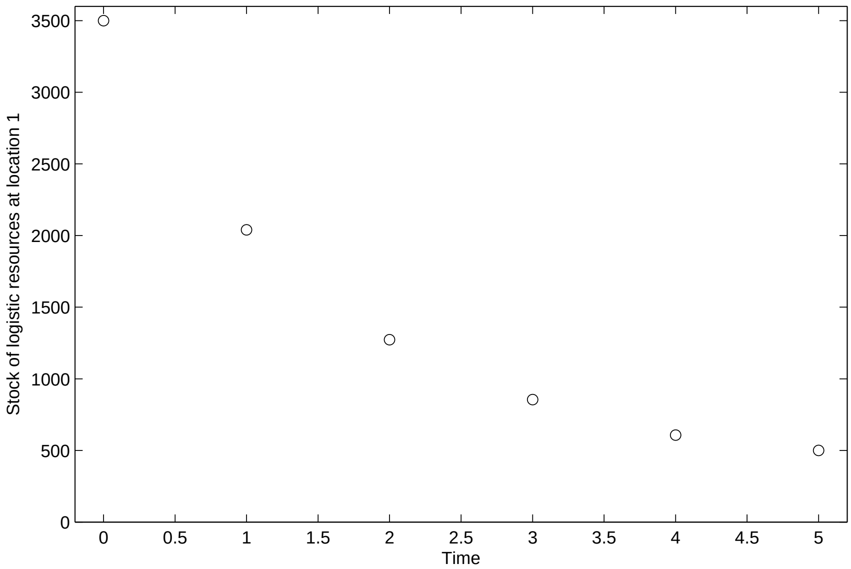

Figure 2.4: Stock at location 1

small.

We also consider the situation when the time horizon is increased to 20. The optimal control and the corresponding optimal state obtained are shown in Figure 2.5, Figure 2.6 and Figure 2.7, respectively. From these figures, we can see that the stock at each node reaches a balance state, i.e., the lower bounds for the stocks at each node, after $t = 7$. It is obvious that the method proposed in [60] fails for this situation. In conclusion, we see the solution obtained by using our method is highly effective.

2.6 Conclusions

In this chapter, we considered a class of discrete time optimal control problem with time delay and subject to all-time-step inequality constraints on both the state and control. It has been shown that, this problem can be approximated by a sequence of discrete time optimal control problems with time delay and subject to canonical constraints. A computational method was then proposed to solve a general class of discrete time optimal control problems with time delay and subject to canonical constraints as a nonlinear optimization problem. This general discrete time optimal control problem contains these approximate problems as special cases. Thus, the computational method developed for solving the general discrete time optimal control problem was used to solve each of these approximate problems. As an application a tactical logistic decision analysis problem was considered. It was solved by using the computational method developed. The results

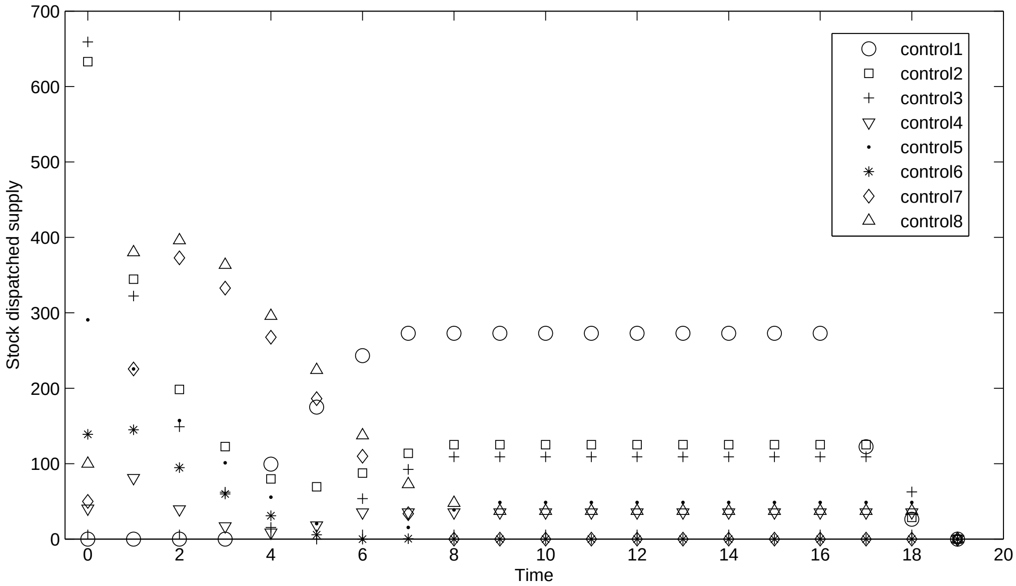

Figure 2.5: Stock dispatched supply for t=20

obtained are much superior to those obtained in [60].

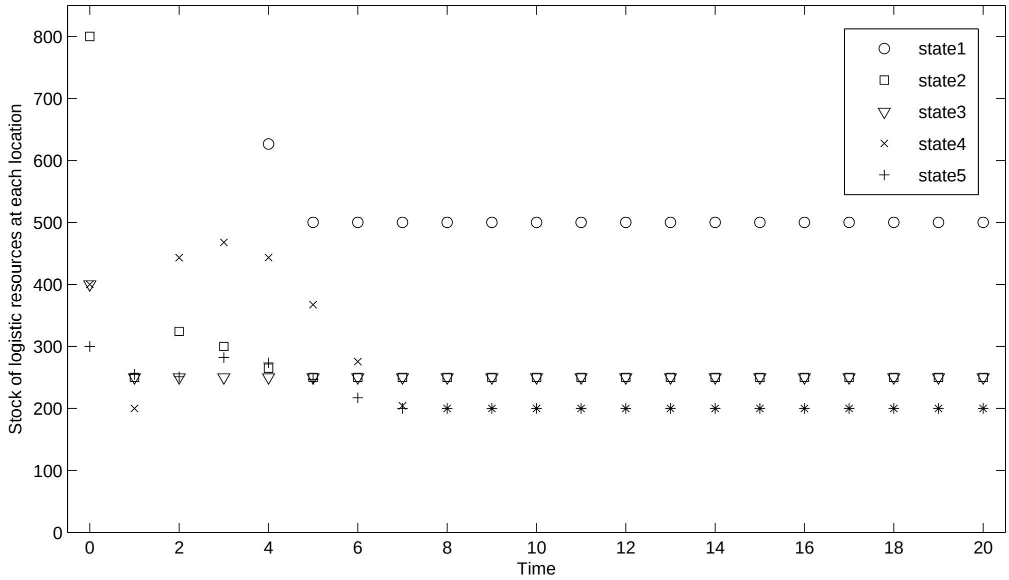

Figure 2.6: Stock at each location for t=20

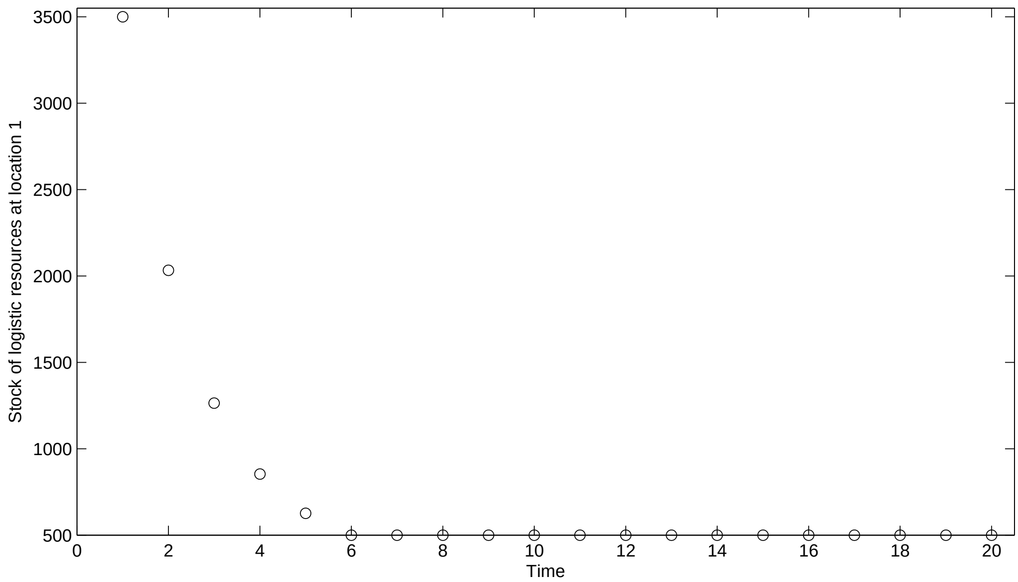

Figure 2.7: Stock at location 1 for t=20

CHAPTER 3

A Maxmin Optimal Control Problem

3.1 Introduction

This chapter is the author's work in [113]. In this chapter, we consider a general class of maxmin optimal control problems, where the violation avoidance of the continuous state constraints is to be maximized. Our aim is to derive an efficient computational method for solving this general maxmin optimal control problem. In this computational method, the constraint transcription method [103] is used to construct a smooth approximate function for each of the continuous state inequality constraints, where the accuracy of the approximation is controlled by an accuracy parameter. We then obtain a sequence of smooth approximate optimal control problems, where the integral of the summation of these smooth approximate functions is taken as the cost function for each of the problems. A necessary condition and a sufficient condition are derived showing the relationship between the original maxmin problem and the sequence of the smooth approximate problems. We then construct a violation avoidance function from the solution of each of the smooth approximate optimal control problems and the original continuous state inequality constraints in such a way that the problem of finding an optimal control of the maxmin optimal control problem is equivalent to the problem of finding the largest root of the violation avoidance function. The control parametrization technique [36] and a time scaling transform [87] are applied to these smooth approximate optimal control problems.

Two practical problems are considered. They are (i) Obstacle avoidance problem of an autonomous mobile robot; and (ii) the abort landing of an aircraft in a windshear downburst. We show that these two practical problems can be formulated as special cases of the general maxmin optimal control problem. The proposed computational method is then applied to solve these problems. The solutions obtained are highly satisfactory.

3.2 Problem Statement

Consider a dynamical system defined on $[0,T]$.

$$\dot{x}(t) = f(x(t), u(t)), \; t \in (0,T] \tag{3.1a}$$

with initial and terminal conditions

$$x(0) = x^0 \tag{3.1b}$$

$$x(T) = x^f \tag{3.1c}$$

where T is the terminal time and

$$x = [x_1, x_2, \ldots, x_n]^\top \in \mathbb{R}^n \text{ and } u = [u_1, u_2, \ldots u_r]^\top \in \mathbb{R}^r$$

are, respectively, state and control vectors, while $f = [f_1, f_2, \ldots, f_n]^\top \in \mathbb{R}^n$ is a given continuously differentiable function of its arguments.

We assume that the following assumption is satisfied.

Assumption 3.1. Let $\mathbb{V}$ be a compact subset of $\mathbb{R}^r$. Then, there exists a positive constant K_1 such that

$$\|f(x,u)\| \leq K_1 (1 + \|x\|)$$

for all $(x,u) \in \mathbb{R}^n \times \mathbb{V}$.

Define

$$U = \left\{ \nu = [v_1, v_2, \ldots v_r]^\top \in \mathbb{R}^r : \alpha_i \leq v_i \leq \beta_i, \; i = 1, 2, \ldots, r \right\} \tag{3.2}$$

where α_i, $i = 1, 2, \ldots, r$, and $\beta_i, i = 1, 2, \ldots, r$, are given real numbers. A piecewise continuous function u is said to be an admissible control if $u(t) \in U$ for all $t \in [0,T]$. Let $\mathcal{U}$ be the class of all such admissible controls. Furthermore, let $x(\cdot|u)$ denote the solution of system (3.1a)-(3.1b) corresponding to $u \in \mathcal{U}$.

Consider the continuous state inequality constraints, given by

$$g_i(x(t|u)) \geq 0, \; t \in [0,T], i = 1, 2, \ldots, N \tag{3.3}$$

where T is the terminal time. It is assumed that the following assumption is satisfied.

Assumption 3.2. g_i, $i = 1, 2, \ldots, N$, are continuously differentiable with respect to x.

Let $\mathcal{F}$ be defined by

$$\mathcal{F} = \{u \in \mathcal{U} : \ g_i(x(t|u)) \geq 0, \ t \in [0, T], \ i = 1, 2, \ldots, N\} \tag{3.4}$$

A $u \in \mathcal{F}$ is called a feasible control and $\mathcal{F}$ is referred to as the set of feasible controls. Then we introduce the violation avoidance parameter δ which is defined as follows:

$$\delta = \min_{\substack{0 \leq t \leq T \\ 1 \leq i \leq N}} g_i(x(t|u)), \ \forall u \in \mathcal{F} \tag{3.5}$$

It is the size parameter in the obstacle-avoidance problem and the height of the aircraft in the abort landing problem to be discussed later.

To proceed further, we define the following violation avoidance constraints

$$\tilde{g}_i(x(t|u), \delta) = g_i(x(t|u)) - \delta \geq 0, \ i = 1, 2, \ldots, N \tag{3.6}$$

Remark 3.1. It is obvious that δ is a nonnegative constant number and the following conditions are satisfied.

(1) For each $i = 1, 2, \ldots, N$, $\tilde{g}_i(x, \delta)$ is a strictly monotonically decreasing function in $\delta \geq 0$.

(2) For each $u \in \mathcal{F}$, there exists a $\delta^u > 0$ such that

$$\min_{1 \leq i \leq N} \min_{0 \leq t \leq T} \tilde{g}_i\left(x(t|u), \delta^u\right) = 0.$$

We may now state our maxmin optimal control problem as follows.

Problem (P3) Given the dynamical system (3.1a)-(3.1b) subject to the terminal constraint (3.1c), find a control $u \in \mathcal{F}$ such that δ is maximized.

Remark 3.2. By (A1) and the definition of $\mathcal{U}$, it follows from an argument similar to that given in the proof of Lemma 6.4.2 in [36] that $x(t|u) \in X$ for all $t \in [0, T]$ and for all $u \in \mathcal{U}$, where $X \subset \mathbb{R}^n$ is a compact subset.

3.3 Computational Method

To solve Problem (P3), we shall apply the control parametrization scheme [36] together with a time scaling transform [87]. It is briefly revealed below. The time horizon $[0, T]$ is partitioned with a monotonically increasing sequence $\{t_0, t_1, \ldots, t_p\}$. Then, the control is approximated by a piecewise constant function as follows.

$$u^p(t) = \sum_{i=1}^{p} \sigma^i \chi_{[t_{i-1}, t_i)}(t), \tag{3.7}$$

where $t_{i-1} \leq t_i,\ i = 1, 2, \ldots, p$, with $t_0 = 0$ and $t_p = T$, and

$$\chi_I(t) = \begin{cases} 1, & \text{if } t \in I, \\ 0, & \text{otherwise.} \end{cases}$$

As $u^p \in \mathcal{U}$, $\sigma^i = [\sigma_1^i, \sigma_2^i, \ldots, \sigma_r^i]^\top \in U$ for $i = 1, 2, \ldots, p$. Denote by Ξ the set of all such $\sigma = \left[(\sigma^1)^\top, (\sigma^2)^\top, \ldots, (\sigma^p)^\top\right]^\top \in \mathbb{R}^{pr}$.

The switching times $t_i,\ 1 \leq i \leq p-1$, are also regarded as decision variables. We shall employ the time scaling transform introduced in [87] to map these switching times into a set of fixed time points $\dfrac{k}{p}$, $k = 1, 2, \ldots, p-1$, on a new time horizon $[0, 1]$. This is easily achieved by the following differential equation

$$\dot{t}(s) = \upsilon^p(s),\ s \in [0, 1], \tag{3.8a}$$

with initial condition

$$t(0) = 0, \tag{3.8b}$$

where

$$\upsilon^p(s) = \sum_{i=1}^{p} \theta_i \chi_{\left[\frac{i-1}{p}, \frac{i}{p}\right)}(s). \tag{3.9}$$

Here, $\theta_i \geq 0,\ i = 1, 2, \ldots p$, and

$$\sum_{i=1}^{p} \frac{\theta_i}{p} = T. \tag{3.10}$$

Let $\theta = [\theta_1, \theta_2, \ldots, \theta_p]^\top$ and let Θ be the set containing all such θ.

Taking integration of (3.8a) with initial condition (3.8b), it is easy to see that, for $s \in \left[\frac{q-1}{p}, \frac{q}{p}\right)$,

$$t(s) = \sum_{k=1}^{q-1} \frac{\theta_k}{p} + \frac{\theta_q}{p}(ps - q + 1), \tag{3.11}$$

where $q = 1, 2, \ldots, p$. Clearly, $t(1) = T$. The approximate control given by (3.7) in the new time horizon $[0, 1]$ becomes

$$\tilde{u}^p(s) = u^p(t(s)) = \sum_{k=1}^{p} \sigma^k \chi_{\left[\frac{k-1}{p}, \frac{k}{p}\right)}(s), \tag{3.12}$$

which has fixed switching times at $s = \frac{1}{p}, \frac{2}{p}, \ldots, \frac{p-1}{p}$. Now, by using the time scaling transform (3.2), the dynamic system (3.1a)-(3.1b) is transformed into

$$\dot{y}(s) = \tilde{f}(s, y(s), \sigma, \theta) \tag{3.13a}$$

$$y(0) = x^0 \tag{3.13b}$$

and the terminal condition (3.1c) becomes

$$y(1) = x^f \tag{3.13c}$$

where $y(s) = x(t(s))$ and

$$\tilde{f}(s, y(s), \sigma, \theta) = v^p(s) f(x(t(s)), \tilde{u}^p(s))$$

Similarly, applying the time scaling transform to the continuous state inequality constraints (3.6) yields:

$$\tilde{g}_i(y(s), \delta) \geq 0, \ s \in [0, T], \ i = 1, 2, \ldots, N. \tag{3.14}$$

To proceed further, let $y(\cdot|\sigma, \theta)$ denote the solution of system (3.13a)-(3.13b) corresponding to $(\sigma, \theta) \in \Xi \times \Theta$.

The approximate problem to Problem (P3) may now be stated formally as follows.

Problem (P3(p)) Given system (3.13a)-(3.13b), find a $(\sigma, \theta) \in \Xi \times \Theta$ such that the violation avoidance parameter δ is maximized subject to (3.13c), (3.14) and (3.10).

To solve Problem (P3(p)), we will re-formulate it as a problem of finding the largest root. First, we will construct a new optimal control problem by applying the constraint transcription technique [103]. Then a necessary condition and a sufficient condition are given showing the relationships between the new problem and the original problem. To derive an effective algorithm, a violation avoidance function of δ is defined. The original problem is equivalent to the problem of finding the largest root of the violation avoidance function. The maximum violation avoidance parameter δ can be located by using the section search method. The optimal control software package MISER 3.3 ([37]) is used at each iteration.

To begin, let us introduce an auxiliary optimal control problem below.

By using the constraint transcription method [103], each of the continuous state inequality constraints is approximated by a smooth function $\mathcal{L}_{i,\varepsilon}(y(s|\sigma,\theta),\delta)$, where

$$\mathcal{L}_{i,\varepsilon}(y(s|\sigma,\theta),\delta) = \begin{cases} -\tilde{g}_i(y(s),\delta), & \text{if } \tilde{g}_i(y(s),\delta) < -\varepsilon \\ \dfrac{(\tilde{g}_i(y(s),\delta)-\varepsilon)^2}{4\varepsilon}, & \text{if } -\varepsilon \le \tilde{g}_i(y(s),\delta) \le \varepsilon \\ 0, & \text{if } \tilde{g}_i(y(s),\delta) > \varepsilon \end{cases} \tag{3.15}$$

Now, for a fixed $\delta \in [0,\infty)$, we define an auxiliary optimal control problem, where a $(\delta,\theta) \in \Xi \times \Theta$ is to be chosen such that the cost function

$$J_\varepsilon(\sigma,\theta|\delta) = \int_0^1 \sum_{i=1}^N (\mathcal{L}_{i,\varepsilon}(y(s),\delta))\mathrm{d}s, \tag{3.16}$$

is minimized over $\Xi \times \Theta$ subject to (3.13c) and (3.10).

Let this problem be referred to as **Problem (P3$_\delta$(p)**.

We will give a necessary condition and a sufficient condition to show the relationships between Problem (P3(p)) and Problem (P3$_\delta$(p)).

Remark 3.3. By Remark 3.1, we can show that

$$\int_0^1 \mathcal{L}_{i,\varepsilon}(y(s),\delta)\mathrm{d}s$$

is well defined for each $\delta \ge 0$.

Remark 3.4. From Remark 3.1, it is clear that, for each δ, $\dfrac{\partial \tilde{g}_j(y(s),\delta)}{\partial y}$, $j = 1,2,\ldots,N$, are continuous in $s \in [0,1]$, and $\dot{y}_i(s)$, $i = 1,2,\ldots,n$, is piecewise continuous in $s \in [0,1]$.

Theorem 3.1. *Let $(\sigma^{(0)},\ \theta^{(0)}) \in \Xi \times \Theta$ and let $y^{(0)}$ be the corresponding solution of system (3.13a)-(3.13b) such that (3.13c) is satisfied. Suppose that $\delta^{(0)}$ is such that (3.14) is satisfied. Then*

$$J_\varepsilon(\sigma^{(0)},\theta^{(0)}|\delta^{(0)}) = \int_0^1 \sum_{i=1}^N \mathcal{L}_{i,\varepsilon}(y^{(0)}(s),\delta^{(0)})\mathrm{d}s \le \frac{N\varepsilon}{4}. \tag{3.17}$$

Proof. From the constraint transcription defined in (3.15), it is clear that

$$\mathcal{L}_{i,\varepsilon}(y^{(0)}(s),\delta^{(0)}) \le \frac{\varepsilon}{4},\ \forall s \in [0,1],\ i = 1,2,\ldots,N. \tag{3.18}$$

Thus, the conclusion follows readily from (3.16). □

Before deriving the sufficient condition, Lemma 3.1 in [86] is quoted without proof as follows.

Lemma 3.1. *Let f be a nonnegative valued function defined on $[0,T]$. If f is continuously differentiable on $[0,T]$, then*

$$\int_0^T f(t)\mathrm{d}t \geq \frac{\widehat{f}}{2}\min\left\{\frac{\widehat{f}}{M}, T\right\}, \tag{3.19}$$

where

$$M = \max_{t\in[0,T]}\left|\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right| \tag{3.20}$$

and

$$\widehat{f} = \max_{t\in[0,T]} f(t). \tag{3.21}$$

With the help of Lemma 3.1, we can derive the desired sufficient condition as in [86] below.

Theorem 3.2. *Let $\left(\sigma^{(0)},\ \theta^{(0)}\right) \in \Xi\times\Theta$ be such that (3.10) is satisfied and let $y^{(0)}$ be the corresponding solution of system (3.13a)-(3.13b) such that (3.13c) is satisfied. Suppose that $\delta^{(0)}$ is such that*

$$J_\varepsilon(\sigma^{(0)},\theta^{(0)}|\delta^{(0)}) \leq \frac{\varepsilon}{8}\min\left\{\frac{\varepsilon}{4M}, 1\right\}, \tag{3.22}$$

where

$$M = \max\left\{\left\|\frac{\partial\tilde{g}_i(y^{(0)}(s),\delta^{(0)})}{\partial y^{(0)}}\frac{\mathrm{d}y^{(0)}(s)}{\mathrm{d}s}\right\| :\ s\in[0,1],\ i=1,2,\ldots,N\right\}. \tag{3.23}$$

Then, the constraints (3.14) are satisfied.

Proof. Suppose that there exits an i, $0\leq i\leq N$, such that the corresponding constraint (3.14) is not satisfied. Then, by Assumption 3.2, there exits an open set ϑ_i with positive measure such that

$$\tilde{g}_i(y^{(0)}(s),\delta^{(0)}) < 0,\ \forall s\in\vartheta_i. \tag{3.24}$$

From Remark 3.3, there exists a positive constant M satisfying (3.23). Then, we define

$$M' = \max\left\{\left\|\frac{\partial\tilde{g}_{i,\varepsilon}(y^{(0)}(s),\delta^{(0)})}{\partial y}\frac{\mathrm{d}y^{(0)}(s)}{\mathrm{d}s}\right\| :\ s\in[0,1],\ i=1,2,\ldots,N\right\}. \tag{3.25}$$

Clearly,

$$\left\|\frac{\partial\mathcal{L}_{i,\varepsilon}(y^{(0)}(s),\delta^{(0)})}{\partial y}\frac{\mathrm{d}y^{(0)}(s)}{\mathrm{d}s}\right\| \leq \left\|\frac{\mathrm{d}\mathcal{L}_{i,\varepsilon}(y^{(0)}(s),\delta^{(0)})}{\mathrm{d}\tilde{g}_j}\right\|\left\|\frac{\partial\tilde{g}_i(y^{(0)}(s),\delta^{(0)})}{\partial y}\frac{\mathrm{d}y^{(0)}(s)}{\mathrm{d}s}\right\|. \tag{3.26}$$

Since

$$\left\|\frac{\mathrm{d}\mathcal{L}_{i,\varepsilon}(y^{(0)}(s),\delta^{(0)})}{\mathrm{d}\tilde{g}_j}\right\| \leq 1,\ \forall s\in[0,1],$$

it is clear that

$$M' \leq M. \tag{3.27}$$

Thus, by Lemma 3.1, we have

$$J_\varepsilon(\sigma^{(0)}, \theta^{(0)}|\delta^{(0)}) = \int_0^1 \mathcal{L}_{i,\varepsilon}(y^{(0)}(s), \delta^{(0)})dt \geq \frac{\widehat{\mathcal{L}_\varepsilon}}{2} \min\left\{\frac{\widehat{\mathcal{L}_\varepsilon}}{M'}, 1\right\}, \tag{3.28}$$

where

$$\widehat{\mathcal{L}_\varepsilon} = \max_{s\in[0,1]} \mathcal{L}_{i,\varepsilon}(y^{(0)}(s), \delta^{(0)}). \tag{3.29}$$

By (3.24) and (3.15), we obtain

$$\widehat{\mathcal{L}_\varepsilon} > \frac{\varepsilon}{4}. \tag{3.30}$$

Thus, it follows from (3.27), (3.28) and (3.30) that

$$J_\varepsilon(\sigma^{(0)}, \theta^{(0)}, \delta^{(0)}) \geq \frac{\widehat{\mathcal{L}_\varepsilon}}{2} \min\left\{\frac{\widehat{\mathcal{L}_\varepsilon}}{M'}, 1\right\} > \frac{\varepsilon}{8} \min\left\{\frac{\varepsilon}{4M'}, 1\right\} \geq \frac{\varepsilon}{8} \min\left\{\frac{\varepsilon}{4M}, 1\right\}. \tag{3.31}$$

This is, however, a contradiction to (3.22). This completes the proof. □

Note that Problem (P3$_\delta$(p)) is an optimal control problem in canonical form. To solve it as a nonlinear optimization problem by using the optimal control software MISER 3.3 (see [37]), we need the gradient formulas of the objective function (3.16), the constraint function $\Phi(y(1)) = y(1) - x^f = 0$ from (3.13c) and the constraint function Φ from (3.10). The gradient of the constraint function

$$\Phi(\theta) = \sum_{i=1}^{p} \frac{\theta_i}{p} - T$$

is given by

$$\frac{\partial \Phi(\theta)}{\partial \theta} = \left[\frac{1}{p}, \frac{2}{p}, \ldots, \frac{1}{p}\right]^\top$$

The other two are given in the following theorem, where $y(\cdot|\sigma, \theta)$ is referred to as the solution of (3.13a)-(3.13b) corresponding to $(\sigma, \theta) \in \Xi \times \Theta$.

Theorem 3.3. *For each $\delta > 0$, the gradients of the cost function $J_\varepsilon(\sigma, \theta|\delta)$ with respect to σ and θ are:*

$$\frac{\partial J_\varepsilon(\sigma, \theta|\delta)}{\partial \sigma} = \int_0^1 \frac{\partial H_0(s, y(s|\sigma, \theta), \sigma, \theta, \lambda_0(s|\sigma, \theta, \varepsilon))}{\partial \sigma} \mathrm{d}s, \tag{3.32}$$

$$\frac{\partial J_\varepsilon(\sigma, \theta|\delta)}{\partial \theta} = \int_0^1 \frac{\partial H_0(s, y(s|\sigma, \theta), \sigma, \theta, \lambda_0(s|\sigma, \theta, \varepsilon))}{\partial \theta} \mathrm{d}s, \tag{3.33}$$

where $H_0(s, y(s), \sigma, \lambda(s))$ is the Hamiltonian function for the cost function (3.16) given by

$$H_0(s, y(s), \sigma, \theta, \lambda(s)) = \sum_{i=1}^{N} \mathcal{L}_{i,\varepsilon}(y(s), \delta) + \lambda_0(s)\tilde{f}(s, y(s), \sigma, \theta), \tag{3.34}$$

and $\lambda_0(\cdot|\sigma, \theta, \varepsilon)$ is the solution of the following co-state differential equation

$$\left(\frac{\mathrm{d}\lambda_0(s)}{\mathrm{d}s}\right)^{\top} = -\frac{\partial H_0(s, y(s|\sigma, \theta), \sigma, \theta, \lambda_0(s))}{\partial y} \tag{3.35a}$$

with the boundary condition

$$(\lambda_0(1))^{\top} = 0. \tag{3.35b}$$

Proof. Let $\sigma \in \mathbb{R}^r$ be given and let $\rho \in \mathbb{R}^r$ be arbitrary but fixed. Define

$$\sigma(\epsilon) = \sigma + \epsilon\rho \tag{3.36}$$

where $\epsilon > 0$ is an arbitrarily small real number. For brevity, let $y(\cdot)$ and $y(\cdot;\epsilon)$ denote, respectively, the solution of the system (3.13a)-(3.13b) corresponding to σ and $\sigma(\epsilon)$ while θ is fixed. Clearly,

$$y(s) = y(0) + \int_0^s \tilde{f}(\tau, y(\tau), \sigma, \theta)\mathrm{d}\tau \tag{3.37}$$

$$y(s;\epsilon) = y(0) + \int_0^s \tilde{f}(\tau, y(\tau;\epsilon), \sigma(\epsilon), \theta)\mathrm{d}\tau \tag{3.38}$$

Thus,

$$\begin{aligned} \triangle y(s) &\equiv \left.\frac{\mathrm{d}y(s;\epsilon)}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \\ &= \int_0^s \left\{ \frac{\partial \tilde{f}(\tau, y(\tau), \sigma, \theta)}{\partial y} \triangle y(\tau) + \frac{\partial \tilde{f}(\tau, y(\tau), \sigma, \theta)}{\partial \sigma}\rho \right\} \mathrm{d}\tau \end{aligned} \tag{3.39}$$

Clearly,

$$\frac{\mathrm{d}(\triangle y(s))}{\mathrm{d}s} = \frac{\partial \tilde{f}(s, y(s), \sigma, \theta)}{\partial y} \triangle y(s) + \frac{\partial \tilde{f}(s, y(s), \sigma, \theta)}{\partial \sigma}\rho \tag{3.40}$$

Now, $J_\varepsilon(\sigma(\epsilon), \theta|\delta)$ can be expressed as:

$$J_\varepsilon(\sigma(\epsilon), \theta|\delta) = \int_0^1 \left\{ H_0(s, y(s;\epsilon), \sigma(\epsilon), \theta, \lambda_0(s)) - \lambda_0(s)\tilde{f}(s, y(s;\epsilon), \sigma(\epsilon), \theta) \right\} \mathrm{d}s \tag{3.41}$$

where $\lambda_0(s)$ is yet arbitrary. Thus, it follows that

$$\begin{aligned}\triangle J_\varepsilon(\sigma(\epsilon),\theta|\delta) &\equiv \left.\frac{\mathrm{d}J_\varepsilon(\sigma(\epsilon),\theta|\delta)}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = \frac{\partial J_\varepsilon(\sigma,\theta|\delta)}{\partial\sigma}\rho \\ &= \int_0^1 \left\{\triangle H_0(s,y(s),\sigma,\theta,\lambda_0(s)) - \lambda_0(s)\triangle\tilde{f}(s,y(s),\sigma,\theta)\right\}\mathrm{d}s \qquad (3.42)\end{aligned}$$

where

$$\triangle\tilde{f}(s,y(s),\sigma,\theta) = \frac{\mathrm{d}\triangle y(s)}{\mathrm{d}s} \qquad (3.43)$$

and

$$\begin{aligned}&\triangle H_0(s,y(s),\sigma,\theta,\lambda_0(s)) \\ &= \frac{\partial H_0(s,y(s),\sigma,\theta,\lambda_0(s))}{\partial y}\triangle y(s) + \frac{\partial H_0(s,y(s),\sigma,\theta,\lambda_0(s))}{\partial\sigma}\rho \qquad (3.44)\end{aligned}$$

Choose λ_0 to be the solution of the costate system (3.35a)-(3.35b) corresponding to σ while θ is fixed. Then, by substituting (3.35a) into (3.44), we obtain

$$\begin{aligned}&\triangle H_0(s,y(s),\sigma,\theta,\lambda_0(s)) \\ &= -\left(\frac{\mathrm{d}\lambda_0(s)}{\mathrm{d}s}\right)^\top \triangle y(s) + \frac{\partial H_0(s,y(s),\sigma,\theta,\lambda_0(s))}{\partial\sigma}\rho \qquad (3.45)\end{aligned}$$

By (3.43) and (3.45), it follows from (3.42) that

$$\begin{aligned}&\frac{\partial J_\varepsilon(\sigma,\theta|\delta)}{\partial\sigma}\rho \\ &= \int_0^1 \left\{-\frac{\mathrm{d}}{\mathrm{d}s}\left[(\lambda_0(s))^\top\triangle y(s)\right] + \frac{\partial H_0(s,y(s),\sigma,\theta,\lambda_0(s))}{\partial\sigma}\rho\right\}\mathrm{d}s \\ &= (\lambda_0(0))^\top\triangle y(0) - (\lambda_0(1))^\top\triangle y(1) + \int_0^1\left\{\frac{\partial H_0(s,y(s),\sigma,\theta,\lambda_0(s))}{\partial\sigma}\rho\right\}\mathrm{d}s \qquad (3.46)\end{aligned}$$

Note that $y(0)$ is constant. Thus, by (3.35b), we deduce from (3.46) that

$$\begin{aligned}&\frac{\partial J_\varepsilon(\sigma,\theta|\delta)}{\partial\sigma}\rho \\ &= \int_0^1\left\{\frac{\partial H_0\left(s,y(s),\sigma,\theta,\lambda_0(s)\right)}{\partial\sigma}\rho\right\}\mathrm{d}s \qquad (3.47)\end{aligned}$$

Since ρ is arbitrary, (3.32) follows readily from (3.47). (3.33) can be derived in the same way. Thus, the proof is completed. □

For the constraint function $\Phi(y(1|\sigma,\theta)) = y(1|\sigma,\theta) - x^f$, we have

Theorem 3.4. *For each δ, the gradients of the constraint function $\Phi(y(1|\sigma,\theta))$ with* *respect to σ and θ are:*

$$\frac{\partial \Phi(y(1|\sigma,\theta))}{\partial \sigma} = \int_0^1 \frac{\partial H_1(s, y(s|\sigma,\theta), \sigma, \theta, \lambda_1(s|\sigma,\theta))}{\partial \sigma} \mathrm{d}s, \tag{3.48}$$

$$\frac{\partial \Phi(y(1|\sigma,\theta))}{\partial \theta} = \int_0^1 \frac{\partial H_1(s, y(s|\sigma,\theta), \sigma, \theta, \lambda_1(s|\sigma,\theta))}{\partial \theta} \mathrm{d}s, \tag{3.49}$$

where $H_1(s, y(s), \sigma, \lambda(s))$ is the Hamiltonian function for the constraint function (3.13c) given by

$$H_1(s, y(s), \sigma, \theta, \lambda(s)) = \lambda_1(s)\tilde{f}(s, y(s), \sigma, \theta), \tag{3.50}$$

and $\lambda_1(\cdot|\sigma,\theta)$ is the solution of the following co-state differential equation

$$\frac{(\mathrm{d}\lambda_1(s))^\top}{\mathrm{d}s} = -\frac{\partial H_1(s, y(s), \sigma, \theta, \lambda_1(s))}{\partial y} \tag{3.51a}$$

with the boundary condition

$$(\lambda_1(1))^\top = \frac{\mathrm{d}\Phi(y(1))}{\mathrm{d}y} \tag{3.51b}$$

Proof. Let $\sigma \in \mathbb{R}^r$ be given and let $\rho \in \mathbb{R}^r$ be arbitrary but fixed. Define

$$\sigma(\epsilon) = \sigma + \epsilon\rho \tag{3.52}$$

where $\epsilon > 0$ is an arbitrarily small real number. $y(\cdot)$ and $y(\cdot;\epsilon)$ denote, respectively, the solutions of the system (3.13a)-(3.13b) corresponding to σ and $\sigma(\epsilon)$, while θ is fixed. As it has been done in the proof for Theorem 3.3, we have

$$\begin{aligned} \triangle y(s) &\equiv \left.\frac{\mathrm{d}y(s;\epsilon)}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \\ &= \int_0^s \left\{ \frac{\partial \tilde{f}(\tau, y(\tau), \sigma, \theta)}{\partial y} \triangle y(\tau) + \frac{\partial \tilde{f}(\tau, y(\tau), \sigma, \theta)}{\partial \sigma}\rho \right\} \mathrm{d}\tau \end{aligned} \tag{3.53}$$

and

$$\frac{\mathrm{d}\,(\triangle y(s))}{\mathrm{d}s} = \frac{\partial \tilde{f}(s, y(s), \sigma, \theta)}{\partial y} \triangle y(s) + \frac{\partial \tilde{f}(s, y(s), \sigma, \theta)}{\partial \sigma}\rho \tag{3.54}$$

From (3.50), $\Phi(y(1|\sigma(\epsilon),\theta))$ can be expressed as:

$$\begin{aligned} &\Phi(y(1|\sigma(\epsilon),\theta)) \\ =\ &\Phi(y(1|\sigma(\epsilon),\theta)) \\ &+ \int_0^1 \left\{ H_1(s, y(s;\epsilon), \sigma(\epsilon), \theta, \lambda_1(s)) - \lambda_1(s)\tilde{f}(s, y(s;\epsilon), \sigma(\epsilon), \theta) \right\} \mathrm{d}s \end{aligned} \tag{3.55}$$

where $\lambda_1(s)$ is yet arbitrary. Thus, it follows that

$$\begin{aligned}
& \triangle\Phi(y(1|\sigma,\theta)) \equiv \left.\frac{\mathrm{d}\Phi(y(1|\sigma(\epsilon),\theta))}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = \frac{\partial\Phi(y(1|\sigma,\theta))}{\partial\sigma}\rho \\
= \; & \triangle\Phi(y(1|\sigma,\theta)) \\
& + \int_0^1 \Big\{ \triangle H_1(s,y(s),\sigma,\theta,\lambda_1(s)) - \lambda_1(s) \triangle \tilde{f}(s,y(s),\sigma,\theta) \Big\}\mathrm{d}s \qquad (3.56)
\end{aligned}$$

where

$$\triangle\Phi(y(1|\sigma,\theta)) = \frac{\mathrm{d}\Phi(y(1))}{\mathrm{d}y} \triangle y(1|\sigma,\theta) \tag{3.57}$$

$$\triangle\tilde{f}(s,y(s),\sigma,\theta) = \frac{\mathrm{d}\triangle y(s)}{\mathrm{d}s} \tag{3.58}$$

and

$$\begin{aligned}
& \triangle H_1(s,y(s),\sigma,\theta,\lambda_1(s)) \\
= \; & \left(\frac{\mathrm{d}\lambda_1(s)}{\mathrm{d}s}\right)^\top \triangle y(s) + \frac{\partial H_1(s,y(s),\sigma,\theta,\lambda_1(s))}{\partial\sigma}\rho \qquad (3.59)
\end{aligned}$$

Choose λ_1 to be the solution of the costate system (3.51a)-(3.51b) corresponding to σ. Then, by substituting (3.57), (3.58), (3.59) into (3.56), we obtain

$$\begin{aligned}
& \frac{\partial\Phi(y(1|\sigma,\theta))}{\partial\sigma}\rho \\
= \; & \frac{\mathrm{d}\Phi(y(1))}{\mathrm{d}y} \triangle y(1|\sigma,\theta) \\
& + \int_0^1 \left\{ -\frac{\mathrm{d}}{\mathrm{d}s}\left[(\lambda_1(s))^\top \triangle y(s)\right] + \frac{\partial H_1(s,y(s),\sigma,\theta,\lambda_1(s))}{\partial\sigma}\rho \right\} \mathrm{d}s \\
= \; & \frac{\mathrm{d}\Phi(y(1))}{\mathrm{d}y} \triangle y(1|\sigma,\theta) \\
& + (\lambda_1(0))^\top \triangle y(0) - (\lambda_1(1))^\top \triangle y(1) + \int_0^1 \left\{ \frac{\partial H_1(s,y(s),\sigma,\theta,\lambda_1(s))}{\partial\sigma}\rho \right\} \qquad (3.60)
\end{aligned}$$

Note that $y(0)$ is a constant number. Thus, by (3.51b), it follows from (3.68) that

$$\begin{aligned}
& \frac{\partial\Phi(y(1|\sigma,\theta))}{\partial\sigma}\rho \\
= \; & \int_0^1 \left\{ \frac{\partial H_1(s,y(s),\sigma,\theta,\lambda_1(s))}{\partial\sigma}\rho \right\} \mathrm{d}s \qquad (3.61)
\end{aligned}$$

Since ρ is arbitrary, (3.48) follows readily from (3.61). (3.49) can be derived in the same way. Thus, the proof is completed. □

Let

$$\Psi_i^p(\delta) = \max_{0\le s\le 1}\left\{\max\left(-\tilde{g}_i(\widehat{y}(s),\delta),0\right)\right\},\ i = 1,2,\ldots,N, \tag{3.62}$$

and define

$$\Psi^p(\delta) = \max_{0\le i\le N}\Psi_i^p(\delta). \tag{3.63}$$

$\Psi^p(\delta)$ is called the violation avoidance function. Here, $\widehat{y}(s)$ is the solution obtained from solving **Problem** $(\mathbf{P3}_\delta\mathbf{(p)})$. Clearly, $\Psi^p(\delta) = 0$ if and only if (3.14) are satisfied. From (1) of Remark 3.1, we can show that the real-valued function $\Psi^p(\delta)$ is a nondecreasing function of δ. By (2) of Remark 3.1, there exits a δ such that $\Psi^p(\delta) = 0$. The maximum value of δ such that $\Psi^p(\delta) = 0$ is the largest zero of the real-valued function $\Psi^p(\delta)$ in $[0,+\infty)$.

Now, we see that Problem (P3(p)) is equivalent to the problem of finding the largest zero of $\Psi^p(\delta)$. We shall use a zero-finding algorithm to generate a sequence of points $\delta_k,\ k = 1,2,\ldots,$ which converges to δ^*, where δ^* is the largest zero of $\Psi^p(\delta)$. For each δ_k, we evaluate $\Psi^p(\delta_k)$ after solving the optimal control problem $(\mathrm{P3}_{\delta_k}(\mathrm{p}))$.

Then, for a fixed integer p, we present the section search method to search for the largest zero $\delta^{p,*}$ of the function Ψ^p. Given two starting values δ_1 and δ_2, the recursive formula for the section search method is

$$\delta_{k+1} = \frac{1}{2}(\delta_k + \delta_{k-1}),\ k = 2,3,\cdots. \tag{3.64}$$

Algorithm 3.1.

1. Choose a positive integer p, a relative accuracy $\epsilon \ge 0$, δ_2, and set $\delta_1 = 0$.
2. Evaluate $\Psi^p(\delta_2)$. If $\Psi^p(\delta_2) = 0$, repeat $\delta_2 + c_0$ until $\Psi^p(\delta_2) > 0$.
3. Compute δ_3 from (3.64), and evaluate $\Psi^p(\delta_3)$. If $|\ \delta_3 - \delta_2\ | < \epsilon$, stop; otherwise, goto Step 4.
4. If $\Psi^p(\delta_3) > 0$, $\delta_2 = \delta_3$. Otherwise, if $\Psi^p(\delta_3) = 0$, $\delta_1 = \delta_3$. Then, goto Step 3.

Remark 3.5. In Step 2, c_0 is chosen so as to generate a point δ_2 such that $\Psi^p(\delta_2) > 0$.

Remark 3.6. $\Psi^p(\delta)$ is calculated by using the formula (3.63) after solving Problem $(P_{\delta_k}(p))$.

Remark 3.7. The interval of uncertainty for this section search method is $(0.5)^k$ of the original interval.

Remark 3.8. Let $\delta^{p,*}$ and δ^* be, respectively, the optimal solutions of Problem (P3(p)) and Problem (P3). Then, $\delta^{p,*} \to \delta^*$, as $p \to \infty$. The proof is similar to that given for Theorem 8.5.2 in [36].

3.4 Obstacle-avoidance Problems

In this section, we consider an autonomous mobile robot in the framework of behavior-based control (see [80]). The robot is required to reach a pre-specified target from a given initial condition (position, orientation) while avoiding obstacles along the way.

The robot dynamics are:

$$\begin{cases} \dot{x} = \upsilon \cos\phi & (3.65a) \\ \dot{y} = \upsilon \sin\phi & (3.65b) \\ \dot{\phi} = \omega & (3.65c) \end{cases}$$

$$|\omega(t)| \leq 1, \ t \in [0, \infty).$$

where (x, y) is the position of the robot and ϕ is its orientation, while υ and ω are its translational and angular velocities. υ is assumed to be a constant value, and ω is a control variable. In our problem, $\upsilon = 1$. The initial point is $(x(0), y(0), \phi(0)) = (4, 0, \pi/2)$ and the target is $(x(T), y(T)) = (0, 0)$. We will discuss the situations with one and two obstacles. Define a circle with the obstacle as its center and δ its radius.

3.4.1 An One-obstacle Avoidance Problem

We first solve the minimum time optimal control problem involving a robot with its dynamic given by (3.1a)-(3.1c), initial condition $(x(0), y(0), \phi(0)) = (4, 0, \pi/2)$ and target condition $(x(T), y(T)) = (0, 0)$. Let this time optimal control be referred to as **Problem (E3)**. We solve this problem by using optimal control software package MISER 3.3 [37]. Let the optimal control u^* and the optimal trajectory be denoted as u^* and x^*, respectively. They are depicted in Figure 3.1 and Figure 3.2, respectively. The minimum time of reaching the target is:

$$T^* = 4.7391.$$

Suppose now that an obstacle is discovered at the location

$$\bar{x} = [1.78682, 0.63301]^\top,$$

which is a point at the optimal path. Our task is to steer the trajectory of the robot in such a way that the minimum distance from the obstacle is maximized while allowing the time of reaching the target to be increased by a small amount, say 5%, from the minimum time T^*. That is, to find a control u such that δ is maximized subject to the following nonlinear inequality constraint

$$(x(t) - 1.78682)^2 + (y(t) - 0.63301)^2 \geq \delta^2, \ \forall t \in [0, T] \tag{3.66}$$

and the time constraint

$$\sum_{i=1}^{p} \frac{\theta_i}{p} = T \leq (1+\alpha)T^*, \tag{3.67}$$

$$T \leq (1 + 5\%)T^*. \tag{3.68}$$

After applying the control parametrization and the time scaling transform outlined in

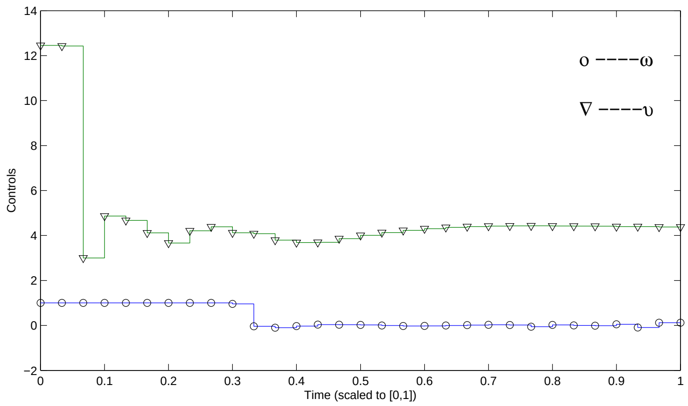

Figure 3.1: The optimal control with no obstacle

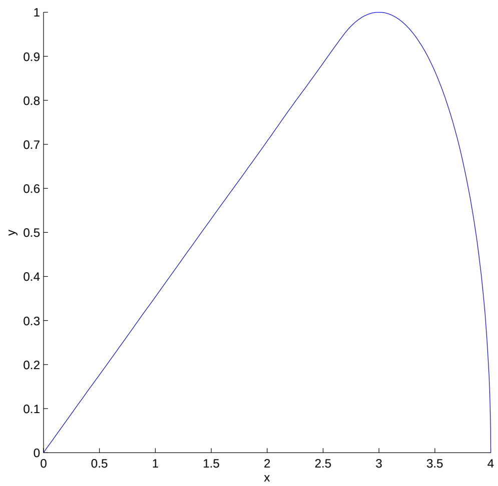

Figure 3.2: The optimal trajectory with no obstacle

Section **??**, the time constraint (3.68) becomes a constraint on the control parameters given below.

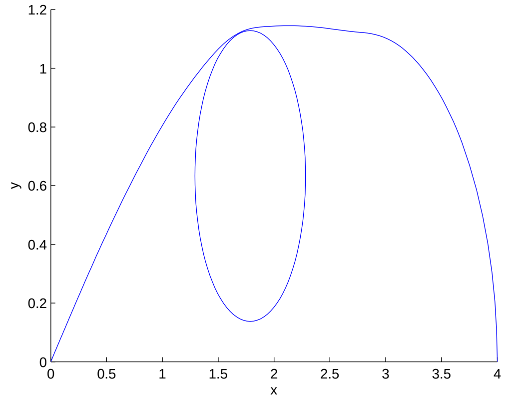

Figure 3.3: The optimal trajectory with one obstacle

We solve this problem by using Algorithm 3.1. The maximum value of $\delta = 0.49539$ is obtained. The corresponding optimal trajectory is shown in Figure 3.3.

3.4.2 A Two-obstacle Avoidance Problem

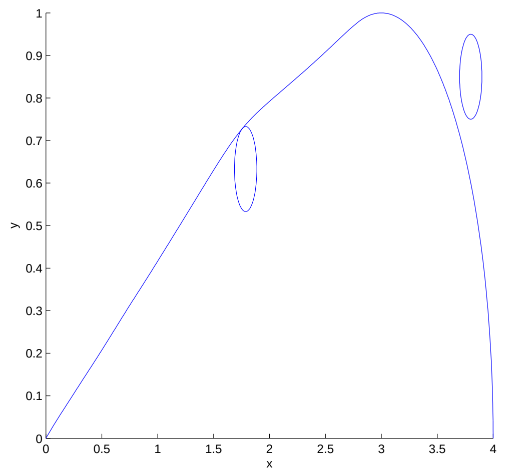

Figure 3.4: The optimal trajectory with two obstacles

Suppose now there are two obstacles. One is located at $(1.78682, 0.63301)^\top$ and the other is located at $(3.80000, 0.85000)^\top$. They are two points at the optimal path. By allowing the time of reaching the target to be increased to less than or equal to $(1+\alpha)T^*$, with $\alpha = 5\%$, we use Algorithm 3.1 to solve the corresponding maximum two-obstacle avoidance problem. The maximum value of δ obtained is 0.0996. The corresponding optimal trajectory is shown in Figure 3.4.

3.5 Abort Landing of an Aircraft in a Windshear Downburst

In this section, we consider the abort landing of a passenger aircraft Boeing 727 in the presence of a windshear downburst which is taken from [83] and [84].

To set up the the equations of motion, we assume that the aircraft is a particle of constant mass, the flight takes place in a vertical plane, and Newton's law is valid in an Earth-fixed system. Moreover, the wind flow field is assumed to be steady. Under these assumptions, the dynamical equations are:

$$\begin{cases} \dot{x} = V\cos\gamma + W_x, & (3.69a) \\ \dot{h} = V\sin\gamma + W_h, & (3.69b) \\ \dot{V} = \dfrac{T}{m}\cos(\alpha+\delta) - D/m - g\sin\gamma & \\ \quad -(\dot{W}_x\cos\gamma + \dot{W}_h\sin\gamma), & (3.69c) \\ \dot{\gamma} = \dfrac{T}{mV}\sin(\alpha+\delta) + L/mV - (1/V)g\cos\gamma & \\ \quad +(1/V)(\dot{W}_x\sin\gamma - \dot{W}_h\cos\gamma), & (3.69d) \end{cases}$$

The state variables are the horizontal distance x, the altitude h, the relative velocity V, and the relative path inclination γ. In the formulation above, the relative attack angle α is chosen as the control variable.

The approximation of the aerodynamic forces are listed below:

$$\begin{cases} T = \beta T_* & (3.70a) \\ T_* = A_0 + A_1 V + A_2 V^2, & (3.70b) \\ D = (1/2)C_D\rho S V^2, & (3.70c) \\ C_D(\alpha) = B_0 + B_1\alpha + B_2\alpha^2, & (3.70d) \\ L = (1/2)C_L\rho S V^2, & (3.70e) \\ C_L(\alpha) = \begin{cases} C_0 + C_1\alpha, & \alpha \le \alpha_*, \\ C_0 + C_1\alpha + C_2(\alpha-\alpha_*)^2, & \alpha_* \le \alpha \le \alpha_{max}. \end{cases} & (3.70f) \end{cases}$$

Here, T, D and L denote the thrust, the drag and the lift, respectively. The power setting β is specified in advance,

$$\beta(t) = \begin{cases} \beta_0 + \dot{\beta}_0 t, & 0 \le t \le t_0, \\ 1, & t_0 \le t \le t_f \end{cases} \tag{3.71}$$

The windshear model, which is valid for $h \leq 1000$ ft, is given below.

$$W_x = kA(x), \tag{3.72a}$$

$$W_h = k(h/h_*)B(x), \tag{3.72b}$$

with

$$A(x) = \begin{cases} -50 + ax^3 + bx^4, & 0 \leq x \leq 500, \\ (1/40)(x - 2300), & 500 \leq x \leq 4100, \\ 50 - a(4600 - x)^3 - b(4600 - x)^4, & 4100 \leq x \leq 4600, \\ 50, & 4600 \leq x \end{cases} \tag{3.72c}$$

$$B(x) = \begin{cases} dx^3 + ex^4, & 0 \leq x \leq 500, \\ -51exp[-c(x - 2300)^4], & 500 \leq x \leq 4100, \\ d(4600 - x)^3 - e(4600 - x)^4, & 4100 \leq x \leq 4600, \\ 0, & 4600 \leq x \end{cases} \tag{3.72d}$$

The angle of attack, α, is subject to the inequality constraints

$$\alpha \leq \alpha_{max}. \tag{3.73}$$

The initial conditions are:

$$\begin{aligned} &x(0) = x_0, \quad h(0) = h_0, \\ &V(0) = V_0, \quad \gamma(0) = \gamma_0, \\ &\alpha(0) = \alpha_0. \end{aligned} \tag{3.74}$$

and the terminal condition is:

$$\gamma(t_f) = \gamma_f. \tag{3.75}$$

All the required data are given in Table 3.1.

To avoid crashing on the ground, the minimal altitude is required to be maximized, i.e.,

$$\max_{\alpha \leq \alpha_{max}} \min_{0 \leq t \leq t_f} h(t). \tag{3.76}$$

(3.76) is equivalent to

$$\max \ \delta \tag{3.77}$$

subject to the continuous state inequality constraint

$$g(h(t), \delta) = h(t) - \delta \geq 0. \tag{3.78}$$

The problem is now a special case of Problem (P3). The computational method developed

Table 3.1: Model data for a Boeing 727 aircraft

Eqs. (3-70), (3-71)	Eqs. (3-71), (3-72)	Eqs. (3-71)
$\rho = 0.2203 \times 10^{-2}$ lb sec^2 ft^{-4}	$A_0 = 0.4456 \times 10^5$ lb	$B_0 = 0.1551$
$S = 0.1560 \times 10^4$ ft^2	$A_1 = -0.2398 \times 10^2$ lb sec ft^{-1}	$B_1 = 0.12369$ rad^{-1}
$g = 3.2171 \times 10^1$ ft sec^{-2}	$A_2 = 0.1442 \times 10^{-1}$ lb sec^2 ft^{-2}	$B_2 = 2.4203$ rad^{-2}
$mg = 150,000$ lb	$\beta^0 = 0.3825$	$C_0 = 0.7125$
$\delta = 2$ deg	$\dot{\beta}_0 = 0.2$ sec^{-1}	$C_1 = 6.0877$ rad^{-1}
	$t_0 = (1 - \beta_0/)\dot{\beta}_0$	$C_2 = -9.0277$ rad^{-2}
	$t_f = 40$ sec	$\alpha_* = 12$ deg
	$\alpha_{max} = 17.2$ deg	
Eqs. (3-73)	**Eqs. (3-75)**	
$k = 1$	$x_0 = 0$ ft	
$h_* = 1000$ ft	$\gamma_0 = -2.249$ deg	
$a = 6 \times 10^{-8}$ sec^{-1} ft^{-2}	$h_0 = 600$ ft	
$b = -4 \times 10^{-11}$ sec^{-1} ft^{-3}	$\alpha_0 = 7.353$ deg	
$c = -\ln(25/30.6) \times 10^{-12}$ ft^{-4}	$V_0 = 239.7$ ft sec^{-1}	
$d = -8.02881 \times 10^{-8}$ sec^{-1} ft^{-2}	$\gamma_f = 7.431$ deg	
$e = 6.28083 \times 10^{-11}$ sec^{-1} ft^{-3}		

in Section **??** is used to solve it. The results obtained are given below. The minimal altitude is 492 ft, the trajectories of the altitude h, the relative attack angle, α, and the relative path inclination angle, γ, are shown, respectively, in the pictures Figure 3.5 to Figure 3.8. The minimal altitude is slightly better than that reported in [85]. Our method does not require a good initial guess and the optimization process converges very rapidly.

3.6 Conclusions

In this chapter, a general class of maxmin optimal control problems was considered. An efficient computational method for solving this general maxmin optimal control problem was developed. It has been shown that, a sequence of smooth approximate optimal control problems can then be constructed by taking the summation of some smooth approximate functions, which were obtained by applying the constraint transcription method [103] to each of the continuous state inequality constraints, as its cost function. A necessary condition and a sufficient condition were derived to show the relationship between the original maxmin problem and the sequence of the smooth approximate problems. A violation avoidance function was then constructed from the solution of each of the smooth approximate optimal control problems and the original continuous state inequality constraints. It shows that the problem of finding an optimal control of the maxmin optimal control

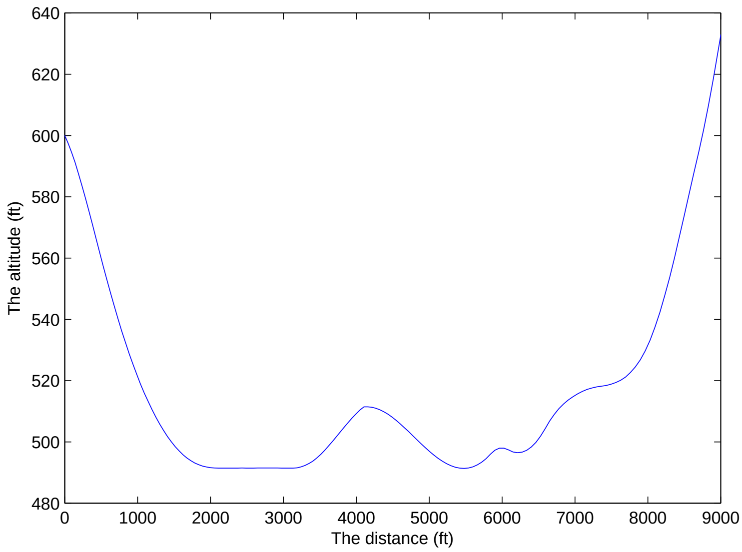

Figure 3.5: The altitude with distance

problem is equivalent to the problem of finding the largest root of the violation avoidance function, and each of these smooth approximate optimal control problems can be solved by applying the control parametrization technique [36] and a time scaling transform [87]. Two applications, an obstacle avoidance problem of an autonomous mobile robot and the abort landing of an aircraft in a windshear downburst, were studied by applying the computational method proposed. The solutions obtained are satisfactory.

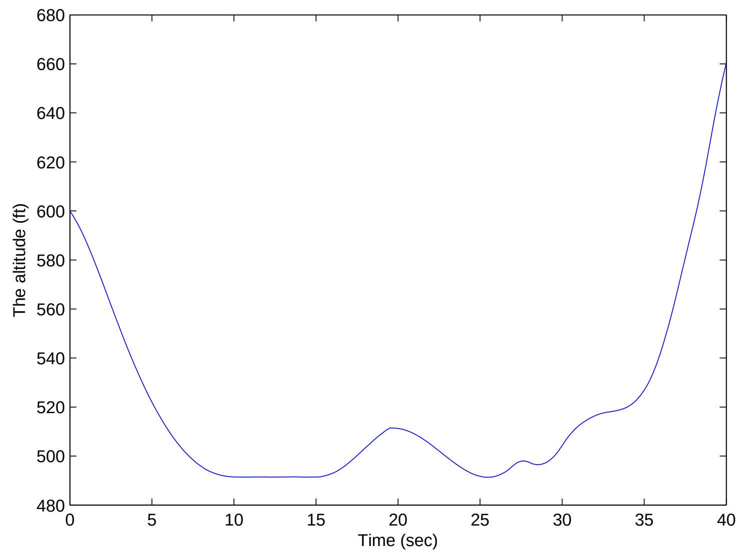

Figure 3.6: The altitude with time

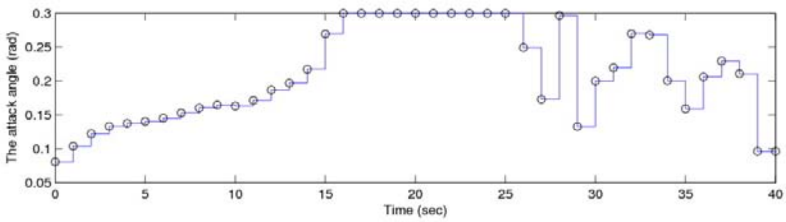

Figure 3.7: The relative angle of attack

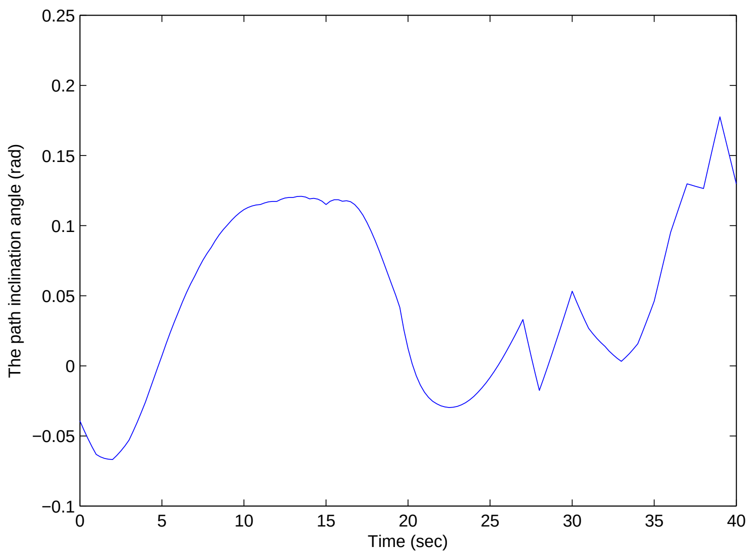

Figure 3.8: The relative path inclination angle

CHAPTER 4

A Nonlinear Optimal PID Tuning Problem

4.1 Introduction

This chapter is the author's work in [114]. In this chapter, we consider a class of optimal PID tuning problems subject to continuous inequality constraints and terminal equality constraint. By applying the constraint transcription method [103] and a local smoothing technique to these continuous inequality constraint functions, we construct the corresponding smooth approximate functions. We use the concept of the penalty function to append these smooth approximate functions to the cost function, forming a new cost function. Then, the constrained optimal PID tuning problem is approximated by a sequence of optimal parameter selection problems subject to only terminal equality constraint. Each of these optimal parameter selection problems can be viewed and hence solved as a nonlinear optimization problem. The gradient formulas of the new appended cost function and the terminal equality constraint function are derived, and a reliable computation algorithm is given. The method proposed is used to solve a ship steering control problem.

4.2 Problem Statement

Consider a dynamical system:

$$\begin{cases} \dot{x}(t) = f(x(t), y(t), u(t)),\ t \in (0, T] & (4.1a) \\ \dot{y}(t) = p(x(t)) & (4.1b) \\ x(0) = x^0 & (4.1c) \\ y(0) = y^0, & (4.1d) \end{cases}$$

where T is the terminal time, and $x = [x_1, x_2, \dots, x_n]^\top \in \mathbb{R}^n$, $u = [u_1, u_2, \dots, u_r]^\top \in \mathbb{R}^r$, $y \in \mathbb{R}$ are, respectively, state, control and output, while $f = [f_1, f_2, \dots, f_n]^\top \in \mathbb{R}^n$ and $p \in \mathbb{R}$ are, respectively, given continuously differentiable functions. $x^0 \in \mathbb{R}^n$ and

$y^0 \in \mathbb{R}$ are a given constant vector and a given scalar, respectively.

We assume that the following conditions are satisfied.

Assumption 4.1. There exists a positive constant C_1 such that

$$\|f(x, y, u)\| \leq C_1 (1 + \|x\| + \|y\|)$$

for all $(x, y, u) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^r$.

Assumption 4.2. There exists a positive constant C_2 such that

$$\|p(x)\| \leq C_2 (1 + \|x\|) .$$

The control u is assumed to take the form of a PID controller given below.

$$\begin{aligned} u(t) &= \sum_{j=1}^{N_1} k_{1,j}(y(t) - r(t))\chi_{I_{1,j}}(t) \\ &\quad + \sum_{j=1}^{N_2} k_{2,j} \int_0^t (y(s) - r(s))\chi_{I_{2,j}}(s)\mathrm{d}s + \sum_{j=1}^{N_3} k_{3,j}\dot{y}(t)\chi_{I_{3,j}}(t), \end{aligned} \tag{4.2}$$

where $r(t)$ denotes a given reference input which is a piecewise continuous function defined on $[0, T]$,

$$I_{i,j} = [t_{i,j-1}, t_{i,j}), \; i = 1, 2, 3; \; j = 1, 2, \ldots, N_i \tag{4.3}$$

while

$$0 = t_{i,0} < t_{i,1} < t_{i,2} < \cdots < t_{i,N_i} < t_{i,N_{i+1}} = T, i = 1, 2, 3, \tag{4.4}$$

are the switching times for the proportional, integral and derivative control actions, respectively, and χ_I denotes the indicator function of I given by

$$\chi_I(t) = \begin{cases} 1, & t \in I, \\ 0, & \text{otherwise} \end{cases} \tag{4.5}$$

here, $\{k_{i,1}, k_{i,2}, \ldots, k_{i,N_i}\}$, $i = 1, 2, 3$, are respective gains for the proportional, integral and derivative terms of the PID controller.

The form of the PID controller is a generalized version of the conventional PID controller, in particular, the form of the integral control. For the conventional integral control, it performs the integral action over a period of time. Because of the accumulation effect, a large value of the gain for the integral control will cause huge overshoot. On the other hand, if the gain for the integral control is chosen to be very small, while the overshoot can become small, the steady state error will take a long time to reduce in presence of constant disturbances. The generalized integral control is in the form for which it is re-set at appropriately chosen fixed switching time points so as to give a well-regulated control operation.

Remark 4.1. Here, we assume that y and r are real-valued functions. It is straightforward to extend the results to the case where y and r are vector-valued functions at the expense of notational complexity.

We now specify the region within which the output trajectory is allowed to move. This region is defined in terms of the following continuous inequality constraints, which arise due to practical requirements, such as constraints on the rise time and for avoiding overshoot. They may also arise due to engineering specification on the PID controller.

$$g_i(t, x(t), y(t), u(t)) \leq 0, \ t \in [0, T], \ i = 1, 2, \dots, M. \tag{4.6}$$

For each $i = 1, 2, \dots, M$, the function g_i is continuously differentiable with respect to x, y and u, while continuous with respect to t.

To ensure a satisfactory tracking of $r(t)$ by $y(t)$, the following terminal state constraint is imposed.

$$\Omega(y(T)) = y(T) - r(T) = 0 \tag{4.7}$$

The optimal control problem may now be stated below. Given system (4.1a)-(4.1d), design a PID controller in the form defined by (4.2) such that the output $y(t)$ of the corresponding closed loop system will move within the specified region defined by the continuous inequality constraints (4.6) and, at the same time, it will track the given reference input such that the terminal condition (4.7) is satisfied. First, we formulate a cost function below.

$$J(k) = \int_0^T \left\{ \alpha_1 \left[y(t) - r(t)\right]^2 + \alpha_2 \left[\dot{y}(t)\right]^2 + \alpha_3 \left[u(t)\right]^2 \right\} \mathrm{d}t, \tag{4.8}$$

where $\alpha_i, \ i = 1, 2, 3$, are the weightings.

For the integral term of the PID controller given by (4.2), we define

$$z_j(t) = \int_0^t [y(s) - r(s)] \chi_{I_{2,j}}(s) \mathrm{d}s, \ j = 1, 2, \dots, N_2 \tag{4.9}$$

Clearly, for each $j = 1, 2, \dots, N_2$, (4.9) is equivalent to

$$\begin{cases} \dot{z}_j(t) = (y(t) - r(t)) \chi_{I_{2,j}}(t) & \text{(4.10a)} \\ z_j(0) = 0. & \text{(4.10b)} \end{cases}$$

Let $z(t) = [z_1(t), z_2(t), \dots, z_{N_2}(t)]^\top$ and $q(t) = [q_1(t), q_2(t), \dots, q_{N_2}(t)]^\top$, where the superscript $\top$ denotes the transpose and

$$q_j(t) = (y(t) - r(t))\chi_{I_{2,j}}(t),\ j = 1, 2, \ldots, N_2 \tag{4.11}$$

Then, system (4.10a)-(4.10b) becomes

$$\begin{cases} \dot{z}(t) = q(t) & (4.12a) \\ z(0) = 0. & (4.12b) \end{cases}$$

Now, it follows from (4.12a)-(4.12b) that system (4.1a)-(4.1d) with $u(t)$ chosen as a PID controller given by (4.2) can be written as:

$$\begin{cases} \dot{x}(t) &= \overline{f}(t, x(t), y(t), z(t), k) \\ \dot{y}(t) &= p(x(t)) \\ \dot{z}(t) &= q(t) \end{cases} \tag{4.13}$$

with initial conditions

$$\begin{cases} x(0) &= x^0 \\ y(0) &= y^0 \\ z(0) &= 0 \end{cases} \tag{4.14}$$

where

$$\overline{f}(t, x(t), y(t), z(t), k) = f(x(t), y(t), u(t)), \tag{4.15}$$

while the PID controller $u(t)$ given by (4.2) becomes

$$\begin{aligned} u(t) &= \sum_{j=1}^{N_1} k_{1,j}(y(t) - r(t))\chi_{I_{1,j}}(t) \\ &\quad + \sum_{j=1}^{N_2} k_{2,j} z_j(t) + \sum_{j=1}^{N_3} k_{3,j} p(x(t))\chi_{I_{3,j}}(t), \end{aligned} \tag{4.16}$$

here

$$k = [k_{1,1}, k_{1,2}, \ldots, k_{1,N_1}, k_{2,1}, k_{2,2}, \ldots, k_{2,N_2}, k_{3,1}, k_{3,2}, \ldots, k_{3,N_3}]^\top \tag{4.17}$$

is the vector containing the gains for the proportional, integral and derivative terms of the PID controller.

The continuous inequality constraints (4.6) become

$$\tilde{g}_i(t, x(t), y(t), z(t), k) \leq 0,\ t \in [0, T],\ i = 1, 2, \ldots, M. \tag{4.18}$$

The cost function (4.8) becomes

$$\begin{aligned}\overline{J}(k) \;=\; & \int_0^T \Bigg\{ \alpha_1 \left[y(t) - r(t)\right]^2 + \alpha_2 \left[p(x(t))\right]^2 + \alpha_3 \Bigg[\sum_{j=1}^{N_1} k_{1,j} \left[y(t)\right. \\ & \left. - r(t)\right] \chi_{I_{1,j}}(t) + \sum_{j=1}^{N_2} k_{2,j} z_j(t) + \sum_{j=1}^{N_3} k_{3,j} p(x(t)) \chi_{I_{3,j}}(t) \Bigg]^2 \Bigg\} \mathrm{d}t. \qquad (4.19)\end{aligned}$$

The problem may now be re-stated as: Given system (4.13) with initial condition (4.14), find a PID control parameter vector k such that the cost function (4.19) is minimized subject to the continuous inequality constraints (4.6) and the terminal equality constraint (4.7). Let this problem be referred to as **Problem (P4)**. Clearly, Problem (P4) is an optimal parameter selection problem.

4.3 Constraint Approximation

The continuous inequality constraints (4.6) cannot be handled directly, because each of which contains infinite number of constraints. Here, the constraint transcription technique (see [36, 94, 101, 103]) is applied to the continuous inequality constraints (4.6), leading to the following equivalent equality constraints:

$$\int_0^T \max\left\{\tilde{g}_i(t, x(t), y(t), z(t), k), 0\right\} \mathrm{d}t = 0, \;\; i = 1, 2, \ldots, M, \qquad (4.20)$$

However, the integrands appeared under the integration in (4.20) are nonsmooth. Thus, for each $i = 1, 2, \ldots, M$, we shall approximate the nonsmooth function $\max\{\tilde{g}_i(t, x(t), y(t), z(t), k), 0\}$ by a smooth function $\mathcal{L}_{i,\varepsilon}(t, x(t), y(t), z(t), k)$ given by

$$\begin{aligned}&\mathcal{L}_{i,\varepsilon}(t, x(t), y(t), z(t), k) \\ &= \begin{cases} 0, & \text{if} \;\; \tilde{g}_i(t, x(t), y(t), z(t), k) < -\varepsilon \\ \dfrac{\left(\tilde{g}_i(t, x(t), y(t), z(t), k) + \varepsilon\right)^2}{4\varepsilon}, & \text{if} \; -\varepsilon \le \tilde{g}_i(t, x(t), y(t), z(t), k) \le \varepsilon \\ \tilde{g}_i(t, x(t), y(t), z(t), k), & \text{if} \;\; \tilde{g}_i(t, x(t), y(t), z(t), k) > \varepsilon, \end{cases} \qquad (4.21)\end{aligned}$$

where $\varepsilon > 0$ is an adjustable constant with small value. Then, for each $i = 1, 2, \ldots, M$, we define

$$g_{i,\varepsilon}(k) = \int_0^T \mathcal{L}_{i,\varepsilon}(t, x(t), y(t), z(t), k) \mathrm{d}t \qquad (4.22)$$

We now use the concept of the penalty function to append the functions $g_{i,\varepsilon}$ given by

(4.22) to the cost function (4.19), forming a new cost function given below.

$$\begin{aligned} \overline{J}_{\varepsilon,\gamma}(k) &= \int_0^T l(t, x(t), y(t), z(t), k)\mathrm{d}t \\ &\quad +\gamma\sum_{i=1}^{M}\int_0^T \mathcal{L}_{i,\varepsilon}(t, x(t), y(t), z(t), k)\mathrm{d}t \end{aligned} \tag{4.23}$$

where

$$l(t, x, y, z, k) = \alpha_1 (y - r)^2 + \alpha_2 [p(x)]^2 + \alpha_3 [u(t)]^2 , \tag{4.24}$$

and $u(t)$ is given by (4.16) and $\gamma > 0$ is a penalty parameter.

We may now state the approximate problem for each $\varepsilon > 0$ and $\gamma > 0$ as follows. Given system (4.13) with initial condition (4.14) and terminal condition (4.7), find a PID control parameter vector k such that the cost function

$$\overline{J}_{\varepsilon,\gamma}(k) = \int_0^T \hat{\mathcal{L}}_{\varepsilon,\gamma}(t, x, y, z, k)\mathrm{d}t, \tag{4.25}$$

is minimized, where

$$\begin{aligned} \hat{\mathcal{L}}_{\varepsilon,\gamma}(t, x, y, z, k) &= l(t, x(t), y(t), z(t), k) \\ &\quad +\gamma\sum_{i=1}^{M}\mathcal{L}_{i,\varepsilon}(t, x(t), y(t), z(t), k) \end{aligned} \tag{4.26}$$

This problem is referred to as **Problem (P$4_{\varepsilon,\gamma}$)**.

To proceed further, we define

$$\Theta = \{k : \Omega(y(T|k)) = 0\}$$

$$\mathcal{F} = \{k \in \Theta : \tilde{g}_i(t, x(t), y(t), z(t), k) \leq 0,\ t \in [0, T],\ i = 1, 2, \ldots, M\}$$

Furthermore, let $\mathring{\mathcal{F}}$ denote the interior of $\mathcal{F}$, in the sense that

$$\mathring{\mathcal{F}} = \{k \in \Theta : \tilde{g}_i(t, x(t), y(t), z(t), k) < 0,\ t \in [0, T],\ i = 1, 2, \ldots, M\}$$

We assume that the following assumptions hold.
Assumption 4.3 $\mathring{\mathcal{F}} \neq \emptyset$.
Assumption 4.4 For every optimal solution k^* of Problem (P4), there exists a $\overline{k}$ such

that

$$\alpha\overline{k} + (1-\alpha)k^* \in \overset{\circ}{\mathcal{F}}$$

for all $\alpha \in (0,1]$.

Let

$$\begin{aligned}\mathcal{F}_\varepsilon &= \{k \in \Theta : \tilde{g}_i(t, x(t), y(t), z(t), k) \leq -\varepsilon,\ t \in [0,T],\ i = 1,2,\ldots,M\} \\ &= \{k \in \Theta : \mathcal{L}_{i,\varepsilon}(t, x(t), y(t), z(t), k) = 0,\ t \in [0,T],\ i = 1,2,\ldots,M\}\end{aligned}$$

where $u(t)$ is given by (4.16).

Thus, Problem (P4) can be also stated as follows: Given system (4.13) with initial condition (4.14), find a PID control parameter vector k over $\mathcal{F}$ such that the cost function (4.19) is minimized.

Before continuing, we recall Lemma 8.6.2 from [36].

Lemma 4.1. *There exists a $\tau(\varepsilon)$ such that for any $0 < \tau < \tau(\varepsilon)$, if $g_{i,\varepsilon}(k) < \tau$, $k \in \Theta$, then $k \in \mathcal{F}$.*

Then, we establish the relationships between Problem (P4$_{\varepsilon,\gamma}$) and Problem (P4) with the following two theorems.

Theorem 4.1. *For any $\varepsilon > 0$, there exists a $\gamma(\varepsilon) > 0$ such that for all γ, $0 < \gamma < \gamma(\varepsilon)$, if $k^*_{\varepsilon,\gamma}$ is an optimal solution of Problem (P4$_{\varepsilon,\gamma}$), then it satisfies the continuous inequality constraints (4.6) of Problem (P4).*

Proof. As $k^*_{\varepsilon,\gamma}$ is an optimal solution of Problem (P4$_{\varepsilon,\gamma}$), we have

$$\overline{J}_{\varepsilon,\gamma}(k^*_{\varepsilon,\gamma}) = \overline{J}(k^*_{\varepsilon,\gamma}) + \gamma\sum_{i=1}^{M} g_{i,\varepsilon}(k^*_{\varepsilon,\gamma}) \leq \overline{J}(k) + \gamma\sum_{i=1}^{M} g_{i,\varepsilon}(k) \tag{4.27}$$

for all $k \in \Theta$, where $g_{i,\varepsilon}(k)$ and $\overline{J}(k)$ are as defined in (4.22) and (4.19), respectively. Let $k_\varepsilon \in \mathcal{F}_\varepsilon$ be fixed. Then, by the definition of $g_{i,\varepsilon}(k)$, $g_{i,\varepsilon}(k) = 0$, $i = 1,2,\ldots,M$. Obviously, there exists a $\overline{k} \in \Theta$ such that $\overline{J}(\overline{k}) \leq \overline{J}(k^*_{\varepsilon,\gamma})$. Then, we have

$$\overline{J}(\overline{k}) + \gamma\sum_{i=1}^{M} g_{i,\varepsilon}(k^*_{\varepsilon,\gamma}) \leq \overline{J}(k^*_{\varepsilon,\gamma}) + \gamma\sum_{i=1}^{M} g_{i,\varepsilon}(k^*_{\varepsilon,\gamma}) \leq \overline{J}(k_\varepsilon) \tag{4.28}$$

Rearranging (4.28), we have

$$\gamma\sum_{i=1}^{M} g_{i,\varepsilon}(k^*_{\varepsilon,\gamma}) \leq \overline{J}(k_\varepsilon) - \overline{J}(\overline{k}) \tag{4.29}$$

Letting $\beta = \overline{J}(k_\varepsilon) - \overline{J}(\overline{k})$ in (4.29), we get

$$\sum_{i=1}^{M} g_{i,\varepsilon}(k^*_{\varepsilon,\gamma}) \le \frac{\beta}{\gamma}$$

By choosing $\gamma(\varepsilon) \ge \beta/\tau(\varepsilon)$, it follows that for all $\gamma > \gamma(\varepsilon)$, $\sum_{i=1}^{M} g_{i,\varepsilon}(k^*_{\varepsilon,\gamma}) < \tau$, $i = 1, 2, \ldots, M$. Hence, from Lemma 4.1, $k^*_{\varepsilon,\gamma} < \mathcal{F}$. This completes the proof. □

Theorem 4.2. *Let k^* and $k^*_{\varepsilon,\gamma(\varepsilon)}$ be, respectively, optimal solutions of Problem (P4) and Problem (P4$_{\varepsilon,\gamma}$), where $\gamma(\varepsilon)$ is chosen such that $k^*_{\varepsilon,\gamma(\varepsilon)}$ satisfies the continuous inequality constraints (4.6) of Problem (P4). Then,*

$$\lim_{\varepsilon \to 0} \overline{J}(k^*_{\varepsilon,\gamma(\varepsilon)}) = \overline{J}(k^*), \tag{4.30}$$

where $\overline{J}$ is defined by (4.19).

Proof. By Assumption 4.4, there exists a $\hat{k} \in \mathring{\mathcal{F}}$ such that $k_\alpha = \alpha\hat{k} + (1-\alpha)k^* = k^* + \alpha(\hat{k} - k^*) \in \mathring{\mathcal{F}}$ for all $\alpha \in (0, 1]$. Now, for any $\delta_1 > 0$, there exists an $\alpha_1 \in (0, 1]$ such that

$$\overline{J}(k^*) \le \overline{J}(k_\alpha) \le \overline{J}(k^*) + \delta_1, \tag{4.31}$$

for all $\alpha \in (0, \alpha_1)$. Choose $\alpha_2 = \alpha_1/2$. Then it is clear that $k_{\alpha_2} \in \mathring{\mathcal{F}}$. Thus, there exists a δ_2 such that $\max_{t\in[0,T]} \tilde{g}_i(t, x(t), y(t), z(t), k_{\alpha_2}) < -\delta_2$, $i = 1, 2, \ldots, M$. If we choose $\varepsilon = \delta_2$, then k_{α_2} satisfies $g_{i,\varepsilon} = 0$, $i = 1, 2, \ldots, M$. Using this and from the definition of $k^*_{\varepsilon,\gamma(\varepsilon)}$, we have

$$\overline{J}(k^*_{\varepsilon,\gamma(\varepsilon)}) + \gamma\sum_{i=1}^{M} g_{i,\varepsilon}(k^*_{\varepsilon,\gamma(\varepsilon)}) \le \overline{J}(k_{\alpha_2}) + \gamma\sum_{i=1}^{M} g_{i,\varepsilon}(k_{\alpha_2}) = \overline{J}(k_{\alpha_2})$$

Note that

$$\gamma\sum_{i=1}^{M} g_{i,\varepsilon}(k^*_{\varepsilon,\gamma(\varepsilon)}) \ge 0$$

we get

$$\overline{J}(k^*_{\varepsilon,\gamma(\varepsilon)}) \le \overline{J}(k_{\alpha_2})$$

Combining (4.31) and remembering that $k^*_{\varepsilon,\gamma(\varepsilon)}$ is feasible for Problem (P4), we have

$$\overline{J}(k^*) \le \overline{J}(k^*_{\varepsilon,\gamma(\varepsilon)}) \le \overline{J}(k^*) + \delta_1.$$

Letting $\varepsilon \to 0$ and noting that $\delta_1 > 0$ is arbitrary, the result follows. □

On the basis of Theorem 4.1 and Theorem 4.2, Problem (P4) can be solved through solving a sequence of optimal parameter selection problems (P4$_{\varepsilon,\gamma}$) subject to only terminal equality condition (4.7). Each of these optimal parameter selection problems can be solved as a nonlinear optimization problem by using a gradient-based optimization method, such as the SQP approximation scheme [36]. Thus, the optimal control software, MISER 3.3 [37], is applicable. Further details are given in the next section.

4.4 Computational Method

In this section, we will propose a reliable computational method for solving Problem (P4) via solving a sequence of Problems ($P4_{\varepsilon,\gamma}$), where for each $\varepsilon > 0$ and $\gamma > 0$, Problem ($P4_{\varepsilon,\gamma}$) is solved as a nonlinear optimization problem. For doing this, it is required to provide, for each k, the value of the cost function $\overline{J}_{\varepsilon,\gamma}(k)$, as well as its gradient $\dfrac{\partial \overline{J}_{\varepsilon,\gamma}(k)}{\partial k}$. Furthermore, we also need the value of the terminal constraint function $\Omega(y(T|k))$ and its gradient $\dfrac{\partial \Omega(y(T|k))}{\partial k}$. It is obvious that the values of the cost function $\overline{J}_{\varepsilon,\gamma}(k)$ and the terminal constraint function $\Omega(y(T|k))$ can be readily obtained after system (4.13) with initial condition (4.14) corresponding to k is solved. For the gradient formulas of the cost function $\overline{J}_{\varepsilon,\gamma}(k)$ and the terminal constraint function $\Omega(y(T|u))$ corresponding to each k, we have the following two theorems. Their proofs are similar to that given for Theorem 5.2.1 in [36].

For the notation simplicity, we define

$$\tilde{x} = \left[x^\top, y, z^\top\right]^\top$$

and

$$\tilde{f} = [(\overline{f})^\top, p, q^\top]^\top.$$

Then, system (4.13)-(4.14) becomes

$$\dot{\tilde{x}}(t) = \tilde{f}(t, \tilde{x}, k) \tag{4.32}$$

with initial conditions

$$\tilde{x}(0) = [(x^0)^\top, y^0, 0]^\top \tag{4.33}$$

Theorem 4.3. *The gradient formula for the cost function $\overline{J}_{\varepsilon,\gamma}(k)$ with respect to k is given by*

$$\frac{\partial \overline{J}_{\varepsilon,\gamma}(k)}{\partial k} = \int_0^T \frac{\partial H_{\varepsilon,\gamma}(t, \tilde{x}(t), k, \lambda_{\varepsilon,\gamma}(t))}{\partial k} \mathrm{d}t. \tag{4.34}$$

Here, $H_{\varepsilon,\gamma}(t, \tilde{x}, k, \lambda)$ is the Hamiltonian function given by

$$H_{\varepsilon,\gamma}(t, \tilde{x}, k, \lambda) = \hat{\mathcal{L}}_{\varepsilon,\gamma}(t, \tilde{x}, k) + \lambda_{\varepsilon,\gamma}^\top \tilde{f}(t, \tilde{x}, k), \tag{4.35}$$

where $\hat{\mathcal{L}}_{\varepsilon,\gamma}$ is as defined by (4.26), and $\lambda_{\varepsilon,\gamma}$ is the solution of following system of co-state differential equations

$$\dot{\lambda}(t) = -\frac{\partial H_{\varepsilon,\gamma}(t, \tilde{x}(t), k, \lambda(t))}{\partial \tilde{x}} \tag{4.36a}$$

with the boundary condition

$$\lambda(T) = 0. \tag{4.36b}$$

Proof. Let $k \in \mathbb{R}^{N_1+N_2+N_3}$ be given and let $\rho \in \mathbb{R}^{N_1+N_2+N_3}$ be arbitrary but fixed. Define

$$k(\epsilon) = k + \epsilon\rho \tag{4.37}$$

where $\epsilon > 0$ is an arbitrarily small real number. For brevity, let $\tilde{x}(\cdot)$ and $\tilde{x}(\cdot;\epsilon)$ denote, respectively, the solution of the system (4.32)-(4.33) corresponding to k and $k(\epsilon)$. Clearly,

$$\tilde{x}(s) = \tilde{x}(0) + \int_0^s \tilde{f}(\tau, \tilde{x}(\tau), k)\mathrm{d}\tau \tag{4.38}$$

$$\tilde{x}(s;\epsilon) = \tilde{x}(0) + \int_0^s \tilde{f}(\tau, \tilde{x}(\tau;\epsilon), k)\mathrm{d}\tau \tag{4.39}$$

Thus,

$$\begin{aligned} \triangle\tilde{x}(s) &\equiv \left.\frac{\mathrm{d}\tilde{x}(s;\epsilon)}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \\ &= \int_0^s \left\{ \frac{\partial \tilde{f}(\tau, \tilde{x}(\tau), k)}{\partial \tilde{x}} \triangle \tilde{x}(\tau) + \frac{\partial \tilde{f}(\tau, \tilde{x}(\tau), k)}{\partial k}\rho \right\} \mathrm{d}\tau \end{aligned} \tag{4.40}$$

Clearly,

$$\frac{\mathrm{d}(\triangle\tilde{x}(s))}{\mathrm{d}s} = \frac{\partial \tilde{f}(s, \tilde{x}(s), k)}{\partial \tilde{x}} \triangle \tilde{x}(s) + \frac{\partial \tilde{f}(s, \tilde{x}(s), k)}{\partial k}\rho \tag{4.41}$$

Now, $\overline{J}_{\varepsilon,\gamma}(k)$ can be expressed as:

$$\overline{J}_{\varepsilon,\gamma}(k(\epsilon)) = \int_0^T \left\{ H_{\varepsilon,\gamma}(s, \tilde{x}(s;\epsilon), k, \lambda(s)) - \lambda^\top(s)\tilde{f}(s, \tilde{x}(s;\epsilon), k) \right\} \mathrm{d}s \tag{4.42}$$

where $\lambda(s)$ is yet arbitrary. Thus, it follows that

$$\begin{aligned} \triangle\overline{J}_{\varepsilon,\gamma}(k(\epsilon)) &\equiv \left.\frac{\mathrm{d}\overline{J}_{\varepsilon,\gamma}(k(\epsilon))}{d\epsilon}\right|_{\epsilon=0} = \frac{\partial \overline{J}_{\varepsilon,\gamma}(k(\epsilon))}{\partial k}\rho \\ &= \int_0^T \left\{ \triangle H_{\varepsilon,\gamma}(s, \tilde{x}(s), k, \lambda(s)) - \lambda^\top(s) \triangle \tilde{f}(s, \tilde{x}(s), k) \right\} \mathrm{d}s \end{aligned} \tag{4.43}$$

where

$$\triangle\tilde{f}(s,\tilde{x}(s),k)=\frac{\mathrm{d}\triangle\tilde{x}(s)}{\mathrm{d}s} \tag{4.44}$$

and

$$\begin{aligned}&\triangle H_{\varepsilon,\gamma}(s,\tilde{x}(s),k,\lambda(s))\\ =\ &\frac{\partial H_{\varepsilon,\gamma}(s,\tilde{x}(s),k,\lambda(s))}{\partial\tilde{x}}\triangle\tilde{x}(s)+\frac{\partial H_{\varepsilon,\gamma}(s,\tilde{x}(s),k,\lambda(s))}{\partial k}\rho\end{aligned} \tag{4.45}$$

Choose λ_0 to be the solution of the costate system (4.36a)-(4.46) corresponding to k. Then, by substituting (4.36a) into (4.45), we obtain

$$\begin{aligned}&\triangle H_{\varepsilon,\gamma}(s,\tilde{x}(s),k,\lambda(s))\\ =\ &-\left(\frac{\mathrm{d}\lambda(s)}{\mathrm{d}s}\right)^{\top}\triangle\tilde{x}(s)+\frac{\partial H_{\varepsilon,\gamma}(s,\tilde{x}(s),k,\lambda(s))}{\partial k}\rho\end{aligned} \tag{4.46}$$

By (4.44) and (4.46), it follows from (4.43) that

$$\begin{aligned}&\frac{\partial\overline{J}_{\varepsilon,\gamma}(k)}{\partial k}\rho\\ =\ &\int_0^T\left\{-\frac{\mathrm{d}}{\mathrm{d}s}\left[(\lambda_0(s))^{\top}\triangle\tilde{x}(s)\right]+\frac{\partial H_{\varepsilon,\gamma}(s,\tilde{x}(s),k,\lambda(s))}{\partial k}\rho\right\}\mathrm{d}s\\ =\ &(\lambda(0))^{\top}\triangle\tilde{x}(0)-(\lambda(T))^{\top}\triangle\tilde{x}(T)+\int_0^T\left\{\frac{\partial H_{\varepsilon,\gamma}(s,\tilde{x}(s),k,\lambda(s))}{\partial k}\rho\right\}\mathrm{d}s\end{aligned} \tag{4.47}$$

Note that $\tilde{x}(0)$ is constant. Thus, by (4.46), we deduce from (4.47) that

$$\begin{aligned}&\frac{\partial\overline{J}_{\varepsilon,\gamma}(k(\epsilon))}{\partial k}\rho\\ =\ &\int_0^T\left\{\frac{\partial H_{\varepsilon,\gamma}(s,\tilde{x}(s),k,\lambda(s))}{\partial k}\rho\right\}\mathrm{d}s\end{aligned} \tag{4.48}$$

Since ρ is arbitrary, (4.34) follows readily from (4.48). Thus, the proof is completed. □

Before giving the gradient formula for the terminal constraint function $\Omega(y(T|k))$, we define

$$\Omega(y(T|k))=\tilde{\Omega}(\tilde{x}(T|k))$$

Theorem 4.4. *The gradient formula for the terminal constraint function $\tilde{\Omega}(\tilde{x}(T|k))$ with respect to k is given by*

$$\frac{\partial\tilde{\Omega}(\tilde{x}(T|k))}{\partial k}=\int_0^T\frac{\partial\tilde{H}_{\varepsilon,\gamma}(t,\tilde{x}(t),k,\tilde{\lambda}_{\varepsilon,\gamma}(t))}{\partial k}\mathrm{d}t, \tag{4.49}$$

where $\tilde{H}_{\varepsilon,\gamma}(t,\tilde{x},k,\lambda)$ *is the Hamiltonian function given by*

$$\tilde{H}_{\varepsilon,\gamma}(t,\tilde{x},k,\lambda) = \tilde{\lambda}_{\varepsilon,\gamma}^{\top}\tilde{f}(t,\tilde{x},k). \tag{4.50}$$

and $\tilde{\lambda}_{\varepsilon,\gamma}$ *is the solution of following system of co-state differential equations*

$$\frac{\mathrm{d}\tilde{\lambda}(t)}{\mathrm{d}t} = -\frac{\partial \tilde{H}_{\varepsilon,\gamma}(t,\tilde{x}(t),k,\tilde{\lambda}(t))}{\partial \tilde{x}} \tag{4.51a}$$

with the boundary condition

$$\tilde{\lambda}(T) = \frac{\mathrm{d}\tilde{\Omega}(\tilde{x}(T|k))}{\mathrm{d}\tilde{x}} \tag{4.51b}$$

Proof. Let $k \in \mathbb{R}^{N_1+N_2+N_3}$ be given and let $\rho \in \mathbb{R}^{N_1+N_2+N_3}$ be arbitrary but fixed. Define

$$k(\epsilon) = k + \epsilon\rho \tag{4.52}$$

where $\epsilon > 0$ is an arbitrarily small real number. $\tilde{x}(\cdot)$ and $\tilde{x}(\cdot;\epsilon)$ denote, respectively, the solutions of the system (4.32)-(4.33) corresponding to k and $k(\epsilon)$. As it has been done in the proof for Theorem 4.3, we have

$$\begin{aligned} \triangle\tilde{x}(s) &\equiv \left.\frac{\mathrm{d}\tilde{x}(s;\epsilon)}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \\ &= \int_0^s \left\{ \frac{\partial \tilde{f}(\tau,\tilde{x}(\tau),k)}{\partial \tilde{x}} \triangle \tilde{x}(\tau) + \frac{\partial \tilde{f}(\tau,\tilde{x}(\tau),k)}{\partial k}\rho \right\} \mathrm{d}s \end{aligned} \tag{4.53}$$

and

$$\frac{\mathrm{d}(\triangle\tilde{x}(s))}{\mathrm{d}s} = \frac{\partial \tilde{f}(s,\tilde{x}(s),k)}{\partial \tilde{x}} \triangle \tilde{x}(s) + \frac{\partial \tilde{f}(s,\tilde{x}(s),k)}{\partial k}\rho \tag{4.54}$$

From (4.50), $\tilde{\Omega}(\tilde{x}(T|k))$ can be expressed as:

$$\begin{aligned} & \tilde{\Omega}(\tilde{x}(T|k)) \\ = \; & \tilde{\Omega}(\tilde{x}(T|k)) \\ & + \int_0^T \left\{ \tilde{H}_{\varepsilon,\gamma}(s,\tilde{x}(s;\epsilon),k(\epsilon),\tilde{\lambda}(s)) - \tilde{\lambda}^{\top}(s)\tilde{f}(s,\tilde{x}(s;\epsilon),k(\epsilon)) \right\} \mathrm{d}s \end{aligned} \tag{4.55}$$

where $\tilde{\lambda}(s)$ is yet arbitrary. Thus, it follows that

$$\begin{aligned} & \triangle\tilde{\Omega}(\tilde{x}(T|k)) \equiv \left.\frac{\mathrm{d}\tilde{\Omega}(\tilde{x}(T|k))}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = \frac{\partial \tilde{\Omega}(\tilde{x}(T|k))}{\partial k}\rho \\ = \; & \triangle\tilde{\Omega}(\tilde{x}(T|k)) \\ & + \int_0^T \left\{ \triangle\tilde{H}_{\varepsilon,\gamma}(s,\tilde{x}(s),k,\tilde{\lambda}(s)) - \tilde{\lambda}^{\top}(s) \triangle \tilde{f}(s,\tilde{x}(s),k) \right\} \mathrm{d}s \end{aligned} \tag{4.56}$$

where

$$\triangle\tilde{\Omega}(\tilde{x}(T|k)) = \frac{\mathrm{d}\tilde{\Omega}(\tilde{x}(T))}{\mathrm{d}\tilde{x}} \triangle \tilde{x}(T|k) \tag{4.57}$$

$$\triangle\tilde{f}(s,\tilde{x}(s),k) = \frac{\mathrm{d} \triangle \tilde{x}(s)}{\mathrm{d}s} \tag{4.58}$$

and

$$\begin{aligned} &\triangle\tilde{H}_{\varepsilon,\gamma}(s,\tilde{x}(s),k,\tilde{\lambda}(s)) \\ = &\left(\frac{\mathrm{d}\tilde{\lambda}(s)}{\mathrm{d}s}\right)^{\top} \triangle \tilde{x}(s) + \frac{\partial\tilde{H}_{\varepsilon,\gamma}(s,\tilde{x}(s),k,\tilde{\lambda}(s))}{\partial k}\rho \end{aligned} \tag{4.59}$$

Choose $\tilde{\lambda}$ to be the solution of the costate system (4.51a)-(4.51b) corresponding to k. Then, by substituting (4.57), (4.58), (4.59) into (4.56), we obtain

$$\begin{aligned} &\frac{\partial\tilde{\Omega}(\tilde{x}(T|k))}{\partial k}\rho \\ = &\frac{\mathrm{d}\tilde{\Omega}(\tilde{x}(T))}{\mathrm{d}\tilde{x}} \triangle \tilde{x}(T|k) \\ &+ \int_0^T \left\{ -\frac{\mathrm{d}}{\mathrm{d}s}\left[(\tilde{\lambda}(s))^{\top} \triangle \tilde{x}(s)\right] + \frac{\partial\tilde{H}_{\varepsilon,\gamma}(s,\tilde{x}(s),k,\tilde{\lambda}(s))}{\partial k}\rho \right\} \mathrm{d}s \\ = &\frac{\mathrm{d}\tilde{\Omega}(\tilde{x}(T))}{\mathrm{d}\tilde{x}} \triangle \tilde{x}(T|k) \\ &+(\tilde{\lambda}(0))^{\top} \triangle \tilde{x}(0) - (\tilde{\lambda}(T))^{\top} \triangle \tilde{x}(T) + \int_0^T \left\{ \frac{\partial\tilde{H}_{\varepsilon,\gamma}(s,\tilde{x}(s),k,\tilde{\lambda}(s))}{\partial k}\rho \right\} \mathrm{d}s \end{aligned} \tag{4.60}$$

Note that $\tilde{x}(0)$ is a constant number. Thus, it follows from (4.60) that

$$\begin{aligned} &\frac{\partial\tilde{\Omega}(\tilde{x}(T|k))}{\partial k}\rho \\ = &\int_0^T \left\{ \frac{\partial\tilde{H}_{\varepsilon,\gamma}(s,\tilde{x}(s),k,\tilde{\lambda}(s))}{\partial k}\rho \right\} \mathrm{d}s \end{aligned} \tag{4.61}$$

Since ρ is arbitrary, (4.49) follows readily from (4.61). Thus, the proof is completed. $\square$

For each $\varepsilon > 0$, $\gamma > 0$, Problem ($P4_{\varepsilon,\gamma}$) is to be solved as a nonlinear optimization problem using the gradient formulas given in Theorem 4.3 and Theorem 4.4. Details are reported in the following as an algorithm.

Algorithm 4.1.

1. Choose $\varepsilon > 0$, $\gamma > 0$ and k.
2. Solve Problem ($P4_{\varepsilon,\gamma}$) as a nonlinear optimization problem, yielding $k^*_{\varepsilon,\gamma}$.
3. Check whether all the continuous inequality constraints (4.6) are satisfied or not.

If they are satisfied, go to Step 4. Otherwise, increase γ to 10γ and go to Step 2 with $k^*_{\varepsilon,\gamma}$ as the initial guess for the new optimization process.

4. If ε is small enough, say, less or equal to a given small number, we have a successful exit. Else, decrease ε to $\varepsilon/10$ and go to Step 2, using $k^*_{\varepsilon,\gamma}$ as the initial guess for the new optimization process.

4.5 Application to A Ship Steering Control Problem

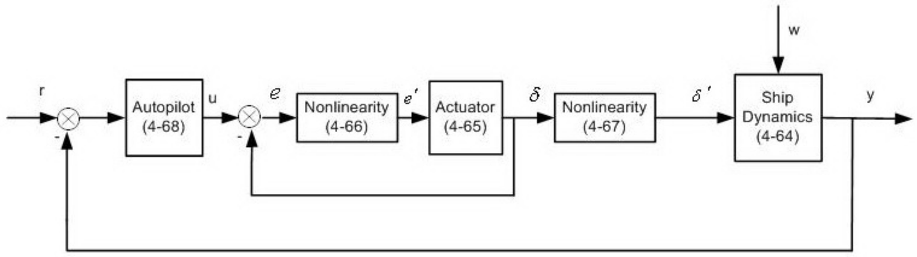

Figure 4.1: The overall control system of the ship

In this section, we apply the proposed method to a ship steering control problem. Our aim is to design a PID controller such that the heading angle, $y(t)$, of the ship will follow the change course set by the reference input signal $r(t)$. The control system is as shown in Figure 4.1. The ship motion can be described by the following differential equations defined on $[0, T]$ (see [91]). In this application, $T = 300$s.

$$\dddot{y}(t) + b_1\ddot{y}(t) + b_2((a_1\dot{y}(t))^3 + a_2\dot{y}(t)) = b_3\dot{\delta}'(t) + b_2\dot{\delta}'(t) + w, \tag{4.62}$$

where

$$\begin{aligned} w &= b_3\dot{d} + b_2 d, \\ \dot{\delta}(t) &= b_4 e'(t), \end{aligned} \tag{4.63}$$

with

$$e' = \begin{cases} e, & \text{if } |e| \le e_{\max} \\ e_{\max}\text{sign}(e), & \text{if } |e| \ge e_{\max} \end{cases} \tag{4.64}$$

where $e = u - \delta$,

$$\delta' = \begin{cases} \delta, & \text{if } |\delta| \le \delta_{\max} \\ \delta_{\max}\text{sign}(\delta), & \text{if } |\delta| \ge \delta_{\max} \end{cases} \tag{4.65}$$

The variable w is to account for sea disturbances acting on the ship with d a constant disturbance, u is the control which is chosen in the form of a PID controller defined by

(4.2), δ is the rudder angle, e is the error signal, e' and δ' are the real inputs to the actuator and ship dynamics, respectively, because of the saturation properties which are defined as (4.64) and (4.65). The ship model is in its full generality without resorting to simplification and linearization. This work develops further some pervious studies of optimal ship steering strategies with time optimal control [99], phase advanced control [92], parameter self-turning [95], adaptive control [90], and constrained optimal model following [98].

For a ship steering problem, it has two phases: course changing and course keeping. During the course changing phase, it is required to manoeuver the ship such that it moves quickly towards the desired course set by the command without violating the constraints arising from performance specifications and physical limitations on the controller. During the course keeping phase, the ship is required to move along the desired course. In this application, the PID controller of the form defined by (4.2) with $N_1 = N_2 = N_3 = 6$ is used. More specifically,

$$\begin{aligned} u(t) \;=\; & \sum_{i=1}^{6} k_{1,i}(y(t)-r(t))\chi_{[t_{i-1},t_i)}(t) \\ & + \sum_{i=1}^{6} k_{2,i}\int_0^t (y(s)-r(s))\chi_{[t_{i-1},t_i)}(s)\mathrm{d}s \\ & + \sum_{i=1}^{6} k_{3,i}\dot{y}(t)\chi_{[t_{i-1},t_i)}(t) \end{aligned} \tag{4.66}$$

where χ_I denotes the indicator function of I defined by (4.5), while t_i, $i = 1, \ldots, 5$, are fixed switching time points to be specified later.

Set

$$x_1(t) = y(t),\ x_2(t) = \dot{y}(t),\ x_3(t) = \ddot{y}(t),\ x_4(t) = \delta(t) \tag{4.67}$$

and

$$x_{5,j}(t) = \int_0^t (y(s)-r(s))\chi_{[t_{j-1},t_j)}(s)\mathrm{d}s,\ j = 1, \ldots, 6. \tag{4.68}$$

Then, the dynamics of the ship can be expressed as:

$$\begin{cases} \dot{x}_1(t) = x_2(t) & \text{(4.69a)} \\ \dot{x}_2(t) = x_3(t) & \text{(4.69b)} \\ \dot{x}_3(t) = -b_1x_3(t) - b_2(a_1x_2^3(t) + a_2x_2(t)) + b_3b_4e + b_2(x_4(t) + d) & \text{(4.69c)} \\ \dot{x}_4(t) = b_4e & \text{(4.69d)} \\ \dot{x}_{5,j}(t) = x_1(t) - r(t)\chi_{[t_{j-1},t_j)}(t),\ j = 1, \ldots, 6 & \text{(4.69e)} \end{cases}$$

Table 4.1: Coefficients for the ship model

a_1	a_2	b_1	b_2	b_3	b_4
-30.0	-5.6	0.1372	-0.0002014	-0.003737	0.5

with the initial condition

$$x(0) = [0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0]^\top, \tag{4.70}$$

where

$$e = u(t) - x_4(t) \tag{4.71}$$

with

$$\begin{aligned} u(t) &= \sum_{i=1}^{6} k_{1,i}(x_1(t) - r(t))\chi_{[t_{i-1},t_i)}(t) \\ &\quad + \sum_{i=1}^{6} k_{2,i}x_{5,j}(t) + \sum_{i=1}^{6} k_{3,i}x_2(t)\chi_{[t_{i-1},t_i)}(t). \end{aligned} \tag{4.72}$$

The values of the coefficients appeared in the equations are given in Table 4.1. The reference input signal $r(t)$ used in our example is: $r(t) = \pi/180$, for $t \in [0, 300\text{s}]$.

This ship steering problem is a special case of (4.1a)-(4.1d), where the output system is:

$$\dot{x}_1(t) = x_2(t)$$

with initial condition

$$x_1(0) = 0.$$

In practice, a large overshoot is undesirable. In this problem, the following constraint is imposed on the upper bound of the heading angle $x_1(t)$.

$$x_1(t) - 1.01r(t) \le 0,\ \ t \in [0, 300\text{s}] \tag{4.73}$$

i.e., the heading angle should not go beyond 1% of the desired reference input $r(t)$. This constraint can be written as:

$$g_1(t) = x_1(t) - 101\% r(t) \le 0,\ \ t \in [0, 300\text{s}] \tag{4.74}$$

We also impose constraint on the rise time of the heading angle such that the heading angle is constrained to reach at least 70% of the desired reference input in 30 seconds and

95% in 60 seconds, i.e.,

$$g_2(t) = h(t) - x_1(t) \leq 0,\ t \in [0, 300\text{s}], \tag{4.75}$$

where

$$h(t) = \begin{cases} 0, & t \in [0,6) \\ 5.1 \times 10^{-4}t - 3.1 \times 10^{-3}, & t \in [6,30) \\ 1.5 \times 10^{-4}t + 7.9 \times 10^{-3}, & t \in [30,60) \\ 2.2 \times 10^{-6}t + 16.4 \times 10^{-3}, & t \in [60,300] \end{cases} \tag{4.76}$$

To cater for the saturation property of the actuator, it is equivalent to impose upper and lower bounds on $x_4(t)$, i.e.,

$$-\pi/6 \leq x_4(t) \leq \pi/6,\ t \in [0, 300\text{s}] \tag{4.77}$$

which are continuous inequality constraints. They can be rewritten as:

$$g_3(t) = -x_4(t) - \pi/6 \leq 0,\ t \in [0, 300\text{s}] \tag{4.78}$$

and

$$g_4(t) = x_4(t) - \pi/6 \leq 0,\ t \in [0, 300\text{s}] \tag{4.79}$$

Similarly, to cater for another saturation property, we have

$$-\pi/30 \leq x_4(t) - u(t) \leq \pi/30,\ t \in [0, 300\text{s}] \tag{4.80}$$

They are again continuous inequality constraints, which can be rewritten as:

$$g_5(t) = -x_4(t) + u(t) - \pi/30 \leq 0,\ t \in [0, 300\text{s}] \tag{4.81}$$

and

$$g_6(t) = x_4(t) - u(t) - \pi/30 \leq 0,\ t \in [0, 300\text{s}] \tag{4.82}$$

where $u(t)$ is given by (4.72).

The terminal equality constraint is:

$$\begin{aligned} \Omega(x_1(300)) &= x_1(300) - r(300) \\ &= x_1(300) - \pi/180 = 0 \end{aligned} \tag{4.83}$$

The optimal PID control problem may now be stated formally as follows.

Given system (4.69a)-(4.70), find a PID control parameter vector $k = [(k^1)^\top, (k^2)^\top, \ldots, (k^6)^\top]^\top$

with $k^i = [k_1^i, k_2^i, k_3^i]^\top,\ i = 1, 2, \ldots, 6$, such that the cost function

$$J = \int_0^{300} \{\alpha_1(x_1(t) - r(t))^2 + \alpha_2 x_2^2(t) + \alpha_3 u^2(t)\}\mathrm{d}t \tag{4.84}$$

is minimized subject to the continuous inequality constraints (4.78), (4.79), (4.81), (4.82), (4.74) and the terminal condition (4.83), where

$$\begin{aligned} u(t) \quad = \quad & \sum_{i=1}^{6} k_{1,i}(x_1(t) - r(t))\chi_{[t_{i-1},t_i)}(t) + \sum_{i=1}^{6} k_{2,i} x_{5,j}(t) \\ & + \sum_{i=1}^{6} k_{3,i} x_2(t)\chi_{[t_{i-1},t_i)}(t). \end{aligned} \tag{4.85}$$

Here, $t_0 = 0$, $t_6 = 300$, and $t_i,\ i = 1, 2, \ldots, 5$, are the switching time points which are chosen to be at the time points where the constraint function g_2 are nondifferentiable. They are: $t_1 = 6$, $t_2 = 18$, $t_3 = 30$, $t_4 = 45$, and $t_5 = 60$.

Let this problem be referred to as Problem (Q4), and it is solvable by the computational method developed in Section **??**.

We then construct Problem ($\mathrm{Q4}_{\varepsilon,\gamma}$) according to the procedure as specified in Section 4.3, where the appended new cost function is given by

$$\begin{aligned} \overline{J}_{\varepsilon,\gamma}(k) = \int_0^{300} & \left\{\alpha_1(x_1(t) - r(t))^2 + \alpha_2 x_2^2(t)\right. \\ & \left. + \alpha_3 u^2(t) + \gamma(g_{1,\varepsilon} + g_{2,\varepsilon} + g_{3,\varepsilon} + g_{4,\varepsilon} + g_{5,\varepsilon} + g_{6,\varepsilon})\right\} \mathrm{d}t, \end{aligned} \tag{4.86}$$

where $g_{i,\varepsilon},\ i = 1, 2, \ldots, 6$, are obtained from $g_i,\ i = 1, 2, \ldots, 6$, respectively, according to (4.22). In this problem, we set $\alpha_1 = 10,\ \alpha_2 = 400,\ \alpha_3 = 0.05$. It is to be minimized subject to terminal equality constraint (4.83).

In real world, disturbances always exist and there are many kinds of disturbances. We consider the case, where the ship is encountered with a constant disturbance d. Assume that $d = 0.3\pi/180$.

Problem ($\mathrm{Q4}_{\varepsilon,\gamma}$) is solved by using Algorithm 4.1, where the final ε and γ are $\varepsilon = 0.01$ and $\gamma = 10$. The optimal parameters for the PID controller u obtained are:

$$\begin{aligned} k^{1,*} &= [5.78685,\ 7.27203,\ 2.62351,\ 6.98467,\ 9.76934,\ 7.39799]^\top \\ k^{2,*} &= [1.03217,\ 0.00000,\ 0.00303,\ 0.00000,\ 0.24398,\ 0.84387]^\top \\ k^{3,*} &= [99.81791,\ 100.52777,\ 100.67912,\ 100.13533,\ 99.75020,\ 99.70358]^\top \end{aligned}$$

The results obtained are shown in Figure 4.2 to Figure 4.5. From the results obtained,

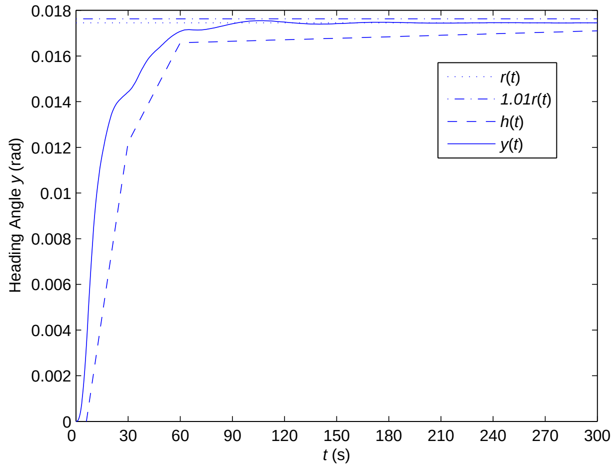

Figure 4.2: The heading angle of the ship

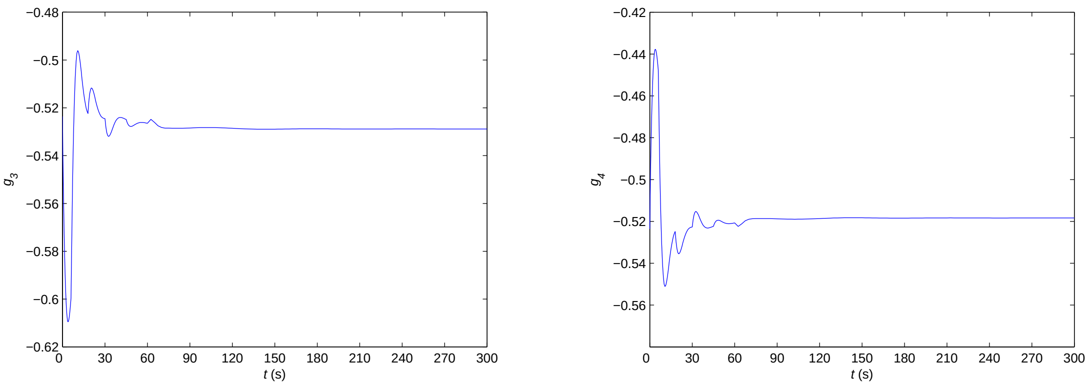

Figure 4.3: The constraints for the saturation of the actuator

we see that all the constraints are satisfied. The heading angle tracks the desired reference input with no steady state error after some small oscillation due to the constant disturbance. The overshooting of the heading angle above the reference input is less than 1%, and hence the constraint $g_1(t) \leq 0,\ t \in [0, 300]$ is satisfied. To test the robustness of this PID controller, we run the model with the optimal PID controller under the following environments. (i) The disturbance is much larger, more specifically, $d = 0.6 \times \pi/180$; and (ii) the disturbance is coming from the initial heading direction, more specifically, $d = -0.3 \times \pi/180$. The results are shown in Figure 4.6 and Figure 4.7. In both cases, we see that the heading angles track the desired reference input with no steady state error after some small oscillations.

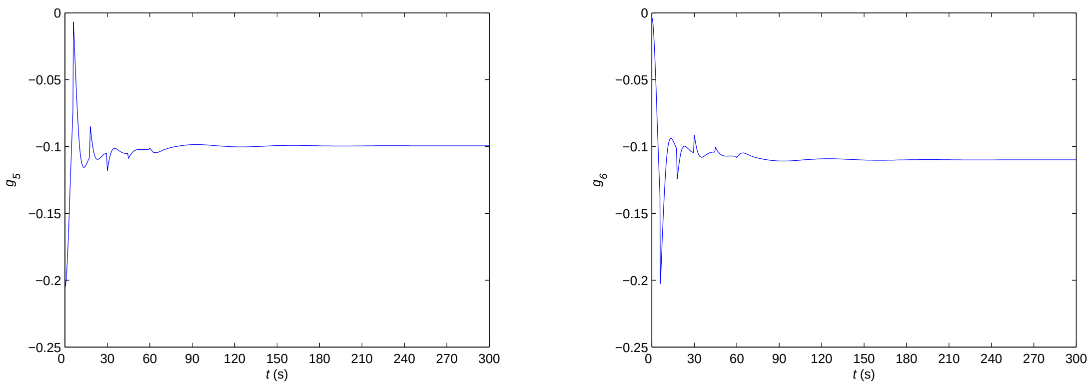

Figure 4.4: The constraints for the saturation of the control

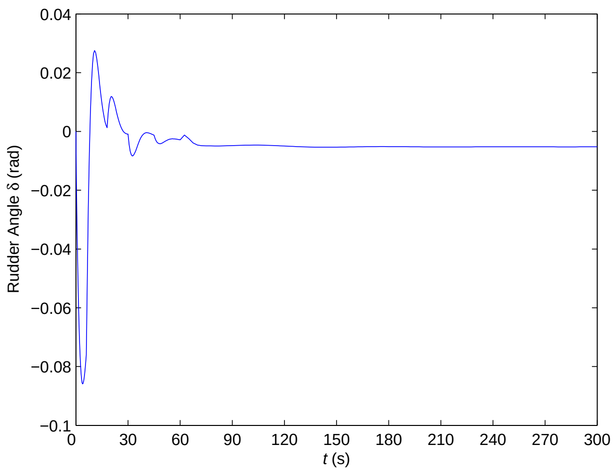

Figure 4.5: The rudder angle of the ship

4.6 Conclusions

This chapter considered an optimal PID tuning problem subject to continuous inequality constraints and terminal state equality constraint. It was shown that the problem can be solved via solving a sequence of nonlinear optimization problems. An efficient computational method was proposed. It was then applied to a ship steering control problem. The results obtained show that the method proposed is reliable and effective.

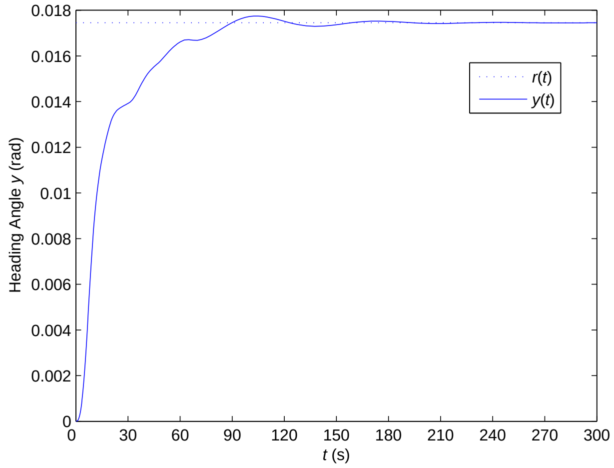

Figure 4.6: The heading angle of the ship with a larger disturbance

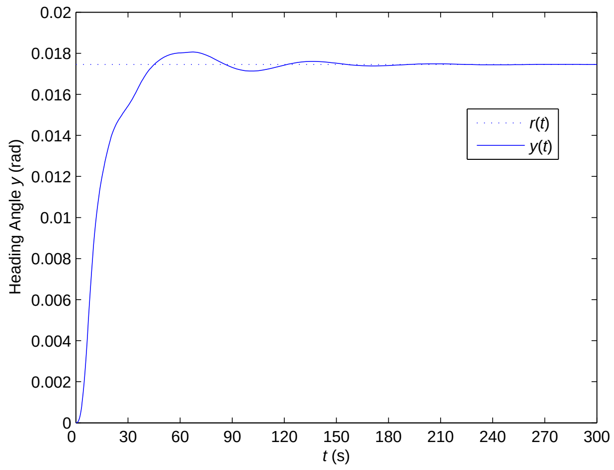

Figure 4.7: The heading angle of the ship with a disturbance coming from the initial heading direction

CHAPTER 5

An Exact Penalty Function Method for Continuous Inequality Constrained Optimal Control Problem

5.1 Introduction

This chapter is the author's work in [115]. In this chapter, we present a computational approach based on an exact penalty function method [104] for solving a class of optimal control problems subject to equality terminal state constraints and continuous inequality constraints on the state and/or control variables. After the control parametrization together with a time scaling transformation, the problem is approximated by a sequence of optimal parameter selection problems with equality terminal state constraints and continuous inequality constraints on the state and/or control. The new exact penalty functions, developed in [?] and [104] (there are several other similar works in the literature, such as [116] and [117]), are constructed for these terminal equality constraints and continuous inequality constraints. They are appended to the cost function to form a new cost function, giving rise to an unconstrained optimal parameter selection problem. The convergence analysis shows that, for a sufficiently large penalty parameter, a local minimizer of the unconstrained optimization problem is a local minimizer of the optimal parameter selection problem with terminal equality constraints and continuous inequality constraints. The relationships between the approximate optimal parameter selection problems and the original optimal control problem are also discussed. Finally, the method proposed is applied to solve three nontrivial optimal control problems.

5.2 Problem Statement

Consider a dynamical system defined on $[0, T]$ given below

$$\dot{x}(t) = f(t, x(t), u(t)),\ t \in (0, T] \tag{5.1a}$$

with initial and terminal conditions

$$x(0) = x^0 \tag{5.1b}$$

$$x(T) = x^f \tag{5.1c}$$

respectively, where T is the terminal time and $x = [x_1, x_2, \ldots, x_n]^\top \in \mathbb{R}^n$ and $u = [u_1, u_2, \ldots u_r]^\top \in \mathbb{R}^r$ are, respectively, state and control vectors, while $f = [f_1, f_2, \ldots, f_n]^\top \in \mathbb{R}^n$.

We assume that the following assumptions are satisfied.

Assumption 5.1. f is continuously differentiable with respect to all it arguments.

Assumption 5.2. Let $\mathbb{V}$ be a compact subset of $\mathbb{R}^r$. We require that there exists a constant K such that

$$\|f(t, x, u)\| \leq K\left(1 + \|x\|\right)$$

for all $(t, x, u) \in [0, \infty) \times \mathbb{R}^n \times \mathbb{V}$.

Define

$$U = \left\{\nu = [v_1, v_2, \ldots, v_r]^\top \in \mathbb{R}^r : \alpha_i \leq v_i \leq \beta_i,\ i = 1, 2, \ldots, r\right\} \tag{5.2}$$

where α_i, $i = 1, 2, \ldots, r$, and $\beta_i, i = 1, 2, \ldots, r$, are given real numbers. A piecewise continuous function u is said to be an admissible control if $u(t) \in U$ for all $t \in [0, T]$. Let

$\mathcal{U}$ be the class of all such admissible controls. Furthermore, let $x(\cdot|u)$ denote the solution of system (5.1a)-(5.1b) corresponding to $u \in \mathcal{U}$.

Consider the continuous state inequality constraints, given by

$$g_i\left(t, x\left(T|u\right), u(t)\right) \leq 0,\ t \in [0, T], i = 1, 2, \ldots, N \tag{5.3}$$

It is assumed that the following assumption is satisfied.

Assumption 5.3. g_i, $i = 1, 2, \ldots, N$, are continuously differentiable with respect to all its arguments.

Define

$$\mathcal{G} = \left\{u \in \mathcal{U} : g_i\left(t, x\left(T|u\right), u(t)\right) \leq 0,\ t \in [0, T], i = 1, 2, \ldots, N\right\} \tag{5.4}$$

and

$$\mathcal{H} = \left\{u \in \mathcal{G} : x\left(T|u\right) = x^f\right\} \tag{5.5}$$

Now we state our problem as follows.

Problem (P5) Given the dynamical system (5.1a)-(5.1b), find a control $u \in \mathcal{H}$ such that the cost function

$$J(u) = \Phi_0\left(x(T)\right) + \int_0^T \mathcal{L}_0\left(t, x(t), u(t)\right) \mathrm{d}t \tag{5.6}$$

is minimized.

We assume that the following assumptions are satisfied.

Assumption 5.4. Φ_0 is continuously differentiable with respect to x.

Assumption 5.5. $\mathcal{L}_0$ is continuously differentiable with respect to all its arguments.

Remark 5.1. By Assumption 5.1 and the definition of $\mathcal{U}$, it follows from an argument similar to that given for the proof of Lemma 6.4.2 in [36] that there exits a compact subset $X \subset \mathbb{R}^n$ such that $x\left(T|u\right) \in X$ for all $t \in [0, T]$ and for all $u \in \mathcal{U}$.

5.3 Computational Method

To solve Problem (P5), as in Chapter 3, we shall apply the control parametrization scheme [36] together with a time scaling transform [87]. The time horizon $[0, T]$ is partitioned with a sequence $\tau = [\tau_0, \tau_1, \ldots, \tau_p]^\top$ of time points $\tau_i,\ i = 0, 1, \ldots, p$. Then, the control is approximated by a piecewise constant function as follows.

$$u^p(t|\sigma, \tau) = \sum_{i=1}^{p} \sigma^i \chi_{[\tau_{i-1}, \tau_i)}(t), \tag{5.7}$$

where $\tau_{i-1} \leq \tau_i,\ i = 1, \ldots, p$, with $\tau_0 = 0$ and $\tau_p = T$, and

$$\chi_I(t) = \begin{cases} 1, & \text{if } t \in I, \\ 0, & \text{otherwise.} \end{cases}$$

As $u^p \in \mathcal{U}$, $\sigma^i = [\sigma_1^i, \sigma_2^i, \ldots, \sigma_r^i]^\top \in U$ for $i = 1, 2, \ldots, p$. Denote by Ξ the set of all such $\sigma = \left[(\sigma^1)^\top, (\sigma^2)^\top, \ldots, (\sigma^p)^\top\right]^\top \in \mathbb{R}^{pr}$, and denote by Γ the set of all $\tau = [\tau_0, \tau_1, \ldots, \tau_p]^\top$ such that $\tau_{i-1} \leq \tau_i,\ i = 1, 2, \ldots, p$, with $\tau_0 = 0$ and $\tau_p = T$.

Let $x^p(\cdot|\sigma, \tau)$ denote the solution of (5.1a)-(5.1b) corresponding to $(\sigma, \tau) \in \Xi \times \Gamma$, where

$$x^p(\cdot|\sigma, \tau) = x(\cdot|u^p(t|\sigma, \tau))$$

Substituting (5.7) into the continuous inequality constraints (5.3) gives

$$g_i\left(t, x^p\left(t|\sigma,\tau\right), u^p\left(t|\sigma,\tau\right)\right) \leq 0,\ t \in [0,T],\ i = 1,2,\ldots,N \tag{5.8}$$

Let Λ be the set containing all those $(\sigma,\tau) \in \Xi \times \Gamma$ such that the constraints (5.8) are satisfied. Furthermore, let

$$\Upsilon = \left\{(\sigma,\tau) \in \Lambda : x^p\left(T|u^p\left(t|\sigma,\tau\right)\right) = x^f\right\} \tag{5.9}$$

The cost function becomes

$$J^p(\sigma,\tau) = J\left(u^p(t|\sigma,\tau)\right) = \Phi_0\left(x^p(T|\sigma,\tau)\right) + \int_0^T \mathcal{L}_0\left(t, x^p(t|\sigma,\tau), u^p(t|\sigma,\tau)\right) \mathrm{d}t \tag{5.10}$$

We may now define the following approximate optimization problem.

Problem (P5(p)) Given system (5.1a)-(5.1b), find a $(\sigma,\tau) \in \Upsilon$ such that the cost function (5.10) is minimized.

In Problem (P5(p)), the switching times $\tau_i,\ 1 \leq i \leq p-1$, are also regarded as decision variables. We shall employ the time scaling transform introduced in [87] to map these switching times into a set of fixed time points $\frac{k}{p}$, $k = 1,2,\ldots,p-1$, on a new time horizon $[0,1]$. This is achieved by the following differential equation

$$\dot{t}(s) = \upsilon^p(s),\ s \in [0,1], \tag{5.11a}$$

with initial condition

$$t(0) = 0, \tag{5.11b}$$

where

$$\upsilon^p(s) = \sum_{i=1}^{p} \theta_i \chi_{\left[\frac{i-1}{p},\frac{i}{p}\right)}(s). \tag{5.12}$$

Here, $\theta_i \geq 0,\ i = 1,2,\ldots,p$.

Let $\theta = [\theta_1,\theta_2,\ldots,\theta_p]^\top$ and let Θ be the set containing all such θ.

Taking integration of (5.11a) with initial condition (5.11b), it is easy to see that, for $s \in \left[\frac{k-1}{p},\frac{k}{p}\right),\ k = 1,2,\ldots,p$,

$$t(s) = \sum_{i=1}^{k-1} \frac{\theta_k}{p} + \frac{\theta_k}{p}\left(ps - k + 1\right), \tag{5.13}$$

where $k = 1, 2, \ldots, p$. Clearly, for each $k = 1, 2, \ldots, p-1$

$$\tau_k = \sum_{i=1}^{k} \frac{\theta_i}{p} \tag{5.14}$$

and

$$t(1) = \sum_{i=1}^{p} \frac{\theta_i}{p} = T. \tag{5.15}$$

Let $\tilde{\Theta}$ be a subset of Θ such that (5.15) is satisfied.

The approximate control given by (5.7) in the new time horizon $[0, 1]$ becomes

$$\tilde{u}^p(s) = u^p\left(t(s)\right) = \sum_{i=1}^{p} \sigma^i \chi_{\left[\frac{i-1}{p}, \frac{i}{p}\right)}(s). \tag{5.16}$$

which has fixed switching times at $s = \frac{1}{p}, \ldots, \frac{p-1}{p}$. Now, by using the time scaling transform (5.11a)-(5.11b), the dynamic system (5.1a)-(5.1b) is transformed into

$$\dot{y}(s) = \theta_k f\left(t(s), y(s), \sigma^k\right), \; s \in \mathcal{J}_k, \; k = 1, 2, \ldots, p \tag{5.17a}$$

$$\dot{t}(s) = v^p(s) \tag{5.17b}$$

$$y(0) = x^0 \text{ and } t(0) = 0 \tag{5.17c}$$

and the terminal conditions (5.1c) and (5.15) become

$$y(1) = x^f \text{ and } t(1) = T \tag{5.17d}$$

respectively, where $y(s) = x\left(t(s)\right)$ and

$$\mathcal{J}_k = \begin{cases} \left[\frac{k-1}{p}, \frac{k}{p}\right), & \text{if } k = 1 \\ \left(\frac{k-1}{p}, \frac{k}{p}\right) & \text{if } k \in \{2, 3, \ldots, p-1\} \\ \left(\frac{k-1}{p}, \frac{k}{p}\right], & \text{if } k = p \end{cases}$$

We then rewrite system (5.17a)-(5.17c) as follows.

$$\dot{\tilde{y}}(s) = \tilde{f}\left(s, \tilde{y}(s), \sigma, \theta\right), \; s \in [0, 1] \tag{5.18a}$$

$$\tilde{y}(0) = \tilde{y}^0 \tag{5.18b}$$

with the terminal conditions

$$\tilde{y}(1) = \tilde{y}^f \tag{5.18c}$$

where

$$\tilde{y}(s) = \left[(y(s))^\top,\ t(s)\right]^\top \tag{5.19}$$

$$\tilde{f}(s, \tilde{y}(s), \sigma, \theta) = \begin{bmatrix} \sum_{k=1}^{p} \theta_k f\left(t(s), y(s), \sigma^k\right) \chi_{\mathcal{J}_k}(s) \\ \upsilon^p(s) \end{bmatrix} \tag{5.20}$$

$$\tilde{y}^0 = \begin{bmatrix} x^0 \\ 0 \end{bmatrix} \tag{5.21}$$

$$\tilde{y}^f = \begin{bmatrix} x^f \\ T \end{bmatrix} \tag{5.22}$$

To proceed further, let $\tilde{y}(\cdot|\sigma, \theta)$ denote the solution of system (5.18a)-(5.18b) corresponding to $(\sigma, \theta) \in \Xi \times \Theta$.

Remark 5.2. As in Remark 5.1, there exists a compact subset $Y \subset \mathbb{R}^{n+1}$ such that $\tilde{y}(s|\sigma, \theta) \in Y$ for all $s \in [0, 1]$ and $(\sigma, \theta) \in \Xi \times \tilde{\Theta}$.

Similarly, applying the time scaling transform to the continuous inequality constraints (5.3) and the cost function (5.6) yields:

$$g_i\left(t(s|\theta), \tilde{y}(s|\sigma, \theta), \sigma^k\right) \leq 0,\ s \in \mathcal{J}_k,\ k = 1, 2, \ldots, p,\ i = 1, 2, \ldots, N \tag{5.23}$$

and

$$\tilde{J}(\sigma, \theta) = \Phi_0\left(y\left(1|\sigma, \theta\right)\right) + \int_0^1 \bar{\mathcal{L}}_0\left(s, \tilde{y}\left(s|\sigma, \theta\right), \sigma, \theta\right) \mathrm{d}s \tag{5.24}$$

respectively, where

$$\bar{\mathcal{L}}_0\left(s, \tilde{y}\left(s|\sigma, \theta\right), \sigma, \theta\right) = \upsilon^p(s)\mathcal{L}_0\left(t\left(s\right), x\left(t(s)\right), \tilde{u}^p\left(s\right)\right) \tag{5.25}$$

Remark 5.3. By respective assumptions specified in Assumption 5.1, Assumption 5.2 and Assumption 5.5, it follows from Remark 5.2 that there exits constants $K_1 > 0$ and $K_2 > 0$ such that

$$\|\Upsilon(s, \tilde{y}(s|\sigma, \theta), \sigma, \theta)\| \leq K_1,\ s \in \mathcal{J}_k,\ k = 1, 2, \ldots, p;\ (\sigma, \theta) \in \Xi \times \tilde{\Theta}$$

$$\left\|\frac{\partial \Upsilon(s, \tilde{y}(s|\sigma, \theta), \sigma, \theta)}{\partial \sigma}\right\| \leq K_2,\ s \in \mathcal{J}_k,\ k = 1, 2, \ldots, p;\ (\sigma, \theta) \in \Xi \times \tilde{\Theta}$$

$$\left\|\frac{\partial \Upsilon(s, \tilde{y}(s|\sigma, \theta), \sigma, \theta)}{\partial \theta}\right\| \leq K_2,\ s \in \mathcal{J}_k,\ k = 1, 2, \ldots, p;\ (\sigma, \theta) \in \Xi \times \tilde{\Theta}$$

$$\left\|\frac{\partial \Upsilon(s, \tilde{y}(s|\sigma, \theta), \sigma, \theta)}{\partial \tilde{y}}\right\| \leq K_2,\ s \in \mathcal{J}_k,\ k = 1, 2, \ldots, p;\ (\sigma, \theta) \in \Xi \times \tilde{\Theta}$$

where Υ is used to denote $\tilde{f}_i$, $i = 1, 2, \ldots, n$, $g_i\left(t(s), \tilde{y}(s|\sigma, \theta), \sigma^k\right)$, $i = 1, 2, \ldots, N$; $k =$

$1, 2, \ldots, p$, or $\bar{\mathcal{L}}_0$.

Problem (P5(p)) is equivalent to the following problem.

Problem ($\tilde{\mathrm{P}}$5(p)) Given system (5.18a)-(5.18b), find a $(\sigma, \theta) \in \Xi \times \Theta$ such that the cost function (5.24) is minimized subject to (5.18c) and (5.23).

Problem ($\tilde{\mathrm{P}}$5(p)) is an optimization problem subject to both the equality constraints (5.18c) and the continuous inequality constraints (5.23). To solve this problem, an exact penalty function method introduced in [**?**] and [104] is used.

First, we define

$$\mathcal{F}_\epsilon = \{(\sigma, \theta, \epsilon) \in \Xi \times \Theta \times \mathbb{R}_+ : g_i\left(t\left(s|\theta\right), y\left(s|\sigma, \theta\right), \sigma^k\right) \leq \epsilon^\gamma W_i, \quad \forall\ s \in \mathcal{J}_k,\ k = 1, 2, \ldots, p,\ i = 1, 2, \ldots, N\} \tag{5.26}$$

where $\mathbb{R}_+ = \{\alpha \in \mathbb{R}\ :\ \alpha \geq 0\}$, $W_i \in (0, 1)$, $i = 1, 2,\ \ldots\ ,\ N$, are fixed constants and γ is a positive real number. In particular, when $\epsilon = 0$, let

$$\mathcal{F}_0 = \left\{(\sigma, \theta) \in \Xi \times \Theta : g_i\left(t\left(s|\theta\right), y\left(s|\sigma, \theta\right), \sigma^k\right) \leq 0,\ \forall\ s \in \mathcal{J}_k,\ k = 1, 2, \ldots, p,\ i = 1, 2, \ldots, N\right\} \tag{5.27}$$

Similarly, we define

$$\Omega_\epsilon = \left\{(\sigma, \theta, \epsilon) \in \mathcal{F}_\epsilon : \tilde{y}\left(1|\sigma, \theta\right) - \tilde{y}^f = 0\right\} \tag{5.28}$$

and

$$\Omega_0 = \left\{(\sigma, \theta) \in \mathcal{F}_0 : \tilde{y}(1|\sigma, \theta) - \tilde{y}^f = 0\right\}$$

Clearly, Problem ($\tilde{\mathrm{P}}$5(p)) is equivalent to the following problem.

Problem ($\hat{\mathrm{P}}$5(p)) Given system (5.18a)-(5.18b), find a $(\sigma, \theta) \in \Omega_0$ such that the cost function (5.24) is minimized.

Then, by applying a new exact penalty functions introduced in [**?**] and [104], we obtain a new cost function defined below.

$$\tilde{J}_\delta(\sigma, \theta, \epsilon) = \begin{cases} \tilde{J}(\sigma, \theta) & \text{if } \epsilon = 0,\ g_i\left(t(s|\theta), \tilde{y}(s|\theta, \sigma), \sigma^k\right) \leq 0 \\ & \ (s \in \mathcal{J}_k, k = 1, 2, \ldots, p) \\ \tilde{J}(\sigma, \theta) + \epsilon^{-\alpha}\left(\Delta(\sigma, \theta, \epsilon) + \Delta_1\right) + \delta\epsilon^\beta & \text{if } \epsilon > 0 \\ +\infty & \text{otherwise} \end{cases} \tag{5.29}$$

Here, $\delta > 0$ is a penalty parameter, $\Delta(\sigma, \theta, \epsilon)$, which is referred to as the continuous inequality constraint violation, is defined by

$$\Delta(\sigma, \theta, \epsilon) = \sum_{i=1}^{N} \int_0^1 \left[\max\left\{0, \bar{g}_i(s, \tilde{y}(s|\sigma, \theta), \sigma) - \epsilon^\gamma W_i\right\}\right]^2 \mathrm{d}s \tag{5.30}$$

where

$$\begin{aligned}\bar{g}_i(s,\tilde{y}(s),\sigma) &= g_i\left(t(s),y(s),\tilde{u}^p(s)\right)\\ &= \sum_{k=1}^{p} g_i\left(t(s),y(s),\sigma^k\right)\chi_{\mathcal{J}_k}(s)\end{aligned} \tag{5.31}$$

and $\tilde{u}^p$ is defined by (5.16). Furthermore, Δ_1, which is referred to as the equality constraint violation, is defined by

$$\begin{aligned}\Delta_1 &= \left\|\tilde{y}(1|\sigma,\theta)-\tilde{y}^f\right\|^2\\ &= \sum_{i=1}^{n+1}\left(\tilde{y}_i\left(1|\sigma,\theta\right)-\tilde{y}_i^f\right)^2\end{aligned} \tag{5.32}$$

where α and γ are positive real numbers, and $\beta > 2$.

Remark 5.4. Note that other types of equality constraints, such as interior point constraints (see [36]) can be dealt with similarly by introducing appropriate equality constraint violation as defined by (5.32).

We now introduce a surrogate optimal parameter selection problem, which is referred to as **Problem (P**5_δ**(p))**, as follows.

Given system (5.18a)-(5.18b), find a $(\sigma,\theta,\epsilon)\in\Xi\times\Theta\times[0,+\infty)$ such that the cost function (5.29) is minimized.

Intuitively, during the process of minimizing $\tilde{J}_\delta(\sigma,\theta,\epsilon)$, if σ is increased, ϵ^β should be reduced. This means that ϵ should be reduced as β is fixed. Thus $\epsilon^{-\alpha}$ will be increased, and hence the constraint violation will be reduced. This means that the values of

$$\sum_{i=1}^{N}\int_0^1\left[\max\left\{0,\bar{g}_i\left(s,\tilde{y}\left(s|\sigma,\theta\right),\sigma\right)-\epsilon^\gamma W_i\right\}\right]^2\mathrm{d}s$$

and

$$\sum_{i=1}^{n+1}\left(\tilde{y}_i(1|\sigma,\theta)-\tilde{y}_i^f\right)^2$$

must go down. In this way, the satisfaction of the continuous inequality constraints (5.23) and the equality constraints (5.17d) will eventually be achieved.

Before presenting the gradient formulas of the cost function of **Problem (P**5_δ**(p))**, we will rewrite the cost function in the canonical form as in [36] below.

$$\begin{aligned}
\tilde{J}_\delta(\sigma,\theta,\epsilon) &= \Phi_0\left(y(1|\sigma,\theta)\right) + \int_0^1 \bar{\mathcal{L}}_0(s,\tilde{y}(s|\sigma,\theta),\sigma,\theta)\mathrm{d}s \\
&\quad + \epsilon^{-\alpha}\left\{\sum_{i=1}^{N}\int_0^1 \left[\max\{0,\bar{g}_i(s,\tilde{y}(s|\sigma,\theta),\sigma)-\epsilon^\gamma W_i\}\right]^2\mathrm{d}s\right. \\
&\quad \left. + \sum_{i=1}^{n+1}\left(\tilde{y}_i(1|\sigma,\theta)-\tilde{y}_i^f\right)^2\right\} + \delta\epsilon^\beta \\
&= \left(\Phi_0(\tilde{y}(1|\sigma,\theta)) + \epsilon^{-\alpha}\sum_{i=1}^{n+1}\left(\tilde{y}_i(1|\sigma,\theta)-\tilde{y}_i^f\right)^2 + \delta\epsilon^\beta\right) \\
&\quad + \int_0^1 \bar{\mathcal{L}}_0(s,\tilde{y}(s|\sigma,\theta),\sigma,\theta)\mathrm{d}s \\
&\quad + \epsilon^{-\alpha}\sum_{i=1}^{N}\int_0^1 \left[\max\left\{0,\bar{g}_i(s,\tilde{y}(s|\sigma,\theta),\sigma)-\epsilon^\gamma W_i\right\}\right]^2\mathrm{d}s
\end{aligned} \tag{5.33}$$

Let

$$\tilde{\Phi}_0\left(\tilde{y}(1|\sigma,\theta),\epsilon\right) = \Phi_0\left(y(1|\sigma,\theta)\right) + \epsilon^{-\alpha}\sum_{i=1}^{n+1}\left(\tilde{y}_i(1|\sigma,\theta)-\tilde{y}_i^f\right)^2 + \delta\epsilon^\beta \tag{5.34}$$

and

$$\begin{aligned}
&\tilde{\mathcal{L}}_0(s,\tilde{y}(s|\sigma,\theta),\sigma,\theta,\epsilon) \\
=\ &\bar{\mathcal{L}}_0(s,\tilde{y}(s|\sigma,\theta),\sigma,\theta) \\
&+\epsilon^{-\alpha}\sum_{i=1}^{N}\left[\max\left\{0,\bar{g}_i(s,\tilde{y}(s|\sigma,\theta),\sigma)-\epsilon^\gamma W_i\right\}\right]^2
\end{aligned} \tag{5.35}$$

We then substitute (5.34) and (5.35) into (5.33) to give

$$\tilde{J}_\delta(\sigma,\theta,\epsilon) = \tilde{\Phi}_0\left(\tilde{y}(1|\sigma,\theta),\epsilon\right) + \int_0^1 \tilde{\mathcal{L}}_0\left(s,\tilde{y}(s|\sigma,\theta),\sigma,\theta,\epsilon\right)\mathrm{d}s \tag{5.36}$$

Now, the cost function of **Problem (P5_δ(p))** is in canonical form. As derived for the proof of Theorem 5.2.1 in [36], the gradient formulas of the cost function (5.36) are given in the following theorem.

Theorem 5.1. *The gradients of the cost function $\tilde{J}_\delta(\sigma,\theta,\epsilon)$ with respect to σ, θ, and ϵ are:*

$$\frac{\partial\tilde{J}_\delta(\sigma,\theta,\epsilon)}{\partial\sigma} = \int_0^1 \frac{\partial H_0\left(s,\tilde{y}\left(s|\sigma,\theta\right),\sigma,\theta,\epsilon,\lambda_0\left(s|\sigma,\theta,\epsilon\right)\right)}{\partial\sigma}\mathrm{d}s \tag{5.37}$$

$$\frac{\partial \tilde{J}_\delta(\sigma,\theta,\epsilon)}{\partial\theta} = \int_0^1 \frac{\partial H_0(s,\tilde{y}(s|\sigma,\theta),\sigma,\theta,\epsilon,\lambda_0(s|\sigma,\theta,\epsilon))}{\partial\theta}\mathrm{d}s \tag{5.38}$$

$$\begin{aligned}
&\frac{\partial \tilde{J}_\delta(\sigma,\theta,\epsilon)}{\partial\epsilon}\\
=\;& -\alpha\epsilon^{-\alpha-1}\left\{\sum_{i=1}^{N}\int_0^1 \left[\max\left\{0,\bar{g}_i(s,\tilde{y}(s|\sigma,\theta),\sigma)-\epsilon^\gamma W_i\right\}\right]^2\mathrm{d}s\right.\\
&\left.+\sum_{i=1}^{n+1}\left(\tilde{y}_i(1|\sigma,\theta)-\tilde{y}_i^f\right)^2\right\}\\
&-2\gamma\epsilon^{\gamma-\alpha-1}\sum_{i=1}^{N}\int_0^1 \max\left\{0,\bar{g}_i(s,\tilde{y}(s|\sigma,\theta),\sigma)-\epsilon^\gamma W_i\right\}W_i\mathrm{d}s\\
&+\delta\beta\epsilon^{\beta-1}\\
=\;& \epsilon^{-\alpha-1}\left\{-\alpha\sum_{i=1}^{N}\int_0^1 \left[\max\left\{0,\bar{g}_i(s,\tilde{y}(s|\sigma,\theta),\sigma)-\epsilon^\gamma W_i\right\}\right]^2 ds\right.\\
&+2\gamma\sum_{i=1}^{N}\int_0^1 \max\left\{0,\bar{g}_i(s,\tilde{y}(s|\sigma,\theta),\sigma)-\epsilon^\gamma W_i\right\}(-\epsilon^\gamma W_i)\mathrm{d}s\\
&\left.-\alpha\sum_{i=1}^{n+1}\left(\tilde{y}_i(1|\sigma,\theta)-\tilde{y}_i^f\right)^2\right\}+\delta\beta\epsilon^{\beta-1},
\end{aligned} \tag{5.39}$$

respectively, where $H_0(s,\tilde{y}(s|\sigma,\theta),\sigma,\theta,\epsilon,\lambda(s|\sigma,\theta,\epsilon))$ *is the Hamiltonian function for the cost function (5.24) given by*

$$\begin{aligned}
&H_0(s,\tilde{y}(s|\sigma,\theta),\sigma,\theta,\epsilon,\lambda(s|\sigma,\theta,\epsilon))\\
=\;&\tilde{\mathcal{L}}_0(s,\tilde{y}(s|\sigma,\theta),\sigma,\theta,\epsilon)+\lambda_0(s|\sigma,\theta,\epsilon)\tilde{f}(s,\tilde{y}(s|\sigma,\theta),\sigma,\theta),
\end{aligned} \tag{5.40}$$

and $\lambda_0(\cdot|\sigma,\theta,\epsilon)$ *is the solution of the following system of co-state differential equations*

$$\frac{(\mathrm{d}\lambda_0(s))^\top}{\mathrm{d}s} = -\frac{\partial H_0(s,\tilde{y}(s|\sigma,\theta),\sigma,\theta,\epsilon,\lambda_0(s))}{\partial\tilde{y}} \tag{5.41a}$$

with the boundary condition

$$(\lambda_0(1))^\top = \frac{\partial\tilde{\Phi}_0(\tilde{y}(1|\sigma,\theta),\epsilon)}{\partial\tilde{y}}. \tag{5.41b}$$

Proof. Let $\sigma\in\mathbb{R}^r$ be given and let $\rho\in\mathbb{R}^r$ be arbitrary but fixed. Define

$$\sigma(\epsilon) = \sigma + \varsigma\rho \tag{5.42}$$

where $\epsilon > 0$ is an arbitrarily small real number. For brevity, let $\tilde{y}(\cdot)$ and $\tilde{y}(\cdot;\varsigma)$ denote, respectively, the solution of the system (5.18a)-(5.18a) corresponding to σ and $\sigma(\varsigma)$ while θ and ϵ are fixed. Clearly,

$$\tilde{y}(s) = \tilde{y}(0) + \int_0^s \tilde{f}(\tau, \tilde{y}(\tau), \sigma, \theta) d\tau \tag{5.43}$$

$$\tilde{y}(s;\varsigma) = \tilde{y}(0) + \int_0^s \tilde{f}(\tau, \tilde{y}(\tau;\varsigma), \sigma(\varsigma), \theta) d\tau \tag{5.44}$$

Thus,

$$\begin{aligned} \triangle \tilde{y}(s) &\equiv \left. \frac{\mathrm{d}\tilde{y}(s;\varsigma)}{\mathrm{d}\varsigma} \right|_{\varsigma=0} \\ &= \int_0^s \left\{ \frac{\partial \tilde{f}(\tau, \tilde{y}(\tau), \sigma, \theta)}{\partial \tilde{y}} \triangle \tilde{y}(\tau) + \frac{\partial \tilde{f}(\tau, \tilde{y}(\tau), \sigma, \theta)}{\partial \sigma} \rho \right\} \mathrm{d}\tau \end{aligned} \tag{5.45}$$

Clearly,

$$\frac{\mathrm{d}(\triangle \tilde{y}(s))}{\mathrm{d}s} = \frac{\partial \tilde{f}(s, \tilde{y}(s), \sigma, \theta)}{\partial \tilde{y}} \triangle \tilde{y}(s) + \frac{\partial \tilde{f}(s, \tilde{y}(s), \sigma, \theta)}{\partial \sigma} \rho \tag{5.46}$$

Now, $\tilde{J}_\delta(\sigma, \theta, \epsilon)$ can be expressed as:

$$\tilde{J}_\delta(\sigma, \theta, \epsilon) = \int_0^1 \{H_0(s, \tilde{y}(s;\varsigma), \sigma(\varsigma), \theta, \epsilon, \lambda_0(s)) - \lambda_0(s) \tilde{f}(s, \tilde{y}(s;\varsigma), \sigma(\varsigma), \theta)\} \mathrm{d}s \tag{5.47}$$

where $\lambda_0(s)$ is yet arbitrary. Thus, it follows that

$$\begin{aligned} \triangle \tilde{J}_\delta(\sigma(\varsigma), \theta, \epsilon) &\equiv \left. \frac{\mathrm{d}\tilde{J}_\delta(\sigma(\varsigma), \theta, \epsilon)}{\mathrm{d}\varsigma} \right|_{\varsigma=0} = \frac{\partial \tilde{J}_\delta(\sigma, \theta, \epsilon)}{\partial \sigma} \rho \\ &= \int_0^1 \{\triangle H_0(s, \tilde{y}(s), \sigma, \theta, \epsilon, \lambda_0(s)) - \lambda_0(s) \triangle \tilde{f}(s, \tilde{y}(s), \sigma, \theta)\} \mathrm{d}s \end{aligned} \tag{5.48}$$

where

$$\triangle \tilde{f}(s, \tilde{y}(s), \sigma, \theta) = \frac{\mathrm{d} \triangle \tilde{y}(s)}{\mathrm{d}s} \tag{5.49}$$

and

$$\begin{aligned} &\triangle H_0(s, \tilde{y}(s), \sigma, \theta, \lambda_0(s)) \\ = \ &\frac{\partial H_0(s, \tilde{y}(s), \sigma, \theta, \epsilon, \lambda_0(s))}{\partial \tilde{y}} \triangle \tilde{y}(s) + \frac{\partial H_0(s, \tilde{y}(s), \sigma, \theta, \epsilon, \lambda_0(s))}{\partial \sigma} \rho \end{aligned} \tag{5.50}$$

Choose λ_0 to be the solution of the costate system (5.41a)-(5.41b) corresponding to σ

while θ and ϵ are fixed. Then, by substituting (5.41a) into (5.50), we obtain

$$\begin{aligned} & \triangle H_0(s,\tilde{y}(s),\sigma,\theta,\epsilon,\lambda_0(s)) \\ = & -\left(\frac{\mathrm{d}\lambda_0(s)}{\mathrm{d}s}\right)^\top \triangle\, \tilde{y}(s) + \frac{\partial H_0(s,\tilde{y}(s),\sigma,\theta,\epsilon,\lambda_0(s))}{\partial\sigma}\rho \end{aligned} \tag{5.51}$$

By (5.49) and (5.51), it follows from (5.48) that

$$\begin{aligned} & \frac{\partial \tilde{J}_\delta(\sigma,\theta,\epsilon)}{\partial\sigma}\rho \\ = & \int_0^1 \left\{-\frac{\mathrm{d}}{\mathrm{d}t}\left[(\lambda_0(t))^\top \triangle\, \tilde{y}(t)\right] + \frac{\partial H_0(s,y(s),\sigma,\theta,\epsilon,\lambda_0(s))}{\partial\sigma}\rho\right\}\mathrm{d}s \\ = & (\lambda_0(0))^\top \triangle\, \tilde{y}(0) - (\lambda_0(1))^\top \triangle\, \tilde{y}(1) \\ & + \int_0^1 \left\{\frac{\partial H_0(s,\tilde{y}(s),\sigma,\theta,\epsilon,\lambda_0(s))}{\partial\sigma}\rho\right\}\mathrm{d}s \end{aligned} \tag{5.52}$$

Note that $\tilde{y}(0)$ is constant. Thus, by (3.35b), we deduce from (5.52) that

$$\begin{aligned} & \frac{\partial \tilde{J}_\delta(\sigma,\theta,\epsilon)}{\partial\sigma}\rho \\ = & \int_0^1 \left\{\frac{\partial H_0(s,\tilde{y}(s),\sigma,\theta,\epsilon,\lambda_0(s))}{\partial\sigma}\rho\right\}\mathrm{d}s \end{aligned} \tag{5.53}$$

Since ρ is arbitrary, (5.37) follows readily from (5.53). (5.38) can be derived in the same way. For, (5.39), it can be derived readily by applying the chain rule. Thus, the proof is completed. □

Remark 5.5. By Assumption 5.1-Assumption 5.5, Remark 5.2 and Remark 5.3, it follows from arguments similar to those given for the proof of lemma 6.4.2 in [36] that there exits a compact set $\mathcal{Z} \subset \mathbb{R}^n$ such that $\lambda_0(s|\sigma,\theta,\epsilon) \in \mathcal{Z}$ for all $s \in [0,1]$, $(\sigma,\theta) \in \Xi \times \tilde{\Theta}$ and $\epsilon \geq 0$.

Our main aim is to show that, under some mild assumptions, if the parameter δ_k is sufficiently large($\delta_k \to +\infty$ as k $\to +\infty$) and $\left(\sigma^{(k),*},\theta^{(k),*},\epsilon^{(k),*}\right)$ is a local minimizer of Problem $(P_{\delta_k}(p))$, then $\epsilon^{(k),*} \to \epsilon^* = 0$, and $\left(\sigma^{(k),*},\theta^{(k),*}\right) \to (\sigma^*,\theta^*)$ with (σ^*,θ^*) being a local minimizer of Problem (P5(p)).

For every positive integer k, let $\left(\sigma^{(k),*},\theta^{(k),*}\right)$ be a local minimizer of Problem $(P_{\delta_k}(p))$. To obtain our main result, we need the following lemma.

Lemma 5.1. *Let $\left(\sigma^{(k),*},\theta^{(k),*},\epsilon^{(k),*}\right)$ be a local minimizer of Problem $(P_{\delta_k}(p))$. Suppose that $\tilde{J}_{\delta_k}\left(\sigma^{(k),*},\theta^{(k),*},\epsilon^{(k),*}\right)$ is finite and that $\epsilon^{(k),*} > 0$. Then*

$$\left(\sigma^{(k),*},\theta^{(k),*},\epsilon^{(k),*}\right) \notin \Omega_{\epsilon_k}$$

where Ω_{ϵ_k} is defined by (5.28).

Proof. Since $\left(\sigma^{(k),*}, \theta^{(k),*}, \epsilon^{(k),*}\right)$ is a local minimizer of Problem $(P_{\delta_k}(p))$ and $\epsilon^{(k),*} > 0$, we have

$$\frac{\partial \tilde{J}_{\delta_k}\left(\sigma^{(k),*}, \theta^{(k),*}, \epsilon^{(k),*}\right)}{\partial \epsilon} = 0 \tag{5.54}$$

On a contrary, we assume that the conclusion of the lemma is false. Then, we have

$$g_i(\tilde{y}(s|\sigma^{(k),*}, \theta^{(k),*}), \sigma^{j(k),*}) \leq (\epsilon^{(k),*})^{\gamma} W_i, \forall\ s \in \mathcal{J}_j,\ j = 1, 2, \ldots, p,\ i = 1, 2, \ldots, N \tag{5.55}$$

and

$$\tilde{y}(1|\sigma^{(k),*}, \theta^{(k),*}) - \tilde{y}^f = 0 \tag{5.56}$$

Thus, by (5.55), (5.56), (5.39) and (5.54), we obtain

$$0 = \frac{\partial \tilde{J}_{\delta_k}\left(\sigma^{(k),*}, \theta^{(k),*}, \epsilon^{(k),*}\right)}{\partial \epsilon} = \beta \delta_k \epsilon^{\beta - 1} > 0$$

This is a contradiction, and hence completing the proof. □

Before we introduce the definition of the constraint qualification, we first define

$$\phi_i(\tilde{y}(1|\sigma, \theta)) = \tilde{y}_i(1|\sigma, \theta) - \tilde{y}_i^f\ ,\ i = 1, 2, \ldots, n+1 \tag{5.57}$$

Definition 5.1. Suppose that $\frac{\partial \bar{g}_i(s, y(s|\sigma,\theta), \sigma)}{\partial \sigma}$, $i = 1, 2,\ \ldots, N$, and $\frac{\partial \phi_i(\tilde{y}(1|\sigma,\theta))}{\partial \sigma}$, $i = 1, 2, \ldots, n+1$, are linearly independent at $(\sigma, \theta) = \left(\bar{\sigma}, \bar{\theta}\right)$ for each $s \in [0, 1]$. Then, it is said that the constraint qualification is satisfied for the continuous inequality constraints (5.23) and the terminal equality constraint (5.18c) at $(\sigma, \theta) = \left(\bar{\sigma}, \bar{\theta}\right)$.

Theorem 5.2. *Suppose that $\left(\sigma^{(k),*}, \theta^{(k),*}, \epsilon^{(k),*}\right)$ is a local minimizer of Problem $(P_{\delta_k}(p))$ such that $\tilde{J}_{\delta_k}\left(\sigma^{(k),*}, \theta^{(k),*}, \epsilon^{(k),*}\right)$ is finite and $\epsilon^{(k),*} > 0$. If $\left(\sigma^{(k),*}, \theta^{(k),*}, \epsilon^{(k),*}\right) \to (\sigma^*, \theta^*, \epsilon^*)$ as $k \to +\infty$, and the constraint qualification is satisfied for the continuous inequality constraints (5.23) at $(\sigma, \theta) = (\sigma^*, \theta^*)$, then $\epsilon^* = 0$ and $(\sigma^*, \theta^*) \in \Omega_0$.*

Proof. From Lemma 5.1, it follows that $\left(\sigma^{(k),*}, \theta^{(k),*}, \epsilon^{(k),*}\right) \notin \Omega_{\epsilon^{(k),*}}$. Now, suppose that the continuous inequality constraints (5.3) are to be satisfied for $t \in \hat{T}$, where $\hat{T}$ is a subset of $\hat{T} \subset [0, T]$, then there exits a corresponding subset $S \subset [0, 1]$ in the new time horizon obtained after the time scaling transform, such that

$$\bar{g}_i\left(s, \tilde{y}\left(s|\sigma, \theta\right), \sigma\right) \leq 0,\ \forall s \in S,\ i = 1, 2, \ldots, N, \tag{5.58}$$

here, S can be chosen arbitrarily through a proper choice of $\hat{T} \subset [0, T]$. Then, by the same token as that given for (5.29), we obtain

$$\tilde{J}_\delta(\sigma,\theta,\epsilon) = \begin{cases} \tilde{J}(\sigma,\theta) & \text{if } \epsilon = 0,\ \bar{g}_i\left(s,\tilde{y}\left(s|\sigma,\theta\right),\sigma\right) \le 0 \\ & \ (s \in S) \\ \tilde{J}(\sigma,\theta) + \epsilon^{-\alpha}\left(\hat{\Delta}(\sigma,\theta,\epsilon) + \Delta_1\right) + \delta\epsilon^\beta & \text{if } \epsilon > 0 \\ +\infty & \text{otherwise} \end{cases}$$

where

$$\hat{\Delta}(\sigma,\theta,\epsilon) = \int_S \sum_{i=1}^N \left[\max\left\{0, \bar{g}_i\left(s,\tilde{y}\left(s|\sigma,\theta\right),\sigma\right) - \epsilon^\gamma W_i\right\}\right]^2 \mathrm{d}s$$

Thus, we have

$$\begin{aligned} &\frac{\partial \tilde{J}_\delta\left(\sigma^{(k),*},\theta^{(k),*},\epsilon^{(k),*}\right)}{\partial\sigma} \\ = & \int_0^1 \frac{\partial H_0\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*},\theta^{(k),*},\epsilon^{(k),*},\lambda_0\left(s|\sigma^{(k),*},\theta^{(k),*},\epsilon^{(k),*}\right)\right)}{\partial\sigma}\mathrm{d}s \\ = & \int_0^1 \frac{\partial \bar{\mathcal{L}}_0\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*},\theta^{(k),*}\right)}{\partial\sigma}\mathrm{d}s \\ & +2\left(\epsilon^{(k),*}\right)^{-\alpha}\int_S \sum_{i=1}^N \max\left\{0, \bar{g}_i(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right) \\ & \qquad -(\epsilon^{(k),*})^\gamma W_i\Big\} \frac{\partial \bar{g}_i(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*})}{\partial\sigma}\mathrm{d}s \\ & +\int_0^1 \lambda_0\left(s|\sigma^{(k),*},\theta^{(k),*},\epsilon^{(k),*}\right)\frac{\partial \tilde{f}\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*},\theta^{(k),*}\right)}{\partial\sigma}\mathrm{d}s \\ = & \ 0 \end{aligned} \tag{5.59}$$

and

$$\begin{aligned} &\frac{\partial \tilde{J}_\delta\left(\sigma^{(k),*},\theta^{(k),*},\epsilon^{(k),*}\right)}{\partial\epsilon} \\ = & \ (\epsilon^{(k),*})^{-\alpha-1}\Bigg\{-\alpha\int_S \sum_{i=1}^N \left[\max\left\{0, \bar{g}_i(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}) - (\epsilon^{(k),*})^\gamma W_i\right\}\right]^2 \mathrm{d}s \\ & +2\gamma\int_S \sum_{i=1}^N \max\left\{0, \bar{g}_i\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right) - \left(\epsilon^{(k),*}\right)^\gamma W_i\right\}\left(-\left(\epsilon^{(k),*}\right)^\gamma W_i\right)\mathrm{d}s \\ & -\alpha\sum_{i=1}^{n+1}\left(\tilde{y}_i\left(1|\sigma^{(k),*},\theta^{(k),*}\right) - \tilde{y}_i^f\right)^2\Bigg\} + \delta_k\beta\left(\epsilon^{(k),*}\right)^{\beta-1} \\ = & \ 0, \end{aligned} \tag{5.60}$$

Suppose that $\epsilon^{(k),*} \to \epsilon^* \neq 0$. Then, by (5.60), it can be shown by using Remark 5.2 and

Remark 5.3 and Lebesgue dominated convergence theorem that its first term tends to a finite value, while the last term tends to infinity as $\delta_k \to +\infty$, when $k \to +\infty$. This is impossible for the validity of (5.60). Thus, $\epsilon^* = 0$.

Now, by (5.59), we have

$$\begin{aligned}
&\int_0^1 \frac{\partial \bar{\mathcal{L}}_0\left(s, \tilde{y}\left(s|\sigma^{(k),*}, \theta^{(k),*}\right), \sigma^{(k),*}, \theta^{(k),*}\right)}{\partial \sigma} \mathrm{d}s \\
&+2\left(\epsilon^{(k),*}\right)^{-\alpha} \int_S \sum_{i=1}^N \max\left\{0, \bar{g}_i\left(s, \tilde{y}\left(s|\sigma^{(k),*}, \theta^{(k),*}\right), \sigma^{(k),*}\right) - \left(\epsilon^{(k),*}\right)^\gamma W_i\right\} \\
&\qquad\qquad \frac{\partial \bar{g}_i\left(s, \tilde{y}\left(s|\sigma^{(k),*}, \theta^{(k),*}\right), \sigma^{(k),*}\right)}{\partial \sigma} \mathrm{d}s \\
&+\int_0^1 \lambda_0\left(s|\sigma^{(k),*}, \theta^{(k),*}, \epsilon^{(k),*}\right) \frac{\partial \tilde{f}\left(\tilde{y}\left(s|\sigma^{(k),*}, \theta^{(k),*}\right), \sigma^{(k),*}, \theta^{(k),*}\right)}{\partial \sigma} \mathrm{d}s \\
&= 0
\end{aligned}$$

Thus,

$$\begin{aligned}
&\lim_{k\to+\infty} \left\{ \int_0^1 \frac{\partial \bar{\mathcal{L}}_0\left(s, \tilde{y}\left(s|\sigma^{(k),*}, \theta^{(k),*}\right), \sigma^{(k),*}, \theta^{(k),*}\right)}{\partial \sigma} \mathrm{d}s \right. \\
&+2\left(\epsilon^{(k),*}\right)^{-\alpha} \int_S \sum_{i=1}^N \max\left\{0, \bar{g}_i\left(s, \tilde{y}\left(s|\sigma^{(k),*}, \theta^{(k),*}\right), \sigma^{(k),*}\right)\right. \\
&\qquad\qquad \left. - \left(\epsilon^{(k),*}\right)^\gamma W_i\right\} \frac{\partial \bar{g}_i\left(s, \tilde{y}\left(s|\sigma^{(k),*}, \theta^{(k),*}\right), \sigma^{(k),*}\right)}{\partial \sigma} \mathrm{d}s \\
&\left. +\int_0^1 \lambda_0\left(s|\sigma^{(k),*}, \theta^{(k),*}, \epsilon^{(k),*}\right) \frac{\partial \tilde{f}\left(s, \tilde{y}\left(s|\sigma^{(k),*}, \theta^{(k),*}\right), \sigma^{(k),*}, \theta^{(k),*}\right)}{\partial \sigma} \mathrm{d}s \right\} \\
&= 0
\end{aligned}$$

By Remark 5.3 and Remark 5.4, it follows from the Lebesgue dominated convergence theorem that the first and third terms converge to some finite values. On the other hand, the second term tends to infinite, which is impossible, and hence

$$\begin{aligned}
&\int_S \lim_{k\to+\infty}\left\{\sum_{i=1}^{N}\max\left\{0,\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right)\right\}\right. \\
&\qquad\qquad\left.\frac{\partial\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right)}{\partial\sigma}\right\}\mathrm{d}s \\
=\ &\sum_{i=1}^{N}\int_S \max\left\{0,\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right)\right\} \\
&\qquad\qquad\frac{\partial\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right)}{\partial\sigma}\mathrm{d}s \\
=\ &0
\end{aligned}\tag{5.61}$$

where $S\subset[0,1]$ is arbitrary.

Let S_0, S_1, S_2 be chosen such that

$$S=\bigcup_{i=0}^{2}S_i\tag{5.62}$$

where

$$A_0=\int_{S_0}\left\{\sum_{i=1}^{N}\max\left\{0,\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^*,\theta^*\right)\right),\sigma^*\right)\right\}\frac{\partial\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^*,\theta^*\right),\sigma^*\right)}{\partial\sigma}\right\}\mathrm{d}s=0\tag{5.63}$$

$$A_1=\int_{S_1}\left\{\sum_{i=1}^{N}\max\left\{0,\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^*,\theta^*\right),\sigma^*\right)\right\}\frac{\partial\bar{g}_i(s,\tilde{y}\left(s|\sigma^*,\theta^*\right),\sigma^*)}{\partial\sigma}\right\}\mathrm{d}s>0\tag{5.64}$$

$$A_2=\int_{S_2}\left\{\sum_{i=1}^{N}\max\left\{0,\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^*,\theta^*\right),\sigma^*\right)\right\}\frac{\partial\bar{g}_i(s,\tilde{y}\left(s|\sigma^*,\theta^*\right),\sigma^*)}{\partial\sigma}\right\}\mathrm{d}s<0\tag{5.65}$$

and $A_1+A_2=0$. Now, choose a non empty subset $S_3\subset S_1$. Let $S_4=S_1\backslash S_3$ and $\hat{S}=S_0\bigcup S_4\bigcup S_2$. Clearly,

$$\int_{\hat{S}}\left\{\sum_{i=1}^{N}\max\left\{0,\bar{g}_i(s,\tilde{y}\left(s|\sigma^*,\theta^*\right),\sigma^*)\right\}\frac{\partial g_i\left(\tilde{y}\left(s|\sigma^*,\theta^*\right),\sigma^*\right)}{\partial\sigma}\right\}\mathrm{d}s\neq 0.\tag{5.66}$$

However, it is a contradiction to (5.61). This implies that S_1 and S_2 must be empty sets. In other words, for each $i=1,\ \ldots,\ N$,

$$\sum_{i=1}^{N}\max\{0,\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^*,\theta^*\right),\sigma^*\right)\}\frac{\partial\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^*,\theta^*\right),\sigma^*\right)}{\partial\sigma}=0$$

for each $s \in [0,1]$. Since the constraint qualification is satisfied for the continuous inequality constraints (5.23) at $(\sigma,\theta) = (\sigma^*,\theta^*)$, it follows that, for each $i = 1,2,\ldots,N$,

$$\max\{0, \bar{g}_i\left(s, \tilde{y}\left(s|\sigma^*,\theta^*\right),\sigma^*\right)\} = 0$$

for each $s \in [0,1]$. This, in turn, implies that, for each $i = 1,2,\ldots,N$,

$$\bar{g}_i\left(s, \tilde{y}\left(s|\sigma^*,\theta^*\right),\sigma^*\right) \leq 0 \tag{5.67}$$

for each $s \in [0,1]$. Next, from (5.67) and (5.60), it is easy to see that, for each $i = 1,2,\ldots,n+1$, when $k \to +\infty$,

$$\tilde{y}_i(1|\sigma^*,\theta^*) - \tilde{y}_i^f = 0 \tag{5.68}$$

The proof is completed. □

Corollary 5.1 Suppose that $\left(\sigma^{(k),*},\theta^{(k),*}\right) \to (\sigma^*,\theta^*) \in \Omega_0$ and that $\epsilon^{(k),*} \to \epsilon^* = 0$. Then, $\Delta\left(\sigma^{(k),*},\theta^{(k),*},\epsilon^{(k),*}\right) \to \Delta\left(\sigma^*,\theta^*,\epsilon^*\right) = 0$, and $\Delta_1 \to 0$

Proof. The conclusion follows readily from the definitions of $\Delta(\sigma,\theta,\epsilon)$ and Δ_1, and the continuity of g_i and $\tilde{f}$. □

In what follows, we shall turn our attention to the exact penalty function constructed in (5.29). We shall see that, under some mild conditions, $\tilde{J}_\delta(\sigma,\theta,\epsilon)$ is continuously differentiable.

We assume that the following assumptions are satisfied.

Assumption 5.6.

$$\max\left\{0, \bar{g}_i\left(s, \tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right)\right\} = o\left(\left(\epsilon^{(k),*}\right)^{\xi}\right),\ \xi > 0,\ s \in [0,1],\ i = 1,2,\ldots,N. \tag{5.69}$$

Assumption 5.7.

$$\phi_i\left(\tilde{y}\left(1|\sigma^{(k),*},\theta^{(k),*}\right)\right) = o\left(\left(\epsilon^{(k),*}\right)^{\xi'}\right),\ \xi' > 0,\ i = 1,2,\ldots,n+1 \tag{5.70}$$

Theorem 5.3. *Suppose that* $\gamma > \alpha,\ \xi > \alpha,\ \xi' > \alpha,\ -\alpha - 1 + 2\xi > 0,\ -\alpha - 1 + 2\xi' > 0,\ 2\gamma - \alpha - 1 > 0$. *Then*

$$\tilde{J}_{\delta_k}\left(\sigma^{(k),*},\theta^{(k),*},\epsilon^{(k),*}\right) \xrightarrow[\left(\sigma^{(k),*},\theta^{(k),*}\right)\to(\sigma^*,\theta^*)\in\Omega_0]{\epsilon^{(k),*}\to\epsilon^*=0} \tilde{J}_{\delta_k}\left(\sigma^{(k),*},\theta^{(k),*},0\right) = \tilde{J}\left(\sigma^*,\theta^*\right) \tag{5.71}$$

$$\nabla_{(\sigma,\theta,\epsilon)}\tilde{J}_{\delta_k}\left(\sigma^{(k),*},\theta^{(k),*},\epsilon^{(k),*}\right)\xrightarrow[\left(\sigma^{(k),*},\theta^{(k),*}\right)\to(\sigma^*,\theta^*)\in\Omega_0]{\epsilon^{(k),*}\to\epsilon^*=0}\nabla_{(\sigma,\theta,\epsilon)}\tilde{J}_{\delta_k}(\sigma^*,\theta^*,0)$$
$$=\left(\nabla_{(\sigma,\theta)}\tilde{J}\left(\sigma^*,\theta^*\right),0\right) \tag{5.72}$$

Proof. Based on the conditions of the theorem, we can show that, for $\epsilon \neq 0$,

$$\begin{aligned}
&\lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0\\ \left(\sigma^{(k),*},\theta^{(k),*}\right)\to(\sigma^*,\theta^*)\in\Omega_0}} \tilde{J}_{\delta_k}\left(\sigma^{(k),*},\theta^{(k),*},\epsilon^{(k),*}\right)\\
=&\lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0\\ \left(\sigma^{(k),*},\theta^{(k),*}\right)\to(\sigma^*,\theta^*)\in\Omega_0}} \Bigg\{\tilde{J}\left(\sigma^{(k),*},\theta^{(k),*}\right)\\
&+\left(\epsilon^{(k),*}\right)^{-\alpha}\sum_{i=1}^{N}\int_0^1\left[\max\left\{0,\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right)-\left(\epsilon^{(k),*}\right)^{\gamma}W_i\right\}\right]^2\mathrm{d}s\\
&+\left(\epsilon^{(k),*}\right)^{-\alpha}\sum_{i=1}^{n+1}\left(\tilde{y}_i\left(1|\sigma^{(k),*},\theta^{(k),*}\right)-\tilde{y}_i^f\right)^2+\delta_k\left(\epsilon^{(k),*}\right)^{\beta}\Bigg\}
\end{aligned} \tag{5.73}$$

By an argument similar to that given for the proof of Lemma 6.4.3 in [36], we can show that when $\left(\sigma^{(k),*},\theta^{(k),*}\right)\to(\sigma^*,\theta^*)$,

$$\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right)\to\tilde{y}\left(s|\sigma^*,\theta^*\right) \tag{5.74}$$

for each $s\in[0,1]$. By (5.74) and (5.24), it follows from an argument similar to that given for the proof of Lemma 6.4.4 in [36] that

$$\lim_{\left(\sigma^{(k),*},\theta^{(k),*}\right)\to(\sigma^*,\theta^*)\in\Omega_0}\tilde{J}\left(\sigma^{(k),*},\theta^{(k),*}\right)=\tilde{J}\left(\sigma^*,\theta^*\right) \tag{5.75}$$

Substituting (5.75) into (5.73), we have

$$
\begin{aligned}
&\lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0 \\ \left(\sigma^{(k),*},\theta^{(k),*}\right)\to(\sigma^*,\theta^*)\in\Omega_0}} \tilde{J}_{\delta_k}\left(\sigma^{(k),*},\theta^{(k),*},\epsilon^{(k),*}\right) \\
=&\tilde{J}\left(\sigma^*,\theta^*\right) \\
&+\lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0 \\ \left(\sigma^{(k),*},\theta^{(k),*}\right)\to(\sigma^*,\theta^*)\in\Omega_0}} \frac{\sum_{i=1}^{N}\int_0^1\left[\max\left\{0,\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right)-\left(\epsilon^{(k),*}\right)^{\gamma}W_i\right\}\right]^2\mathrm{d}s}{\left(\epsilon^{(k),*}\right)^{\alpha}} \\
&+\lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0 \\ \left(\sigma^{(k),*},\theta^{(k),*}\right)\to(\sigma^*,\theta^*)\in\Omega_0}} \frac{\sum_{i=1}^{n+1}\left(\tilde{y}_i\left(1|\sigma^{(k),*},\theta^{(k),*}\right)-\tilde{y}_i^f\right)^2}{\left(\epsilon^{(k),*}\right)^{\alpha}}
\end{aligned}
\tag{5.76}
$$

For the second term of (5.76), we have

$$
\begin{aligned}
&\lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0 \\ \left(\sigma^{(k),*},\theta^{(k),*}\right)\to(\sigma^*,\theta^*)\in\Omega_0}} \frac{\sum_{i=1}^{N}\int_0^1\left[\max\left\{0,\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right)-\left(\epsilon^{(k),*}\right)^{\gamma}W_i\right\}\right]^2\mathrm{d}s}{\left(\epsilon^{(k),*}\right)^{\alpha}} \\
=&\lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0 \\ \left(\sigma^{(k),*},\theta^{(k),*}\right)\to(\sigma^*,\theta^*)\in\Omega_0}} \sum_{i=1}^{N}\int_0^1\left[\max\left\{0,\left(\epsilon^{(k),*}\right)^{-\frac{\alpha}{2}}\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right)\right.\right. \\
&\qquad\left.\left.-\left(\epsilon^{(k),*}\right)^{\gamma-\frac{\alpha}{2}}W_i\right\}\right]^2\mathrm{d}s
\end{aligned}
$$

Since $\xi > \alpha,\ \gamma > \alpha$, it follows from Assumption 5.6 that, for any $s \in [0,1]$,

$$
\begin{aligned}
&\lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0 \\ \left(\sigma^{(k),*},\theta^{(k),*}\right)\to(\sigma^*,\theta^*)\in\Omega_0}} \max\left\{0,\left(\epsilon^{(k),*}\right)^{-\frac{\alpha}{2}}\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right)\right. \\
&\qquad\left.-\left(\epsilon^{(k),*}\right)^{\gamma-\frac{\alpha}{2}}W_i\right\}=0
\end{aligned}
\tag{5.77}
$$

Thus, by Remark 5.3 and Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}
&\lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0\\ \left(\sigma^{(k),*},\theta^{(k),*}\right)\to(\sigma^*,\theta^*)\in\Omega_0}} \sum_{i=1}^{N}\int_0^1 \Big[\max\Big\{0,\left(\epsilon^{(k),*}\right)^{-\frac{\alpha}{2}}\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right) \\
&\qquad\qquad - \left(\epsilon^{(k),*}\right)^{\gamma-\frac{\alpha}{2}}W_i\Big\}\Big]^2\mathrm{d}s \\
= &\sum_{i=1}^{N}\int_0^1 \lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0\\ \left(\sigma^{(k),*},\theta^{(k),*}\right)\to(\sigma^*,\theta^*)\in\Omega_0}} \Big[\max\Big\{0,\left(\epsilon^{(k),*}\right)^{-\frac{\alpha}{2}}\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right) \\
&\qquad\qquad - \left(\epsilon^{(k),*}\right)^{\gamma-\frac{\alpha}{2}}W_i\Big\}\Big]^2\mathrm{d}s \\
= &\ 0
\end{aligned} \tag{5.78}$$

Similarly, for the third term of (5.76), it is clear from Assumption 5.7 that

$$\lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0\\ \left(\sigma^{(k),*},\theta^{(k),*}\right)\to(\sigma^*,\theta^*)\in\Omega_0}} \frac{\sum_{i=1}^{n+1}\left(\tilde{y}_i\left(1|\sigma^{(k),*},\theta^{(k),*}\right)-\tilde{y}_i^f\right)^2}{\left(\epsilon^{(k),*}\right)^{\alpha}} = 0 \tag{5.79}$$

Combining (5.76), (5.78) and (5.79) gives

$$\lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0\\ \left(\sigma^{(k),*},\theta^{(k),*}\right)\to(\sigma^*,\theta^*)\in\Omega_0}} \tilde{J}_{\delta_k}\left(\sigma^{(k),*},\theta^{(k),*},\epsilon^{(k),*}\right) = \tilde{J}_{\delta_k}\left(\sigma^*,\theta^*,0\right) = \tilde{J}\left(\sigma^*,\theta^*\right) \tag{5.80}$$

For the second part of the theorem, we need the gradient formulas of $\tilde{J}(\sigma,\theta)$, which can be derived in the same way as that given for the proof of Theorem 5.2.1 in [36]. These gradient formulas are given as follows.

$$\frac{\partial\tilde{J}(\sigma,\theta)}{\partial\sigma} = \int_0^1 \frac{\partial\bar{H}_0\left(s,\tilde{y}\left(s|\sigma,\theta\right),\sigma,\theta,\bar{\lambda}_0\left(s|\sigma,\theta\right)\right)}{\partial\sigma}\mathrm{d}s \tag{5.81}$$

$$\frac{\partial\tilde{J}(\sigma,\theta)}{\partial\theta} = \int_0^1 \frac{\partial\bar{H}_0\left(s,\tilde{y}\left(s|\sigma,\theta\right),\sigma,\theta,\bar{\lambda}_0\left(s|\sigma,\theta\right)\right)}{\partial\theta}\mathrm{d}s \tag{5.82}$$

where $\bar{H}_0\left(s,\tilde{y}\left(s|\sigma,\theta\right),\sigma,\theta,\bar{\lambda}_0\left(s|\sigma,\theta\right)\right)$ is the Hamiltonian function defined by

$$\begin{aligned}
&\bar{H}_0\left(s,\tilde{y}\left(s|\sigma,\theta\right),\sigma,\theta,\bar{\lambda}_0\left(s|\sigma,\theta\right)\right) \\
= &\ \bar{\mathcal{L}}_0(s,\tilde{y}\left(s|\sigma,\theta\right),\sigma,\theta) + \bar{\lambda}_0\left(s|\sigma,\theta\right)\tilde{f}(s,\tilde{y}\left(s|\sigma,\theta\right),\sigma,\theta),
\end{aligned} \tag{5.83}$$

and $\bar{\lambda}_0(\cdot|\sigma,\theta)$ is the solution of the following system of co-state differential equations

$$\frac{\left(\mathrm{d}\bar{\lambda}_0(s)\right)^\top}{\mathrm{d}s} = -\frac{\partial \bar{H}_0\left(s, \tilde{y}\left(s|\sigma,\theta\right), \sigma, \theta, \bar{\lambda}_0\left(s|\sigma,\theta\right)\right)}{\partial \tilde{y}} \tag{5.84a}$$

with the boundary condition

$$\left(\bar{\lambda}_0(1)\right)^\top = \frac{\partial \Phi_0\left(\tilde{y}(1|\sigma,\theta)\right)}{\partial \tilde{y}}. \tag{5.84b}$$

By (5.83), we can rewrite (5.84a) as:

$$\frac{(\mathrm{d}\bar{\lambda}_0(s))^\top}{\mathrm{d}s} = -\frac{\partial \bar{\mathcal{L}}_0(s, \tilde{y}\left(s|\sigma,\theta\right), \sigma, \theta)}{\partial \tilde{y}} - \bar{\lambda}_0(s)\frac{\partial \tilde{f}(s, \tilde{y}\left(s|\sigma,\theta\right), \sigma, \theta)}{\partial \tilde{y}} \tag{5.85}$$

By (5.85) with terminal condition (5.84b) and (5.41a) with terminal condition (5.41b), we obtain

$$\begin{aligned}
& \left\|\bar{\lambda}_0\left(s|\sigma^{(k),*}, \theta^{(k),*}\right) - \lambda_0\left(s|\sigma^{(k),*}, \theta^{(k),*}, \epsilon^{(k),*}\right)\right\| \\
\leq\ & \left\|\bar{\lambda}_0\left(1|\sigma^{(k),*}, \theta^{(k),*}\right) - \lambda_0(1|\sigma^{(k),*}, \theta^{(k),*}, \epsilon^{(k),*})\right\| \\
& + \int_1^0 \left\| -\frac{\partial \bar{\mathcal{L}}_0\left(\omega, \tilde{y}\left(\omega|\sigma^{(k),*}, \theta^{(k),*}\right), \sigma^{(k),*}, \theta^{(k),*}\right)}{\partial \tilde{y}} \right. \\
& \left. + \frac{\partial \tilde{\mathcal{L}}_0\left(\omega, \tilde{y}\left(\omega|\sigma^{(k),*}, \theta^{(k),*}\right), \sigma^{(k),*}, \theta^{(k),*}\right)}{\partial \tilde{y}} \right\| \mathrm{d}\omega \\
& + \int_1^s \left\| \frac{\partial \tilde{f}\left(s, \tilde{y}\left(\omega|\sigma^{(k),*}, \theta^{(k),*}\right), \sigma^{(k),*}, \theta^{(k),*}\right)}{\partial \tilde{y}} \right\| \\
& \qquad \left\| -\bar{\lambda}_0\left(\omega|\sigma^{(k),*}, \theta^{(k),*}\right) + \lambda_0\left(\omega|\sigma^{(k),*}, \theta^{(k),*}, \epsilon^{(k),*}\right)\right\| \mathrm{d}\omega
\end{aligned} \tag{5.86}$$

By the definitions of (5.41b), (5.84b), (5.34) and $\xi' > \alpha$, it follows from Assumption 5.7 that

$$\begin{aligned}
& \lim_{\substack{\epsilon^{(k),*} \to \epsilon^* = 0 \\ \left(\sigma^{(k),*}, \theta^{(k),*}\right) \to (\sigma^*, \theta^*) \in \Omega_0}} \left\|\bar{\lambda}_0\left(1|\sigma^{(k),*}, \theta^{(k),*}\right) - \lambda_0\left(1|\sigma^{(k),*}, \theta^{(k),*}, \epsilon^{(k),*}\right)\right\| \\
=\ & \lim_{\substack{\epsilon^{(k),*} \to \epsilon^* = 0 \\ \left(\sigma^{(k),*}, \theta^{(k),*}\right) \to (\sigma^*, \theta^*) \in \Omega_0}} \left\| \left(\epsilon^{(k),*}\right)^{-\alpha} \frac{\partial}{\partial y} \sum_{i=1}^{n+1} \left(\tilde{y}_i\left(1|\sigma^{(k),*}, \theta^{(k),*}\right) - \tilde{y}_i^f\right)^2 \right\| \\
=\ & 0
\end{aligned} \tag{5.87}$$

On the other hand, by (5.35), $\xi > \alpha$ and $\gamma > \alpha$, it follows from Assumption 5.6, Remark

5.3 and Lebesgue dominated convergence theorem that, for each $s \in [0,1]$,

$$
\begin{aligned}
& \lim_{\substack{\epsilon^{(k),*} \to \epsilon^* = 0 \\ \left(\sigma^{(k),*}, \theta^{(k),*}\right) \to (\sigma^*, \theta^*) \in \Omega_0}} \int_1^0 \Bigg\| - \frac{\partial \bar{\mathcal{L}}_0\left(\omega, \tilde{y}\left(\omega | \sigma^{(k),*}, \theta^{(k),*}\right), \sigma^{(k),*}, \theta^{(k),*}\right)}{\partial \tilde{y}} \\
& \qquad + \frac{\partial \tilde{\mathcal{L}}_0\left(\omega, \tilde{y}\left(\omega | \sigma^{(k),*}, \theta^{(k),*}\right), \sigma^{(k),*}, \theta^{(k),*}, \epsilon^{(k),*}\right)}{\partial \tilde{y}} \Bigg\| \mathrm{d}\omega \\
= & \lim_{\substack{\epsilon^{(k),*} \to \epsilon^* = 0 \\ \left(\sigma^{(k),*}, \theta^{(k),*}\right) \to (\sigma^*, \theta^*) \in \Omega_0}} \int_1^0 2\left(\epsilon^{(k),*}\right)^{-\alpha} \sum_{i=1}^{N} \Bigg\| \left[\max\left\{0, \bar{g}_i\left(\omega, \tilde{y}\left(\omega | \sigma^{(k),*}, \theta^{(k),*}\right), \sigma^{(k),*}\right) - \epsilon^{\gamma} W_i\right\}\right] \\
& \qquad \frac{\partial \bar{g}_i\left(\omega, \tilde{y}\left(\omega | \sigma^{(k),*}, \theta^{(k),*}\right), \sigma^{(k),*}\right)}{\partial \tilde{y}} \Bigg\| \mathrm{d}\omega \\
= & \ 0
\end{aligned} \tag{5.88}
$$

Thus, by applying Gronwall-Bellman's lemma (Theorem 2.8.6 in [36]) to (5.86), it follows from (5.87), (5.88), Remark 5.3 and Lebesgue dominated theorem that, for each $s \in [0,1]$,

$$
\begin{aligned}
& \lim_{\substack{\epsilon^{(k),*} \to \epsilon^* = 0 \\ \left(\sigma^{(k),*}, \theta^{(k),*}\right) \to (\sigma^*, \theta^*) \in \Omega_0}} \left\| \bar{\lambda}_0\left(s | \sigma^{(k),*}, \theta^{(k),*}\right) - \lambda_0\left(s | \sigma^{(k),*}, \theta^{(k),*}, \epsilon^{(k),*}\right) \right\| \\
= & \ 0
\end{aligned} \tag{5.89}
$$

By (5.37), (5.132), and (5.35), we have

$$
\begin{aligned}
&\lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0\\ \left(\sigma^{(k),*},\theta^{(k),*}\right)\to(\sigma^*,\theta^*)\in\Omega_0}} \nabla_\sigma \tilde{J}_{\delta_k}\left(\sigma^{(k),*},\theta^{(k),*},\epsilon^{(k),*}\right)\\
&=\lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0\\ \left(\sigma^{(k),*},\theta^{(k),*}\right)\to(\sigma^*,\theta^*)\in\Omega_0}} \left\{\int_0^1 \frac{\partial\bar{\mathcal{L}}_0\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right)}{\partial\sigma}\mathrm{d}s\right.\\
&+2\left(\epsilon^{(k),*}\right)^{-\alpha}\sum_{i=1}^N\int_0^1\max\left\{0,\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right)-\left(\epsilon^{(k),*}\right)^{\gamma}W_i\right\}\\
&\qquad\frac{\partial\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*},\theta^{(k),*}\right)}{\partial\sigma}\mathrm{d}s\\
&\left.+\int_0^1\lambda_0\left(s|\sigma^{(k),*},\theta^{(k),*},\epsilon^{(k),*}\right)\frac{\partial\tilde{f}\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*},\theta^{(k),*}\right)}{\partial\sigma}\mathrm{d}s\right\}\\
&=\lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0\\ \left(\sigma^{(k),*},\theta^{(k),*}\right)\to(\sigma^*,\theta^*)\in\Omega_0}} \left\{\int_0^1 \frac{\partial\bar{\mathcal{L}}_0\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*},\theta^{(k),*}\right)}{\partial\sigma}\mathrm{d}s\right.\\
&\left.+\int_0^1\lambda_0\left(s|\sigma^{(k),*},\theta^{(k),*},\epsilon^{(k),*}\right)\frac{\partial\tilde{f}\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*},\theta^{(k),*}\right)}{\partial\sigma}\mathrm{d}s\right\}\\
&+\lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0\\ \left(\sigma^{(k),*},\theta^{(k),*}\right)\to(\sigma^*,\theta^*)\in\Omega_0}} \left\{2\sum_{i=1}^N\int_0^1\max\left\{0,\left(\epsilon^{(k),*}\right)^{-\alpha}\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right)\right.\right.\\
&\qquad\left.\left.-\left(\epsilon^{(k),*}\right)^{\gamma-\alpha}W_i\right\}\frac{\partial\bar{g}_i(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*})}{\partial\sigma}\mathrm{d}s\right\}
\end{aligned}
\tag{5.90}
$$

By Remark 5.3 and Lebesgue dominated convergence theorem, it follows from (5.89) that

$$\begin{aligned}
&\lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0\\ \left(\sigma^{(k),*},\theta^{(k),*}\right)\to(\sigma^*,\theta^*)\in\Omega_0}} \left\{\int_0^1 \frac{\partial\bar{\mathcal{L}}_0\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*},\theta^{(k),*}\right)}{\partial\sigma}\mathrm{d}s\right.\\
&\left.+\int_0^1 \lambda_0\left(s|\sigma^{(k),*},\theta^{(k),*},\epsilon^{(k),*}\right)\frac{\partial\tilde{f}\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*},\theta^{(k),*}\right)}{\partial\sigma}\mathrm{d}s\right\}\\
= &\int_0^1 \lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0\\ \left(\sigma^{(k),*},\theta^{(k),*}\right)\to(\sigma^*,\theta^*)\in\Omega_0}} \frac{\partial\bar{\mathcal{L}}_0\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*},\theta^{(k),*}\right)}{\partial\sigma}\mathrm{d}s\\
&+\int_0^1 \lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0\\ \left(\sigma^{(k),*},\theta^{(k),*}\right)\to(\sigma^*,\theta^*)\in\Omega_0}} \lambda_0\left(s|\sigma^{(k),*},\theta^{(k),*},\epsilon^{(k),*}\right)\\
&\qquad\frac{\partial\tilde{f}\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*},\theta^{(k),*}\right)}{\partial\sigma}\mathrm{d}s\\
= &\int_0^1 \frac{\partial\bar{\mathcal{L}}_0\left(s,\tilde{y}\left(s|\sigma^*,\theta^*\right),\sigma^*,\theta^*\right)}{\partial\sigma}\mathrm{d}s\\
&+\int_0^1 \bar{\lambda}_0\left(s|\sigma^*,\theta^*\right)\frac{\partial\tilde{f}(s,\tilde{y}\left(s|\sigma^*,\theta^*\right),\sigma^*,\theta^*)}{\partial\sigma}\mathrm{d}s\\
= &\nabla_\sigma\tilde{J}\left(\sigma^*,\theta^*\right)
\end{aligned}\tag{5.91}$$

Similarly, by Remark 5.3, Assumption 5.6 and $\xi>\alpha$, $\gamma>\alpha$, it follows from Lebesgue dominated convergence theorem that

$$\begin{aligned}
&\lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0\\ \left(\sigma^{(k),*},\theta^{(k),*}\right)\to(\sigma^*,\theta^*)\in\Omega_0}} \left\{2\sum_{i=1}^N\int_0^1 \max\left\{0,\left(\epsilon^{(k),*}\right)^{-\alpha}\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right)\right.\right.\\
&\qquad\left.\left.-\left(\epsilon^{(k),*}\right)^{\gamma-\alpha}W_i\right\}\frac{\partial\bar{g}_i(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*})}{\partial\sigma}\mathrm{d}s\right\}\\
= &\ 2\sum_{i=1}^N\int_0^1 \lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0\\ \left(\sigma^{(k),*},\theta^{(k),*}\right)\to(\sigma^*,\theta^*)\in\Omega_0}} \left\{\max\left\{0,\left(\epsilon^{(k),*}\right)^{-\alpha}\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right)\right.\right.\\
&\qquad\left.\left.-\left(\epsilon^{(k),*}\right)^{\gamma-\alpha}W_i\right\}\frac{\partial\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right)}{\partial\sigma}\right\}\mathrm{d}s\\
= &\ 0
\end{aligned}\tag{5.92}$$

We substitute (5.91) and (5.92) into (5.90) to give

$$\lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0\\ \left(\sigma^{(k),*},\theta^{(k),*}\right)\to(\sigma^*,\theta^*)\in\Omega_0}} \nabla_\sigma\tilde{J}_{\delta_k}\left(\sigma^{(k),*},\theta^{(k),*},\epsilon^{(k),*}\right)=\nabla_\sigma\tilde{J}\left(\sigma^*,\theta^*\right)\tag{5.93}$$

Similarly, we can show that

$$\lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0\\ \left(\sigma^{(k),*},\theta^{(k),*}\right)\to(\sigma^*,\theta^*)\in\Omega_0}} \nabla_\theta \tilde{J}_{\delta_k}\left(\sigma^{(k),*},\theta^{(k),*},\epsilon^{(k),*}\right) = \nabla_\theta \tilde{J}\left(\sigma^*,\theta^*\right) \tag{5.94}$$

On the other hand, we note that

$$\begin{aligned}
&\lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0\\ \left(\sigma^{(k),*},\theta^{(k),*}\right)\to(\sigma^*,\theta^*)\in\Omega_0}} \nabla_\epsilon \tilde{J}_{\delta_k}\left(\sigma^{(k),*},\theta^{(k),*},\epsilon^{(k),*}\right)\\
=&\lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0\\ \left(\sigma^{(k),*},\theta^{(k),*}\right)\to(\sigma^*,\theta^*)\in\Omega_0}} \left\{\left(\epsilon^{(k),*}\right)^{-\alpha-1}\left\{-\alpha\sum_{i=1}^{N}\int_0^1\left[\max\left\{0,\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right)\right.\right.\right.\right.\\
&\qquad\left.\left.-\left(\epsilon^{(k),*}\right)^{\gamma}W_i\right\}\right]^2\mathrm{d}s\\
&+2\gamma\sum_{i=1}^{N}\int_0^1\max\left\{0,\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right)-\left(\epsilon^{(k),*}\right)^{\gamma}W_i\right\}\left(\left(-\epsilon^{(k),*}\right)^{\gamma}W_i\right)\mathrm{d}s\\
&\left.\left.+\sum_{i=1}^{n+1}\left[\phi_i\left(\tilde{y}\left(1|\sigma^{(k),*},\theta^{(k),*}\right)\right)\right]^2\right\}+\sigma_k\beta\left(\epsilon^{(k),*}\right)^{\beta-1}\right\}\\
=&\lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0\\ \left(\sigma^{(k),*},\theta^{(k),*}\right)\to(\sigma^*,\theta^*)\in\Omega_0}} \left\{-\alpha\sum_{i=1}^{N}\int_0^1\max\left\{0,\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right)\left(\epsilon^{(k),*}\right)^{-\frac{\alpha+1}{2}}\right.\right.\\
&\qquad\left.-\left(\epsilon^{(k),*}\right)^{\gamma-\frac{\alpha+1}{2}}W_i\right\}^2\mathrm{d}s\\
&+2\gamma\sum_{i=1}^{N}\int_0^1\max\left\{0,\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right)-\left(\epsilon^{(k),*}\right)^{\gamma}W_i\right\}\\
&\qquad\left(\left(-\epsilon^{(k),*}\right)^{\gamma}W_i\right)\left(\epsilon^{(k),*}\right)^{-\alpha-1}\mathrm{d}s\\
&\left.+\sum_{i=1}^{n+1}\left[\phi_i\left(\tilde{y}\left(1|\sigma^{(k),*},\theta^{(k),*}\right)\left(\epsilon^{(k),*}\right)^{-\frac{\alpha+1}{2}}\right)\right]^2\right\}
\end{aligned} \tag{5.95}$$

Similarly, by Remark 5.3, Assumption 5.6, Assumption 5.7 and $-\alpha-1+2\xi>0,\ -\alpha-1+2\xi'>0,\ 2\gamma-\alpha-1>0$, it follows from Lebesgue dominated convergence theorem that

$$
\begin{aligned}
& \lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0 \\ \left(\sigma^{(k),*},\theta^{(k),*}\right)\to(\sigma^*,\theta^*)\in\Omega_0}} \nabla_\epsilon \tilde{J}_{\delta_k}\left(\sigma^{(k),*},\theta^{(k),*},\epsilon^{(k),*}\right) \\
= & -\alpha\sum_{i=1}^{N}\int_0^1 \lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0 \\ \left(\sigma^{(k),*},\theta^{(k),*}\right)\to(\sigma^*,\theta^*)\in\Omega_0}} \left[\max\left\{0,\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right)\left(\epsilon^{(k),*}\right)^{-\frac{\alpha+1}{2}}\right.\right. \\
& \left.\left. -\left(\epsilon^{(k),*}\right)^{\gamma-\frac{\alpha+1}{2}}W_i\right\}\right]^2 \mathrm{d}s \\
& +2\gamma\sum_{i=1}^{N}\int_0^1 \lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0 \\ \left(\sigma^{(k),*},\theta^{(k),*}\right)\to(\sigma^*,\theta^*)\in\Omega_0}} \left[\max\left\{0,\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*},\theta^{(k),*}\right)\right.\right. \\
& \left.\left. -\left(\epsilon^{(k),*}\right)^{\gamma}W_i\right\}\left(\left(-\epsilon^{(k),*}\right)^{\gamma}W_i\right)\left(\epsilon^{(k),*}\right)^{-\alpha-1}\mathrm{d}s\right] \\
& +\sum_{i=1}^{n+1} \lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0 \\ \left(\sigma^{(k),*},\theta^{(k),*}\right)\to(\sigma^*,\theta^*)\in\Omega_0}} \left[\phi_i\left(\tilde{y}\left(1|\sigma^{(k),*},\theta^{(k),*}\right)\left(\epsilon^{(k),*}\right)^{-\frac{\alpha+1}{2}}\right)\right]^2 \\
= & \ 0
\end{aligned} \tag{5.96}
$$

The proof is completed. □

Theorem 5.4. *Let $\epsilon^{(k),*} \to \epsilon^* = 0$ and $\left(\sigma^{(k),*},\theta^{(k),*}\right) \to (\sigma^*,\theta^*) \in \Omega_0$ be such that $\tilde{J}_{\delta_k}\left(\sigma^{(k),*},\theta^{(k),*},\epsilon^{(k),*}\right)$ is finite. Then, (σ^*,θ^*) is a local minimizer of Problem (P5(p)).*

Proof. On a contrary, assume that (σ^*,θ^*) is not a local minimizer of Problem (P5(p)). Then, there must exist a feasible point $(\hat{\sigma}^*,\hat{\theta}^*) \in \mathcal{N}_\Lambda(\sigma^*,\theta^*)$ of Problem (P5(p)) such that

$$\tilde{J}\left(\hat{\sigma}^*,\hat{\theta}^*\right) < \tilde{J}(\sigma^*,\theta^*) \tag{5.97}$$

where $\mathcal{N}_\Lambda(\sigma^*,\theta^*)$ is a Λ-neighborhood of $\left(\hat{\sigma}^*,\hat{\theta}^*\right)$ in Ω_0 for some $\Lambda > 0$. Since $\left(\sigma^{(k),*},\theta^{(k),*},\epsilon^{(k),*}\right)$ is a local minimizer of Problem (P_{δ_k}), there exists a sequence $\{\xi_k\}$, such that

$$\tilde{J}_{\delta_k}\left(\sigma,\theta,\epsilon^{(k),*}\right) \geq \tilde{J}_{\delta_k}\left(\sigma^{(k),*},\theta^{(k),*},\epsilon^{(k),*}\right)$$

for any $(\sigma,\theta) \in \mathcal{N}_{\xi_k}\left(\sigma^{(k),*},\theta^{(k),*}\right)$. Now, we construct a sequence of feasible points $\{(\hat{\sigma}^{(k),*},\hat{\theta}^{(k),*})\}$ of Problem (P5(p)) satisfying

$$\left\|\left(\hat{\sigma}^{(k),*},\hat{\theta}^{(k),*}\right)-\left(\sigma^{(k),*},\theta^{(k),*}\right)\right\| \leq \frac{\xi_k}{k}$$

Clearly,

$$\tilde{J}_{\delta_k}\left(\hat{\sigma}^{(k),*},\hat{\theta}^{(k),*},\epsilon^{(k),*}\right) \geq \tilde{J}_{\delta_k}\left(\sigma^{(k),*},\theta^{(k),*},\epsilon^{(k),*}\right) \tag{5.98}$$

Letting $k \to +\infty$, we have

$$\begin{aligned}\lim_{k\to+\infty}\left\|\left(\hat{\sigma}^{(k),*},\hat{\theta}^{(k),*}\right)-\left(\hat{\sigma}^{*},\hat{\theta}^{*}\right)\right\| \leq\ & \lim_{k\to+\infty}\left\|\left(\hat{\sigma}^{(k),*},\hat{\theta}^{(k),*}\right)-\left(\sigma^{(k),*},\theta^{(k),*}\right)\right\| \\ &+\lim_{k\to+\infty}\left\|\left(\sigma^{(k),*},\theta^{(k),*}\right)-\left(\sigma^{*},\theta^{*}\right)\right\| \\ &+\left\|\left(\sigma^{*},\theta^{*}\right)-\left(\hat{\sigma}^{*},\hat{\theta}^{*}\right)\right\| \\ &\leq 0+0+\Lambda\end{aligned} \tag{5.99}$$

However, $\Lambda > 0$ is arbitrary. Thus,

$$\lim_{k\to+\infty}\left(\hat{\sigma}^{(k),*},\hat{\theta}^{(k),*}\right)=\left(\hat{\sigma}^{*},\hat{\theta}^{*}\right) \tag{5.100}$$

Letting $k \to +\infty$ in (5.98), it follows from the first part of Theorem 5.3 and (5.100) that

$$\begin{aligned}&\lim_{k\to+\infty}\tilde{J}_{\delta_k}\left(\hat{\sigma}^{(k),*},\hat{\theta}^{(k),*},\epsilon^{(k),*}\right)\\ &=\tilde{J}\left(\hat{\sigma}^{*},\hat{\theta}^{*}\right)\geq\lim_{k\to+\infty}\tilde{J}_{\delta_k}\left(\sigma^{(k),*},\theta^{(k),*},\epsilon^{(k),*}\right)\\ &=\tilde{J}\left(\sigma^{*},\theta^{*}\right)\end{aligned} \tag{5.101}$$

This is a contradiction to (5.97), and hence it completes the proof. □

Theorem 5.5. *Let $-\alpha-\beta+2\xi>0$, $-\alpha-\beta+2\xi'>0$ and $-\alpha-\beta+2\gamma>0$. Then, there exists a $k_0>0$, such that $\epsilon^{(k),*}=0$, $\left(\sigma^{(k),*},\theta^{(k),*}\right)$ is local minimizer of Problem (P5(p)), for $k\geq k_0$.*

Proof. On a contrary, we assume that the conclusion is false. Then, there exists a subsequence of $\left\{\left(\sigma^{(k),*},\theta^{(k),*},\epsilon^{(k),*}\right)\right\}$, which is denoted by the original sequence, such that for any $k_0>0$, there exists a $k'>k_0$ satisfying $\epsilon^{(k'),*}\neq 0$. By Theorem 5.2, we have

$$\epsilon^{(k),*}\to\epsilon^{*}=0,\ \left(\sigma^{(k),*},\theta^{(k),*}\right)\to\left(\sigma^{*},\theta^{*}\right)\in\Omega_0,\ \text{as } k\to+\infty$$

Since $\epsilon^{(k),*}\neq 0$ for all k, it follows from dividing (5.39) by $\left(\epsilon^{(k),*}\right)^{\beta-1}$ that

$$\begin{aligned}&\left(\epsilon^{(k),*}\right)^{-\alpha-\beta}\left\{-\alpha\sum_{i=1}^{N}\int_0^1\left[\max\left\{0,\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right)-\left(\epsilon^{(k),*}\right)^{\gamma}W_i\right\}\right]^2\mathrm{d}s\right.\\ &+2\gamma\sum_{i=1}^{N}\int_0^1\max\left\{0,\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right)-\left(\epsilon^{(k),*}\right)^{\gamma}W_i\right\}\left(\left(-\epsilon^{(k),*}\right)^{\gamma}W_i\right)\mathrm{d}s\\ &\left.-\alpha\sum_{i=1}^{n+1}\left(\tilde{y}_i\left(1|\sigma^{(k),*},\theta^{(k),*}\right)-\tilde{y}_i^{f}\right)^2\right\}+\delta_k\beta=0\end{aligned} \tag{5.102}$$

This is equivalent to

$$\begin{aligned}
&\left(\epsilon^{(k),*}\right)^{-\alpha-\beta}\Bigg\{-\alpha\sum_{i=1}^{N}\int_0^1\left[\max\left\{0,\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right)-\left(\epsilon^{(k),*}\right)^{\gamma}W_i\right\}\right]^2\mathrm{d}s\\
&+2\gamma\sum_{i=1}^{N}\int_0^1\left[\max\left\{0,\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right)-\left(\epsilon^{(k),*}\right)^{\gamma}W_i\right\}\left(\left(-\epsilon^{(k),*}\right)^{\gamma}W_i\right)\right.\\
&+\max\left\{0,\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right)-\left(\epsilon^{(k),*}\right)^{\gamma}W_i\right\}\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right)\\
&\left.-\max\left\{0,\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right)-\left(\epsilon^{(k),*}\right)^{\gamma}W_i\right\}\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right)\right]\mathrm{d}s\\
&-\alpha\sum_{i=1}^{n+1}\left(\tilde{y}_i\left(1|\sigma^{(k),*},\theta^{(k),*}\right)-\tilde{y}_i^f\right)^2\Bigg\}+\delta_k\beta=0
\end{aligned} \tag{5.103}$$

Rearranging (5.103) yields

$$\begin{aligned}
&\left(\epsilon^{(k),*}\right)^{-\alpha-\beta}\Bigg\{(2\gamma-\alpha)\sum_{i=1}^{N}\int_0^1\left[\max\left\{0,\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right)-\left(\epsilon^{(k),*}\right)^{\gamma}W_i\right\}\right]^2\mathrm{d}s\\
&-\alpha\sum_{i=1}^{n+1}\left(\tilde{y}_i\left(1|\sigma^{(k),*},\theta^{(k),*}\right)-\tilde{y}_i^f\right)^2\Bigg\}+\delta_k\beta\\
=\;&2\gamma\left(\epsilon^{(k),*}\right)^{-\alpha-\beta}\sum_{i=1}^{N}\int_0^1\Big\{\max\left\{0,\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right)-\left(\epsilon^{(k),*}\right)^{\gamma}W_i\right\}\\
&\qquad\qquad\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right)\Big\}\mathrm{d}s
\end{aligned} \tag{5.104}$$

Note that $-\alpha-\beta+2\xi>0$ and $-\alpha-\beta+2\xi'>0$. Then, by Remark 5.3 and Lebesgue dominated convergence theorem, we can show that the left hand side of (5.104) yields

$$\begin{aligned}
&\left(\epsilon^{(k),*}\right)^{-\alpha-\beta}\Bigg\{(2\gamma-\alpha)\sum_{i=1}^{N}\int_0^1\left[\max\left\{0,\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right)-\left(\epsilon^{(k),*}\right)^{\gamma}W_i\right\}\right]^2\mathrm{d}s\\
&\qquad-\alpha\sum_{i=1}^{n+1}\left(\tilde{y}_i\left(1|\sigma^{(k),*},\theta^{(k),*}\right)-\tilde{y}_i^f\right)^2\Bigg\}+\delta_k\beta\to\infty
\end{aligned} \tag{5.105}$$

However, under the same conditions and $-\alpha-\beta+2\gamma>0$, we can show, also by Remark 5.3, Assumption 5.6 and Lebesgue dominated convergence theorem, that the right hand side of (5.104) gives

$$\begin{aligned}
&2\gamma\left(\epsilon^{(k),*}\right)^{-\alpha-\beta}\sum_{i=1}^{N}\int_0^1\max\left\{0,\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right)-\left(\epsilon^{(k),*}\right)^{\gamma}W_i\right\}\\
&\qquad\bar{g}_i\left(s,\tilde{y}\left(s|\sigma^{(k),*},\theta^{(k),*}\right),\sigma^{(k),*}\right)\mathrm{d}s\to 0
\end{aligned} \tag{5.106}$$

This is a contradiction. Thus, the proof is completed. □

To proceed further, we will define another two new problems. Before we define the first problem, we define

$$\Upsilon^{p,\zeta} = \left\{(\sigma,\tau) \in \Lambda : \left\|x(T|\sigma,\tau) - x^f\right\| \le \zeta\right\} \tag{5.107}$$

where Λ is defined by (5.27), and

$$\overset{\circ}{\Upsilon}^{p,\zeta} = \left\{(\sigma,\tau) \in \overset{\circ}{\Lambda} : \left\|x\left(T|\sigma,\tau\right) - x^f\right\| < \zeta\right\}$$

where

$$\overset{\circ}{\Lambda} = \left\{(\sigma,\tau) \in \Xi \times \Gamma : g_i\left(t, x^p\left(T|\sigma,\tau\right), u^p\left(T|\sigma,\tau\right)\right) < 0,\ t \in [0,T],\ i = 1,2,\dots,N\right\}$$

We now define ζ-tolerated version of the approximate problem (P5(p)) as follows:
Problem (P5$^{\zeta}$(p)) Given system (5.1a)-(5.1b), find a $(\sigma,\tau) \in \Upsilon^{p,\zeta}$ such that cost function (5.24) is minimized.

We let $\left(\sigma^{p,\zeta,*}, \tau^{p,\zeta,*}\right)$ and $\left(\sigma^{p,*}, \tau^{p,*}\right)$ be optimal parameter vector of the problems (P5$^{\zeta}$(p)) and (P5(p)), respectively.

Similarly, before we define the second problem, we define

$$\mathcal{G} = \left\{u \in \mathcal{U} : g_i\left(t, x\left(T|u\right), u(t)\right) \le 0,\ t \in [0,T], i = 1,2,\dots,N\right\} \tag{5.108}$$

$$\mathcal{H}^{\zeta} = \left\{u \in \mathcal{G} : \left\|x\left(T|u\right) - x^f\right\| \le \zeta\right\} \tag{5.109}$$

and

$$\overset{\circ}{\mathcal{H}}^{\zeta} = \left\{u \in \overset{\circ}{\mathcal{G}} : \left\|x\left(T|u\right) - x^f\right\| < \zeta\right\} \tag{5.110}$$

where

$$\overset{\circ}{\mathcal{G}} = \left\{u \in \mathcal{U} : g_i\left(x\left(T|u\right), u(t)\right) < 0,\ t \in [0,T], i = 1,2,\dots,N\right\} \tag{5.111}$$

We then define ζ-tolerated version of the approximate problem (P5) as follows:
Problem (P5$^{\zeta}$) Given system (5.1a)-(5.1b), find a $u \in \mathcal{H}^{\zeta}$ such that the cost function (5.6) is minimized.

We let $u^{\zeta,*}$ and u^* be optimal control of the problems (P5$^{\zeta}$) and (P5), respectively.

To continue, we assume the following assumptions are satisfied.

Assumption 5.8. For each $(\sigma,\tau) \in \Upsilon^{p,\zeta}$, there exits a $(\bar{\sigma},\bar{\tau}) \in \overset{\circ}{\Upsilon}^{p,\zeta}$ such that

$$\alpha(\bar{\sigma},\bar{\tau}) + (1-\alpha)(\sigma,\tau) \in \overset{\circ}{\Upsilon}^{p,\zeta},\ \alpha \in (0,1]; \tag{5.112}$$

Assumption 5.9. there exits a $p_1 > 0$ such that

$$\lim_{\zeta \to 0} J^p\left(\sigma^{p,\zeta,*}, \tau^{p,\zeta,*}\right) = J^p\left(\sigma^{p,*}, \tau^{p,*}\right) \tag{5.113}$$

uniformly with respect to $p \geq p_1$.

Assumption 5.10. For each $u \in \mathcal{H}^{\zeta}$, there exits a $\bar{u} \in \overset{\circ}{\mathcal{H}}^{\zeta}$ such that

$$\alpha \bar{u} + (1-\alpha) u \in \overset{\circ}{\mathcal{H}}^{\zeta}, \ \alpha \in (0,1]; \tag{5.114}$$

Assumption 5.11.

$$\lim_{\zeta \to 0} J\left(u^{\zeta,*}\right) = J\left(u^*\right) \tag{5.115}$$

uniformly with respect to $p \geq p_1$

Before continuing, we define another two optimal control problems.

Problem (Q5) Given system (5.1a)-(5.1b), find a $u \in \mathcal{G}$ such that the cost function (5.6) is minimized.

Problem (Q5(p)) Given system (5.1a)-(5.1b), find a $(\sigma, \tau) \in \Lambda$ such that cost function (5.10) is minimized.

Then, we recall a result (Theorem 8.5.1) from Chapter 8 of [87]

Lemma 5.2. *Let $\hat{u}^{p,*}$ be an optimal control, which is constructed from the optimal solution $(\sigma^{p,*}, \tau^{p,*})$ of Problem (Q5(p)) according to (5.7). Suppose Problem (Q5) has an optimal control $\hat{u}^*$, Then,*

$$\lim_{p \to \infty} J\left(\hat{u}^{p,*}\right) = J\left(\hat{u}^*\right). \tag{5.116}$$

Theorem 5.6. *Let $u^{p,*}$ (respectively, $u^{p,\zeta,*}$), which is constructed from the corresponding optimal solution $(\sigma^{p,*}, \tau^{p,*})$ (respectively, $\left(\sigma^{p,\zeta,*}, \tau^{p,\zeta,*}\right)$) according to (5.7), be an optimal control of the approximate problem (P5(p)) (respectively, (P5$^{\zeta}$(p))). Suppose that u^* is an optimal control of the Problem (P5). Then,*

$$\lim_{p \to +\infty} J\left(u^{p,*}\right) = J\left(u^*\right) \tag{5.117}$$

Proof. From Assumption 5.9 and Assumption 5.11, it is clear that for any $c > 0$, there exits a $\hat{\zeta} > 0$ such that

$$0 \leq J\left(u^*\right) - J\left(u^{\hat{\zeta},*}\right) \leq c \tag{5.118}$$

and

$$0 \leq J\left(u^{p,*}\right) - J\left(u^{p,\hat{\zeta},*}\right) \leq c \tag{5.119}$$

for all $p \geq p_1$.

Note that, the terminal state inequality constraints specified in (5.27) (respectively, (5.5)) of Problem (P5(p)) (respectively, Problem (P5)) are relaxed to terminal state inequality constraints specified in (5.107) (respectively, (5.109)) of Problem (P5$^\varsigma$(p)) (respectively, Problem (P5$^\varsigma$). Thus, it has the same structure as Problem (Q5(p)) (respectively, Problem (Q5)). From Lemma 5.2, we have

$$\lim_{p\to\infty} J\left(u^{p,\hat{\zeta},*}\right) = J\left(u^{\hat{\zeta},*}\right) \tag{5.120}$$

Therefore, it follows (5.118), (5.119) and (5.120) that

$$\left\|\lim_{p\to\infty} J\left(u^{p,*}\right) - J\left(u^{*}\right)\right\| \leq 2c \tag{5.121}$$

Since $c \geq 0$ is arbitrary, we have

$$\lim_{p\to\infty} J\left(u^{p,*}\right) = J\left(u^{*}\right) \tag{5.122}$$

This completes the proof. □

Before continuing, we recall three results (Lemma 6.4.2, Lemma 6.4.4 and Theorem 2.6.4) from [87].

Lemma 5.3. *Let $\{u^p\}_{p=1}^{\infty}$ be a bounded sequence in L_∞^r. Then, the sequence $\{x(\cdot|u^p)\}_{p=1}^{\infty}$ of the corresponding solutions of the system (5.1a)-(5.1b) is also bounded in L_∞^n.*

Lemma 5.4. *Let $\{u^p\}_{p=1}^{\infty}$ be a bounded sequence in L_∞^r that converges to a function u a.e. in $[0,T]$. Then,*

$$\lim_{p\to\infty} J\left(u^{p}\right) = J\left(u\right). \tag{5.123}$$

Lemma 5.5. *Let $\{f^{(k)}\} \subset L_1(I) \equiv L_1(I,\mathbb{R})$, where I is a finite interval. Suppose that and let there exits a function $g \in L_1(I)$ such that $|f^{(k)}(t)| \leq |g(t)|$ a.e. on I for all $k = 1,2,3,\cdots$ and that*

$$f^{(k)}(t) \to f^{0}(t) \text{ a.e. on } I.$$

Then $f^0 \in L_1(I)$ and

$$\int_I f^{(k)}(t)\mathrm{d}t \to \int_I f^{0}(t)\mathrm{d}t \text{ as } k \to \infty. \tag{5.124}$$

Theorem 5.7. *Let $u^{p,*}$ be an optimal control which is constructed from the optimal solution of Problem (P5(p)) according to (5.7), and let u^* be an optimal control of Problem (P5). Suppose that*

$$\lim_{p\to+\infty} u^{p,*} = \bar{u}, \text{ a.e. on } [0,\, T] \tag{5.125}$$

Then, $\bar{u}$ is an optimal control of the problem (P5)

$$\lim_{p\to+\infty} J\left(u^{p,*}\right) = J\left(u^{*}\right) \tag{5.126}$$

Proof. Since $u^{p,*} \to \bar{u}$, a.e. on $[0, T]$, it follows from Lemma 5.4 that

$$\lim_{p\to+\infty} J\left(u^{p,*}\right) = J\left(\bar{u}\right) \tag{5.127}$$

Next, we shall show that $\bar{u}$ is feasible for Problem (P5). Assume the contrary. Then, we have

$$\left\|x\left(t|\bar{u}\right) - x^f\right\| \neq 0 \tag{5.128}$$

or there exits a $i \in \{1, \ldots, N\}$ and a non-zero interval $I \subset [0, T]$ such that

$$g_i\left(t, x\left(t|\bar{u}\right), \bar{u}(t)\right) > 0 \ \forall t \in I \tag{5.129}$$

However, we know that

$$\left\|x\left(t|u^{p,*}\right) - x^f\right\| = 0 \tag{5.130}$$

and

$$g_i\left(t, x\left(t|u^{p,*}\right), u^{p,*}(t)\right) \leq 0 \ \forall t \in I \tag{5.131}$$

From Lemma 5.3, we note that, as $p \to \infty$

$$x\left(t|u^{p,*}\right) \to x\left(t|\bar{u}\right) \tag{5.132}$$

for each $t \in [0, T]$. From (5.130) and (5.132), we see that (5.128) is false. Now, it remains to falsify the validity of (5.129). For this, if follows from Assumption 5.3 and Lemma 5.3 that

$$\left\|g_i\left(t, x\left(t|u^{p,*}\right), u^{p,*}(t)\right) - g_i\left(t, x\left(t|\bar{u}\right), \bar{u}(t)\right)\right\| \tag{5.133}$$

is uniformly bounded on $I \times \mathbb{R}^n \times \mathbb{R}^r$. Thus, by (5.132), it follows from Lemma 5.5 that

$$\lim_{p\to\infty} \int_I \left\|g_i\left(t, x\left(t|u^{p,*}\right), u^{p,*}(t)\right) - g_i\left(t, x\left(t|\bar{u}\right), \bar{u}(t)\right)\right\| \mathrm{d}t = 0. \tag{5.134}$$

Therefore, if (5.129) is valid, then

$$\begin{aligned} 0 &< \int_I g_i\left(t, x\left(t|\bar{u}\right), \bar{u}(t)\right) \mathrm{d}t \\ &= \int_I \left[g_i\left(t, x\left(t|\bar{u}\right), \bar{u}(t)\right) - g_i\left(t, x\left(t|u^{p,*}\right), u^{p,*}(t)\right)\right] \mathrm{d}t + \int_I g_i\left(t, x\left(t|u^{p,*}\right), u^{p,*}(t)\right) \mathrm{d}t \end{aligned}$$

Thus,

$$\begin{aligned} 0 &< \int_I g_i\left(t, x\left(t|\bar{u}\right), \bar{u}(t)\right) \mathrm{d}t \\ &\leq \int_I \left[g_i(t, x(t|\bar{u}), \bar{u}(t)) - g_i\left(t, x\left(t|u^{p,*}\right), u^{p,*}(t)\right)\right] \mathrm{d}t = 0 \end{aligned}$$

This is a contradiction. Thus, $\bar{u}$ is feasible for the problem (P5). On this basis, if follows from Theorem 5.6 and (5.127) that $\bar{u}$ is also an optimal control of the problem (P5). $\square$

Algorithm 5.1.
Step 1 Set $\delta^{(1)} = 10$, $\epsilon^{(1)} = 0.1$, $\epsilon^* = 10^{-9}$, $\beta > 2$, choose an initial point $(\sigma^0, \theta^0, \epsilon^0)$, the iteration index $k = 0$. The values of γ and α are chosen depending on the specific structure of Problem (P5) concerned.
Step 2 Solve Problem (P_{δ_k}), and let $\left(\sigma^{(k),*}, \theta^{(k),*}, \epsilon^{(k),*}\right)$ be the minimizer obtained.
Step 3 If $\epsilon^{(k),*} > \epsilon^*$, $\delta^{(k)} < 10^8$,
set $\delta^{(k+1)} = 10 \times \sigma^{(k)}$, $k := k+1$. Go to **Step 2** with $\left(\sigma^{(k),*}, \theta^{(k),*}, \epsilon^{(k),*}\right)$ as the new initial point in the new optimization process
Else set $\epsilon^{(k),*} := \epsilon^*$, then go to **Step 4**
Step 4 Check the feasibility of $\sigma^{(k),*}, \theta^{(k),*}$ (i.e., check whether or not $\max\limits_{1 \le i \le N} \max\limits_{s \in [0,1]} \bar{g}_i \left(s, \tilde{y}\left(s|\sigma^{(k),*}, \theta^{(k),*}\right), \sigma^{(k),*}\right) \le 0$).
If $\left(\sigma^{(k),*}, \theta^{(k),*}\right)$ is feasible, then it is a local minimizer of Problem (P5(p)). Exit.
Else go to **Step 5**
Step 5: Adjust the parameters α, β and γ such that the conditions of Lemma 5.1 are satisfied. Set $\delta^{(k+1)} = 10\delta^{(k)}$, $\epsilon^{(k+1)} = 0.1\epsilon^{(k)}$, $k := k+1$. Go to **Step 2**.

Remark 5.6. In **Step 3**, if $\epsilon^{(k),*} > \epsilon^*$, it follows from Theorem 5.2 and Theorem 5.3 that $\left(\sigma^{(k),*}, \theta^{(k),*}\right)$ cannot be a feasible point. This means that the penalty parameter δ is not chosen large enough. Thus we need to increase δ. If $\delta_k > 10^8$, but still $\epsilon^{(k),*} > \epsilon^*$, then we should adjust the value of α, β and γ, such that the conditions of Theorem 5.3 are satisfied. Then, go to **Step 2**.

Remark 5.7. Clearly, we cannot check the feasibility of $g_i(y(s), \sigma) \le 0$, $i = 1, 2, \ldots, N$, for every $s \in [0, 1]$. In practice, we choose a set which contains a dense enough of points in $[0, 1]$. Check the feasibility of $g_i(y(s), \sigma) \le 0$ over this set for each $i = 1, 2, \ldots, N$.

Remark 5.8. Although we have proved that a local minimizer of the exact penalty function optimization problem $(\mathrm{P5}_{\delta_k})$ will converge to a local minimizer of the original problem (P5(p)), we need, in actual computation, set a lower bound $\epsilon^* = 10^{-9}$ for $\epsilon^{(k),*}$ so as to avoid the situation of being divided by $\epsilon^{(k),*} = 0$, leading to infinity.

5.4 Examples

5.4.1 Example 5.1

The following optimal control problem is taken from [36] and [105]:

$$\min \mathrm{g}_0 \tag{5.135}$$

where

$$g_0 = \int_0^1 \left\{ [x_1(t)]^2 + [x_2(t)]^2 + 0.005\,[u(t)]^2 \right\} \mathrm{d}t \tag{5.136}$$

subject to

$$\begin{cases} \dot{x}_1(t) = x_2(t) & (5.137a) \\ \dot{x}_2(t) = -x_2(t) + u(t) & (5.137b) \end{cases}$$

with initial conditions

$$x_1(0) = 0,\ x_2(0) = -1 \tag{5.138}$$

and the continuous state inequality constraint

$$g_1 = 8(t-0.5)^2 - 0.5 - x_2(t) \geq 0,\ \forall t \in [0,1] \tag{5.139}$$

together with the control constraints

$$-20 \leq u(t) \leq 20,\ \forall t \in [0,1]. \tag{5.140}$$

In this problem, we set $p = 20$, $\gamma = 3$ and $W_1 = 0.3$. The result is shown below.

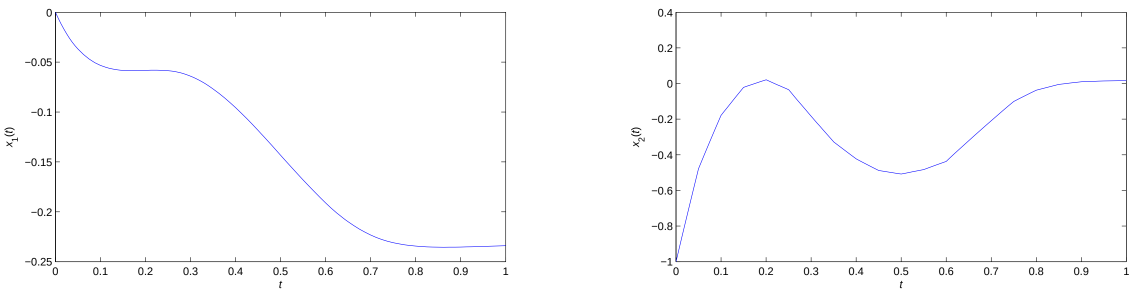

Figure 5.1: Optimal state variables for Example 5.1

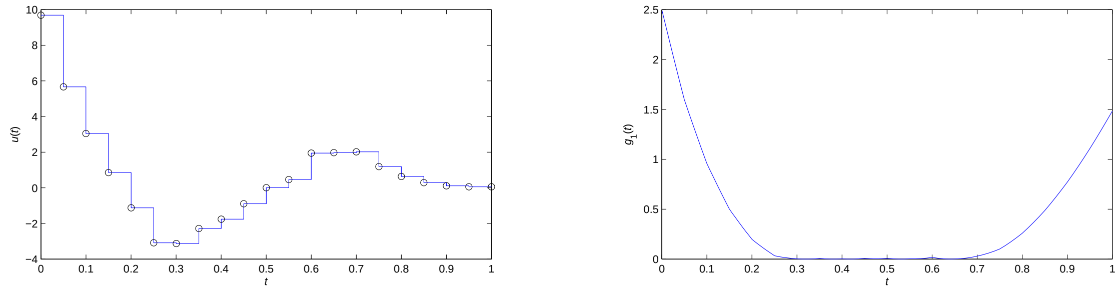

Figure 5.2: Optimal control and the resulting constraint function for Example 5.1

The optimal objective function value is: $g_0^* = 1.75101803 \times 10^{-1}$, where $\delta = 1.0 \times 10^6$

and $\epsilon = 1.89531 \times 10^{-5}$. The continuous inequality constraints (5.139) is satisfied for all t$\in [0,1]$. Comparing with the results obtained for Example 6.7.2 in [36], the minimum value of the objective function is almost the same (it is 0.1730 in [36]). However, in [36], the continuous inequality constraints (5.139) are slightly violated at some t$\in [0,1]$. The optimal control, the optimal state and the constraint are shown in Figure 5.1 and Figure 5.2.

5.4.2 Example 5.2

We consider a realistic and complex problem of transferring containers from a ship to a cargo truck at the port of Kobe. It is taken from [106]. The containers crane is driven by a hoist motor and a trolley drive motor. For safety reason, the objective is to minimize the swing during and at the end of the transfer. The problem is summarized after appropriate normalization as follows:

$$\text{minimize} \left\{ g_0 = 4.5 \int_0^1 \left[(x_3(t))^2 + (x_6(t))^2 \right] \mathrm{d}t \right\} \tag{5.141}$$

subject to the dynamical equations

$$\begin{cases} \dot{x}_1(t) = 9x_4(t) & (5.142a) \\ \dot{x}_2(t) = 9x_5(t) & (5.142b) \\ \dot{x}_3(t) = 9x_6(t) & (5.142c) \\ \dot{x}_4(t) = 9(u_1(t) + 17.2656x_3(t)) & (5.142d) \\ \dot{x}_5(t) = 9u_2(t) & (5.142e) \\ \dot{x}_6(t) = -\dfrac{9}{x_2(t)}[u_1(t) + 27.0756x_3(t) + 2x_5(t)x_6(t)] & (5.142f) \end{cases}$$

where

$$\begin{cases} x(0) = [0, 22, 0, 0, -1, 0]^\top & (5.143a) \\ x(1) = [10, 14, 0, 2.5, 0, 0]^\top & (5.143b) \end{cases}$$

and

$$|u_1(t)| \leq 2.83374 \tag{5.144}$$

$$-0.80865 \leq u_2(t) \leq 0.71265, \ \forall t \in [0,1]. \tag{5.145}$$

with continuous state inequality constraints

$$|x_4(t)| \leq 2.5, \ \forall t \in [0,1], \tag{5.146a}$$

$$|x_5(t)| \leq 1.0, \ \forall t \in [0,1]. \tag{5.146b}$$

The bounds on the states can be formulated as the continuous inequality constraints as follows:

$$g_1 = -x_4(t) + 2.5 \geq 0 \tag{5.147}$$

$$g_2 = x_4(t) + 2.5 \geq 0 \tag{5.148}$$

$$g_3 = -x_5(t) + 1.0 \geq 0 \tag{5.149}$$

$$g_4 = x_5(t) + 1.0 \geq 0 \tag{5.150}$$

In this problem, we set $p = 20$, $\gamma = 3$ and $W_1 = W_2 = W_3 = W_4 = 0.3$. The result

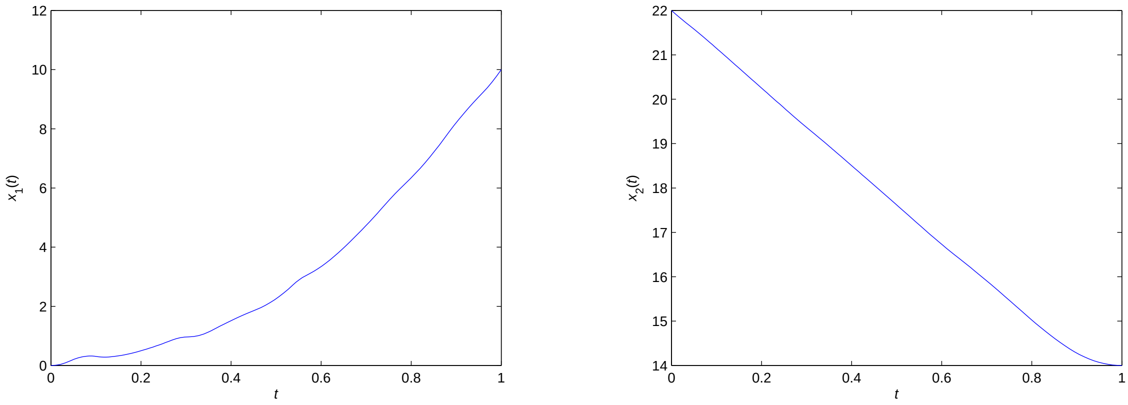

Figure 5.3: Optimal state variables for Example 5.2

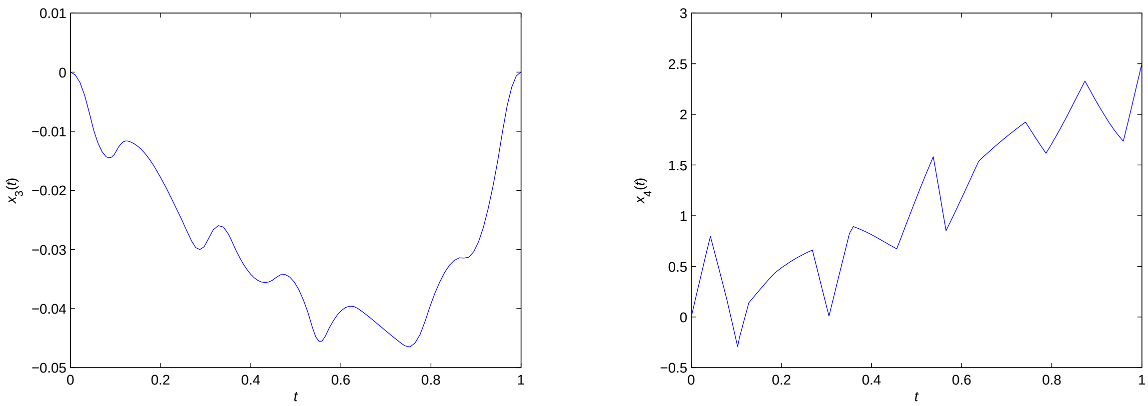

Figure 5.4: The optimal state variables for Example 5.2

obtained is shown below. The optimal objective function value is: $g_0^* = 5.75921513 \times 10^{-3}$, where $\delta = 1.0 \times 10^5$ and $\epsilon = 1.00057 \times 10^{-7}$. All the continuous inequality constraints are

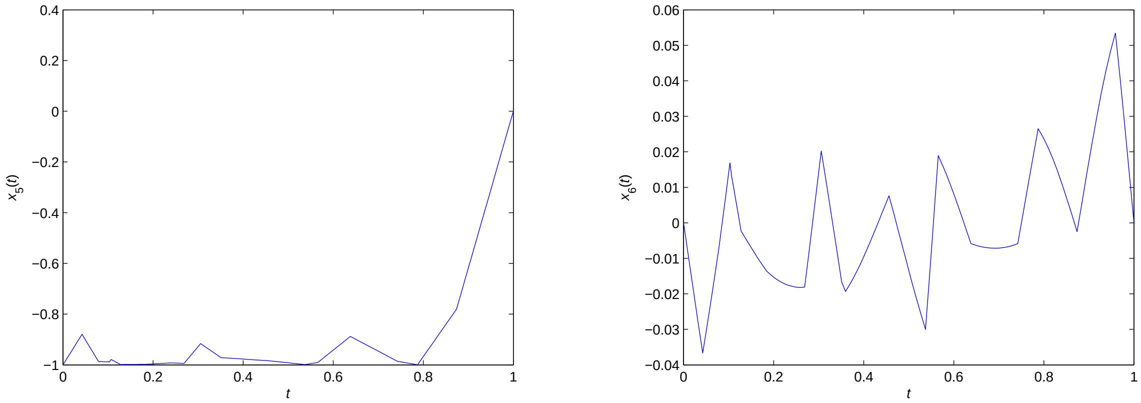

Figure 5.5: Optimal state variables for Example 5.2

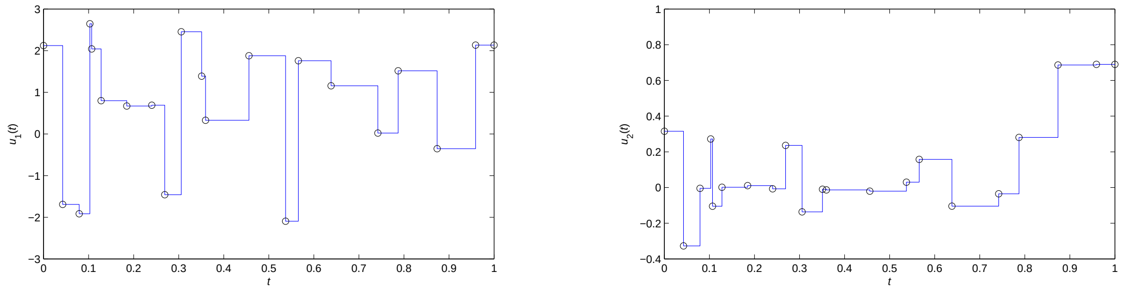

Figure 5.6: Optimal controls for Example 5.2

satisfied for all t$\in [0,1]$. Comparing with the results obtained for Example 6.7.3 in [36], our minimum value of the objective function is slightly larger (it is 4.684×10^{-3} in [36]). However, in [36], the continuous inequality constraints are not completely satisfied for all t$\in [0,1]$. The optimal state variables, the optimal control and the constraints are shown in Figure 5.3-Figure 5.8, respectively.

5.4.3 Example 5.3

The following problem is taken from [107]: Find a control $u : [0, 4.5] \to \mathbb{R}$ that minimizes the cost function

$$\int_0^{4.5} \left\{ u^2(t) + x_1^2(t) \right\} \mathrm{d}t \tag{5.151}$$

subject to the following dynamical equations

$$\begin{cases} \dot{x}_1(t) = x_2(t) & (5.152\text{a}) \\ \dot{x}_2(t) = -x_1(t) + x_2(t)(1.4 - 0.14x_2^2(t)) + 4u(t) & (5.152\text{b}) \end{cases}$$

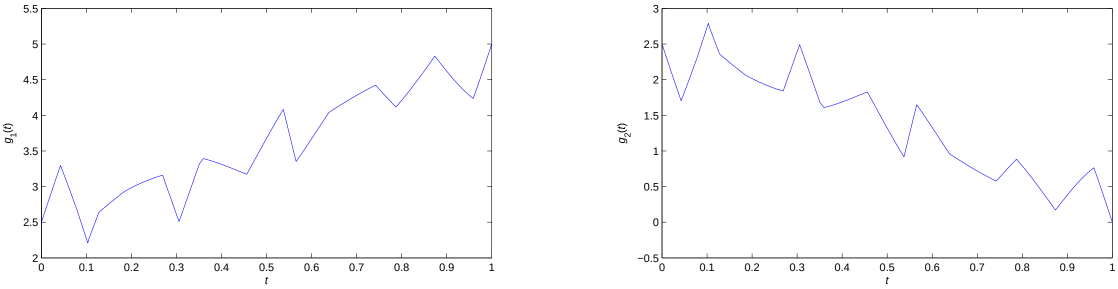

Figure 5.7: The constraints function under the optimal control for Example 5.2

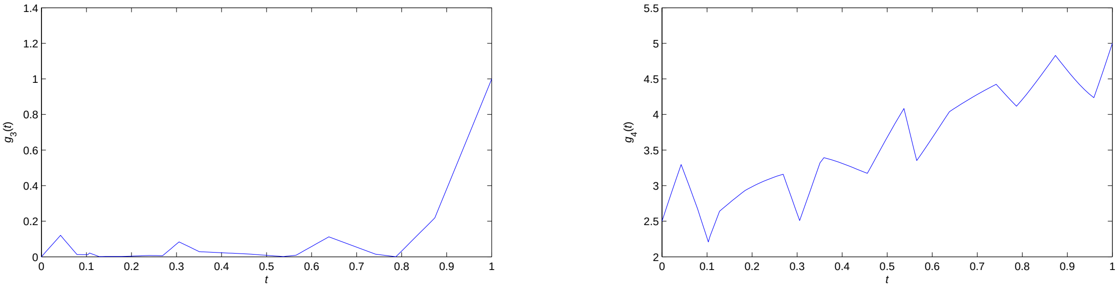

Figure 5.8: The constraints function under the optimal control for Example 5.2

with the initial conditions

$$\begin{cases} x_1(0) = -5 & (5.153a) \\ x_2(0) = -5 & (5.153b) \end{cases}$$

and the continuous inequality constraint

$$g_1 = -u(t) - \frac{1}{6}x_1(t) \geq 0, \;\; t \in [0, 4.5]. \tag{5.154}$$

In this problem, we set $p = 10$, $\gamma = 3$ and $W_1 = 0.3$. The result is shown below. The optimal objective function value obtained is $g_0^* = 4.58048380 \times 10^1$, where $\delta = 1.0 \times 10^4$ and $\epsilon = 9.99998 \times 10^{-5}$. The continuous inequality constraint (5.154) is satisfied for all t$\in [0, 4.5]$. In [97], the optimal objective function value is about $4.6961921e \times 10^1$, which is slightly larger than our result. The optimal state variables, the optimal control and the constraint are shown in Figure 5.9 and Figure 5.10.

5.5 Conclusions

In this chapter, we present a new exact penalty function method for optimal control problems subject to continuous inequality constraints and terminal equality constraints.

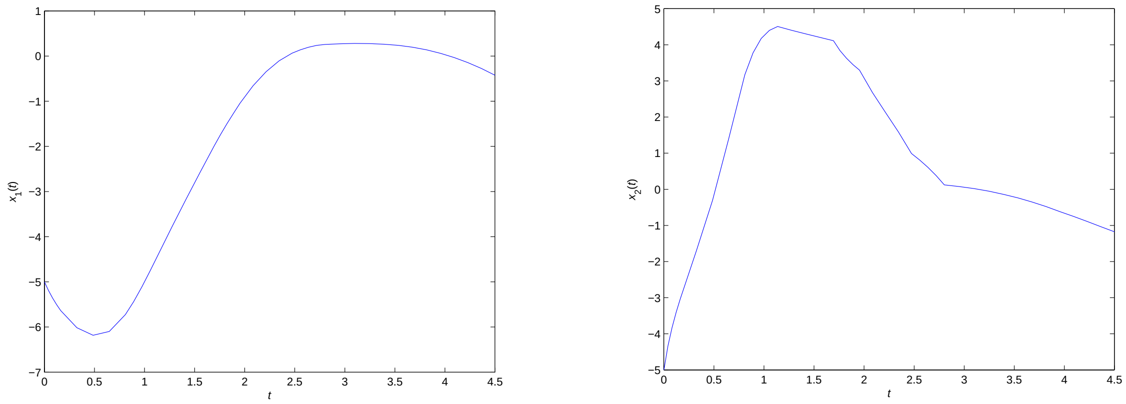

Figure 5.9: Optimal state variables for Example 5.3

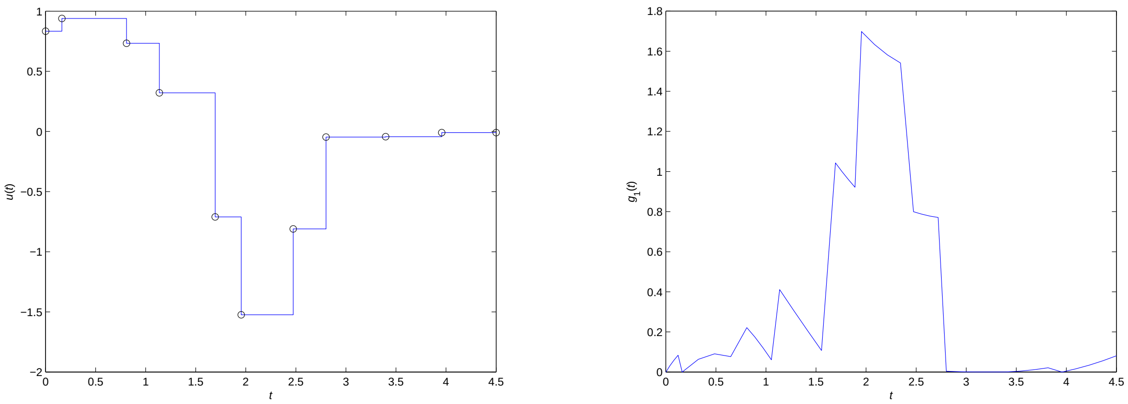

Figure 5.10: Optimal control and the resulting constraint function for Example 5.3

It shows that, for a sufficiently large penalty parameter value any local minimizer of the transformed problem is a local minimizer of the original problem. From results obtained for the three examples, we see that the method proposed is effective. In particular, the optimal controls obtained are feasible controls.

CHAPTER 6

Conclusions and Future Research Directions

6.1 Main Contributions of the Thesis

In this thesis, we studied several optimal control problems with constraints on the state and control. New numerical methods are developed. We summarize our main contributions below.

In Chapter 2, we considered a class of discrete time optimal control problem with time delay and subject to all-time-step inequality constraints on both the state and control. This problem was approximated by a sequence of discrete time optimal control problems with time delay and subject to canonical constraints. We have shown that these approximate problems are special cases of a general discrete time optimal control problem with time delay and subject to canonical constraints as a nonlinear optimization problem. A computational method was then developed to solve this general discrete time optimal control problem. It was then used to solve each of these approximate problems. This approach was applied to study a tactical logistic decision analysis problem. The results obtained are satisfactory.

In Chapter 3, we developed a new efficient computational method for solving a general class of maxmin optimal control problems. We constructed a sequence of smooth approximate optimal control problems by taking the summation of some smooth approximate functions, which were obtained by applying the constraint transcription method [103] to each of the continuous state inequality constraints, as its cost function. A necessary condition and a sufficient condition were derived to show the relationship between the original maxmin problem and the sequence of the smooth approximate problems. We then introduced a violation avoidance function from the solution of each of the smooth approximate optimal control problems and the original continuous state inequality constraints. We showed that the problem of finding an optimal control of the maxmin optimal control problem is equivalent to the problem of finding the largest root of this violation avoidance function. A largest root finding algorithm was then developed based on the control parameterization technique [36], the time scaling transform [87] and the bisection search algorithm. The method was applied to study two practical problems. The first one is an obstacle avoidance problem of an autonomous robot. The second one is the abort landing of an aircraft in a windshear downburst. The results obtained are satisfactory.

In Chapter 4, we presented an optimal PID tuning method for a class of optimal control problems with continuous inequality constraints and terminal state equality constraints, where the control is in the form of a PID controller. Two theorems were given to show that the problem can be solved via solving a sequence of nonlinear optimization problems. An efficient computational method for tuning the parameters of the PID controller optimally was proposed. This method was applied to a ship steering control problem. The results obtained showed that the method proposed is reliable and effective.

In Chapter 5, we present a new exact penalty function method for a class of optimal control problems subject to continuous inequality constraints and terminal equality constraints. This problem was approximated by a sequence of approximate optimal parameter selection problems through the application of the control parameterization technique and the time scaling transform. An exact penalty function method was then used to transform the approximated problems to unconstrained optimal control problems. It was shown that, for a sufficiently large penalty parameter value, any local minimizer of the transformed problem is a local minimizer of the original approximate problem. Convergence analysis was also carried out. Three examples were studied by using the method proposed. The results obtained are satisfactory.

6.2 Future Research Directions

The computational methods developed in Chapters 2, 3 and 4 are based on the control parameterization technique, the time scaling transform and the constraint transcription method. For the constraint transcription method, there are two adjustable parameters involved. One of these two parameters is to control the accuracy of the approximation, while the other controls the feasibility of the constraints. Due to the need for adjusting these two parameters, the burden of these computational methods can be heavy. They may also encounter difficulty in finding a control such that the continuous inequality constraints are satisfied at each time point. Furthermore, there is no theoretical result showing that a local minimizer of the approximate optimization problem is a local minimizer of the original optimal control problem concerned. Thus, an interesting future research direction is to develop computational methods with the constraint transcription method replaced by the exact penalty function method introduced in Chapter 5. This is to be supported by a comparison study showing the advantages and disadvantages between these two approaches.

Except for Chapter 4, the computational methods developed in this thesis are for finding optimal open loop controllers. It is well known that feedback controls are preferred in engineering applications. However, the feedback controllers are very difficult to construct for nonlinear optimal control problems. This is particularly so when there are constraints involved such as those considered in this thesis. One potential approach to deal with feedback control is to regard the optimal open loop controller obtained as for the planning of the optimal trajectory. Then, a linearization is carried out on the nonlinear control system around the optimal open loop control and its corresponding trajectory, giving rise to a linear system. A new cost function in quadratic form is introduced to regulate the deviation between the actual trajectory and the optimal trajectory. The optimal feedback control for this linearized optimal control problem can be obtained. However, the constraints may not be satisfied with such a composite controller, where the feedback control is acting on top of the optimal open loop control. An intensive study is required to carry out for dealing with this issue. The second approach is to use the neighboring extremal approach to construct a linearized optimal feedback controller. The constraint issue remains and is required to be dealt with carefully. The third approach is to assume that the control is in the form of a PID controller. The method proposed in Chapter 4 with the constraint transcription method replaced by the exact penalty function method could be used to develop new computational methods to tune the parameters of the PID controller optimally for these constrained optimal control problems. This is an interesting research project.

Other future research topics include new engineering applications. This will certainly give rise to new challenging theoretical questions in optimal control problems to be investigated.

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