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Iterative algorithm and estimation of solution for a fractional order differential equation

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Abstract
In this paper, we establish an iterative algorithm and estimation of solutions for a fractional turbulent flow model in a porous medium under a suitable growth condition. Our main tool is the monotone iterative technique.

Keywords: iterated algorithm; fractional order turbulent flow model; *p*-Laplacian operator; nonlocal boundary value problem

1 Introduction

Over the past decades, a large number of investigations have been carried out to study various natural and engineering systems and processes that involve fluid flow through porous media [1–5], such as petroleum extraction, where the flow accelerates toward the pumping well while crossing regions of variable porosity, and a turbulent regime eventually occurs and affects the overall pressure drop and well performance. Dybbs and Edwards [1] conducted a flow visualization study, and they showed that fluid flow in a porous medium exhibits turbulent characteristics when the pore-Reynolds number (based on the pore scale and velocity) becomes higher than a few hundred. Leibenson [5] introduced a p-Laplacian equation to describe turbulent flows in a porous medium. Inspired by the above work, many authors studied the existence and uniqueness of solution for differential equation involving p-Laplacian under various boundary conditions, here we refer the reader to the work of Li *et al.* [6], Chen *et al.* [7], Zhang *et al.* [8–10], Goodrich [11, 12], Ding *et al.* [13], and the references therein.

In this paper, we study a fractional order differential equation involving the p-Laplacian

$$-\mathcal{D}_t^{\beta}\big(\varphi_p\big(-\mathcal{D}_t^{\alpha}x\big)\big)(t)=f\big(x(t)\big), \tag{1.1}$$

subject to the following nonlocal boundary conditions:

$$\mathcal{D}_t^{\alpha}x(0)=\mathcal{D}_t^{\alpha}x(1)=\mathcal{D}_t^{\gamma}x(0)=0,\qquad \mathcal{D}_t^{\gamma}x(1)=\int_0^1\mathcal{D}_t^{\gamma}x(s)\,dA(s),$$

where $\mathcal{D}_t^{\alpha}$, $\mathcal{D}_t^{\beta}$, $\mathcal{D}_t^{\gamma}$ are the standard Riemann-Liouville derivatives, $\int_0^1 x(s)\,dA(s)$ denotes a Riemann-Stieltjes integral and $0<\gamma\le 1<\alpha$, $\beta\le 2$, $\alpha-\gamma>1$, A is a function of the bounded variation and dA can be a signed measure, φ_p is the p-Laplacian operator defined by $\varphi_p(s) = |s|^{p-2}s$, $p > 1$. Obviously, $\varphi_p(s)$ is invertible and its inverse operator is $\varphi_q(s)$, where $q > 1$ is a constant such that $\frac{1}{p} + \frac{1}{q} = 1$.

A fractional derivative gives a perfect aid to characterize the memory and hereditary properties of various processes and materials. Therefore differential equations of fractional order are being used in modeling of the electrical and mechanical properties of various real materials, the rheological properties of rock, as well as the hereditary properties of various biological processes [14–20], and many extensive researches have been carried out to develop numerical schemes and analytical solutions [21–32].

In this paper, we focus on the iterative algorithm and estimation of positive solutions for the problem (1.1) by introducing a new growth condition on the nonlinearity f:

(F) $f : [0, +\infty) \to [0, +\infty)$ is continuous and non-decreasing, and there exists a constant $\epsilon > 0$ such that

$$\sup_{s>0} \frac{f(s)}{s^\epsilon} < +\infty. \tag{1.2}$$

The assumption (F) is a relatively weaker condition than the superlinear or sublinear or the mixed superlinear and sublinear condition, which includes many interesting cases [6, 33–37]. Furthermore, (1.2) can be adopted to extend some results of partial differential equations [35, 36]. Some basic examples satisfying (F) are

(1) $f(s) = as^\kappa$, where $a, \kappa > 0$.
(2) $f(s) = s^\mu \arctan s$, $\mu > 0$.
(3) $f(s) = s^\mu (s + 1) \ln(1 + \frac{1}{s+1}) + s^\mu$, $\mu > 0$.
(4) $f(s) = \ln(a + s)$, $a > 0$.

Here we also comment that there are relatively few results on fractional order equations with nonlocal Riemann-Stieltjes integral boundary conditions, and no work has been done concerning the iterative algorithm of solutions of Equation (1.1). The rest of the paper is organized as follows. In Section 2, we present some preliminaries and lemmas which are used in the rest of the paper. In Section 3, we establish the existence of the maximal and minimal solutions, estimation of the lower and upper bounds of the extremal solutions and an iterative scheme converging to the exact extremal solutions.

2 Preliminaries and lemmas

This work restricts attention to Riemann-Liouville fractional derivatives; for details, see the monographs [14, 15]. First of all, according to the definition and properties of the Riemann-Liouville fractional calculus, we make a change of variable, $x(t) = I^\gamma v(t)$, $v \in C[0, 1]$, then we have

$$\begin{aligned}
&\mathcal{D}_{\mathbf{t}}^{\alpha} x(t) = \frac{d^n}{dt^n} I^{n-\alpha} x(t) = \frac{d^n}{dt^n} I^{n-\alpha} I^\gamma v(t) = \frac{d^n}{dt^n} I^{n-\alpha+\gamma} v(t) = \mathcal{D}_{\mathbf{t}}^{\alpha-\gamma} v(t), \\
&\mathcal{D}_{\mathbf{t}}^{\alpha+1} x(t) = \frac{d^n}{dt^n} I^{n-\alpha-1} x(t) = \frac{d^n}{dt^n} I^{n-\alpha-1} I^\gamma v(t) = \frac{d^n}{dt^n} I^{n-\alpha-1+\gamma} v(t) \\
&\qquad = \mathcal{D}_{\mathbf{t}}^{\alpha-\gamma+1} v(t), \\
&\mathcal{D}_{\mathbf{t}}^{\gamma} x(t) = \mathcal{D}_{\mathbf{t}}^{\gamma} I^\gamma v(t) = v(t).
\end{aligned} \tag{2.1}$$

Thus, using (2.1), the BVP (1.1) reduces to the following modified fractional differential equation:

$$-\mathscr{D}_{\mathbf{t}}^{\beta}\varphi_p\bigl(-\mathscr{D}_{\mathbf{t}}^{\alpha-\gamma}v(t)\bigr)=f\bigl(I^{\gamma}v(t)\bigr), \tag{2.2}$$

with boundary condition

$$\mathscr{D}_{\mathbf{t}}^{\alpha-\gamma}v(0)=\mathscr{D}_{\mathbf{t}}^{\alpha-\gamma}v(1)=0,\qquad v(0)=0, v(1)=\int_0^1 v(s)\,dA(s).$$

As the inverse is also valid, the BVP (2.2) is indeed equivalent to the BVP (1.1).

Now denote

$$G(\beta,t,s)=\frac{1}{\Gamma(\beta)}\begin{cases}[t(1-s)]^{\beta-1}, & 0\le t\le s\le 1,\\ [t(1-s)]^{\beta-1}-(t-s)^{\beta-1}, & 0\le s\le t\le 1,\end{cases} \tag{2.3}$$

$$\mathcal{A}=\int_0^1 t^{\alpha-\gamma-1}\,dA(t),\qquad \mathcal{G}_A(s)=\int_0^1 G(\alpha-\gamma,t,s)\,dA(t), \tag{2.4}$$

and assume $0\le\mathcal{A}<1$ and $\mathcal{G}_A(s)\ge 0$ for $s\in[0,1]$.

Suppose $0<\gamma\le 1<\alpha$, $\beta\le 2$, $\alpha-\gamma>1$. The following results have been obtained in [8, 38].

Lemma 2.1 (see [38]) *Given $h\in L^1(0,1)$, then the problem*

$$\begin{cases}-\mathscr{D}_{\mathbf{t}}^{\alpha-\gamma}v(t)=h(t), & 0<t<1,\\ v(0)=v(1)=0,\end{cases}$$

has the unique solution $v(t)=\int_0^1 G(\alpha-\gamma,t,s)h(s)\,ds$.

Lemma 2.2 (see [8]) *Let $h\in L^1(0,1)$, then the fractional boundary value problem*

$$\begin{cases}-\mathscr{D}_{\mathbf{t}}^{\beta}\varphi_p(-\mathscr{D}_{\mathbf{t}}^{\alpha-\gamma}v(t))=h(t),\\ \mathscr{D}_{\mathbf{t}}^{\alpha-\gamma}v(0)=\mathscr{D}_{\mathbf{t}}^{\alpha-\gamma}v(1)=0, \quad v(0)=0, v(1)=\int_0^1 v(s)\,dA(s),\end{cases}$$

has the unique solution

$$v(t)=\int_0^1 H(t,s)\varphi_q\biggl(\int_0^1 G(\beta,s,\tau)h(\tau)\,d\tau\biggr)\,ds,$$

where

$$H(t,s)=\frac{t^{\alpha-\gamma-1}}{1-\mathcal{A}}\mathcal{G}_A(s)+G(\alpha-\gamma,t,s). \tag{2.5}$$

Lemma 2.3 (see [8]) *The function $G(\beta,t,s)$ and $H(t,s)$ have the following properties*:

(1) $G(\beta,t,s)>0$, $H(t,s)>0$, *for* $t,s\in(0,1)$.

(2) $\frac{t^{\beta-1}(1-t)s(1-s)^{\beta-1}}{\Gamma(\beta)}\le G(\beta,t,s)\le\frac{\beta-1}{\Gamma(\beta)}t^{\beta-1}(1-t)$, *for* $t,s\in[0,1]$, *and* $\frac{t^{\beta-1}(1-t)s(1-s)^{\beta-1}}{\Gamma(\beta)}\le G(\beta,t,s)\le\frac{\beta-1}{\Gamma(\beta)}s(1-s)^{\beta-1}$, *for* $t,s\in[0,1]$.

(3) *There exist two positive constants* d, e *such that*

$$dt^{\alpha-\gamma-1}\mathcal{G}_A(s) \le H(t,s) \le et^{\alpha-\gamma-1}, \quad t,s \in [0,1]. \tag{2.6}$$

Let $E = C[0,1]$ be the Banach space of all continuous functions equipped the norm $\|v\| = \max\{v(t) : t \in [0,1]\}$. Define a cone P of E by

$$\begin{aligned} P = \bigl\{ & v \in E : \text{there exist nonnegative numbers } l_v < L_v \text{ such that} \\ & l_v t^{\alpha-\gamma-1} \le v(t) \le L_v t^{\alpha-\gamma-1}, t \in [0,1] \bigr\} \end{aligned}$$

and an operator T by

$$(Tv)(t) = \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G(\beta,s,\tau)f\bigl(I^\gamma v(\tau)\bigr)\,d\tau\right)ds,$$

then the fixed point of operator T in E is the solution of Equation (2.2), and $x = I^\gamma v(\tau)$ is the solution of Equation (1.1).

Now we define the constant

$$M = \sup_{s>0}\frac{f(s)}{s^\epsilon}.$$

We have the following lemma.

Lemma 2.4 *Assume that* (F) *holds. Then* $T : P \to P$ *is a continuous, compact, and increasing operator.*

Proof Let $v \in P$. Then there exist two nonnegative numbers $L_v > l_v \ge 0$ such that

$$l_v t^{\alpha-\gamma-1} \le v(t) \le L_v t^{\alpha-\gamma-1}, \quad t \in [0,1]. \tag{2.7}$$

Since

$$\frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}s^{\alpha-\gamma-1}\,ds = \frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}t^{\alpha-1},$$

we have

$$\frac{\Gamma(\alpha-\gamma)l_v}{\Gamma(\alpha)}t^{\alpha-1} \le I^\gamma v(\tau) \le \frac{\Gamma(\alpha-\gamma)L_v}{\Gamma(\alpha)}t^{\alpha-1}. \tag{2.8}$$

If $v \equiv 0$, then it follows from (2.6) that

$$\begin{aligned} (Tv)(t) &\le et^{\alpha-\gamma-1}\int_0^1 \varphi_q\left(\int_0^1 G(\beta,s,\tau)f(0)\,d\tau\right)ds \\ &\le e\left[\frac{(\beta-1)f(0)}{\Gamma(\beta)}\right]^{q-1} t^{\alpha-\gamma-1} \end{aligned} \tag{2.9}$$

and

$$(Tv)(t) \geq dt^{\alpha-\gamma-1} \int_0^1 \mathcal{G}_A(s)\varphi_q\left(\int_0^1 G(\beta,s,\tau)f(0)\,d\tau\right)ds$$
$$\geq d\int_0^1 \mathcal{G}_A(s)s^{(\beta-1)(q-1)}(1-s)^{q-1}\,ds\left[\frac{f(0)}{\Gamma(\beta+2)}\right]^{q-1} t^{\alpha-\gamma-1}. \tag{2.10}$$

Otherwise, by (F), T is increasing on x, and thus from (F) and (2.8), we have

$$(Tv)(t) \leq et^{\alpha-\gamma-1} \int_0^1 \varphi_q\left(\int_0^1 G(\beta,s,\tau)f\big(I^\gamma v(\tau)\big)\,d\tau\right)ds$$
$$\leq et^{\alpha-\gamma-1} \int_0^1 \varphi_q\left(\int_0^1 G(\beta,s,\tau)M\big(I^\gamma v(\tau)\big)^\epsilon\,d\tau\right)ds$$
$$\leq et^{\alpha-\gamma-1}\left[\frac{(\beta-1)M}{\Gamma(\beta)}\left(\frac{\Gamma(\alpha-\gamma)L_v}{\Gamma(\alpha)}\right)^\epsilon\right]^{q-1}. \tag{2.11}$$

On the other hand,

$$(Tv)(t) \geq dt^{\alpha-\gamma-1} \int_0^1 \mathcal{G}_A(s)\varphi_q\left(\int_0^1 G(\beta,s,\tau)f\big(I^\gamma v(\tau)\big)\,d\tau\right)ds$$
$$\geq d\int_0^1 \mathcal{G}_A(s)s^{(\beta-1)(q-1)}(1-s)^{q-1}\,ds\left[\frac{f(0)}{\Gamma(\beta+2)}\right]^{q-1} t^{\alpha-\gamma-1}. \tag{2.12}$$

Take

$$L_v^* = \max\left\{e\left[\frac{(\beta-1)M}{\Gamma(\beta)}\left(\frac{\Gamma(\alpha-\gamma)L_v}{\Gamma(\alpha)}\right)^\epsilon\right]^{q-1}, e\left[\frac{(\beta-1)f(0)}{\Gamma(\beta)}\right]^{q-1}\right\},$$
$$l_v^* = d\int_0^1 \mathcal{G}_A(s)s^{(\beta-1)(q-1)}(1-s)^{q-1}\,ds\left[\frac{f(0)}{\Gamma(\beta+2)}\right]^{q-1},$$

then by (2.9)-(2.12), we have $l_v^* t^{\alpha-\gamma-1} \leq (Tv)(t) \leq L_v^* t^{\alpha-\gamma-1}$. Thus T is well defined and uniformly bounded and $T(P) \subset P$.

On the other hand, according to the Arezela-Ascoli theorem and the Lebesgue dominated convergence theorem, it is easy to see that $T: P \to P$ is completely continuous. □

3 Main results

Denote

$$r_q = \left[e\varphi_q\left(\frac{(\beta-1)M}{\Gamma(\beta)\Gamma^\epsilon(\gamma+1)}\right)\right]^{\frac{1}{1-\epsilon(q-1)}}, \qquad \rho_q = \left[\frac{1}{M}\left(\frac{\Gamma(\gamma+1)}{e}\varphi_q\left(\frac{\Gamma(\beta)}{\beta-1}\right)\right)^\epsilon\right]^{\frac{1}{\epsilon(q-1)-1}},$$

then we have the following main results.

Theorem 3.1 *Assume* (F) *holds, and the following conditions are satisfied*:

$$\epsilon(q-1) > 1, \qquad f(0) \leq \rho_q. \tag{3.1}$$

Then Equation (1.1) *has a minimal solution* x^* *and a maximal solution* y^*, *and there exist some nonnegative numbers* $m_i < n_i$, $i = 1, 2$, *such that*

$$m_1 t^{\alpha-1} \le x^*(t) \le n_1 t^{\alpha-1}, \qquad m_2 t^{\alpha-1} \le y^*(t) \le n_2 t^{\alpha-1}, \quad t \in [0,1]. \tag{3.2}$$

Moreover, for initial values $v^{(0)} = 0$, $w^{(0)} = r$, *let* $\{v^{(n)}\}$ *and* $\{w^{(n)}\}$ *be the iterative sequences generated by*

$$\begin{aligned} v^{(n)}(t) &= \bigl(Tv^{(n-1)}\bigr)(t) = \bigl(T^n v^{(0)}\bigr)(t), \\ w^{(n)}(t) &= \bigl(Tw^{(n-1)}\bigr)(t) = \bigl(T^n w^{(0)}\bigr)(t). \end{aligned} \tag{3.3}$$

Then

$$\lim_{n\to+\infty} v^{(n)} = \boldsymbol{\mathcal{D}}_{\mathbf{t}}^{\gamma} x^*, \qquad \lim_{n\to+\infty} w^{(n)} = \boldsymbol{\mathcal{D}}_{\mathbf{t}}^{\gamma} y^*,$$

uniformly for $t \in [0,1]$.

Proof We construct a closed convex set of P by $P[0, r_q] = \{v \in P : \|v\| \le r_q\}$, and we prove $T(P[0, r_q]) \subset P[0, r_q]$.

In fact, for any $v \in P[0, r_q]$, if $v \equiv 0$, it follows from Lemma 2.3 and (3.1) that

$$\begin{aligned} \bigl\|(Tv)\bigr\| &= \max_{t\in[0,1]} \left\{ \int_0^1 H(t,s)\varphi_q\left(\int_0^1 G(\beta,s,\tau) f\bigl(I^\gamma v(\tau)\bigr)\,d\tau\right) ds \right\} \\ &\le e\int_0^1 \varphi_q\left(\int_0^1 \frac{\beta-1}{\Gamma(\beta)}\tau(1-\tau)^{\beta-1} f(0)\,d\tau\right) ds \le e\varphi_q\left(\frac{\beta-1}{\Gamma(\beta)} f(0)\right) \\ &\le \varphi_q\left(\frac{\beta-1}{\Gamma(\beta)}\right)\left[\frac{1}{M}\left(\frac{\Gamma(\gamma+1)}{e}\varphi_q\left(\frac{\Gamma(\beta)}{\beta-1}\right)\right)^{\epsilon}\right]^{\frac{q-1}{\epsilon(q-1)-1}} \\ &= e\varphi_q\left(\frac{\beta-1}{\Gamma(\beta)}\right)\left[M^{q-1}\left(\frac{e}{\Gamma(\gamma+1)}\varphi_q\left(\frac{\beta-1}{\Gamma(\beta)}\right)\right)^{\epsilon(q-1)}\right]^{\frac{1}{1-\epsilon(q-1)}} \\ &= \left[\left(e\varphi_q\left(\frac{\beta-1}{\Gamma(\beta)}\right)\right)^{1-\epsilon(q-1)} M^{q-1}\left(\frac{e}{\Gamma(\gamma+1)}\varphi_q\left(\frac{\beta-1}{\Gamma(\beta)}\right)\right)^{\epsilon(q-1)}\right]^{\frac{1}{1-\epsilon(q-1)}} \\ &= r_q. \end{aligned} \tag{3.4}$$

Otherwise, we have

$$0 < v(t) \le \max_{t\in[0,1]} v(t) \le r_q, \quad t \in (0,1), \tag{3.5}$$

and

$$0 < I^\gamma v(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} v(s)\,ds \le \frac{r_q}{\Gamma(\gamma+1)}, \quad t \in (0,1). \tag{3.6}$$

So by (F), (2) and (3) of Lemma 2.3, and (3.6), we get

$$\begin{aligned}\|(T\nu)\| &= \max_{t\in[0,1]}\left\{\int_0^1 H(t,s)\varphi_q\left(\int_0^1 G(\beta,s,\tau)f\big(I^\gamma\nu(\tau)\big)\,d\tau\right)ds\right\}\\ &\le et^{\alpha-\gamma-1}\int_0^1\varphi_q\left(\int_0^1 G(\beta,s,\tau)\frac{f(I^\gamma\nu)}{(I^\gamma\nu)^\epsilon}\big(I^\gamma\nu\big)^\epsilon\,d\tau\right)ds\\ &\le e\int_0^1\varphi_q\left(\int_0^1\frac{\beta-1}{\Gamma(\beta)}\tau(1-\tau)^{\beta-1}M\frac{r_q^\epsilon}{\Gamma^\epsilon(\gamma+1)}\,d\tau\right)ds\\ &\le e\varphi_q\left(\frac{(\beta-1)Mr_q^\epsilon}{\Gamma(\beta)\Gamma^\epsilon(\gamma+1)}\right)=r_q.\end{aligned}\tag{3.7}$$

Equations (3.4) and (3.7) imply that $T(P[0,r_q])\subset P[0,r_q]$.

Let $\nu^{(0)}(t)=0$ and $\nu^{(1)}(t)=(T\nu^{(0)})(t)=(T0)(t)$, $t\in[0,1]$, then it follows from $0\in P([0,r_q])$ that $\nu^{(1)}(t)\in T(P[0,r_q])$. Denote

$$\nu^{(n+1)}=T\nu^{(n)}=T^{n+1}\nu^{(0)},\quad n=1,2,\ldots.$$

It follows from $T(P[0,r_q])\subset P[0,r_q]$ that $\nu_n\in P[0,r_q]$ for $n\ge 1$. Noticing that T is compact, we see that $\{\nu^{(n)}\}$ is a sequentially compact set.

On the other hand, since $\nu^{(1)}\ge 0=\nu^{(0)}$, we have

$$\nu^{(2)}(t)=\big(T\nu^{(1)}\big)(t)\ge\big(T\nu^{(0)}\big)(t)=\nu^{(1)}(t),\quad t\in[0,1].$$

By induction, we get

$$\nu^{(n+1)}\ge\nu^{(n)},\quad n=1,2,\ldots.$$

Consequently, there exists $\nu^*\in P[0,r_q]$ such that $\nu^{(n)}\to\nu^*$. Letting $n\to+\infty$, from the continuity of T and $T\nu^{(n)}=\nu^{(n-1)}$, we obtain $T\nu^*=\nu^*$, which implies that ν^* is a nonnegative solution of the BVP (2.2), and thus $x^*=I^\gamma\nu^*(t)$ is a nonnegative solution of the BVP (1.1). By (2.8), there exist constants $0\le m_1<n_1$ such that

$$m_1t^{\alpha-1}\le x^*(t)\le n_1t^{\alpha-1},\quad t\in(0,1).$$

Now let $w^{(0)}(t)=r_q$ and

$$w^{(1)}(t)=\big(Tw^{(0)}\big)(t),\quad t\in[0,1].$$

Since $w^{(0)}(t)=r_q\in P[0,r_q]$, $w^{(1)}\in P[0,r_q]$. Let

$$w^{(n+1)}=Tw^{(n)}=T^{(n+1)}w^{(0)},\quad n=1,2,\ldots.$$

It follows from $T(P[0,r_q])\subset P[0,r_q]$ that

$$w^{(n)}\in P[0,r_q],\quad n=0,1,2,\ldots.$$

From Lemma 2.4, T is compact, and consequently $\{w^{(n)}\}$ is a sequentially compact set.

Now, since $w^{(1)} \in P[0, r_q]$, we get

$$0 \le w^{(1)}(t) \le \|w^{(1)}\| \le r_q = w^{(0)}(t).$$

It follows from (F) that $w^{(2)} = Tw^{(1)} \le Tw^{(0)} = w^{(1)}$. By induction, we obtain

$$w^{(n+1)} \le w^{(n)}, \quad n = 0, 1, 2, \dots.$$

Consequently, there exists $w^* \in P[0, r_q]$ such that $w^{(n)} \to w^*$. Letting $n \to +\infty$, from the continuity of T and $Tw^{(n)} = w^{(n-1)}$, we have $Tw^* = w^*$, which implies that $y^* = I^\gamma w^*(t)$ is another nonnegative solution of the boundary value problem (1.1) and y^* also satisfies (3.2) since $w^* \in P$.

At the end, we prove that x^* and y^* are extremal solutions for Equation (1.1). Let $\widetilde{u}$ be any positive solution of Equation (2.2), then $v^{(0)} = 0 \le \widetilde{u} \le r_q = w^{(0)}$, and $v^{(1)} = Tv^{(0)} \le T\widetilde{u} = \widetilde{u} \le T(w^{(0)}) = w^{(1)}$. By induction, we have $v^{(n)} \le \widetilde{u} \le w^{(n)}$, $n = 1, 2, 3, \dots$. Taking the limit, we have $v^* \le \widetilde{u} \le w^*$, which implies that x^* and y^* are the maximal and minimal solutions of Equation (1.1), respectively. The proof is completed. □

Corollary 3.1 *Assume* (F) *holds, and*

$$\epsilon(q-1) > 1, \quad f(0) = 0. \tag{3.8}$$

Then Equation (1.1) *has a minimal solution which is trivial and a positive maximal solution* y^*, *and there exist positive numbers* $0 < m_2 < n_2$, *such that*

$$m_2 t^{\alpha-1} \le y^*(t) \le n_2 t^{\alpha-1}, \quad t \in [0,1]. \tag{3.9}$$

Moreover, for the initial value $w^{(0)} = r$, *let* $\{w^{(n)}\}$ *be the iterative sequence generated by*

$$w^{(n)}(t) = \left(Tw^{(n-1)}\right)(t) = \left(T^n w^{(0)}\right)(t). \tag{3.10}$$

Then

$$\lim_{n \to +\infty} w^{(n)} = \mathcal{D}_{\mathbf{t}}^{\gamma} y^*$$

uniformly for $t \in [0,1]$.

Corollary 3.2 *If* $p = 2$ (φ_p *reduces to the linear operator*), *assume that* (F) *holds and satisfies the following conditions*:

$$\epsilon > 1, \quad f(0) \le \rho_2. \tag{3.11}$$

Then Equation (1.1) *has a minimal solution* x^* *and a maximal solution* y^*, *and there exist some nonnegative numbers* $m_i < n_i$, $i = 1, 2$, *such that*

$$m_1 t^{\alpha-1} \le x^*(t) \le n_1 t^{\alpha-1}, \qquad m_2 t^{\alpha-1} \le y^*(t) \le n_2 t^{\alpha-1}, \quad t \in [0,1]. \tag{3.12}$$

Moreover, for initial values $v^{(0)} = 0$, $w^{(0)} = r_2$, *let* $\{v^{(n)}\}$ *and* $\{w^{(n)}\}$ *be the iterative sequences generated by*

$$v^{(n)} = Tv^{(n-1)} = T^n v^{(0)}, \qquad w^{(n)} = Tw^{(n-1)} = T^n w^{(0)}. \tag{3.13}$$

Then $\lim_{n\to+\infty} v^{(n)} = \boldsymbol{\mathcal{D}}_{\mathbf{t}}^{\gamma} x^*$, $\lim_{n\to+\infty} w^{(n)} = \boldsymbol{\mathcal{D}}_{\mathbf{t}}^{\gamma} y^*$ *uniformly for* $t \in [0,1]$.

Corollary 3.3 *Suppose* (3.1) *holds and one of the following assumptions is satisfied*:

(f1) $f : [0,+\infty) \to [0,+\infty)$ *is continuous and non-decreasing, there exists a constant* $\epsilon > 0$ *such that* $\frac{f(x)}{x^\epsilon}$ *is increasing with respect to* x *and* $\lim_{x\to+\infty} \frac{f(x)}{x^\epsilon} = M > 0$.

(f2) $f : [0,+\infty) \to [0,+\infty)$ *is continuous and non-decreasing, there exists a constant* $\epsilon > 0$ *such that* $\frac{f(x)}{x^\epsilon}$ *is nonincreasing with respect to* x *and* $\lim_{x\to+0} \frac{f(x)}{x^\epsilon} = M > 0$.

(f3) $f : [0,+\infty) \to [0,+\infty)$ *is continuous and non-decreasing, and there exists a constant* $\epsilon > 0$ *such that* $\lim_{x\to+0} \frac{f(x)}{x^\epsilon} = a > 0$, $\lim_{x\to+\infty} \frac{f(x)}{x^\epsilon} = b > 0$, $M = \max\{a,b\}$.

Then Equation (1.1) *has a minimal solution* x^* *and a maximal solution* y^*, *and there exist some nonnegative numbers* $m_i < n_i$, $i = 1,2$, *such that*

$$m_1 t^{\alpha-1} \le x^*(t) \le n_1 t^{\alpha-1}, \qquad m_2 t^{\alpha-1} \le y^*(t) \le n_2 t^{\alpha-1}, \quad t \in [0,1]. \tag{3.14}$$

Moreover, for initial values $v^{(0)} = 0$, $w^{(0)} = r$, *let* $\{v^{(n)}\}$ *and* $\{w^{(n)}\}$ *be the iterative sequences generated by*

$$v^{(n)} = Tv^{(n-1)} = T^n v^{(0)}, \qquad w^{(n)} = Tw^{(n-1)} = T^n w^{(0)}. \tag{3.15}$$

Then

$$\lim_{n\to+\infty} v^{(n)} = \boldsymbol{\mathcal{D}}_{\mathbf{t}}^{\gamma} x^*, \qquad \lim_{n\to+\infty} w^{(n)} = \boldsymbol{\mathcal{D}}_{\mathbf{t}}^{\gamma} y^*,$$

uniformly for $t \in [0,1]$.

An example Consider the following nonlocal boundary value problem of the fractional p-Laplacian equation:

$$\begin{cases} -\boldsymbol{\mathcal{D}}_{\mathbf{t}}^{\frac{4}{3}}(\varphi_{\frac{5}{2}}(-\boldsymbol{\mathcal{D}}_{\mathbf{t}}^{\frac{3}{2}}x))(t) = x^2(t), \quad t \in (0,1), \\ \boldsymbol{\mathcal{D}}_{\mathbf{t}}^{\frac{3}{2}}x(0) = \boldsymbol{\mathcal{D}}_{\mathbf{t}}^{\frac{3}{2}}x(1) = \boldsymbol{\mathcal{D}}_{\mathbf{t}}^{\frac{1}{4}}x(0) = 0, \qquad \boldsymbol{\mathcal{D}}_{\mathbf{t}}^{\frac{1}{4}}(1) = \int_0^1 \boldsymbol{\mathcal{D}}_{\mathbf{t}}^{\frac{1}{4}}x(s)\,dA(s), \end{cases} \tag{3.16}$$

where A is a bounded variation function satisfying $A(t) = 0$ for $t \in [0,\frac{1}{2})$, $A(t) = 2$ for $t \in [\frac{1}{2},\frac{3}{4})$, $A(t) = 1$ for $t \in [\frac{3}{4},1]$. Then Equation (3.16) has a positive maximal solution y^*; and there exist positive numbers $0 < m < n$, such that

$$mt^{\frac{1}{2}} \le y^*(t) \le nt^{\frac{1}{2}}, \quad t \in [0,1]. \tag{3.17}$$

Proof By a simple computation, the problem (3.16) reduces to the following multi-point boundary value problem:

$$\begin{cases} -\boldsymbol{\mathcal{D}}_{\mathbf{t}}^{\frac{4}{3}}(\varphi_{\frac{5}{2}}(-\boldsymbol{\mathcal{D}}_{\mathbf{t}}^{\frac{3}{2}}x))(t) = x^2(t), \quad t \in (0,1), \\ \boldsymbol{\mathcal{D}}_{\mathbf{t}}^{\frac{3}{2}}x(0) = \boldsymbol{\mathcal{D}}_{\mathbf{t}}^{\frac{3}{2}}x(1) = \boldsymbol{\mathcal{D}}_{\mathbf{t}}^{\frac{1}{4}}x(0) = 0, \qquad \boldsymbol{\mathcal{D}}_{\mathbf{t}}^{\frac{1}{4}}x(1) = 2\boldsymbol{\mathcal{D}}_{\mathbf{t}}^{\frac{1}{4}}x(\frac{1}{2}) - \boldsymbol{\mathcal{D}}_{\mathbf{t}}^{\frac{1}{4}}x(\frac{3}{4}). \end{cases}$$

Let

$$\alpha = \frac{3}{2}, \qquad \beta = \frac{4}{3}, \qquad p = \frac{5}{2}, \qquad f(x) = x^2.$$

First, we have

$$\mathcal{A} = \int_0^1 t^{\alpha-1}\,dA(t) = 2 \times \left(\frac{1}{2}\right)^{\frac{1}{2}} - \left(\frac{3}{4}\right)^{\frac{1}{2}} = 0.5482 < 1,$$

and by a simple computation, we have $\mathcal{G}_A(s) \geq 0$.

Obviously, $f(0) = 0$ and

$$\sup_{s>0} \frac{f(s)}{s^2} = 1, \qquad \epsilon(q-1) = 2\left(\frac{5}{3} - 1\right) = \frac{4}{3} > 1.$$

By Corollary 3.2, Equation (3.16) has a positive maximal solution which satisfies (3.17). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors wrote, read, and approved the final manuscript.

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