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3 **Variability of Soil**

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Reliability-Based Semi-Analytical Solution for Ground Improvement by PVDs Incorporating Inherent (Spatial) Variability of Soil

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Abstract: The design of soil consolidation via prefabricated vertical drains (PVDs) has been traditionally carried out deterministically and thus can be misleading due to the ignorance of the uncertainty associated with the inherent (spatial) variation of soil properties. To treat such uncertainty in the design process of soil consolidation by PVDs, stochastic approaches that combine the finite element method with the Monte Carlo technique (FEMC) have been usually used. However, such approaches are complex, computationally intensive and time consuming. In this paper, a simpler reliability-based semi-analytical (RBSA) method is proposed as an alternative tool to the complex FEMC approach for soil consolidation by PVDs, considering soil spatial variability. The RBSA method is found to give similar results to those obtained from the FEMC approach and can thus be used with confidence in practice.

Keywords: Reliability-based design; Soil consolidation; Prefabricated vertical drains; Finite element method; Monte Carlo technique; Soil spatial variability.

1. Introduction

Traditionally, to predict soil consolidation by PVDs using available deterministic methods [e.g., 1, 2, 3], it has been usually assumed that the consolidating soil surrounding the

49 PVDs is homogeneous. In reality, however, the degree of consolidation achieved via PVDs is
50 strongly dependent on soil properties that are spatially variable in nature, such as soil
51 permeability, k , and volume compressibility, m_v . Consequently, the rate of soil consolidation
52 is difficult to predict deterministically, especially for heterogeneous soil deposits. Therefore,
53 it is crucial to develop more realistic solutions that can accommodate the true nature of the
54 inherent (spatial) variability of soil in the course of design of soil consolidation by PVDs.

55 In recent years, a few attempts have been made to quantify and assess the uncertainty
56 associated with soil consolidation. For example, some studies [i.e., 4, 5, 6] focussed on the
57 impact of soil variability in one dimensional consolidation due to vertical drainage (i.e., no
58 PVDs). A few more studies [i.e., 7, 8] focussed on the uncertainty associated with the
59 measurement errors of soil testing for PVD-improved ground but soil spatial variability has
60 not been explicitly investigated. More recently, Walker and Indraratna [9] proposed an
61 analytical model incorporating a parabolic permeability distribution in the smear zone, and
62 Basu et al. [10] performed a study to include a transition zone of linearly varying permeability
63 between the smear and undisturbed zones with constant permeability. The above solutions,
64 despite of being useful, failed to accommodate the true nature of soil spatial variability in
65 design of ground improvement by PVDs and more alternative realistic solutions are needed.

66 In order to treat soil spatial variability in most geotechnical engineering problems,
67 stochastic computational schemes that combine the finite element method and Monte Carlo
68 technique [e.g., 6, 11, 12] have been often used. Despite the fact that such schemes offer
69 successful solutions, they require a large number of simulations that are computationally
70 intensive and time consuming. In the current study, an alternative simplified reliability-based
71 semi-analytical (RBSA) approach is introduced for design of soil consolidation by PVDs,
72 considering the spatial variations of soil permeability, k , and volume compressibility, m_v . The
73 developed RBSA method is verified by comparing its results with those obtained from the

74 complex stochastic 3D finite-element Monte-Carlo (FEMC) approach and the results are
75 found to be in a good agreement. In the sections that follow, the stochastic FEMC approach is
76 demonstrated first followed by detailed description of the alternative RBSA method.

77

78 **2. Stochastic finite-element Monte-Carlo approach**

79

80 For the purpose of examining the proposed RBSA method which will be discussed later
81 in detail in the following section, a series of stochastic FEMC analyses are performed and
82 their results are used for comparison with the RBSA method. The FEMC approach merges the
83 local average subdivision (LAS) technique [13] and finite element (FE) modelling into a
84 Monte Carlo framework using the following steps:

- 85 1. Identify the spatially variable soil properties affecting soil consolidation by PVDs;
- 86 2. Create a virtual soil profile that contains random fields of designated soil properties;
- 87 3. Incorporate the generated random fields of soil profile into FE modelling; and
- 88 4. Repeat Steps 2 and 3 many times using the Monte Carlo technique so that a series of
89 consolidation responses is obtained from which probabilistic solution for soil consolidation
90 can be derived.

91 The above steps, as well as the numerical procedures, are described below.

92

93 *2.1 Identification of significant spatially variable soil properties*

94

95 As indicated earlier, spatial variability of several soil properties can affect soil
96 consolidation by PVDs. However, as confirmed by several researchers [e.g., 6, 14], soil
97 permeability, k , and volume compressibility, m_v , are the most significant factors affecting soil
98 consolidation by PVDs. Although the coefficients of permeability in the vertical and

99 horizontal directions (i.e., k_v and k_h , respectively) may vary in the ground, the impact of k_h is
100 dominant [8]. Consequently, in the current study, only k_h and m_v are considered to be spatially
101 variables, while the other soil properties are held constant and treated deterministically so as
102 to reduce the superfluous complexity to the problem.

103

104 *2.2 Generation of random fields of soil properties*

105

106 In this study, the LAS method [13] extracted from the random field theory [15] are used
107 to generate virtual random fields that allow rational random distributions of k_h and m_v , which
108 are then implemented in the FEM modelling. Based on the random field theory, a random
109 field of certain probability distribution of spatially variable soil property can be characterised
110 by the soil property mean value, μ , variance, σ^2 (can also be represented by the standard
111 deviation, σ , or coefficient of variation, v , where $v = \sigma/\mu$) and correlation length or scale of
112 fluctuation, θ . The value of θ describes the limits of spatial continuity and can simply be
113 defined as the distance over which a soil property shows considerable correlation between
114 two spatial points. Therefore, a large value of θ indicates strong correlation (i.e., uniform soil
115 property field), whereas a small value of θ implies weak correlation (i.e., erratic soil property
116 field).

117 In the current study, lognormally distributed random fields are assumed for simulating
118 the spatial variability of k_h and m_v , because this distribution is extensively used in the literature
119 both for k_h and m_v [5, 6, 16]. To create a random field of soil property X , the following
120 process is followed. A correlated local (arithmetic) average of normally distributed random
121 field $G_X(i)$ over the domain of the i th element are first generated for 3D grid of soil mass with
122 values of soil property of zero mean, unit variance and scale of fluctuation θ_X . The required

123 lognormally distributed random field defined by μ_X and σ_X is then obtained using the
 124 following transformation function [17]:

$$125 \quad X_i = \exp \left\{ \mu_{\ln X} + \sigma_{\ln X} G_X(i) \right\} \quad (1)$$

126 where X_i is the soil property value assigned to the i th element; $\mu_{\ln X}$ and $\sigma_{\ln X}$ are, respectively,
 127 the mean and standard deviation of the underlying normally distributed $\ln(X)$ evaluated from
 128 the specified μ_X and σ_X of the lognormally distributed X as follows:

$$129 \quad \mu_{\ln X} = \ln \mu_X - \frac{1}{2} \sigma_{\ln X}^2 \quad (2)$$

$$130 \quad \sigma_{\ln X} = \sqrt{\ln \left(1 + \frac{\sigma_X^2}{\mu_X^2} \right)} = \sqrt{\ln(1 + \nu_X^2)} \quad (3)$$

131 Rearranging Eqs. (2) and (3) gives the following inverse relationships for the mean and
 132 standard deviation of the lognormally distributed X :

$$133 \quad \mu_X = \exp \left(\mu_{\ln X} + \frac{1}{2} \sigma_{\ln X}^2 \right) \quad (4)$$

$$134 \quad \sigma_X^2 = \mu_X^2 \left(\exp(\sigma_{\ln X}^2) - 1 \right) \quad (5)$$

135 The correlation coefficient for a soil property between two spatial points within the
 136 soil domain is specified by an exponentially decaying ellipsoidal Markov spatial correlation
 137 function, $\rho(\tau)$, as follows:

$$138 \quad \rho(\tau) = \exp \left\{ - \sqrt{ \left(\frac{2\tau_x}{\theta_x} \right)^2 + \left(\frac{2\tau_y}{\theta_y} \right)^2 + \left(\frac{2\tau_z}{\theta_z} \right)^2 } \right\} \quad (6)$$

139 where τ_x , τ_y and τ_z are, respectively, the distances between two points in x , y and z directions;
 140 and θ_x , θ_y and θ_z are, respectively, the scales of fluctuation in x , y and z directions. It should be
 141 noted that the spatial correlation function in Eq. (6) becomes statistically isotropic when $\theta_x =$
 142 $\theta_y = \theta_z$. It is worthy to note that the scale of fluctuation is estimated with respect to the

143 underlying normally distributed random field (i.e., $\ln X$). Details on the estimation of the scale
144 of fluctuation can be found in Lloret-Cabot et al. [18].

145

146 *2.3 Finite element modelling incorporating soil spatial variability*

147

148 The subsurface profile simulated in the previous step with the specified spatial variation
149 of k_h and m_v can now be employed as inputs into a FE consolidation modelling of soil
150 improvement by PVDs. In this study, all numerical analyses are carried out using a modified
151 version of the FE computational scheme “Program 8.6” from the book by Smith and
152 Griffiths [19] in which soil consolidation is treated as 3D uncoupled problem solved using
153 implicit time integration with the “theta” method. The authors modified the source code of
154 “Program 8.6” to incorporate the volume compressibility and allow for repetitive Monte-
155 Carlo analyses.

156 To demonstrate the validity of the proposed RBSA method against the FEMC approach,
157 a consolidation problem is considered for comparison implying a unit cell of soil with central
158 cylindrical drain of dimensions $L = 1.0\text{m}$, $r_e = 0.536\text{m}$, $r_s = 0.197\text{m}$ and $r_w = 0.032\text{m}$ (see Fig.
159 1a). In the FE analyses, the circular influence area of the cylindrical unit cell is transformed to
160 an equivalent square influence area (see Fig. 1b) of a side length $S = \sqrt{\pi r_e^2}$ (i.e., $S = 0.95\text{m}$).
161 The selection of the equivalent square influence geometry in the FE modelling is convenient
162 because it avoids the unfavourable mesh shape for the LAS method which requires square (or
163 rectangular) elements to accurately compute locally averaged values of k_h and m_v for each
164 element across the soil mass. For the same reason, a square shaped smear zone of side length
165 $S_s = 0.35\text{m}$ and PVD of a side length $S_w = 0.05\text{m}$ are also employed in the FE modelling.

166 It is well known that the overall consolidation of PVD-improved ground is governed by
167 the radial (horizontal) flow of water rather than the vertical flow as the drainage length in the

168 horizontal direction is much less than that of the vertical direction and k_h is often much higher
169 than that of k_v [2]. Under this reasoning, soil consolidation due to the horizontal drainage only
170 is considered in the current study. Neglecting the vertical flow in the FE analyses is matched
171 by setting k_v to be equal to zero and since the permeability variance is often described without
172 referring to any direction, the two components of k_h (i.e., k_x and k_y) are assumed to be
173 isotropic (i.e., $k_x = k_y = k_h$).

174 Although the accuracy of the FE solutions increases with the increase of the number of
175 elements in the FE mesh, a trade-off between accuracy and run-time efficiency is necessary.
176 Previous literature includes some recommendations regarding the optimum ratio of the scale
177 of fluctuation to the finite element size. For example, Ching and Phoon [20] stated that this
178 ratio should be ≥ 20 , whereas Harada and Shinozuka [21] pointed out that it should be ≥ 2 . In
179 the current study, a sensitivity analysis on various FE mesh dimensions is conducted and it is
180 found that a discretization of the FE mesh with an element of size $0.05\text{m} \times 0.05\text{m} \times 0.05\text{m}$
181 gives a reasonable precision and complies with the recommendation given by Harada and
182 Shinozuka [21]. The 3D mesh used consists of 7220 eight node hexahedral elements (see Fig.
183 1b). The initial condition for the uncoupled analysis (i.e., no displacement degrees of freedom
184 and only pore pressure degrees of freedom) is such that the excess pore pressure at all nodes
185 (except at the nodes of the drain boundary) is set equal to 100 kPa, while the excess pore
186 pressure at each node of the drain boundary is set equal to zero.

187 During the mandrel installation of PVDs, a disturbed zone surrounding the drain (i.e.,
188 smear zone) of reduced k_h and increased m_v is produced. However, soil spatial variability in
189 the smear zone persists [22], albeit no longer fully natural. Under this reasoning, two groups
190 of RBSA models are developed in this study under various assumed ground conditions. In the
191 first group, the spatially variable soil properties are assumed to be continuous over the whole
192 unit cell. However, non-stationary mean for the spatially variable soil properties are used to

193 take into account the smear effect. In this case, the random fields of the spatially variable soil
 194 properties in the smear zone are generated separately from those of the undisturbed zone;
 195 however, this is carried out in such a way that the ratio of soil permeability in the undisturbed
 196 zone to the smear zone (i.e., k_h/k'_h) and the ratio of volume compressibility in the smear zone
 197 to the undisturbed zone (i.e., m'_v/m_v) are held constant. In the second group, it is assumed that
 198 the spatially variable soil properties in the smear zone are completely independent of those of
 199 the undisturbed zone. In this case, the random fields of the spatially variable soil properties
 200 for the undisturbed and smear zones are generated separately with their corresponding
 201 dimension and specified random field parameters. The well resistance is another factor that
 202 may affect the efficiency of PVD-improved ground, which is caused due to the deformation
 203 of the drain (i.e., folding, bending, crimping) and infiltration of fine soil particles through the
 204 drain filter. However, the discharge capacity of most available PVDs in the market is
 205 relatively high and well resistance can thus be practically ignored [23], which is the case in
 206 the presented example herein. It should be noted though that the proposed RBSA method can
 207 also take into account the well resistance effect, if needed.

208 In the current study, μ_{k_h} and μ_{m_v} are taken to be equal to 0.15m/year and $1.0 \times 10^{-3} \text{m}^2/\text{kN}$,
 209 respectively. The ratio $\mu_{k_h} / \mu_{k'_h}$, which may vary from 2 to 6 as reported by various
 210 researchers [e.g., 2, 23], is taken to be equal to 3. The ratio $\mu_{m'_v} / \mu_{m_v}$ is taken to be 1.2, which
 211 is in accordance with the value reported by Walker [24]. In order to validate the proposed
 212 RBSA, it is decided to conduct the study over the following range of v and θ for k_h and m_v :

- 213 • v_{k_h} (for both the smear and undisturbed zones) = 100, 200, 300 (%)
- 214 • v_{m_v} (for both the smear and undisturbed zones) = 10, 20, 30 (%)
- 215 • $\theta_{k_h} = \theta_{m_v} = \theta$ (for both the smear and undisturbed zones) = 0.1, 0.25, 0.5, 1.0, 5.0, 10.0 (m)

216 It should be noted that the abovementioned selected ranges of v and θ for k_h and m_v are typical
217 to those reported in the literature [e.g., 25, 26] and are believed to represent sufficiently the
218 practical values that can establish general trends for the stochastic soil consolidation
219 behaviour. A single generation of the random fields and the subsequent FE analysis is termed
220 “realization”. For an individual realization, the degree of consolidation at any certain
221 consolidation time, t , is expressed as $U(t)$ and can be calculated with the help of the following
222 expression:

$$223 \quad U(t) = 1 - \frac{\bar{u}(t)}{u_0} \quad (7)$$

224 where u_0 is the initial pore pressure and $\bar{u}(t)$ is the average pore pressures at any t (calculated
225 by numerically integrating the excess pore pressure across the mesh and dividing by the total
226 mesh volume), of the consolidation process.

227

228 *2.4 Repetition of process based on the Monte Carlo technique*

229

230 Following the procedures of the Monte Carlo technique, the process of generating
231 random fields of k_h and m_v and the subsequent FE analysis is repeated numerous times with
232 the same v and θ until an acceptable accuracy of estimated statistics of $U(t)$ is achieved. It is
233 found that 2000 Monte Carlo simulations are sufficient to yield reliable and reproducible
234 estimates. One single case of FE analysis with 2000 Monte-Carlo simulations typically takes
235 6 days on an Intel core i5 CPU @ 3.4 GHz computer. The above repetitive process is
236 performed for each combination of the selected v and θ , and the obtained outputs from each
237 realization of the Monte Carlo procedures are collated and statistically analyzed to make a
238 comparative study between the FEMC approach and the proposed RBSA method, as will be
239 seen later.

240

241 **3. Reliability-based semi-analytical model**

242

243 It is not uncommon that practicing engineers have neither the time nor the resources to
244 perform full scale FEMC simulations of soil consolidation by PVDs including spatially
245 random properties. Therefore, in this study, an approximate, easy to use reliability-based
246 semi-analytical (RBSA) model is introduced from which direct estimates of the probability of
247 achieving certain $U(t)$ can be readily obtained. The development of the RBSA model requires
248 a performance function or a theoretical (deterministic) model as the commencing point to
249 travel through to the reliability (stochastic) solution. Available deterministic analytical
250 solutions for soil consolidation by PVDs are based on the unit cell concept for a single drain,
251 which is also adopted in the RBSA model. It should be noted that the unit cell concept is
252 deemed to be valid for stochastic analysis of PVD-improved ground because it was found in a
253 recent study carried out by the authors using the FEMC approach that the multi-drain
254 behaviour can be well represented by an idealized unit cell analysis, provided that certain
255 factorized statistical parameters, computed by taking into account the size of the unit cell, are
256 used so as to give equivalent solutions to those of the multi-drain. Detailed description of the
257 validity of the unit cell concept for stochastic analyses of PVD-improved ground as compared
258 to the multi-drain solution is beyond the scope of this paper and can be found elsewhere [see,
259 27].

260 In the current study, the commonly used radial consolidation equation of Hansbo [2] is
261 used as the commencing point towards the RBSA model. This equation considers the unit cell
262 concept and has gained a wide acceptance in practical application. The solution is based on
263 the equal strain hypothesis and can estimate the degree of consolidation due to the horizontal
264 drainage, $U_h(t)$, at any time, t , as given in the following equation (note that as the vertical flow

265 is ignored in the FE solution, $U_h(t)$ from Hansbo's theory will be equal to $U(t)$ of the FE
 266 analysis and can be simply denoted as $U(t)$).

$$267 \quad U(t) = 1 - \exp\left(-\frac{2k_h t}{m_v \gamma_w r_e^2 \alpha}\right) \quad (8)$$

$$268 \quad \alpha = F_n + F_s + F_r \quad (9)$$

269 where F_n , F_s and F_r are the drain spacing factor, smear factor and well-resistance factor,
 270 respectively, and can be determined by:

$$271 \quad F_n = \frac{n^2}{(n^2 - 1)} \left[\ln(n) - \frac{3}{4} + \frac{1}{n^2} - \frac{1}{4n^2} \right] \approx \ln(n) - \frac{3}{4} \quad (10)$$

$$272 \quad F_s = \left(\frac{k_h}{k'_h} - 1 \right) \ln\left(\frac{r_s}{r_w} \right) = \left(\frac{k_h}{k'_h} - 1 \right) \ln(s) \quad (11)$$

$$273 \quad F_r = \pi z (2L - z) \frac{k_h}{q_w} \quad (12)$$

274 where γ_w is the unit weight of water; r_e is the radius of the equivalent soil cylinder with
 275 impermeable perimeter (or the radius of zone of influence); t is the consolidation time; α is a
 276 group parameter representing the smear effect and geometry of the PVD system; $n = r_e/r_w$ is
 277 the drain spacing ratio (r_w is the equivalent radius of the drain); $s = r_s/r_w$ is the smear ratio (r_s
 278 is the radius of the smear zone); k'_h is the horizontal permeability of the smear zone; q_w is the
 279 vertical discharge capacity of the drain; L is the maximum vertical drainage distance; and z is
 280 the depth from the top of the consolidating layer. All parameters shown in Eqs. (8–12) are
 281 illustrated in Fig. 1.

282 It is mentioned earlier that the installation procedure of PVDs not only reduces k_h but also
 283 increases m_v within the smear zone, leading to different volume compressibility in the smear
 284 zone that is denoted earlier as m'_v . The ignorance of the increased m_v in the smear zone may
 285 lead to a lack of precision in the analysis. However, α parameter in Eq. (8) proposed by

286 Hansbo [2] disregards m'_v . In an effort to rectify this situation, Walker [24] introduced a new
 287 parameter termed as the smear zone volume compressibility parameter, α_{m_v} , is included in
 288 Eq. (8) to take into account m'_v . For a single smear zone with constant increased volume
 289 compressibility, α_{m_v} is given by Walker [24] as follows:

$$290 \quad \alpha_{m_v} = \frac{n^2 - s^2}{n^2 - 1} + \frac{s^2 - 1}{n^2 - 1} \frac{m'_v}{m_v} \quad (13)$$

291 By including α_{m_v} into Eq. (8), a modified form of this equation is thus:

$$292 \quad U(t) = 1 - \exp\left(-\frac{2k_h t}{m_v \gamma_w r_e^2 \alpha_{m_v}}\right) \quad (14)$$

293 If the changes of m_v in the smear zone are not considered, then α_{m_v} in Eq. (14) will be equal
 294 to 1.0. That is, Eq. (14) will return back to its original form of Hansbo's [2] formula presented
 295 in Eq. (8). Since k_h and m_v are the only random variables, rearranging Eq. (14) and defining
 296 $\ln[1/\{1-U(t)\}]$ as $U^*(t)$ gives:

$$297 \quad U^*(t) = \frac{2t}{r_e^2 \gamma_w} \frac{k_h}{m_v \alpha_{m_v}} \quad (15)$$

298 The above conversion of Eq. (14) to Eq. (15) is necessary as it simplifies the process of
 299 obtaining a closed form solution for the mean and variance of the degree of consolidation
 300 function $U^*(t)$ directly from the statistically defined input data (i.e., mean and variance) of k_h
 301 and m_v .

302 The reliability-based solution requires determination of a reasonable probability
 303 distribution of $U^*(t)$, once found, the statistical parameters of the distribution of $U^*(t)$ can be
 304 estimated. In this regard, simple semi-analytical relationships are derived to aid the designer
 305 in estimating the statistical parameters of the distribution of $U^*(t)$ directly from the random
 306 field parameters. This involves considering an approximate model where the geometric

307 averages of k_h and m_v (i.e., \bar{k}_h and \bar{m}_v , respectively) over the influence zone surrounding the
 308 PVD are used in Eq. (15). If the consolidating soil domain surrounding the PVD is termed D
 309 and discretized into an assembly of non-overlapping rectangular (or square) elements, then \bar{k}_h
 310 and \bar{m}_v over D can be defined as:

$$311 \quad \bar{k}_h = \left[\prod_{i=1}^j k_{h_i} \right]^{1/j} = \exp \left[\frac{1}{j} \sum_{i=1}^j \ln k_{h_i} \right] = \exp \left\{ \mu_{\ln k_h} + \sigma_{\ln k_h} \bar{G}_{k_h}(D) \right\} \quad (16)$$

$$312 \quad \bar{m}_v = \left[\prod_{i=1}^j m_{v_i} \right]^{1/j} = \exp \left[\frac{1}{j} \sum_{i=1}^j \ln m_{v_i} \right] = \exp \left\{ \mu_{\ln m_v} + \sigma_{\ln m_v} \bar{G}_{m_v}(D) \right\} \quad (17)$$

313 where $i = 1, 2, \dots, j$ represents the element number, $\bar{G}_{k_h}(D)$ and $\bar{G}_{m_v}(D)$ are the arithmetic
 314 averages of $G_{k_h}(i)$ and $G_{m_v}(i)$, respectively, over the domain D . It should be noted that k_h and
 315 m_v are assumed to be uncorrelated in the proposed RBSA model, which is due to the lack of
 316 data available in the literature to identify the degree and nature of the cross-correlation
 317 between k and m_v . For the problem of one dimensional consolidation, Freeze [5] reported that
 318 non-zero cross-correlation between k and m_v has a minor impact on the stochastic results of
 319 soil consolidation. Prior to finding the distribution and statistical parameters of $U^*(t)$, a brief
 320 discussion in regard to the underlying equivalent normally distributed mean and variance of
 321 the lognormally distributed soil property X (i.e., $\mu_{\ln \bar{X}}$ and $\sigma_{\ln \bar{X}}^2$) is essential, as follows.

322 As mentioned earlier, the overall behaviour of PVD system is not governed by the soil
 323 properties at discrete points but rather by the average soil properties of the soil volume within
 324 the soil domain. For example, in a consolidating heterogeneous soil mass, high flow rates in
 325 some regions of high k are offset by lower flow rates in other regions of low k , meaning that
 326 the total flow from the vicinity of PVD is effectively an averaging process. Despite the fact
 327 that the input statistics (i.e., μ , σ and θ) characterizing the random soil property of interest
 328 is defined at the point level, soil properties are rarely measured at a point and most

329 engineering measurements concerned with soil properties are performed on samples of some
330 finite volume, thus actually locally averaged over the sample volume. In light of this, the flow
331 of water through spatially variable soil into the drain is essentially a process governed by the
332 locally averaged soil properties. The local averaging is performed on the underlying point
333 distribution (i.e., normal distribution) of the soil property of interest, which will lead to a
334 reduction in the underlying point variance but the underlying mean will not be affected. For
335 the lognormal distribution, however, both the mean and variance will be reduced by the local
336 averaging, as the mean of a lognormal distribution depends on both the mean and variance of
337 the underlying normal distribution. On the basis of the above discussion, the locally averaged
338 mean of the underlying equivalent log-soil property field ($\ln X$), $\mu_{\ln \bar{X}}$, which is unaltered by
339 the local averaging can be given by:

$$340 \quad \mu_{\ln \bar{X}} = \mu_{\ln X} \quad (18)$$

341 Using Eqs. (2) and (3), $\mu_{\ln \bar{X}}$ can be expressed in terms of the input statistics of X , as follows:

$$342 \quad \mu_{\ln \bar{X}} = \mu_{\ln X} = \ln \mu_X - \frac{1}{2} \ln(1 + \nu_X^2) \quad (19)$$

343 According to the local averaging theory [15], the variance, $\sigma_{\ln \bar{X}}^2$, which is affected by the
344 local averaging, is given by:

$$345 \quad \sigma_{\ln \bar{X}}^2 = \gamma(D) \sigma_{\ln X}^2 \quad (20)$$

346 where $\gamma(D)$ is the ‘‘variance function’’ that defines the amount by which the variance is
347 reduced as a result of the local (arithmetic) averaging over a domain D and is a function of the
348 size of the averaging domain and correlation function. The detailed calculation procedure of
349 the variance reduction factor from the correlation function is given in Appendix A. It should
350 be noted that, since the spatial variability of both k_h and m_v are modelled using 3D random
351 fields and the FEMC results are obtained from 3D FEM analyses, $\gamma(D)$ in this study is also

352 calculated using the 3D variance reduction function. By substituting Eq. (3) into Eq. (20),

353 $\sigma_{\ln \bar{X}}^2$ can be expressed in terms of the prescribed statistics of X , as follows:

354
$$\sigma_{\ln \bar{X}}^2 = \gamma(D) \ln(1 + \nu_X^2) \quad (21)$$

355 For the purpose of comparing the proposed RBSA method with the FEMC approach, two
356 groups of RBSA models are developed. The random soil properties are considered to be
357 continuous over the whole unit cell in the first group, whereas random soil properties of the
358 smear zone in the second group are assumed to be independent of the undisturbed zone. For
359 each group, two RBSA models are developed to comply with the cases of considering both k_h
360 and m_v as random variables, while only k_h is considered to be a random variable in the second
361 case. For convenience, the RBSA models are denoted as G1C1 and G1C2 for the first group,
362 whereas they are denoted as G2C1 and G2C2 for the second group. Considering the
363 readership of the paper, only the two most general RBSA models, namely G1C1 and G2C2,
364 are presented in th section below, whereas the other two RBSA models (i.e., G1C2 and
365 G2C2) are presented in Appendix C. To facilitate the use of the RBSA models, an illustrated
366 worked example will follow.

367

368 *3.1 G1C1: RBSA model considering k_h and m_v as continuous random variables over the entire*
369 *unit cell*

370

371 In the development of the RBSA–G1C1 model, it is assumed that both k_h and m_v vary
372 spatially in such a way that their second moment structures (variance, covariance, etc.) in the
373 undisturbed and smear zones are identical with respect to the mean (i.e., $\nu_{k_k} = \nu_{k'_h}$, $\theta_{k_h} = \theta_{k'_h}$
374 and $\nu_{m_v} = \nu_{m'_v}$, $\theta_{m_v} = \theta_{m'_v}$). This means that the variance and covariance structure is assumed to
375 be stationary. However, non-stationary means for k_h and m_v are used to take into account the
376 smear effect. This is considered because non-stationary correlation structures are uncommon

377 in geotechnical engineering due to the prohibitive volumes of data required to estimate their
378 parameters. In geotechnical engineering, random-field models are often non-stationary in their
379 mean; however, the variance and covariance structure is generally assumed to be stationary
380 (Fenton and Griffiths 2008). As k_h and m_v are continuous over the entire soil domain, each
381 point in the unit cell is correlated to each other. Therefore, it can be assumed that μ_{k_h} / μ_{k_h} and
382 μ_{m_v} / μ_{m_v} remain constant in the unit cell. In other words, α and α_{m_v} contribute with little or no
383 variability to $U^*(t)$. Considering k_h and m_v as the only random variables and using their
384 geometric averages, Eq. (15) becomes:

$$385 \quad U^*(t) = C \frac{\bar{k}_h}{\bar{m}_v} \quad (22)$$

386 where \bar{k}_h and \bar{m}_v are, respectively, the geometric averages of soil permeability and volume
387 compressibility;

$$388 \quad C = \frac{2t}{r_e^2 \gamma_w \alpha \alpha_{m_v}} \quad (23)$$

$$389 \quad \alpha = \ln\left(\frac{n}{s}\right) - \frac{3}{4} + \frac{\mu_{k_h}}{\mu_{k_h'}} \ln(s) + F_r' \quad (24)$$

$$390 \quad \alpha_{m_v} = \frac{n^2 - s^2}{n^2 - 1} + \frac{s^2 - 1}{n^2 - 1} \frac{\mu_{m_v'}}{\mu_{m_v}} \quad (25)$$

391 Since the random variation of well resistance effect is not considered in this study, F_r' in Eq.
392 (24), which represents the average well resistance effect over the entire drain length, can be
393 estimated as [28]:

$$394 \quad F_r' = \frac{2\pi L^2}{3} \frac{\mu_{k_h}}{q_w} \quad (26)$$

395 Now a reasonable distribution for $U^*(t)$ can be found. Since both k_h and m_v are assumed to be
396 lognormally distributed, then \bar{k}_h and \bar{m}_v are also lognormally distributed (based on the central

397 limit theorem, the geometric average of a random variable tends to have a lognormal
 398 distribution), and therefore $U^*(t)$ will be lognormally distributed. In such a case, taking the
 399 logarithm of Eq. (22) yields:

$$400 \quad \ln U^*(t) = \ln C + \ln \bar{k}_h - \ln \bar{m}_v \quad (27)$$

401 To evaluate the probability of achieving a certain $U(t)$, the mean $\mu_{\ln U^*(t)}$ and variance $\sigma_{\ln U^*(t)}^2$
 402 of $\ln U^*(t)$ need to be estimated. The mean $\mu_{\ln U^*(t)}$ of $\ln U^*(t)$ can be obtained by taking the
 403 expectation of Eq. (27), as follows:

$$404 \quad \mu_{\ln U^*(t)} = \ln C + \mu_{\ln \bar{k}_h} - \mu_{\ln \bar{m}_v} \quad (28)$$

405 Assuming no cross-correlation between k_h and m_v , the variance $\sigma_{\ln U^*(t)}^2$ of $\ln U^*(t)$ can be
 406 simply estimated, as follows:

$$407 \quad \sigma_{\ln U^*(t)}^2 = \sigma_{\ln \bar{k}_h}^2 + \sigma_{\ln \bar{m}_v}^2 \quad (29)$$

408 The four unknown parameters: $\mu_{\ln \bar{k}_h}$, $\mu_{\ln \bar{m}_v}$, $\sigma_{\ln \bar{k}_h}^2$ and $\sigma_{\ln \bar{m}_v}^2$ in Eqs. (28) and (29) are now need
 409 to be expressed in terms of the known statistical input parameters of k_h and m_v . With reference
 410 to Eq. (19), the following expressions of $\mu_{\ln \bar{k}_h}$ and $\mu_{\ln \bar{m}_v}$ are obtained:

$$411 \quad \mu_{\ln \bar{k}_h} = \mu_{\ln k_h} = \ln \mu_{k_h} - \frac{1}{2} \ln(1 + \nu_{k_h}^2) \quad (30)$$

$$412 \quad \mu_{\ln \bar{m}_v} = \mu_{\ln m_v} = \ln \mu_{m_v} - \frac{1}{2} \ln(1 + \nu_{m_v}^2) \quad (31)$$

413 With reference to Eq. (21), $\sigma_{\ln \bar{k}_h}^2$ and $\sigma_{\ln \bar{m}_v}^2$ can then be expressed with the specified statistical
 414 parameters of k_h and m_v , as follows:

$$415 \quad \sigma_{\ln \bar{k}_h}^2 = \gamma(D)_{k_h} \left(\ln(1 + \nu_{k_h}^2) \right) \quad (32)$$

$$416 \quad \sigma_{\ln \bar{m}_v}^2 = \gamma(D)_{m_v} \left(\ln(1 + \nu_{m_v}^2) \right) \quad (33)$$

417 where $\gamma(D)_{k_h}$ and $\gamma(D)_{m_v}$ are the variance reduction factors for k_h and m_v , respectively. As the
 418 inherent spatial variability of both k_h and m_v is pertinent over the whole unit cell, the entire
 419 soil domain, D , is used for estimating $\gamma(D)_{k_h}$ and $\gamma(D)_{m_v}$.

420 Now $\mu_{\ln U^{*(t)}}$ and $\sigma_{\ln U^{*(t)}}^2$ can be evaluated by substituting $\mu_{\ln \bar{k}_h}$ and $\mu_{\ln \bar{m}_v}$ in Eq. (28), and
 421 $\sigma_{\ln \bar{k}_h}^2$ and $\sigma_{\ln \bar{m}_v}^2$ in Eq. (29), as follows:

$$422 \quad \mu_{\ln U^{*(t)}} = \ln C + \left[\ln \mu_{k_h} - \frac{1}{2} \ln(1 + \nu_{k_h}^2) \right] - \left[\ln \mu_{m_v} - \frac{1}{2} \ln(1 + \nu_{m_v}^2) \right] \quad (34)$$

$$423 \quad \sigma_{\ln U^{*(t)}}^2 = \gamma(D)_{k_h} \left(\ln(1 + \nu_{k_h}^2) \right) + \gamma(D)_{m_v} \left(\ln(1 + \nu_{m_v}^2) \right) \quad (35)$$

424 Using the developed semi-analytical relationships shown in Eqs. (34) and (35), the procedure
 425 for evaluating $\mu_{\ln U^{*(t)}}$ and $\sigma_{\ln U^{*(t)}}^2$ can then be summarized as follows:

- 426 1. Determine the mean, standard deviation and scale of fluctuation of k_h and m_v (i.e., μ_{k_h} ,
 427 σ_{k_h} and θ_{k_h} ; and μ_{m_v} , σ_{m_v} and θ_{m_v});
- 428 2. Calculate $\nu_{k_h} = \sigma_{k_h} / \mu_{k_h}$ and $\nu_{m_v} = \sigma_{m_v} / \mu_{m_v}$;
- 429 3. Evaluate all constant parameters involved in the RBSA method (i.e., α , α_{m_v} , C , $\gamma(D)_{k_h}$ and
 430 $\gamma(D)_{m_v}$); and
- 431 4. Estimate $\mu_{\ln U^{*(t)}}$ and $\sigma_{\ln U^{*(t)}}^2$ by substituting C , μ_{k_h} , μ_{m_v} , ν_{k_h} and ν_{m_v} in Eq. (34), and
 432 $\gamma(D)_{k_h}$, $\gamma(D)_{m_v}$, ν_{k_h} and ν_{m_v} in Eq. (35).

433

434 3.2 G2C1: RBSA model considering k_h , k'_h , m_v and m'_v as independent random variables

435

436 As k_h , k'_h , m_v and m'_v are independent random variables, α and α_{m_v} are no longer constant

437 parameters. Eq. (15) is therefore becomes:

438
$$U^*(t) = C \frac{\bar{k}_h}{\bar{m}_v \bar{\alpha} \bar{\alpha}_{m_v}} \quad (36)$$

439 where

440
$$C = \frac{2t}{r_e^2 \gamma_w} \quad (37)$$

441 $\bar{\alpha}$ and $\bar{\alpha}_{m_v}$ are, respectively, the equivalent α and α_{m_v} parameters of the spatially variable soil
 442 and can be expressed by the following equations:

443
$$\bar{\alpha} = \ln\left(\frac{n}{s}\right) - \frac{3}{4} + \frac{\bar{k}_h}{k'_h} \ln(s) + F'_r \quad (38)$$

444
$$\bar{\alpha}_{m_v} = \frac{n^2 - s^2}{n^2 - 1} + \frac{s^2 - 1}{n^2 - 1} \frac{\bar{m}_v}{m'_v} \quad (39)$$

445 Assuming that: $\ln\left(\frac{n}{s}\right) - \frac{3}{4} + F'_r = a$; $\ln(s) = b$ and $\frac{\bar{k}_h}{k'_h} = W$, Eq. (38) becomes:

446
$$\bar{\alpha} = a + bW \quad (40)$$

447 Similarly, by assuming $\frac{n^2 - s^2}{n^2 - 1} = g$; $\frac{s^2 - 1}{n^2 - 1} = h$ and $\frac{\bar{m}_v}{m'_v} = V$, Eq. (39) becomes:

448
$$\bar{\alpha}_{m_v} = g + hV \quad (41)$$

449 The parameters α and α_{m_v} are respectively the function of permeability and volume
 450 compressibility. Therefore, $\bar{\alpha}$ and $\bar{\alpha}_{m_v}$, and in turn $U^*(t)$ will also be approximately
 451 lognormally distributed. In such a case, the mean $\mu_{\ln U^*(t)}$ of $\ln U^*(t)$ can be obtained by taking
 452 logarithm and subsequent expectation of Eq. (36):

453
$$\mu_{\ln U^*(t)} = \ln C + \mu_{\ln \bar{k}_h} - \mu_{\ln \bar{m}_v} - \mu_{\ln \bar{\alpha}} - \mu_{\ln \bar{\alpha}_{m_v}} \quad (42)$$

454 The variance of k_h , k'_h , m_v and m'_v contribute to the variance of $\ln U^*(t)$. As $\bar{\alpha}$ and $\bar{\alpha}_{m_v}$
 455 involve k_h , k'_h , m_v and m'_v , then by assuming no cross-correlation between any of the random
 456 variables, the variance $\sigma_{\ln U^*(t)}^2$ of $\ln U^*(t)$ is thus:

$$457 \quad \sigma_{\ln U^*(t)}^2 = \sigma_{\ln \bar{\alpha}}^2 + \sigma_{\ln \bar{\alpha}_{m_v}}^2 \quad (43)$$

458 In order to obtain $\mu_{\ln U^*(t)}$ and $\sigma_{\ln U^*(t)}^2$ in Eqs. (42) and (43) above, the six unknown
 459 parameters $\mu_{\ln \bar{k}_h}$, $\mu_{\ln \bar{m}_v}$, $\mu_{\ln \bar{\alpha}}$, $\mu_{\ln \bar{\alpha}_{m_v}}$, $\sigma_{\ln \bar{\alpha}}^2$ and $\sigma_{\ln \bar{\alpha}_{m_v}}^2$ must be obtained in terms of the
 460 known statistical input parameters of k_h , k'_h , m_v and m'_v . The formulations of all unknown
 461 parameters are presented in Appendix B, as they are large enough not to be included in the
 462 main text so as to avoid any possible disruption to the readership of the paper. At the end of
 463 Appendix B, a procedure for calculating these unknown parameters is summarised from
 464 which $\mu_{\ln U^*(t)}$ and $\sigma_{\ln U^*(t)}^2$ can be estimated by substituting them in Eqs. (42) and (43).

465 Having established with reasonable accuracy the distribution parameters of $\ln U^*(t)$ for
 466 the RBSA method, the probabilities of achieving a target degree of consolidation at any
 467 specified time, $U_s(t)$, can be obtained from the following lognormal probability distribution
 468 transformation:

$$469 \quad P[U^*(t) \geq U_s^*(t)] = 1 - \Phi \left(\frac{\ln U_s^*(t) - \mu_{\ln U^*(t)}}{\sigma_{\ln U^*(t)}} \right) \quad (44)$$

470 where: $P[.]$ = probability of its argument, $\Phi(.)$ is the standard normal cumulative distribution
 471 function and $U_s^*(t)$ is the target $U^*(t)$ that needs to be achieved. Since $U^*(t)$ is a
 472 monotonically increasing function of $U(t)$, the following equation holds [29]:

$$473 \quad P[U^*(t) \geq U_s^*(t)] = P[U(t) \geq U_s(t)] \quad (45)$$

474 Assuming the target degree of consolidation is 90% (i.e., $U_s(t) = 0.9$) and denoting it as U_{90} ,
 475 the probability of achieving U_{90} at any time, t , can be estimated as follows:

$$476 \quad P[U(t) \geq U_{90}] = P[U^*(t) \geq 2.3026] = 1 - \Phi \left(\frac{\ln 2.3026 - \mu_{\ln U^*(t)}}{\sigma_{\ln U^*(t)}} \right) \quad (46)$$

477 Note that when $U_s(t) = U_{90} = 0.9$, then $U_s^*(t) = \ln[1/(1-0.9)] = 2.3026$. In the following
 478 section, detailed comparison between the results obtained from the stochastic FEMC approach
 479 and proposed RBSA method is presented and discussed.

480

481 **4. Comparison between finite-element Monte-Carlo approach and reliability-** 482 **based semi-analytical method**

483

484 In this section, a comparison between the proposed RBSA method and FEMC approach
 485 is demonstrated through an illustrative worked example. For brevity and because of the good
 486 agreement between the results of the four proposed RBSA models (i.e., G1C1, G1C2, G2C1
 487 and G2C2) and their corresponding FEMC solutions, only G1C1 and G2C1 models are
 488 presented herein. It is to be reminded that k_h and m_v in G1C1 model are considered as
 489 continuous random variables over the entire unit cell, whereas in G2C1 model k_h and m_v in the
 490 smear and undisturbed zones are considered to be independent random variables. Prior to
 491 comparison, the rationality of the assumption of lognormal distribution for $U^*(t)$ under
 492 various assumed ground conditions is assessed through the frequency density plot of $U^*(t)$ on
 493 the basis of 2000 realizations for each combination of the variability parameters ν and θ for
 494 the spatially variable soil properties at several different consolidation time. The chi-square
 495 goodness-of-fit tests for all cases are performed and yielded p -values between 0.15–0.96.
 496 Such high p -values indicate that there is a very little evidence in the simulated $U^*(t)$ sample
 497 against the null hypothesis of the assumed lognormal distribution. By accepting the lognormal
 498 distribution, all subsequent statistics of the underlying normally distributed $\ln U^*(t)$ are
 499 estimated by the method of moments from the suite of 2000 realizations using the following
 500 transformations:

$$501 \quad \mu_{\ln U^*(t)} = \frac{1}{n_{sim}} \sum_{i=1}^{n_{sim}} \ln U^*_i(t) \quad (47)$$

$$\sigma_{\ln U^*(t)} = \sqrt{\frac{1}{n_{sim} - 1} \sum_{i=1}^{n_{sim}} [\ln U^*_i(t) - \mu_{\ln U^*(t)}]^2} \quad (48)$$

503 where: $U^*_i(t)$ is the $U^*(t)$ from the i th realization ($i = 1, 2, 3, \dots, n_{sim}$) and n_{sim} is the total
504 number of realizations (i.e., 2000).

505

506 4.1 FEMC approach versus RBSA-G1C1 model

507

508 The illustrative example used for comparison between the FEMC approach and RBSA-
509 G1C1 model involves the same unit cell consolidation problem illustrated earlier (i.e., $L =$
510 1.0m , $r_w = 0.032\text{m}$, $r_e = 0.536\text{m}$, $r_s = 0.197\text{m}$, $n = 16.75$ and $s = 6.156$). The spatial variability
511 of k_h and m_v is assumed to have $\mu_{k_h} = 0.15$ m/year, $k_h/k'_h = 3.0$, $\mu_{m_v} = 1.0 \times 10^{-3}$ m²/kN, m'_v/m_v
512 $= 1.2$, $\nu_{k_h} = 200\%$, $\nu_{m_v} = 20\%$ and $\theta_{k_h} = \theta_{m_v} = 1.0\text{m}$. Armed with the above information,
513 $\mu_{\ln U^*(t)}$ (see Eq. (34)) and $\sigma_{\ln U^*(t)}^2$ (see Eq. (35)) are calculated by following the steps
514 described earlier in developing the RBSA-G1C1 model, as explained below.

515 Since no well resistance is considered, the constant parameters involved in the RBSA
516 method can be calculated using the following equations:

$$517 \alpha = \ln(16.75) - 0.75 + (3 - 1) \times \ln(6.156) = 5.703$$

$$518 \alpha_{m_v} = \frac{16.75^2 - 6.156^2}{16.75^2 - 1} + \frac{6.156^2 - 1}{16.75^2 - 1} \times 1.2 = 1.026$$

519 If the probability of achieving 90% consolidation is to be determined at 0.75 year, then the
520 parameter C will be:

$$521 C = \frac{2 \times 0.75}{0.536^2 \times 9.8 \times 5.703 \times 1.026} = 0.091 \text{ m year/kN}$$

522 Now using the algorithm presented in Appendix A, the variance reduction factor for k_h and m_v
523 is given by:

524 $\gamma(D)_{k_h} = \gamma(D)_{m_v} = \gamma(S, S, L) = 0.312$

525 Substituting the given μ , ν and the calculated constant parameters in Eqs. (34) and (35) yield:

526 $\mu_{\ln U^*(t)} = \ln(0.091) + [\ln 0.15 - 0.5 \ln(1 + 2.0^2)] - [\ln 1.0 \times 10^{-3} - 0.5 \ln(1 + 0.2^2)] = 1.83$

527 $\sigma_{\ln U^*(t)}^2 = 0.312 \times \ln(1 + 2.0^2) + 0.312 \times \ln(1 + 0.2^2) = 0.514$, therefore, $\sigma_{\ln U^*(t)} = 0.717$

528 Using the computed values of $\mu_{\ln U^*(t)}$ and $\sigma_{\ln U^*(t)}$ in Eq. (46), the probability of achieving

529 90% consolidation from the RBSA–G2C1 model can be computed as follows:

530 $P[U(t = 0.75) \geq U_{90}] = 1 - \Phi\left(\frac{\ln(2.3026) - 1.83}{0.717}\right) = 0.92$

531 The FEMC approach of the above problem yields $P[U(t = 0.75) \geq U_{90}] = 0.94$, thus

532 demonstrating an excellent agreement between the FEMC approach and proposed RBSA–

533 G2C1 method. Following the above procedure, $\mu_{\ln U^*}$, $\sigma_{\ln U^*}$ and $P[U \geq U_{90}]$ at each time step

534 over each combinations of the spatial variability parameters are evaluated for both solution

535 approaches and the results are compared in Figs. 2–3. It should be noted that, for brevity, the

536 results of only a few tests are presented.

537 The agreement between $\mu_{\ln U^*}$ and $\sigma_{\ln U^*}$ derived from the FEMC simulation and predicted

538 by the RBSA-G1C1 model is examined in Fig. 2. The influence of ν on $\mu_{\ln U^*}$ is illustrated in

539 Fig. 2(a) for a constant $\theta = 0.5\text{m}$. It can be seen that, in general, the predicted values of $\mu_{\ln U^*}$

540 obtained from the RBSA model and the FEMC approach match exceptionally well. In both

541 methods, the estimated $\mu_{\ln U^*}$ decreases with the increase of ν , as expected. The relationships

542 between the estimated $\mu_{\ln U^*}$ versus the consolidation time, t , for various θ at constant

543 $\nu_{k_h} = 200\%$ and $\nu_{m_v} = 20\%$ are shown in Fig. 2(b). It can be seen that the results obtained

544 from both the FEMC approach and RBSA-G1C1 model are almost identical. In each solution

545 method, even though the results for various θ are drawn in the plot, they are embodied into a

546 single curve, implying that the obtained results at different θ are very close and cannot be

547 distinguished. The virtually identical curves for all θ obtained from each method of analysis
548 demonstrate that $\mu_{\ln U^*}$ is largely independent of θ . This is expected as in principle θ does not
549 affect the local average mean of the normally distributed process.

550 The effect of v on $\sigma_{\ln U^*}$ for a fixed value of $\theta = 0.5m$ is shown in Fig. 2(c), which shows
551 that, in general, $\sigma_{\ln U^*}$ increases with the increase of v and the agreement between the FEMC
552 approach and RBSA model is very good. The influence of θ on $\sigma_{\ln U^*}$ at constant $v_{k_h} = 200\%$
553 and $v_{m_v} = 20\%$ is shown in Fig. 2(d). It can be seen that $\sigma_{\ln U^*}$ increases with the increase of θ
554 for both approaches, and apart from some slight discrepancy at high $\theta \geq 5m$, the agreement
555 between the FEMC approach and RBSA model is reasonable and shows good compliance.
556 This behaviour can be explained by noting that, when $\theta \rightarrow 0$, the simulated soil profile is
557 consisted of an infinite number of independent ‘observations’, thus there is a decrease in the
558 average variance of the consolidation rate and the averaging process almost perfectly predicts
559 the condition in the unit cell. Conversely, when θ is large, the average variance of the
560 consolidation rate is also expected to be large due to the decrease in the number of
561 independent ‘observations’, resulting in less averaging variance reduction within each
562 realization.

563 The agreement between the FEMC approach and RBSA-G1C1 model is examined in
564 terms of $P[U \geq U_{90}]$ in Fig. 3. The effect of v on $P[U \geq U_{90}]$ at a fixed value of $\theta = 0.5m$ is
565 shown in Fig. 3(a). It can be seen that the two solutions are in a good agreement despite some
566 slight discrepancy at the earlier stage of consolidation. This may be attributed to the fact that
567 the FEMC approach relies on the free strain concept, while the RBSA method is based on
568 Hansbo’s solution of an equal strain assumption. As the probability of achieving a target
569 degree of consolidation of usual interest is greater than 50%, any discrepancy in this range has
570 a little implication from the practical point of view. In Fig. 3(b), the compliance between the
571 FEMC approach and proposed RBSA method shows a good agreement for various θ at

572 constant $\nu_{k_h} = 200\%$ and $\nu_{m_v} = 20\%$, although a slight discrepancy in $P[U \geq U_{90}]$ exists
573 when θ is as small as 0.1m (i.e., for erratic soil). It can also be seen that for any $P[U \geq U_{90}] \geq$
574 50%, the RBSA–G1C1 model yields slightly higher (unconservative) estimation of $P[U \geq$
575 $U_{90}]$ than that calculated by the FEMC approach when θ is as low as 0.1m. On the other hand,
576 $P[U \geq U_{90}]$ derived from the RBSA–G1C1 model is slightly lower (conservative) than those
577 obtained from the FEMC approach when θ is as high as 1.0m.

578

579 *4.2 FEMC approach versus RBSA–G2C1 model*

580

581 Following the procedure set out in Appendix B, curves for $\mu_{\ln U^*}$, $\sigma_{\ln U^*}$ and in turn $P[U \geq$
582 $U_{90}]$ with time over some selected combinations of the spatial variability parameters are
583 obtained for RBSA-G2C1 method. The agreement between $\mu_{\ln U^*}$, $\sigma_{\ln U^*}$ and $P[U \geq U_{90}]$ derived
584 from the FEMC simulation and predicted by the RBSA–G2C1 model are then examined in
585 Figs. 4–9. As mentioned earlier, two independent random fields for k_h and m_v are generated
586 for the undisturbed and smear zones. For convenience of presentation, the statistical
587 parameters in the smear and undisturbed zones (i.e., ν and θ of k_h and m_v) are denoted with
588 appropriate subscripts “s” and “u” depending on whether they are specified for the smear zone
589 or undisturbed zone, where s refers to the smear zone while u refers to the undisturbed zone.

590 The influence of increasing ν on the agreement between the FEMC approach and RBSA–
591 G2C1 model in terms of $\mu_{\ln U^*}$ at a fixed value of $\theta_u = \theta_s = 1.0\text{m}$ is shown in Fig. 4. It can be
592 seen that, in general, the predicted values of $\mu_{\ln U^*}$ obtained from the RBSA model match those
593 obtained from the FEMC approach reasonably well. In both methods, the estimated $\mu_{\ln U^*}$
594 decreases with the increase of ν , as expected. However, the identical curves for all cases of ν_u
595 (ν_{k_h} and ν_{m_v} are fixed at 100% and 10%, respectively) for both methods in Fig. 4(a) indicate
596 that the effect of increasing ν_u on $\mu_{\ln U^*}$ remains marginal. The effect of ν_s on $\mu_{\ln U^*}$ at fixed

597 values of $v_{k_h} = 100\%$ and $v_{m_v} = 10\%$ is illustrated in Fig. 4(b). It can be seen that, although
598 the agreement between the RBSA–G2C1 model and the FEMC approach is reasonably well,
599 the discrepancy in $\mu_{\ln U^*}$ between the two methods becomes higher as t increases. Fig. 4 also
600 demonstrates that the decreasing rate of $\mu_{\ln U^*}$ is higher for an increase in v_s than v_u .

601 The matching of $\mu_{\ln U^*}$ obtained from the RBSA-G2C1 model and FEMC approach is
602 examined in Fig. 5 for an increasing θ at constant values of $v_{k_k} = v_{k'_h} = 200\%$ and $v_{m_v} = v_{m'_v} =$
603 20% . The effect of θ_u on $\mu_{\ln U^*}$ for a constant value of $\theta_s = 0.25\text{m}$ is shown Fig. 5(a), whereas
604 the effect of θ_s on $\mu_{\ln U^*}$ for a fixed value of $\theta_u = 0.25\text{m}$ is shown in Fig. 5(b). It can be seen
605 that the results obtained from both the FEMC approach and RBSA–G2C1 model are nearly
606 identical. However, a slight discrepancy in $\mu_{\ln U^*}$ from the two solution approaches is found
607 when the consolidation time t is as large as 1 year. In each solution method, the single curve
608 for all θ confirms that $\mu_{\ln U^*}$ is independent of θ .

609 The agreement between the FEMC approach and RBSA–G2C1 model is further
610 illustrated by matching the estimated $\sigma_{\ln U^*}$ at different values of v_u and v_s , and at a constant θ_u
611 $= \theta_s = 1.0\text{m}$ (see Fig. 6). It can be seen that, in general, $\sigma_{\ln U^*}$ increases with the increase of v
612 and the agreement between the two solution approaches is reasonably well. However, for a
613 certain v at any particular consolidation time t , the estimated values of $\sigma_{\ln U^*}$ derived from the
614 RBSA–G2C1 model are slightly higher than those obtained from the FEMC approach. The
615 above observation is more accurate for v_s (see Fig. 6(b)) than v_u (see Fig. 6(a)). The
616 comparison shown in Fig. 6 reveals that $\sigma_{\ln U^*}$ is largely insensitive to varying v_u and highly
617 sensitive to increasing v_s .

618 The effect of θ derived from the FEMC approach and RBSA–G2C1 model in terms of
619 $\sigma_{\ln U^*}$ for fixed values of $v_{k_k} = v_{k'_h} = 200\%$ and $v_{m_v} = v_{m'_v} = 20\%$ is demonstrated in Fig. 7. It can
620 be seen that $\sigma_{\ln U^*}$ increases with the increase of θ , and apart from some slight discrepancy at
621 large θ (i.e., at $\theta \geq 5.0\text{m}$), the agreement between the two methods is again reasonably well. In

622 Fig. 7(a), it can be seen that varying θ_u (θ_s is fixed at 0.25m) has a marginal effect on $\sigma_{\ln U}$,
623 while varying θ_s has a considerable impact on the estimated values of $\sigma_{\ln U^*}$ (see Fig. 7(b)).

624 The influence of v on the agreement between the FEMC approach and RBSA–G2C1
625 model in terms of $P[U \geq U_{90}]$ at a fixed value of $\theta_u = \theta_s = 1.0\text{m}$ is shown in Fig. 8. The effect
626 of increasing v_u on $P[U \geq U_{90}]$ is illustrated in Fig. 8(a). It can be seen that the predicted $P[U$
627 $\geq U_{90}]$ obtained from the proposed RBSA–G2C1 model agrees exceptionally well with those
628 obtained from the FEMC approach for all cases of v_u (v_{k_h} and v_{m_v} are fixed at 100% and 10%,
629 respectively). The virtually identical curves of $P[U \geq U_{90}]$ in Fig. 8(a) for all v_u indicate that
630 $P[U \geq U_{90}]$ is largely independent of v_u . Fig. 8(b) illustrates the effect of v_s on $P[U \geq U_{90}]$ at a
631 fixed value of $\theta_u = \theta_s = 1.0\text{m}$. Although the overall agreement between the estimated $P[U \geq$
632 $U_{90}]$ by the two methods is very good, the caveat, however, is that the RBSA–G2C1 model
633 gives slightly unconservative estimate of $P[U \geq U_{90}]$ for any $P[U \geq U_{90}] > 50\%$ and
634 particularly when $v_{k_h} \geq 200\%$ with $v_{m_v} \geq 20\%$. This higher values of predicted $P[U \geq U_{90}]$
635 given by the RBSA–G2C1 model is due to the higher predicted $\mu_{\ln U^*}$, as shown in Fig. 4(b).
636 Fig. 8 also illustrates that the increasing rate of $P[U \geq U_{90}]$ with respect to t decreases as v
637 increases and this effect is more pronounced for an increase in v_s than v_u .

638 Apart from some slight discrepancy particularly when $P[U \geq U_{90}]$ in the range between
639 70% – 90%, the FEMC approach and proposed RBSA–G2C1 model show good agreement
640 for various θ_u (see Fig. 9(a)) and θ_s (see Fig. 9(b)) at constant values of $v_k = 100\%$ and $v_{m_v} =$
641 25% as illustrated in Fig 9. This discrepancy between the two solutions is expected because of
642 the fact that the variability in $U(t)$ is zero at the beginning of consolidation (i.e., at $t = 0.0$),
643 and gradually increases with the increase in the consolidation time until it reaches a maximum
644 value at certain intermediate t , then decreases with further increase in time until it approaches
645 zero again after the full consolidation is occurred. It can be seen that for any $P[U \geq U_{90}] \geq$
646 50%, the values of $P[U \geq U_{90}]$ derived from the RBSA–G2C1 model are slightly higher

647 (unconservative) than those obtained from the FEMC approach when θ is as low as 0.25m,
648 while this trend becomes opposite (conservative) when θ is as high as 1.0m. The comparison
649 in Fig. 9(a & b) reveals that the effect of θ_s on $P[U \geq U_{90}]$ is more significant than θ_u .

650 The overall conclusion from the above comparison in Figs. 4–9 is that the RBSA-G2C1
651 model and FEMC approach agree reasonably well despite some discrepancies in the results of
652 $\mu_{\ln U^*}$, $\sigma_{\ln U^*}$ and $P[U \geq U_{90}]$. This is attributed mostly to the empirical adjustment of the
653 RSBA model which is necessary due to the fact that the sum of two lognormally distributed
654 random variables does not have a simple closed form solution. In addition, for both solution
655 methods it is found that the probabilistic behavior of soil consolidation is governed by the
656 spatial variation of the soil properties of the smear zone. This behavior is expected because all
657 expelled water from the PVD must pass through the smear zone.

658

659 **5. Discussion**

660

661 It is noteworthy that the agreement between the proposed RBSA method and FEMC
662 approach shown above was examined for a consolidation problem of a soil layer having a
663 thickness of 1m and isotropic scale of fluctuation. Therefore, to arrive at a general conclusion
664 regarding the validity of the proposed RBSA method compared to the FEMC approach for
665 thicker soil layers of anisotropic correlation structure, the comparison is also tested for a more
666 practical example of a unit cell of thickness of geometry $L = 4.25\text{m}$, $r_e = 0.48\text{m}$, $r_s = 0.197\text{m}$
667 and $r_w = 0.032\text{m}$, and parameters $\mu_{k_h} = 0.15\text{m/year}$, $k_h/k'_h = 3.0$, $\mu_{m_v} = 1.0 \times 10^{-3}\text{m}^2/\text{kN}$ and
668 $m'_v/m_v = 1.2$. The 3D FE mesh of such problem consisted of 24,565 eight node hexahedral
669 elements of size $0.05\text{m} \times 0.05\text{m} \times 0.05\text{m}$. The FEMC approach of the problem needed an
670 intensive computational time of 28 days to run 2000 realizations on an Intel core i5 CPU @
671 3.4 GHz computer. Therefore, only two FEMC simulation tests, named as FEMC1 and

672 FEMC2, are performed considering anisotropic θ . FEMC1 and FEMC2 stand for comparison
673 with RBSA–G1C1 and FEMC2 and RBSA–G2C1 models, respectively. For FEMC1 and its
674 counterpart RBSA–G1C1 model, the random field parameters are assumed to be as follows:
675 $\nu_{k_h} = 200\%$, $\nu_{m_v} = 20\%$, $\theta_x = \theta_y = 10.0\text{m}$ and $\theta_z = 1.0\text{m}$. For FEMC2 and its counterpart
676 RBSA–G2C1 model, the spatial variability of k_h and m_v is assumed to have $\nu_{k_k} = \nu_{k_h} = 200\%$,
677 $\nu_{m_v} = \nu_{m'_v} = 20\%$, $\theta_x = \theta_y = 10.0\text{m}$ and $\theta_z = 1.0\text{m}$. The same θ for k_h and m_v for the smear and
678 undisturbed zones are used in this investigation. The computed $\mu_{\ln U^{*(t)}}$, $\sigma_{\ln U^{*(t)}}$ and $P[U \geq U_{90}]$
679 from the two methods are compared in Fig. 10. It can be seen that $\mu_{\ln U^{*(t)}}$ (Fig. 10a), $\sigma_{\ln U^{*(t)}}$
680 (Fig. 10b) and $P[U \geq U_{90}]$ (Fig. 10c) obtained from both the FEMC approach and RBSA
681 method are almost identical, implying very good agreement between the two methods. This is
682 due to the fact that the stochastic response of soil consolidation by PVDs is dependent on the
683 ratio of the scale of fluctuation to the dimensions of the influence zone surrounding the PVD,
684 which can be readily taken into account by the use of a variance reduction function.
685 Therefore, the proposed RBSA method can be utilized with confidence as an easy-to-use
686 alternative to the computationally intensive FEMC approach for assessing the reliability of
687 soil consolidation by PVDs in spatially variable soils. Despite the fact that the proposed
688 RBSA method is suitable for hand calculations, it is coded by the authors in FORTRAN to
689 provide a user friendly executable program that can be readily used by practitioners, and the
690 program is available for interested readers upon request.

691

692 **6. Conclusions**

693

694 Simple reliability-based semi-analytical (RBSA) models for predicting the statistics and
695 probability of achieving a target degree of consolidation for PVD-improved ground were

696 developed incorporating the inherent (spatial) variability of soils. The performance function
697 of the proposed RBSA models was based on the well-known deterministic equation of
698 Hansbo [2], which considers soil consolidation due to the horizontal drainage only. Under
699 various ground conditions, the proposed RBSA models account for the spatial variability of
700 soil volume compressibility and/or soil permeability, which are considered to be the most
701 significant spatial random variables affecting soil consolidation by PVDs.

702 The results confirm that there is good agreement between the proposed RBSA method
703 and the finite-element Monte-Carlo (FEMC) approach, implying that the simpler RBSA
704 method negates the need for the computationally intensive and time consuming FEMC
705 technique. The results also indicate that, for given coefficients of variation of soil
706 permeability and volume compressibility, the stochastic response of soil consolidation by
707 PVDs is dependent on the ratio of the scale of fluctuation to the dimensions of the influence
708 zone surrounding the PVD, which can be readily taken into account by the use of a variance
709 reduction function. Therefore, the proposed RBSA model can be confidently employed to
710 assess the reliability of consolidation problems implying arbitrary dimensions.

711 Despite the success of the proposed RBSA method for design of PVD-improved ground,
712 it has some limitations compared to the FEMC approach which can deal with more general
713 cases and offers the ability to solve problems with less restrictive conditions. For example, the
714 RBSA method does not consider soil consolidation due to the vertical drainage; hence, the
715 computed probability of achieving a target degree of consolidation would be slightly
716 conservative. However, it should be emphasised that, in practice, the contribution of soil
717 consolidation due to the vertical drainage is only a small fraction of the overall soil
718 consolidation and can thus be neglected without significant impact on the design results. In
719 addition, soil permeability and volume compressibility were assumed to be uncorrelated,
720 which again may lead to somewhat conservative solutions. However, it was reported by

721 Freeze [5] that the impact of non-zero correlation between k and m_v on problems of one
722 dimensional consolidation is quite minor and the uncorrelated assumption adopted in the
723 RBSA method is thus reasonable. The overall conclusion is that despite the abovementioned
724 minor limitations of the proposed RBSA method compared to the FEMC approach, the RBSA
725 provides more practical design for PVD-improved ground with an acceptable accuracy, which
726 negates the need for the more sophisticated FEMC approach that requires impractical
727 intensive computational time.

728

729 **Appendix A. Variance reduction function**

730

731 Considering the averaging domain D is a cube of dimension $X \times Y \times Z$, then $\gamma(D)$
732 corresponding to the Markov correlation function (see Eq. (6)) can be defined by Eq.
733 (A.1), as follows [17]:

$$734 \quad \gamma(X, Y, Z) = \frac{1}{X^2 Y^2 Z^2} \times \int_0^X \int_0^X \int_0^Y \int_0^Y \int_0^Z \int_0^Z \rho(\varsigma_1 - \xi_1, \varsigma_2 - \xi_2, \varsigma_3 - \xi_3) d\varsigma_1 d\xi_1 d\varsigma_2 d\xi_2 d\varsigma_3 d\xi_3 \quad (\text{A.1})$$

735 The sixfold integration in Eq. (A.1) can be condensed to a threefold integration by taking
736 advantage of the quadrant symmetry of the Eq. (6) as follows [17]:

$$737 \quad \gamma(X, Y, Z) = \frac{8}{X^2 Y^2 Z^2} \times \int_0^X \int_0^Y \int_0^Z (X - \tau_1)(Y - \tau_2)(Z - \tau_3) \rho(\tau_1, \tau_2, \tau_3) d\tau_1 d\tau_2 d\tau_3 \quad (\text{A.2})$$

738 Eq. (A.2) can be computed numerically with reasonable accuracy using Gaussian quadrature
739 integration scheme as follows [17]:

$$740 \quad \gamma(X, Y, Z) = \frac{1}{8} \sum_{i=1}^{n_g} \omega_i (1 - \psi_i) \sum_{j=1}^{n_g} \omega_j (1 - \psi_j) \sum_{k=1}^{n_g} \omega_k (1 - \psi_k) \rho(\varsigma_i, \xi_j, \vartheta_k) \quad (\text{A.3})$$

741 where

$$742 \quad \varsigma_i = X(1 + \psi_i)/2, \xi_j = Y(1 + \psi_j)/2, \vartheta_k = Z(1 + \psi_k)/2 \quad (\text{A.4})$$

743 In which, ω_i , ϑ_i , and n_g are the weights, Gauss points, and their total number, respectively.

744 **Appendix B. Computation of $\mu_{\ln \bar{k}_h}$, $\mu_{\ln \bar{m}_v}$, $\mu_{\ln \bar{\alpha}}$, $\mu_{\ln \bar{\alpha}_{m_v}}$, $\sigma_{\ln \bar{\alpha}}^2$ and $\sigma_{\ln \bar{\alpha}_{m_v}}^2$**

745

746 With reference to Eqs. (18) and (19), $\mu_{\ln \bar{k}_h}$ and $\mu_{\ln \bar{m}_v}$ can be calculated as follows:

747
$$\mu_{\ln \bar{k}_h} = \mu_{\ln k_h} = \ln \mu_{k_h} - \frac{1}{2} \ln(1 + \nu_{k_h}^2) \quad (\text{B.1})$$

748
$$\mu_{\ln \bar{m}_v} = \mu_{\ln m_v} = \ln \mu_{m_v} - \frac{1}{2} \ln(1 + \nu_{m_v}^2) \quad (\text{B.2})$$

749 Taking expectation of Eqs. (40) and (41) yield the following equations of the mean of $\bar{\alpha}$ (i.e.,
750 $\mu_{\bar{\alpha}}$) and $\bar{\alpha}_{m_v}$ (i.e., $\mu_{\bar{\alpha}_{m_v}}$):

751
$$\mu_{\bar{\alpha}} = a + b\mu_W \quad (\text{B.3})$$

752
$$\mu_{\bar{\alpha}_{m_v}} = g + h\mu_V \quad (\text{B.4})$$

753 The variance of $\bar{\alpha}$ (i.e., $\sigma_{\bar{\alpha}}^2$) and $\bar{\alpha}_{m_v}$ (i.e., $\mu_{\bar{\alpha}_{m_v}}$) are thus:

754
$$\sigma_{\bar{\alpha}}^2 = b^2 \sigma_W^2 \quad (\text{B.5})$$

755
$$\sigma_{\bar{\alpha}_{m_v}}^2 = h^2 \sigma_V^2 \quad (\text{B.6})$$

756 Recalling that, $W = \bar{k}_h / \bar{k}'_h$ and $V = \bar{m}_v / \bar{m}'_v$. Since both \bar{k}_h , \bar{k}'_h , \bar{m}_v and \bar{m}'_v are lognormally

757 distributed, W and V will also be approximately lognormally distributed. According to Eqs.

758 (18) and (19), the following expressions of $\mu_{\ln W}$ and $\mu_{\ln V}$ with the known parameters are

759 derived:

760
$$\mu_{\ln W} = \mu_{\ln \bar{k}_h} - \mu_{\ln \bar{k}'_h} = \left(\ln \mu_{k_h} - \frac{1}{2} \ln(1 + \nu_{k_h}^2) \right) - \left(\ln \mu_{k'_h} - \frac{1}{2} \ln(1 + \nu_{k'_h}^2) \right) \quad (\text{B.7})$$

761
$$\mu_{\ln V} = \mu_{\ln \bar{m}'_v} - \mu_{\ln \bar{m}_v} = \left(\ln \mu_{m'_v} - \frac{1}{2} \ln(1 + \nu_{m'_v}^2) \right) - \left(\ln \mu_{m_v} - \frac{1}{2} \ln(1 + \nu_{m_v}^2) \right) \quad (\text{B.8})$$

762 Since k_h and k'_h are independent random variables (no correlation between k_h and k'_h) and
 763 pertinent only over the undisturbed soil domain, D_u , and the smear zone, D_s , respectively, the
 764 overall variance $\sigma_{\ln W}^2$ of $\ln W$ can be estimated with reference to Eqs. (20) and (21) as follows:

$$765 \quad \sigma_{\ln W}^2 = \frac{(\sigma_{\ln \bar{k}_h}^2) + (\sigma_{\ln \bar{k}'_h}^2)}{2} = \frac{\gamma(D_u)_{k_h} \ln(1 + \nu_{k_h}^2) + \gamma(D_s)_{k'_h} \ln(1 + \nu_{k'_h}^2)}{2} \quad (\text{B.9})$$

766 and for the same reason,

$$767 \quad \sigma_{\ln V}^2 = \frac{\gamma(D_u)_{m_v} \ln(1 + \nu_{m_v}^2) + \gamma(D_s)_{m'_v} \ln(1 + \nu_{m'_v}^2)}{2} \quad (\text{B.10})$$

768 where $\gamma(D_u)_{k_h}$, $\gamma(D_s)_{k'_h}$, $\gamma(D_u)_{m_v}$ and $\gamma(D_s)_{m'_v}$ are the variance reduction factors for k_h , k'_h ,
 769 m_v and m'_v , respectively.

770 It can be noticed that both $\mu_{\ln W}$ and $\sigma_{\ln W}^2$ of underlying normally distributed $\ln W$ are now
 771 known. So μ_W and σ_W^2 of lognormally distributed W can readily be obtained with reference to
 772 Eqs. (4) and (5). However, as k_h and k'_h are not distributed over the entire soil domain and do
 773 not have the same influence on the overall behaviour of soil consolidation, the true μ_W and
 774 σ_W^2 will be somewhat different from those calculated directly using $\mu_{\ln W}$ and $\sigma_{\ln W}^2$. For this
 775 reason, the expressions for μ_W and σ_W^2 are empirically adjusted to obtain these two parameters
 776 of lognormally distributed W with reasonable accuracy as follows:

$$777 \quad \mu_W = \left(\frac{2\mu_{\bar{k}_h}}{\mu_{\bar{k}_h} + \mu_{\bar{k}'_h}} \right)^\chi \exp\left(\mu_{\ln W} + \frac{1}{2}\sigma_{\ln W}^2 \right) \quad (\text{B.11})$$

778 where $\chi = \frac{\nu_{\bar{k}'_h}}{\nu_{\bar{k}_h}}$ if $\nu_{\bar{k}_h} > \nu_{\bar{k}'_h}$ and $\chi = \frac{\nu_{\bar{k}_h} - \nu_{\bar{k}'_h}}{\nu_{\bar{k}'_h}}$ if $\nu_{\bar{k}'_h} > \nu_{\bar{k}_h}$

$$779 \quad \sigma_W^2 = \left(\frac{2\nu_{\bar{k}'_h}}{\nu_{\bar{k}_h} + \nu_{\bar{k}'_h}} \right) \mu_W^2 [\exp(\sigma_{\ln W}^2) - 1] \quad (\text{B.12})$$

780 The reason as stated above for the empirical adjustment of μ_w and σ_w^2 is also applicable for μ_v
 781 and σ_w^2 , therefore

$$782 \quad \mu_v = \left(\frac{2\mu_{\bar{m}_v}}{\mu_{\bar{m}_v} + \mu_{\bar{m}'_v}} \right)^\gamma \exp\left(\mu_{\ln v} + \frac{1}{2} \sigma_{\ln v}^2 \right) \quad (\text{B.13})$$

783 where $\gamma = \frac{\nu_{\bar{m}'_v}}{\nu_{\bar{m}_v}}$ if $\nu_{\bar{m}_v} > \nu_{\bar{m}'_v}$ and $\gamma = \frac{\nu_{\bar{m}'_v} - \nu_{\bar{m}_v}}{\nu_{\bar{m}'_v}}$ if $\nu_{\bar{m}'_v} > \nu_{\bar{m}_v}$.

$$784 \quad \sigma_v^2 = \left(\frac{2\nu_{\bar{m}'_v}}{\nu_{\bar{m}_v} + \nu_{\bar{m}'_v}} \right) \mu_v^2 [\exp(\sigma_{\ln v}^2) - 1] \quad (\text{B.14})$$

785 In Eqs. (B.11–B.14), $\nu_{\bar{k}_h} = \sigma_{\bar{k}_h}^- / \mu_{\bar{k}_h}^-$ and $\nu_{\bar{k}'_h} = \sigma_{\bar{k}'_h}^- / \mu_{\bar{k}'_h}^-$ are the coefficients of variation of the
 786 equivalent permeability in the undisturbed and smear zones, respectively ($\sigma_{\bar{k}_h}^-$, $\mu_{\bar{k}_h}^-$, $\sigma_{\bar{k}'_h}^-$ and
 787 $\mu_{\bar{k}'_h}^-$ are the standard deviation and mean of \bar{k}_h and \bar{k}'_h , respectively); $\nu_{\bar{m}_v} = \sigma_{\bar{m}_v} / \mu_{\bar{m}_v}$ and
 788 $\nu_{\bar{m}'_v} = \sigma_{\bar{m}'_v} / \mu_{\bar{m}'_v}$ are the coefficients of variation of the equivalent volume compressibility in the
 789 undisturbed and smear zones, respectively ($\sigma_{\bar{m}_v}$, $\mu_{\bar{m}_v}$, $\sigma_{\bar{m}'_v}$ and $\mu_{\bar{m}'_v}$ are the standard deviation
 790 and mean of \bar{m}_v and \bar{m}'_v , respectively). With reference to Eqs. (4) and (5), and making use of
 791 Eqs. (19) and (21) lead to the following equations of the mean and standard deviation of \bar{k}_h
 792 and \bar{k}'_h :

$$793 \quad \mu_{\bar{k}_h}^- = \exp\left(\mu_{\ln \bar{k}_h} + \frac{1}{2} \sigma_{\ln \bar{k}_h}^2 \right) = \exp\left(\ln \mu_{k_h} - \frac{1}{2} \ln(1 + \nu_{k_h}^2) + \frac{1}{2} \gamma(D_u)_{k_h} \ln(1 + \nu_{k_h}^2) \right) \quad (\text{B.15})$$

$$794 \quad \sigma_{\bar{k}_h}^- = \mu_{\bar{k}_h}^- \sqrt{[\exp(\sigma_{\ln \bar{k}_h}^2) - 1]} = \mu_{\bar{k}_h}^- \sqrt{[\exp(\gamma(D_u)_{k_h} \ln(1 + \nu_{k_h}^2)) - 1]} \quad (\text{B.16})$$

$$795 \quad \mu_{\bar{k}'_h}^- = \exp\left(\mu_{\ln \bar{k}'_h} + \frac{1}{2} \sigma_{\ln \bar{k}'_h}^2 \right) = \exp\left(\ln \mu_{k'_h} - \frac{1}{2} \ln(1 + \nu_{k'_h}^2) + \frac{1}{2} \gamma(D_s)_{k'_h} \ln(1 + \nu_{k'_h}^2) \right) \quad (\text{B.17})$$

$$796 \quad \sigma_{\bar{k}'_h}^- = \mu_{\bar{k}'_h}^- \sqrt{[\exp(\sigma_{\ln \bar{k}'_h}^2) - 1]} = \mu_{\bar{k}'_h}^- \sqrt{[\exp(\gamma(D_s)_{k'_h} \ln(1 + \nu_{k'_h}^2)) - 1]} \quad (\text{B.18})$$

797 Similarly, the mean and standard deviation of \bar{m}_v and \bar{m}'_v are thus:

$$798 \quad \mu_{\bar{m}_v} = \exp\left(\ln \mu_{m_v} - \frac{1}{2} \ln(1 + v_{m_v}^2) + \frac{1}{2} \gamma(D_u)_{m_v} \ln(1 + v_{m_v}^2)\right) \quad (\text{B.19})$$

$$799 \quad \sigma_{\bar{m}_v} = \mu_{\bar{m}_v} \sqrt{\left[\exp\left(\gamma(D_u)_{m_v} \ln(1 + v_{m_v}^2)\right) - 1\right]} \quad (\text{B.20})$$

$$800 \quad \mu_{\bar{m}'_v} = \exp\left(\ln \mu_{m'_v} - \frac{1}{2} \ln(1 + v_{m'_v}^2) + \frac{1}{2} \gamma(D_s)_{m'_v} \ln(1 + v_{m'_v}^2)\right) \quad (\text{B.21})$$

$$801 \quad \sigma_{\bar{m}'_v} = \mu_{\bar{m}'_v} \sqrt{\left[\exp\left(\gamma(D_s)_{m'_v} \ln(1 + v_{m'_v}^2)\right) - 1\right]} \quad (\text{B.22})$$

802 Substituting Eqs. (B.11) and (B.12) into Eqs. (B.3) and (B.5) lead to the following equations

803 of $\mu_{\bar{\alpha}}$ and $\sigma_{\bar{\alpha}}^2$:

$$804 \quad \mu_{\bar{\alpha}} = a + b \left(\frac{2\mu_{\bar{k}_h}}{\mu_{\bar{k}_h} + \mu_{\bar{k}'_h}} \right)^z \left(\exp\left(\mu_{\ln W} + \frac{1}{2} \sigma_{\ln W}^2\right) \right) \quad (\text{B.23})$$

$$805 \quad \sigma_{\bar{\alpha}}^2 = \left(\frac{2v_{\bar{k}'_h}}{v_{\bar{k}_h} + v_{\bar{k}'_h}} \right) b^2 \mu_W^2 \left[\exp(\sigma_{\ln W}^2) - 1 \right] \quad (\text{B.24})$$

806 Again substituting Eqs. (B.13) and (B.14) into Eqs. (B.4) and (B.6) gives:

$$807 \quad \mu_{\bar{\alpha}_{m_v}} = g + h \left(\frac{2\mu_{\bar{m}_v}}{\mu_{\bar{m}_v} + \mu_{\bar{m}'_v}} \right)^y \left(\exp\left(\mu_{\ln V} + \frac{1}{2} \sigma_{\ln V}^2\right) \right) \quad (\text{B.25})$$

$$808 \quad \sigma_{\bar{\alpha}_{m_v}}^2 = \left(\frac{2v_{\bar{m}'_v}}{v_{\bar{m}_v} + v_{\bar{m}'_v}} \right) h^2 \mu_V^2 \left[\exp(\sigma_{\ln V}^2) - 1 \right] \quad (\text{B.26})$$

809 Finally, the statistics (i.e., $\mu_{\ln \bar{\alpha}}$, $\sigma_{\ln \bar{\alpha}}^2$, $\mu_{\ln \bar{\alpha}_{m_v}}$ and $\sigma_{\ln \bar{\alpha}_{m_v}}^2$) of the underlying normally

810 distributed $\ln \bar{\alpha}$ and $\ln \bar{\alpha}_{m_v}$ can be obtained from the obtained values of $\mu_{\bar{\alpha}}$, $\sigma_{\bar{\alpha}}^2$, $\mu_{\bar{\alpha}_{m_v}}$ and

811 $\sigma_{\bar{\alpha}_{m_v}}^2$ by using Eqs. (2) and (3).

812 All six requested parameters (i.e., $\mu_{\ln \bar{k}_h}$, $\mu_{\ln \bar{m}_v}$, $\mu_{\ln \bar{\alpha}}$, $\mu_{\ln \bar{\alpha}_{m_v}}$, $\sigma_{\ln \bar{\alpha}}^2$ and $\sigma_{\ln \bar{\alpha}_{m_v}}^2$) are now

813 known and can be used in Eqs. (42) and (43) for the estimation of $\mu_{\ln U^{*(t)}}$ and $\sigma_{\ln U^{*(t)}}^2$. Using

814 the developed semi-analytical relationships, the procedure for calculating $\mu_{\ln U^*(t)}$ and $\sigma_{\ln U^*(t)}^2$
 815 can be summarized as follows:

- 816 1. Determine all constant parameters involved in the RBSA–G2C1 (i.e., C , a , b , g , h ,
 817 $\gamma(D_u)_{k_h}$, $\gamma(D_s)_{k_h}$, $\gamma(D_u)_{m_v}$ and $\gamma(D_s)_{m_v}$);
- 818 2. Calculate $\mu_{\ln \bar{k}_h}$ and $\mu_{\ln \bar{m}_v}$ from Eqs. (B.1) and (B.2);
- 819 3. Calculate $\mu_{\ln W}$, $\sigma_{\ln W}^2$, $\mu_{\ln V}$ and $\sigma_{\ln V}^2$ using Eqs. (B.7)–(B.10);
- 820 4. Calculate $\mu_{\bar{k}_h}$, $\sigma_{\bar{k}_h}^2$, $\mu_{\bar{k}_h'}$, $\sigma_{\bar{k}_h'}^2$, $\mu_{\bar{m}_v}$, $\sigma_{\bar{m}_v}^2$, $\mu_{\bar{m}_v'}$ and $\sigma_{\bar{m}_v'}^2$ using Eqs. (B.15)–(B.22), then
 821 determine $U_{\bar{k}_h}$, $U_{\bar{k}_h'}$, $U_{\bar{m}_v}$ and $U_{\bar{m}_v'}$;
- 822 5. Calculate μ_W and μ_V using Eqs. (B.11) and (B.13);
- 823 6. Using the values of $\mu_{\ln W}$, $\sigma_{\ln W}^2$, μ_W , $\mu_{\ln V}$, $\sigma_{\ln V}^2$, μ_V , $U_{\bar{k}_h}$, $U_{\bar{k}_h'}$, $U_{\bar{m}_v}$ and $U_{\bar{m}_v'}$ obtained in
 824 Steps 3–5, calculate $\mu_{\bar{\alpha}}$, $\sigma_{\bar{\alpha}}^2$, $\mu_{\bar{\alpha}_{m_v}}$ and $\sigma_{\bar{\alpha}_{m_v}}^2$ from Eqs. (B.23)–(B.26);
- 825 7. Use Eqs. (2) and (3) to determine $\sigma_{\ln \bar{\alpha}}^2$, $\mu_{\ln \bar{\alpha}}$, $\sigma_{\ln \bar{\alpha}_{m_v}}^2$ and $\mu_{\ln \bar{\alpha}_{m_v}}$ from the obtained
 826 values of $\mu_{\bar{\alpha}}$, $\sigma_{\bar{\alpha}}^2$, $\mu_{\bar{\alpha}_{m_v}}$ and $\sigma_{\bar{\alpha}_{m_v}}^2$ in Step 6; and
- 827 8. Evaluate $\mu_{\ln U^*(t)}$ and $\sigma_{\ln U^*(t)}^2$ by substituting $\mu_{\ln \bar{k}_h}$, $\mu_{\ln \bar{m}_v}$, $\mu_{\ln \bar{\alpha}}$ and $\mu_{\ln \bar{\alpha}_{m_v}}$ in Eq. (42),
 828 and $\sigma_{\ln \bar{\alpha}}^2$ and $\sigma_{\ln \bar{\alpha}_{m_v}}^2$ in Eq. (43).

829

830 **Appendix C. RBSA model considering permeability as the only random variable**

831

832 *G1C2: RBSA model considering k_h as continuous random variables over the entire unit cell*
 833 *and m_v deterministic*

834 The spatial variability of m_v is generally much less than that of k_h . Therefore, it is not
 835 unlikely to encounter soil with no or very little variability in m_v . For such condition, m_v can be

836 considered as spatially constant without significantly affecting the final results. Treating k_h as
 837 spatially random and m_v as spatially constant, the C parameter in Eq. (23) is transformed to:

$$838 \quad C = \frac{2t}{r_e^2 m_v \gamma_w \alpha \alpha_{m_v}} \quad (C.1)$$

839 The expressions for $\mu_{\ln U^{*(t)}}$ and $\sigma_{\ln U^{*(t)}}^2$ given in Eqs. (33) and (34) are then reduced to:

$$840 \quad \mu_{\ln U^{*(t)}} = \ln C + \ln \mu_{k_h} - \frac{1}{2} \ln(1 + \nu_{k_h}^2) \quad (C.2)$$

$$841 \quad \sigma_{\ln U^{*(t)}}^2 = \gamma(D)_{k_h} \ln(1 + \nu_{k_h}^2) \quad (C.3)$$

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843 *G2C2: RBSA model considering k_h and k'_h as independent random variables and m_v*
 844 *deterministic*

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846 By considering k_h and k'_h as independent random variables and volume compressibility as
 847 spatially constant, the C parameter in Eq. (37) now becomes:

$$848 \quad C = \frac{2t}{r_e^2 \gamma_w m_v \alpha_{m_v}} \quad (C.4)$$

849 The equations for $\mu_{\ln U^{*(t)}}$ and $\sigma_{\ln U^{*(t)}}^2$ given in Eqs. (42) and (43) are then reduced to:

$$850 \quad \mu_{\ln U^{*(t)}} = \ln C + \mu_{\ln \bar{k}_h} - \mu_{\ln \bar{\alpha}} \quad (C.5)$$

$$851 \quad \sigma_{\ln U^{*(t)}}^2 = \sigma_{\ln \bar{\alpha}}^2 \quad (C.6)$$

852 The three unknown parameters: $\mu_{\ln \bar{k}_h}$, $\mu_{\ln \bar{\alpha}}$ and $\sigma_{\ln \bar{\alpha}}^2$ in Eqs. (C.5) and (C.6) are already
 853 determined during the course of the development of the RBSA–G2C1 model as presented in
 854 Appendix B and can be readily used for estimation of $\mu_{\ln U^{*(t)}}$ and $\sigma_{\ln U^{*(t)}}^2$.

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932 **Figure Captions:**

933 **Fig. 1.** Schematic diagram of soil consolidation with prefabricated vertical drain: (a)
934 cylindrical unit cell; (b) equivalent square geometry with FE mesh discretization

935 **Fig. 2.** Comparison between FEMC and RBSA–G1C1 for the effect of: (a) v on $\mu_{\ln U^*}$ for $\theta =$
936 0.5m (b) θ on $\mu_{\ln U^*}$ for $v_{k_h} = 200\%$, $v_{m_v} = 20\%$ (c) v on $\sigma_{\ln U^*}$ for $\theta = 0.5\text{m}$ and (d) θ on $\sigma_{\ln U^*}$
937 for $v_{k_h} = 200\%$, $v_{m_v} = 20\%$

938 **Fig. 3.** Comparison between FEMC and RBSA–G1C1 for the effect of: (a) v on $P[U \geq U_{90}]$
939 for $\theta = 0.5\text{m}$ (b) θ on $P[U \geq U_{90}]$ for $v_{k_h} = 200\%$, $v_{m_v} = 20\%$

940 **Fig. 4.** Comparison between FEMC and RBSA–G2C1 for the effect of: (a) v_u on $\mu_{\ln U^*}$ at fixed
941 value of $v_{k'_h} = 100\%$, $v_{m'_v} = 10\%$, $\theta_u = \theta_s = 1.0\text{m}$; (b) v_s on $\mu_{\ln U^*}$ at fixed value of $v_{k_h} = 100\%$,
942 $v_{m_v} = 10\%$, $\theta_u = \theta_s = 1.0\text{m}$

943 **Fig. 5.** Comparison between FEMC and RBSA–G2C1 for the effect of: (a) θ_u on $\mu_{\ln U^*}$ at fixed
944 value of $v_{k_k} = v_{k'_h} = 200\%$, $v_{m_v} = v_{m'_v} = 20\%$, $\theta_s = 0.25\text{m}$; (b) θ_s on $\mu_{\ln U^*}$ at fixed value of $v_{k_k} =$
945 $v_{k'_h} = 200\%$, $v_{m_v} = v_{m'_v} = 20\%$, $\theta_u = 0.25\text{m}$

946 **Fig. 6.** Comparison between FEMC and RBSA–G2C1 for the effect of: (a) v_u on $\sigma_{\ln U^*}$ at fixed
947 value of $v_{k'_h} = 100\%$, $v_{m'_v} = 10\%$, $\theta_u = \theta_s = 1.0\text{m}$; (b) v_s on $\sigma_{\ln U^*}$ at fixed value of $v_{k_h} = 100\%$,
948 $v_{m_v} = 10\%$, $\theta_u = \theta_s = 1.0\text{m}$

949 **Fig. 7.** Comparison between FEMC and RBSA–G2C1 for the effect of: (a) θ_u on $\sigma_{\ln U^*}$ at fixed
950 value of $v_{k_k} = v_{k'_h} = 200\%$, $v_{m_v} = v_{m'_v} = 20\%$, $\theta_s = 0.25\text{m}$; (b) θ_s on $\sigma_{\ln U^*}$ at fixed value $v_{k_k} = v_{k'_h}$
951 $= 200\%$, $v_{m_v} = v_{m'_v} = 20\%$, $\theta_u = 0.25\text{m}$

952 **Fig. 8.** Comparison between FEMC and RBSA–G2C1 for the effect of: (a) v_u on $P[U \geq U_{90}]$
953 at fixed value of $v_{k'_h} = 100\%$, $v_{m'_v} = 10\%$, $\theta_u = \theta_s = 1.0\text{m}$; (b) v_s on $P[U \geq U_{90}]$ at fixed value
954 of $v_{k_h} = 100\%$, $v_{m_v} = 10\%$, $\theta_u = \theta_s = 1.0\text{m}$

955 **Fig. 9.** Comparison between FEMC and RBSA–G2C1 for the effect of: (a) θ_u on $P[U \geq U_{90}]$
956 at fixed value of $\nu_{k_k} = \nu_{k'_h} = 200\%$, $\nu_{m_v} = \nu_{m'_v} = 20\%$, $\theta_s = 0.25m$ (b) θ_s on $P[U \geq U_{90}]$ at fixed
957 value $\nu_{k_k} = \nu_{k'_h} = 200\%$, $\nu_{m_v} = \nu_{m'_v} = 20\%$, $\theta_u = 0.25m$

958 **Fig. 10.** Comparison between (a) $\mu_{\ln U^*}$ (b) $\sigma_{\ln U^*}$ and (c) $P[U \geq U_{90}]$ obtained from FEMC and
959 RBSA methods for a unit cell having $L/S = 5$ and anisotropic θ

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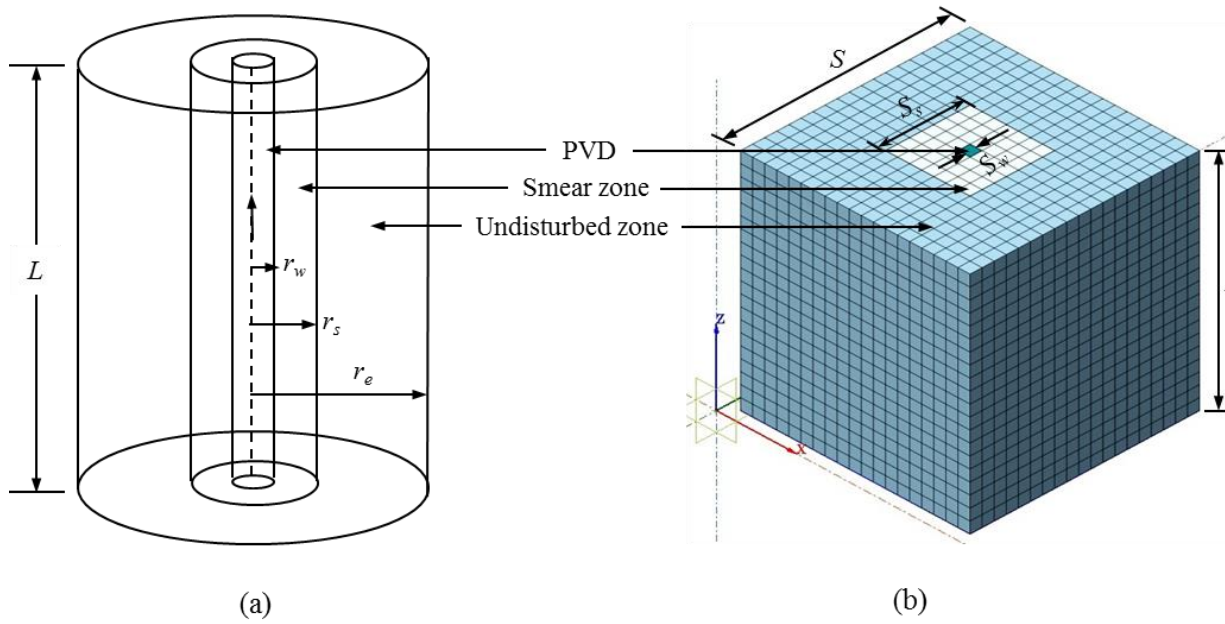


Fig. 1. Schematic diagram of soil consolidation with prefabricated vertical drain: (a) cylindrical unit cell; (b) equivalent square geometry with FE mesh discretization

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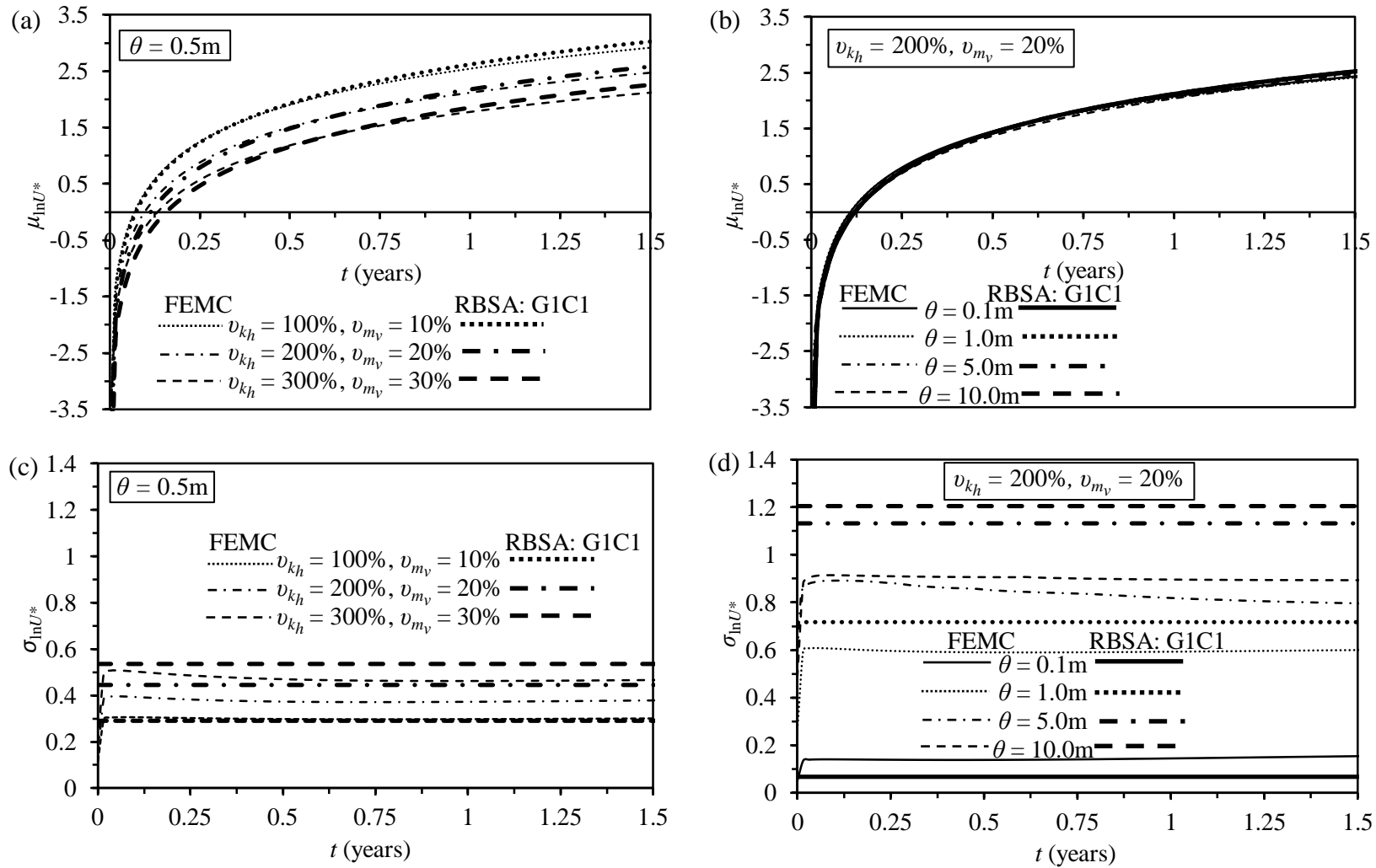


Fig. 2. Comparison between FEMC and RBSA-G1C1 for the effect of: (a) v on $\mu_{\ln U^*}$ for $\theta = 0.5\text{m}$ (b) θ on $\mu_{\ln U^*}$ for $v_{k_h} = 200\%$, $v_{m_v} = 20\%$ (c) v on $\sigma_{\ln U^*}$ for $\theta = 0.5\text{m}$ and (d) θ on $\sigma_{\ln U^*}$ for $v_{k_h} = 200\%$, $v_{m_v} = 20\%$

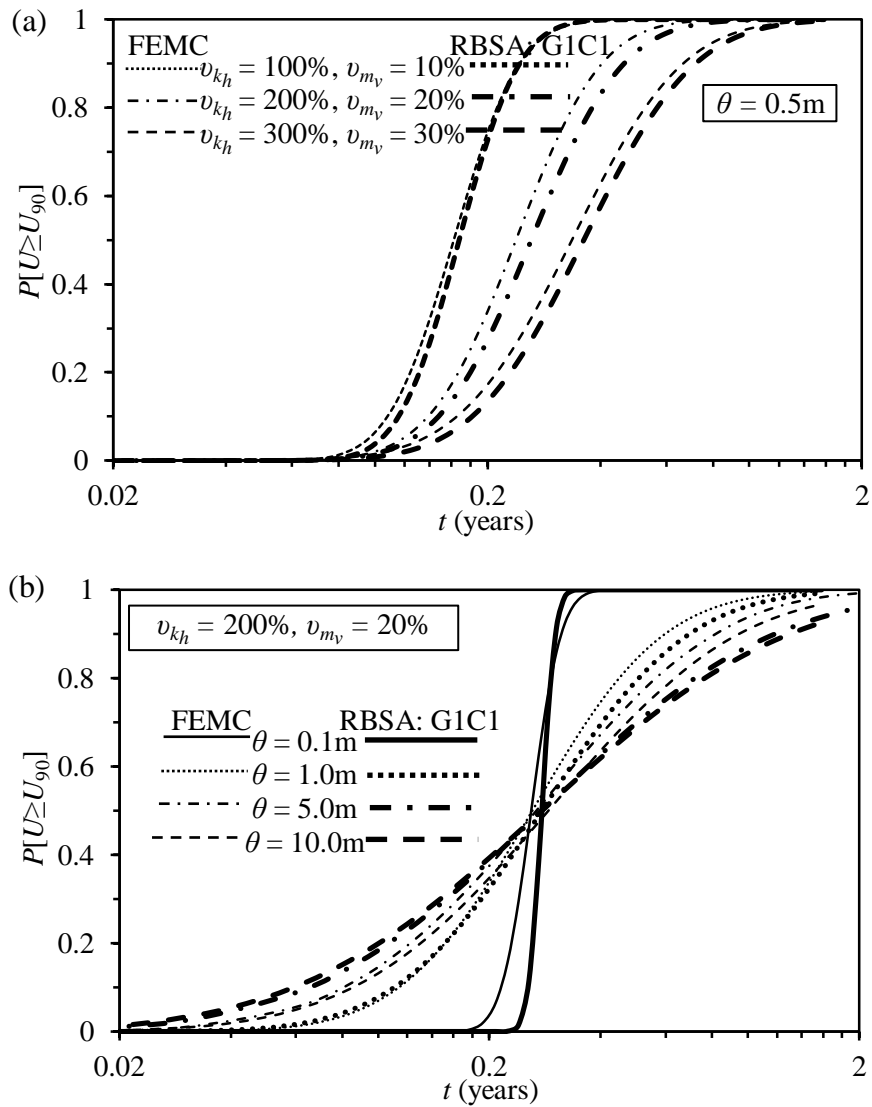


Fig. 3. Comparison between FEMC and RBSA–G1C1 for the effect of: (a) v on $P[U \geq U_{90}]$ for $\theta = 0.5\text{m}$ (b) θ on $P[U \geq U_{90}]$ for $v_{kh} = 200\%$, $v_{mv} = 20\%$

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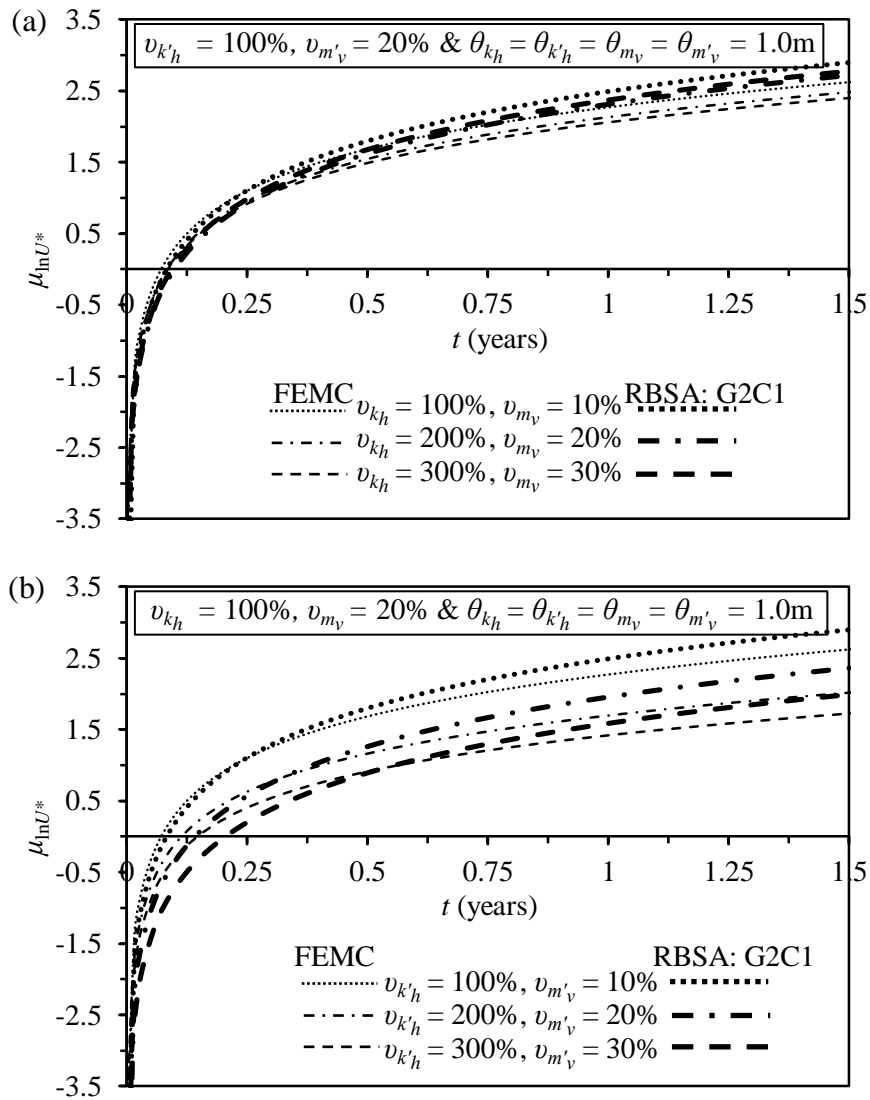


Fig. 4. Comparison between FEMC and RBSA–G2C1 for the effect of: (a) v_u on $\mu_{\ln U^*}$ at fixed value of $v_{k_h}' = 100\%$, $v_{m_v}' = 10\%$, $\theta_u = \theta_s = 1.0\text{m}$; (b) v_s on $\mu_{\ln U^*}$ at fixed value of $v_{k_h} = 100\%$, $v_{m_v} = 10\%$, $\theta_u = \theta_s = 1.0\text{m}$

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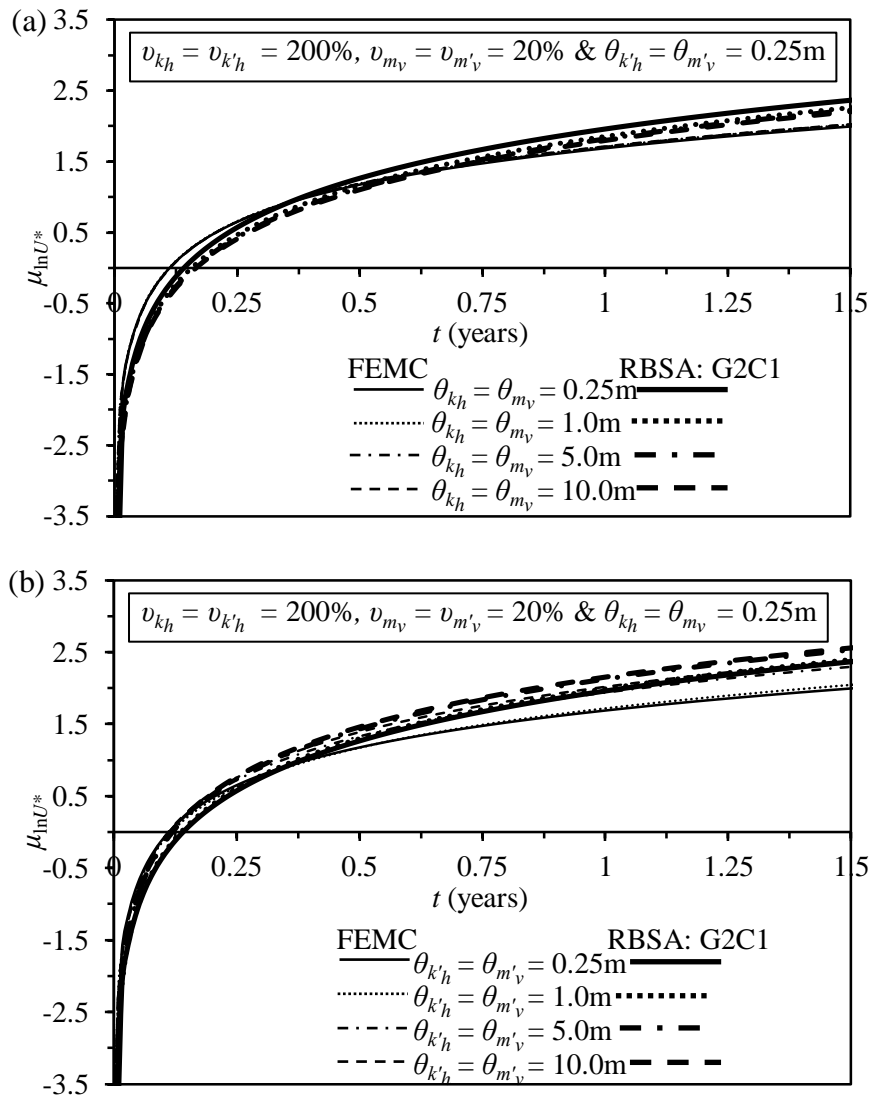


Fig. 5. Comparison between FEMC and RBSA–G2C1 for the effect of: (a) θ_u on $\mu_{\ln U^*}$ at fixed value of $\nu_{k_k} = \nu_{k'_k} = 200\%$, $\nu_{m_v} = \nu_{m'_v} = 20\%$, $\theta_s = 0.25\text{m}$; (b) θ_s on $\mu_{\ln U^*}$ at fixed value of $\nu_{k_k} = \nu_{k'_k} = 200\%$, $\nu_{m_v} = \nu_{m'_v} = 20\%$, $\theta_u = 0.25\text{m}$

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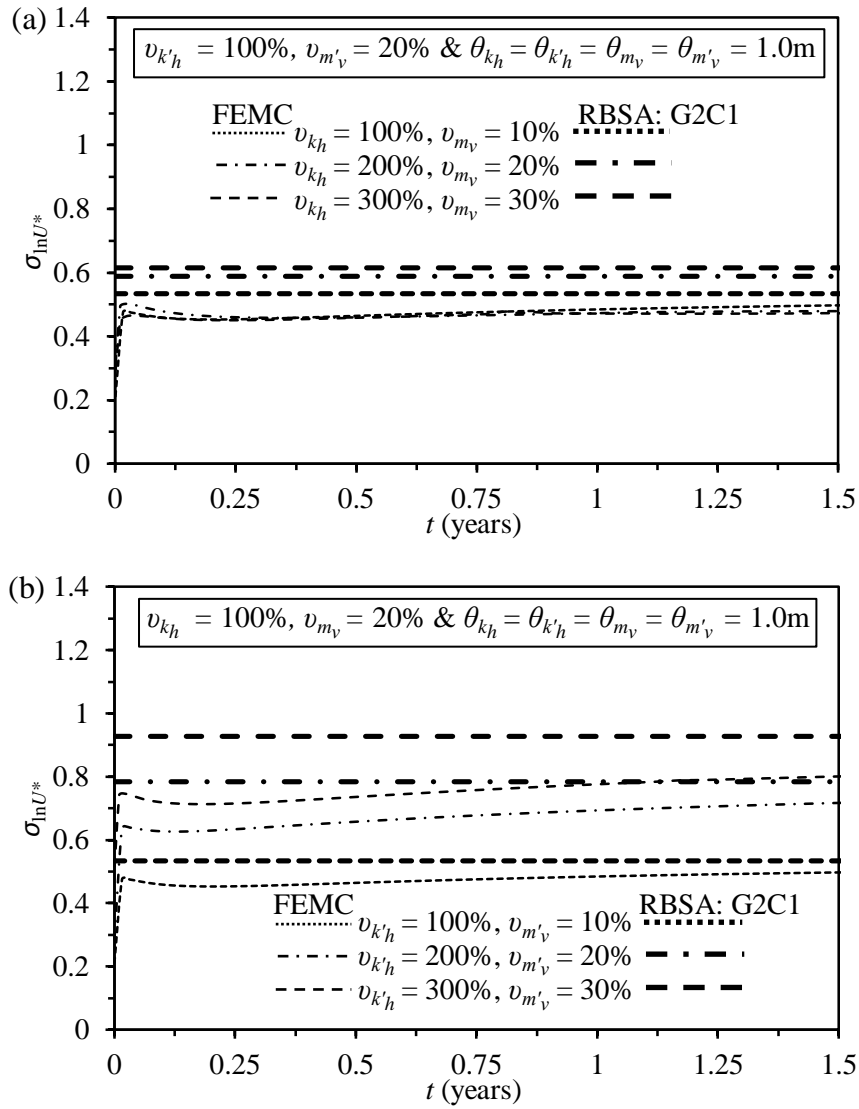


Fig. 6. Comparison between FEMC and RBSA–G2C1 for the effect of: (a) v_u on $\sigma_{\ln U^*}$ at fixed value of $v_{k_h} = 100\%$, $v_{m'_v} = 10\%$, $\theta_u = \theta_s = 1.0m$; (b) v_s on $\sigma_{\ln U^*}$ at fixed value of $v_{k_h} = 100\%$, $v_{m'_v} = 10\%$, $\theta_u = \theta_s = 1.0m$

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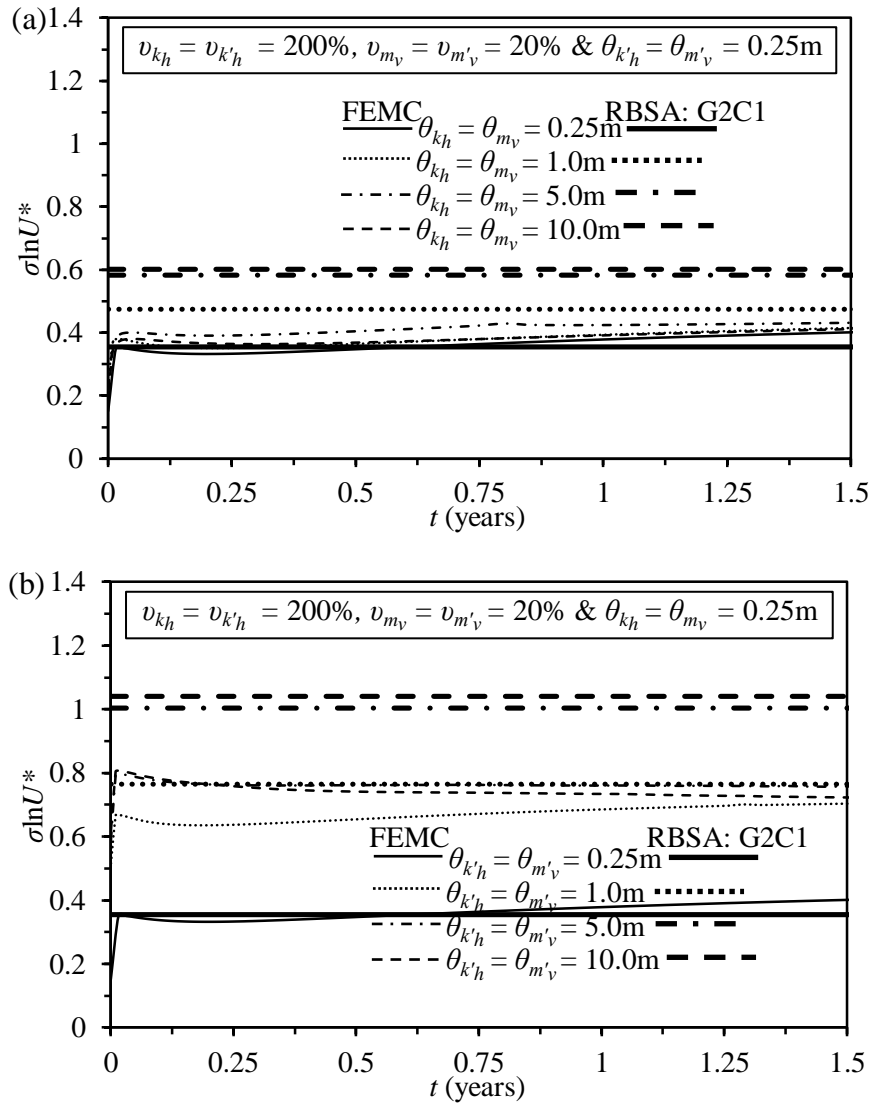


Fig. 7. Comparison between FEMC and RBSA–G2C1 for the effect of: (a) θ_u on $\sigma_{\ln U^*}$ at fixed value of $v_{k_k} = v_{k'_h} = 200\%$, $v_{m_v} = v_{m'_v} = 20\%$, $\theta_s = 0.25\text{m}$; (b) θ_s on $\sigma_{\ln U^*}$ at fixed value $v_{k_k} = v_{k'_h} = 200\%$, $v_{m_v} = v_{m'_v} = 20\%$, $\theta_u = 0.25\text{m}$

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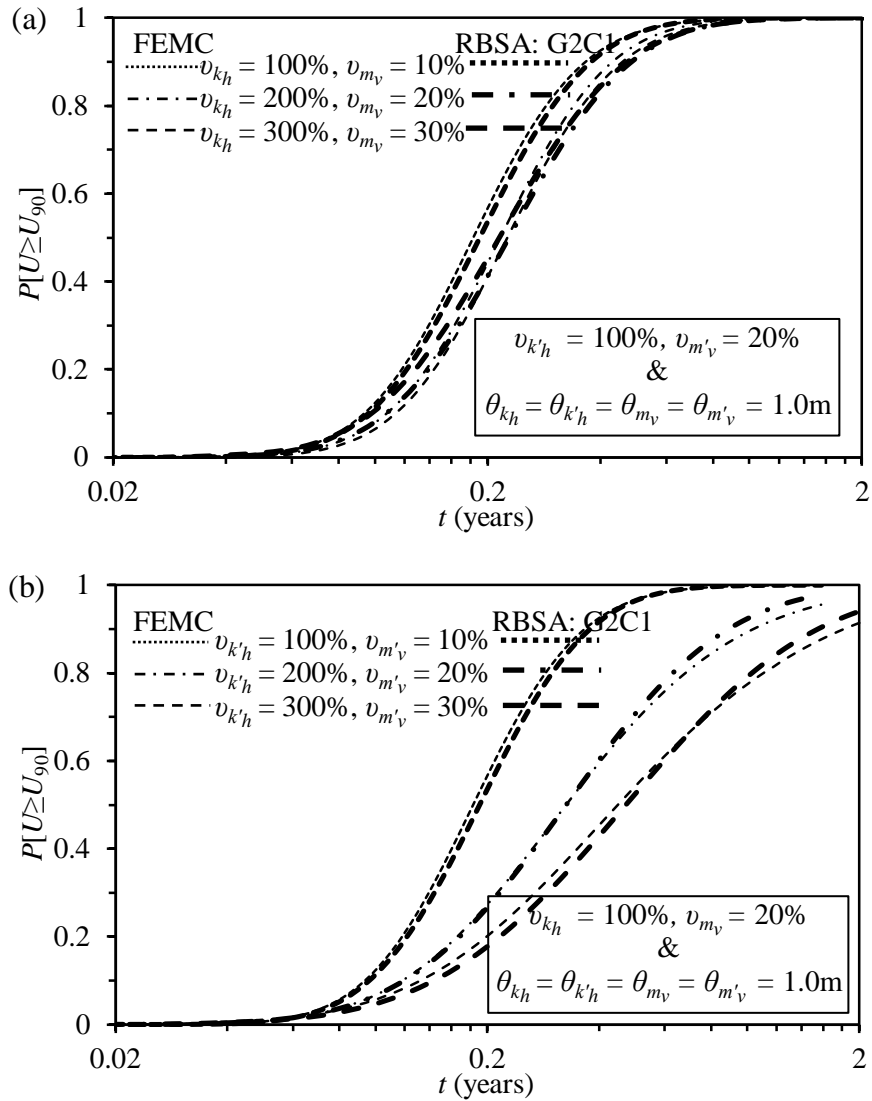


Fig. 8. Comparison between FEMC and RBSA-G2C1 for the effect of: (a) v_u on $P[U \geq U_{90}]$ at fixed value of $v_{k'h} = 100\%$, $v_{m'v} = 10\%$, $\theta_u = \theta_s = 1.0\text{m}$; (b) v_s on $P[U \geq U_{90}]$ at fixed value of $v_{kh} = 100\%$, $v_{m_v} = 10\%$, $\theta_u = \theta_s = 1.0\text{m}$

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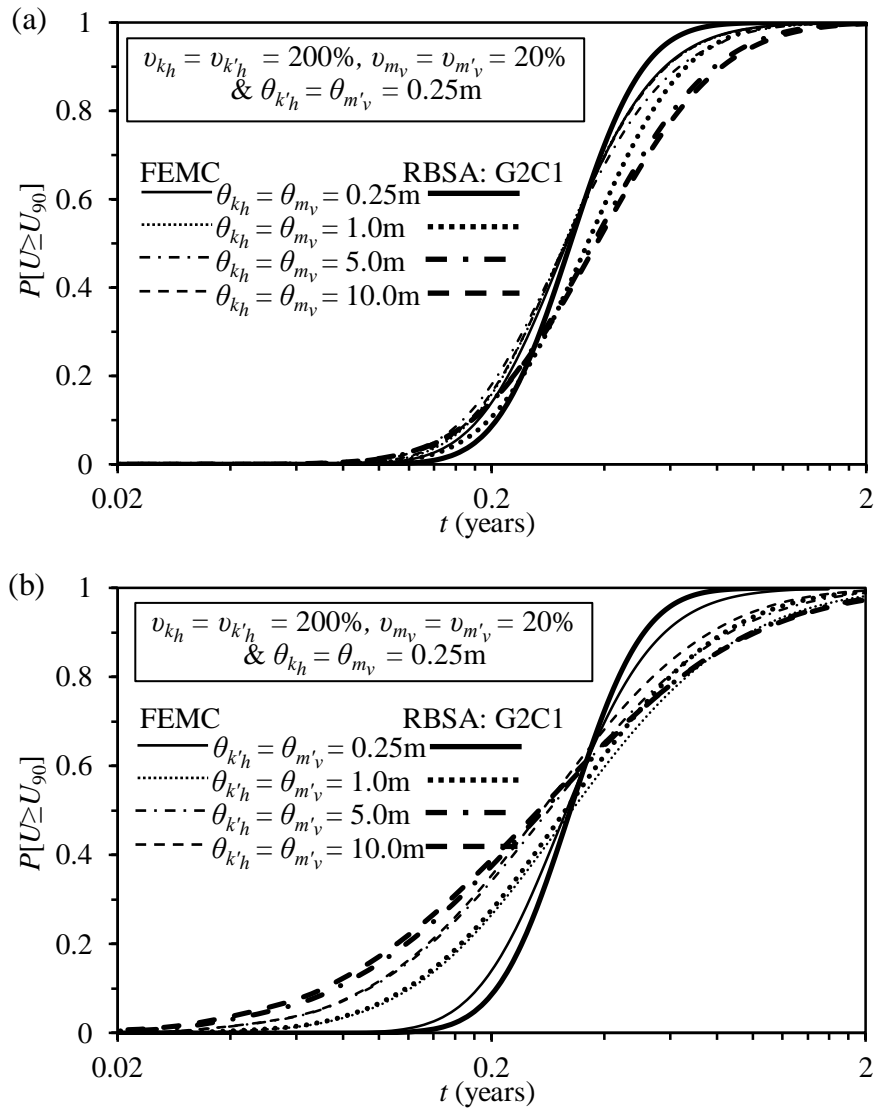


Fig. 9. Comparison between FEMC and RBSA-G2C1 for the effect of: (a) θ_u on $P[U \geq U_{90}]$ at fixed value of $v_{k_k} = v_{k'_h} = 200\%$, $v_{m_v} = v_{m'_v} = 20\%$, $\theta_s = 0.25\text{m}$ (b) θ_s on $P[U \geq U_{90}]$ at fixed value

$$v_{k_k} = v_{k'_h} = 200\%, v_{m_v} = v_{m'_v} = 20\%, \theta_u = 0.25\text{m}$$

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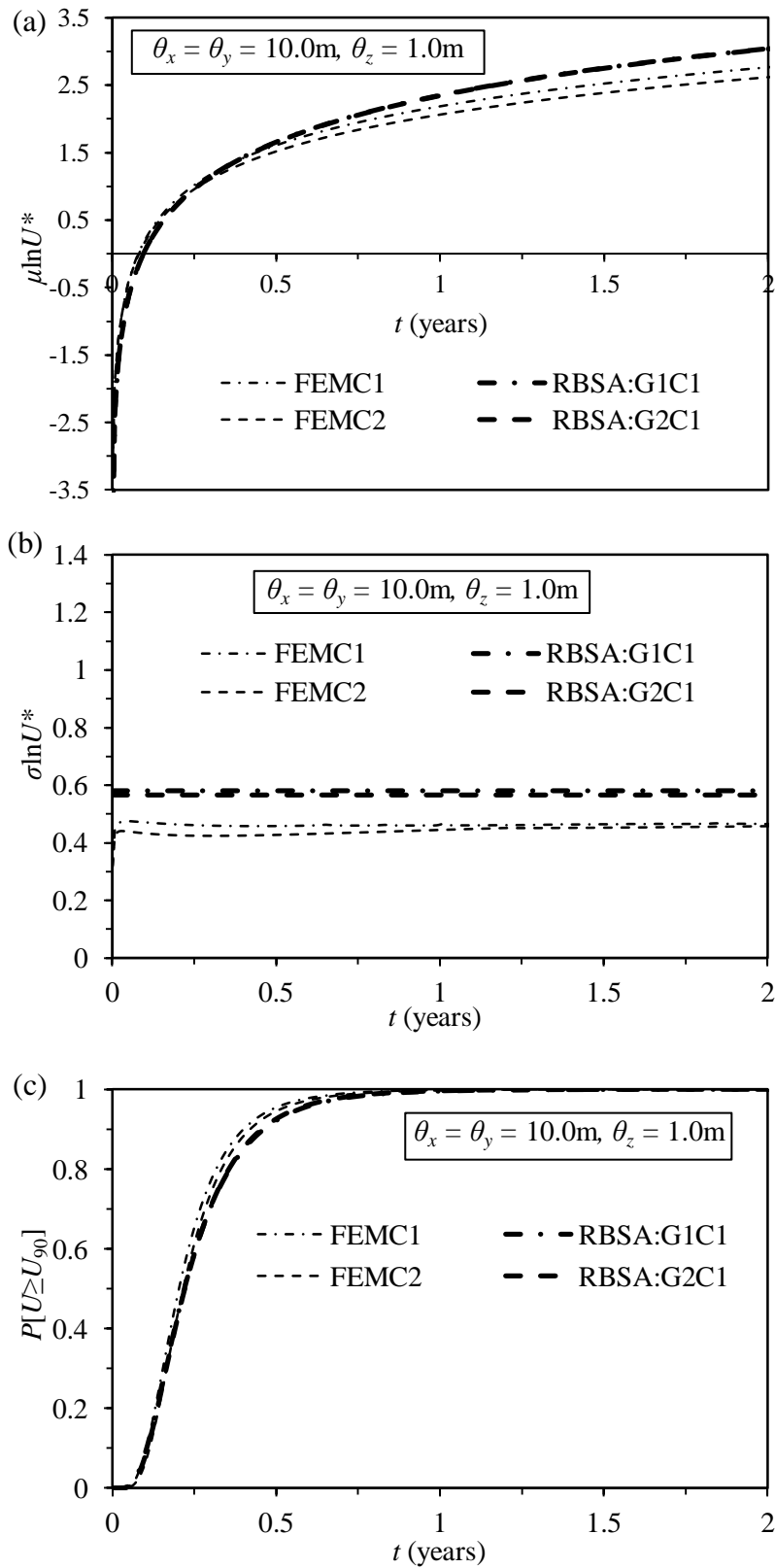


Fig. 10. Comparison between (a) $\mu_{\ln U^*}$ (b) $\sigma_{\ln U^*}$ and (c) $P[U \geq U_{90}]$ obtained from FEMC and RBSA methods for unit cell having $L/S = 5$ and anisotropic θ

