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QSMSOR Iterative Method for the Solution of 2D Homogeneous Helmholtz Equations

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Abstract

In this paper, we consider the numerical solutions of homogeneous Helmholtz equations of the second order. The Quarter-Sweep Modified Successive Over-Relaxation (QSMSOR) iterative method is applied to solve linear systems generated from discretization of the second order homogeneous Helmholtz equations using quarter sweep finite difference (FD) scheme. The formulation and implementation of the method are also discussed. In addition, numerical results by solving several test problems are included and compared with the conventional iterative methods.

Keywords: Helmholtz equations; Quarter-sweep iteration; Finite Difference; Modified Successive Over-Relaxation

1 Introduction

Many problems in engineering and science involve Helmholtz equation, occur in real time application. On the other hand, the applications of Helmholtz equation are encountered in many fields such as time harmonic acoustic and electromagnetic fields, optical waveguide, acoustic wave scattering, noise reduction in silencer, water wave propagation, radar scattering and lightwave propagation problems (Muthuvalu et al., 2014a; Nabavi et al., 2007; Kassim et al., 2006; Yokota and Sugio, 2002). There is a high important in improving the performance of the methods for solving Helmholtz equation. Hence, the development of fast methods is essential in this research area.

Consider the second order Helmholtz equation which is the elliptic equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - \alpha U = f(x, y), \quad (1)$$

with Dirichlet boundary conditions and function f are given. Here, we assume that the domain is the square unit. Assume that the grid spacing is $h=1/n$ with $x_i = ih$ and $y_j = jh$ where $(i, j=1, 2, \dots, n)$. Eq. (1) can be approximated at point (x_i, y_j) by the most commonly used approximation, the full-sweep FD approximation equation we get

$$U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - (4 + \alpha h^2) U_{i,j} = h^2 f_{i,j} \quad (2)$$

Eq. (2) can also can be discretized using the same formula with grid spacing $2h$

and leads to the following formula

$$U_{i+2,j} + U_{i-2,j} + U_{i,j+2} + U_{i,j-2} - (4 + 4h^2\alpha)U_{i,j} = 4h^2 f_{i,j} \tag{3}$$

Eq. (5) is also known as the quarter-sweep FD approximate equation Othman and Abdullah (1998). Another type of approximation derived from the rotated FD approximate equation (Abdullah 1991; Dahlquist and Bjork, 1974) can be constructed by the following transformation

$$\begin{aligned} i, j \pm 1 &\rightarrow i \pm 1, j \pm 1 \\ i \pm 1, j &\rightarrow i \pm 1, j \mp 1 \\ \Delta x, \Delta y &\rightarrow \sqrt{2}h \end{aligned}$$

Therefore, the scheme of central difference using the rotated FD approximate equation (Dahlquist and Bjorck, 1974) can be expressed as

$$U_{i+1,j+1} + U_{i-1,j-1} + U_{i+1,j-1} + U_{i-1,j+1} - (4 + 2h^2\alpha)U_{i,j} = 2h^2 f_{i,j} \tag{4}$$

The standard rotated FD approximation (4) is also called half-sweep FD approximate equation. The article is organized in the following form. The latter section of this article will discuss the formulations of the Full-Sweep Modified Successive Over-Relaxation (FSMSOR), Half-Sweep Modified Successive Over-Relaxation (HSMSOR) and QSMSOR iterative methods in solving the SLE obtained from discretization of the two-dimensional Helmholtz equations. The computational complexity analysis will be shown in Section 4 to assert the performance of the proposed methods. Then, the numerical results and discussion are given in the final section

2 Point MSOR Methods

2.1 FSMSOR Method for Helmholtz Equation

To derive the FSMSOR point iterative method, we use full-sweep approach, in which the domains are divided into two types of points (i.e., ● and ●) as shown in Fig. 1. By applying MSOR method (Akhir et al., 2011a; Kincaid and Young, 1972; De Vogelaere, 1958) into Eq. (2), we will obtain the FSMSOR method for Helmholtz equation as

$$U_{i,j}^{(k+1)} = \frac{\theta_{r,b}}{\rho_0} \left(U_{i+1,j}^{(k)} + U_{i-1,j}^{(k+1)} + U_{i,j+1}^{(k)} + U_{i,j-1}^{(k+1)} - h^2 f_{i,j} \right) + (1 - \theta_{r,b}) U_{i,j}^{(k)} \tag{5}$$

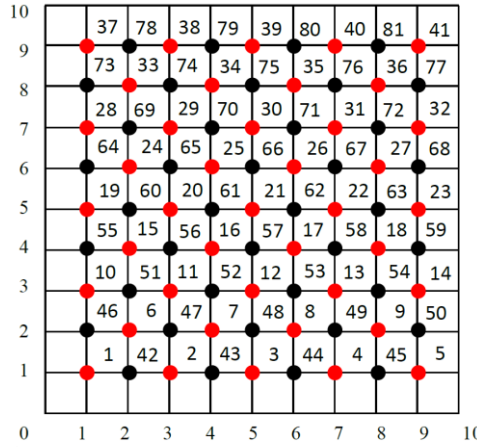


Fig. 1 FMSOR algorithm domain for $n=10$.

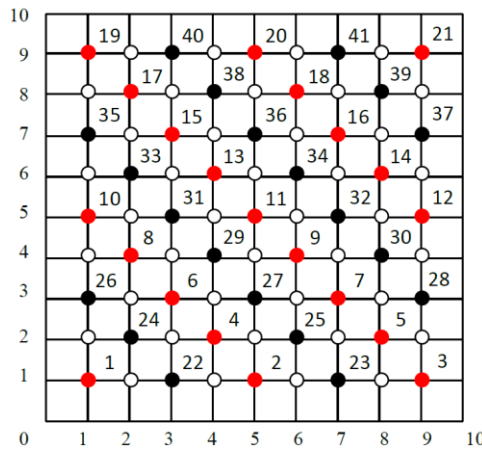


Fig. 2 HSMSOR algorithm domain for $n=10$.

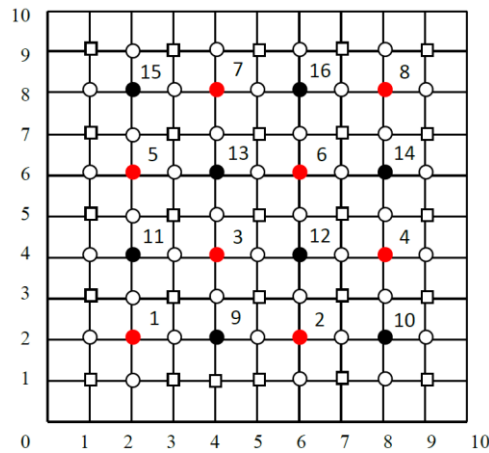


Fig. 3 QSMSOR algorithm domain for $n=10$.

where $\rho_0 = 4 + h^2\alpha$. Eq. (5) allows us to iterate through all of the points, lying on the h -grid. It can be observed that Eq. (5) involves points of type ● and ●. Therefore the iteration can be carried out independently involving only this types of point. The algorithm of FSMSOR method is display in Algorithm 2.1:

Algorithm 2.1

Discretize the solution domain into point of two types ● and ● as shown in Fig. 1.

Step 1: The iterations (using Eq. (5)) implemented on the red point first using the relaxation parameter θ_r .

Step 2: After the red points sweep are completed, the iterations are done on the black points using the relaxation parameter θ_b .

Step 3: Check the convergence. If converge go to **Step 4**, otherwise repeat the iteration cycle, (i.e **Step 1**).

Step 4: Display approximate solutions.

2.2 HSMSOR Method for Helmholtz Equation

To derive the HSMSOR iterative method, we use half-sweep approach, in which the domains are divided into three type of points (i.e., ●, ● and ○) as shown in Fig. 2. By applying MSOR method (Akhir et al., 2011b, 2011c) into Eq. (4), we get the HSMSOR method for Helmholtz equation as

$$U_{i,j}^{(k+1)} = \frac{\theta_{r,b}}{\rho_1} \left(U_{i+1,j+1}^{(k)} + U_{i-1,j-1}^{(k+1)} + U_{i+1,j-1}^{(k+1)} + U_{i-1,j+1}^{(k)} - 2h^2 f_{i,j} \right) + (1 - \theta_{r,b}) U_{i,j}^{(k)} \tag{6}$$

where $\rho_1 = 4 + 2h^2\alpha$. Eq. (6) allows us to iterate through half of the points, lying on the $\sqrt{2}h$ -grid. Again, it can be observed that Eq. (6) involves points of type ● and ●. Therefore the iteration can be carried out autonomously involving only this type of point. The algorithm of HSMSOR method is display in Algorithm 2.2:

Algorithm 2.2

Discretize the solution domain into point of three types ●, ● and ○ as shown in Figure 2.

Step 1. The iterations (using Eq. 6)) implemented on the red point first using the relaxation parameter θ_r .

Step 2. After the red points sweep are completed, the iterations are done on the

black points using the relaxation parameter θ_b .

Step 3. Check the convergence. If converge go to **Step 4**, otherwise repeat the iteration cycle, (i.e **Step 1**).

Step 4. Evaluate the solutions at the remaining points type \bigcirc using the full-sweep FD approximate formula (3) on the grid h (Akhir et al., 2012a, b; 2011b, c).

$$U_{i,j} = \frac{1}{\rho_0} (U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - h^2 f_{i,j})$$

Step 5. Display approximate solutions.

2.3 QSMSOR Method for Helmholtz Equation

To derive the QSMSOR iterative method, we use quarter-sweep approach, in which the domains are divided into four type of points (i.e., \bullet , \blacksquare , \bigcirc and \square) as shown in Fig. 3. By applying MSOR method (Akhir et al., 2012b) into Eq. (3), we will obtain the QSMSOR method for Helmholtz equation as

$$U_{i,j}^{(k+1)} = \frac{\theta_{r,b}}{\rho_2} (U_{i+2,j}^{(k)} + U_{i-2,j}^{(k+1)} + U_{i,j+2}^{(k)} + U_{i,j-2}^{(k+1)} - h^2 f_{i,j}) + (1 - \theta_{r,b}) U_{i,j}^{(k)} \quad (7)$$

where $\rho_2 = 4 + 4h^2\alpha$. Eq. (7) allows us to iterate through quarter of the points, lying on the $2h$ -grid. Again, it can be observed that Eq. (7) involves points of type \bullet and \blacksquare . Therefore, the iteration can be carried out autonomously involving only this type of point. The algorithm of QSMSOR method is display in Algorithm 2.3:

Algorithm 2.3

The solution domain must be labeled for the four types of points (i.e., \bullet , \blacksquare , \bigcirc and \square), as shown in Fig. 3.

Step 1. The iterations (using Eq. (7)) implemented on the red point first using the relaxation parameter θ_r .

Step 2. After the red points sweep are completed, the iterations are done on the black points using the relaxation parameter θ_b .

Step 3. Check the convergence. If converge go to **Step 4**, otherwise repeat the iteration cycle, (i.e **Step 1**).

Step 4. Evaluate the solutions at the remaining points according to the following sequence. (Aruchuan et al., 2014; Akhir et al., 2012b; Othman and Abdullah, 1998).

- a. points of type \square using the full-sweep FD approximate formula (7) on the grid $\sqrt{2h}$.

$$U_{i,j} = U_{i+1,j+1} + U_{i-1,j-1} + U_{i+1,j-1} + U_{i-1,j+1} - 2h^2 f_{i,j}$$

- b. points of type \square using the full-sweep FD approximate formula (3) on the grid h .

$$U_{i,j} = \frac{1}{\rho_0} (U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - h^2 f_{i,j})$$

Step 6. Display approximate solutions.

3 Numerical Results

In this section, we exemplify two numerical examples to illustrate the effectiveness of the methods prescribed in previous section. The algorithms were tested on the following model problems:

Problem 1 (El-Sayed and Kaya, 2004)

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - U = 0$$

with boundary conditions

$$U(x,0) = x, \quad U(x,1) = \exp(x) + 1.543x, \quad 0 \leq x \leq 1.$$

$$U(0,y) = y, \quad U(1,y) = 2.718y + \cosh(y), \quad 0 \leq y \leq 1.$$

and exact solution of this problem is

$$U(x,y) = y \exp(x) + x \cosh(y)$$

Problem 2 (El-Sayed and Kaya, 2004)

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + 5U = 0$$

with boundary conditions

$$\begin{aligned}
 U(x,0) &= 0, & U(x,1) &= 3.627 \sin(3x), & 0 \leq x \leq 1. \\
 U(0,y) &= 0, & U(1,y) &= 0.141 \sinh(2y), & 0 \leq y \leq 1.
 \end{aligned}$$

and exact solution of this problem is

$$U(x, y) = \sin(3x) \sinh(2y)$$

Throughout the experiments, there are three important parameters to be measured, such as the number of iterations (k), maximum absolute error (Abs. Error) and execution time (in seconds) (t). All the methods were tested on several mesh sizes i.e 64, 128, 256 and 512; and the convergence test used was the maximum absolute error (Abs. Error) tolerance by taking $\varepsilon = 10^{-10}$. Each experiment is implemented by choosing $\theta_{r,b}$ value close to the optimal parameter θ of the corresponding SOR iterative method.

The relaxation parameter $\theta_{r,b}$ was chosen to within ± 0.01 that gave the minimum number of iterations. The computer language used for the programming is C++, and the program performed on a personal PC Intel(R) Core (TM) i7 CPU 860@3.00Ghz, 6.00GB RAM. The operation system used was Window 7 with the installation Borland C++ compiler version 5.5. The numerical results of the experiment for different value of mesh size are given in Tables 1 and 2, respectively

4 Computational Complexity Analysis of MSOR Methods for Helmholtz Equation

The computational effort measured by number of computer operations needed to obtain a solution by the three methods discussed for solving problem (1) can be assessed. Undertake the solution domain is large with m^2 number of internal mesh points with $m = n - 1$. In their iterative manner, the FSMSOR and HSMSOR methods require $(m-1)^2$ and $(m-1)^2/2$ internal mesh points respectively. However QSMSOR method require $(m-1)^2/4$ internal mesh points. Note that our valuation on this computational complexity is based on the arithmetic operations performed per iteration and execution time for the additions/subtraction (ADD/SUB) and multiplications/divisions (MUL/DIV) operations. Therefore the number of operations of operations required (excluding red and black equations, convergence test and direct solution) for FSMSOR, HSMSOR and QSMSOR methods as described in Section 3 are correspondingly given as follows in Table 3

Table 1 Number of iterations, execution time and maximum absolute error for the proposed iterative methods in solving Problem 1.

n	Method	θ_r	θ_b	k	t	Abs. Error
64	FSMSOR	1.91	1.91	254	0.14	2.1637e-6
	HSMSOR	1.87	1.27	198	0.11	2.1651e-6
	QSMSOR	1.82	1.83	137	0.05	8.6474e-6
128	FSMSOR	1.95	1.96	529	0.52	5.4143e-7
	HSMSOR	1.93	1.93	412	0.4	5.4075e-7
	QSMSOR	1.91	1.92	282	0.15	2.1637e-6
256	FSMSOR	1.98	1.97	991	3.57	1.3299e-7
	HSMSOR	1.97	1.97	880	3.29	3.3532e-7
	QSMSOR	1.95	1.96	579	1.07	5.4058e-7
512	FSMSOR	1.99	1.98	2855	40.95	2.1701e-8
	HSMSOR	1.98	1.98	1971	34.65	2.9851e-8
	QSMSOR	1.98	1.99	1355	10.67	1.3529e-7

k is the number of iterations; t is the computation timings.

Table 2 Number of iterations, execution time and maximum absolute error for the proposed iterative methods in solving Problem 2.

n	Method	θ_r	θ_b	k	t	Abs. Error
64	FSMSOR	1.92	1.92	283	0.19	2.2777e-4
	HSMSOR	1.89	1.89	221	0.13	2.7964e-4
	QSMSOR	1.85	1.86	162	0.07	9.0996e-4
128	FSMSOR	1.96	1.96	565	0.56	5.6953e-5
	HSMSOR	1.94	1.94	498	0.49	6.9883e-5
	QSMSOR	1.92	1.93	315	0.19	2.7227e-4
256	FSMSOR	1.98	1.99	1694	6.57	1.4239e-5
	HSMSOR	1.97	1.97	921	3.55	1.7471e-5
	QSMSOR	1.96	1.97	625	1.21	5.6952e-5
512	FSMSOR	1.98	1.99	4215	65.63	3.5391e-6
	HSMSOR	1.99	1.99	2561	45.89	4.3674e-6
	QSMSOR	1.98	1.99	1272	10.25	1.4239e-5

k is the number of iterations; t is the computation timings.

Table 3 The total computing costs for the three MSOR methods.

Methods	Arithmetic Operation	
	Per Iteration	
	(ADD/SUB)	(MULT/DIV)
FSMSOR	$(6n^2 - 24n + 24)k$	$(6n^2 - 24n + 24)k$
HSMSOR	$(3n^2 - 12n + 12)k$	$(3n^2 - 12n + 12)k$
QSMSOR	$((3/2)n^2 - 6n + 6)k$	$((3/2)n^2 - 6n + 6)k$
Methods	After Convergence	
	(ADD/SUB)	(MULT/DIV)
	FSMSOR	-
HSMSOR	$(5/2)n^2 - 10n + 10$	$(5/2)n^2 - 10n + 10$
QSMSOR	$(5/4)n^2 - 15n + 15$	$(5/4)n^2 - 15n + 15$

k is the number of iterations; $m = n - 2$.

5 Discussions of Results

In Section 5, three types of pointwise MSOR methods are applied into a Helmholtz equation model to check the execution times and number of iterations. From the numerical result, QSMSOR methods is the fastest method among the other two MSOR methods (HSMSOR and FSMSOR) if we compare with either number of iterations or execution time. This can also be verified if we compared the computational complexity of all three MSOR methods where QSMSOR method has the least computational complexity.

It can be pragmatic that the accuracies of the QSMSOR methods remain as good as the HSMSOR and FSMSOR methods but they oblige lesser number of iterations and computing timing to attain the result. For example, the number of iterations of QSMSOR is merely about 22-31% and 22-39% as well as 46-52% and 43-70% compared to HSMSOR and FSMSOR methods in Problems 1 and 2, respectively. Again, the execution times of QSMSOR are much faster just about 15-21% and 30-32% along with 64-73% and 63-84% compared to HSMSOR and FSMSOR methods in Problems 1 and 2 respectively.

Experimental results also show promising results that make them as alternative to conventional FD scheme. From the number of iterations and timing obtained, it can be seen that among three MSOR iterative methods presented, the QSMSOR method requires the least time for all n compared with the other two MSOR iterative methods. This is due to the fact that among the three methods, the

QSMSOR method requires least number of numbers of iterations and computational operations. This is reflected by total arithmetic operations required by the method given in Table 3.

Moreover, the accuracy are significantly good, since all the method used the descriptive stencil $O(h^2)$. Overall, the numerical results show that the QSMSOR method is superior than HSMSOR and FSMSOR methods. This is mainly because of computational complexity of the QSMSOR method which is approximately 50% and 75% less than HSMSOR and FSMSOR methods respectively. For future works, the capability of octo-sweep iteration (Akhir et al., 2012d) should be scrutinized for solving homogeneous (El-Sayed and Kaya, 2004) and nonhomogeneous Helmholtz equations (Akhir et al., 2012c). Also, advance studies for innumerable point block iterative methods can be also scrutinized (Akhir et al., 2012e).

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