Characteristics of Radiated Waves in Viscoelstic Soil Layer due to Lateral Flexural Vibration of Single Pile

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Synopsis: Radiated waves from a laterally vibrating pile in a homogeneous-elastic soil layer as media within limits of small deformations are studied. It is shown that zero traction surface of soil layer together with fixed bedrock establish conditions which can only be satisfied via certain set of wave frequencies. It is shown that radiation of pressure waves closely related to the amount of vertical deformation induced in soil media due to lateral pile movements. Shear waves however, are radiated from pile independent from the vertical deformations. It is concluded that in cases of very smooth pile surface where relative vertical slippage between soil and pile skin may occur, the formation of P-waves might be limited. Complex valued amplitudes of shear and pressure waves radiated from a rigid circular pile under known lateral vibration is studied.

Keywords: soil-pile interaction, soil pile continuum mechanics, pile lateral vibrations, shear and pressure waves in soil, radiated waves from laterally vibrating pile.

1. Introduction

Continuum mechanics approach has been one of the major theoretical approaches towards the problem of laterally deflected piles due to the insight that can be gained through its mathematical rigor. Baranov [1] solved the problem of laterally vibrating pile, assuming soil is composed from infinitesimal layers under plain strain conditions. Novak [2] used Baranov's results to establish dynamic stiffness and damping of laterally vibrating piles. Nogami & Novak [3] and Novak & Nogami [4] more generalized their mathematical approach by considering soil as three dimensional visco-elastic continuum with zero vertical deformation. Poulos & Davis [5] published a state of the art book in pile analysis and design based on results from the theory of elasticity. Kaynia [6] has presented formulations for the analysis of pile groups in layered semi infinite media. Pak & Jennings [7] formulated the interaction of one dimensional pile with three dimensional elastic continuum as Fredholm integral equation of the second kind.

Numerical methods found extensive application in the problem of soil-pile interaction, especially where nonlinear behavior of soil and pile are involved or when deformations cross small strains limitations. Alongside with finite element method as the backbone of the majority of present research and practical approaches, boundary element method has been developed successfully in the field of soil-pile interaction analysis within past few years. As a brief (and very incomplete) reference to this approach, [8], [9] and [10] can be named here.

In this paper the characteristics of radiated waves from a laterally vibrating pile is approached by continuum mechanics formulation with special attention to mathematical implications of presence of a zero traction surface (ground surface) and its physical conclusions on the nature of such waves and displacement field in soil media. The theoretical study is supported by finite element analyses wherever it is possible.

2. Problem Definition

The problem is to study radiated waves in a semi infinite elastic/ visco-elastic continuum due to lateral vibration of a pile. The semi infinite soil media is assumed to be linearly elastic, homogeneous and is under laid by fixed rigid bedrock. Special attention to the unstressed (zero traction) free surface of soil layer is made. The layer has a finite thickness *"H"*. Deformation components in media in polar coordinates are named $u(r,\theta,z,t)$, $v(r,\theta,z,t)$ and $w(r,\theta,z,t)$ for radial, tangential and vertical directions respectively which are assumed to lie within the limits of small deformations. The soil-pile system is assumed to be in a steady state dynamic equilibrium condition. The pile is connected to the bed rock and undergoes lateral deflections in "y" direction only. Its deflection to the vertical axis is referred to as "$Y(z,t)$". It is assumed to have constant radius "r_0" and is connected to soil continuously in every contact point on its skin. Coordinate axis origin is set on pile tip and central axis of pile is chosen as vertical axis "z" with positive direction upwards. Elasto-dynamic differential equations within small strain limits are aimed to be solved in cylindrical coordinates [e.g. 1]:

$$\left.\begin{aligned}(\lambda+2\mu)\frac{\partial\Delta}{\partial r}-\frac{2\mu}{r}\frac{\partial\omega_z}{\partial\theta}+2\mu\frac{\partial\omega_\theta}{\partial z}=\rho\frac{\partial^2 u}{\partial t^2}\\(\lambda+2\mu)\frac{1}{r}\frac{\partial\Delta}{\partial\theta}-2\mu\frac{\partial\omega_r}{\partial z}+2\mu\frac{\partial\omega_z}{\partial r}=\rho\frac{\partial^2 v}{\partial t^2}\\(\lambda+2\mu)\frac{\partial\Delta}{\partial z}-\frac{2\mu}{r}\frac{\partial(r\omega_\theta)}{\partial r}+\frac{2\mu}{r}\frac{\partial\omega_r}{\partial\theta}=\rho\frac{\partial^2 w}{\partial t^2}\end{aligned}\right\}\tag{1}$$

where

$$\left.\begin{aligned}&\rho=Material\quad density\quad of\ soil\quad media\\&\Delta=\frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{1}{r}\frac{\partial v}{\partial\theta}+\frac{\partial w}{\partial z}\\&\omega_r=\frac{1}{2}\left(\frac{1}{r}\frac{\partial w}{\partial\theta}-\frac{\partial v}{\partial z}\right)\quad\omega_\theta=\frac{1}{2}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial r}\right)\quad\omega_z=\frac{1}{2r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv)-\frac{\partial u}{\partial\theta}\right)\end{aligned}\right\}$$

Continuity of deformations in the media is assumed to be maintained everywhere, including on the contact surface of pile and soil. It is assumed that velocity of lateral movement of pile is very small such that the effect of moving boundary can be neglected. Due to continuity between the pile and media, the followings can be concluded on the pile's surface, i.e. at $r=r_0$ [e.g. 11]

$$\left.\begin{aligned}u(r_0,\theta,z,t)=Y(z,t)\sin\theta\\v(r_0,\theta,z,t)=Y(z,t)\cos\theta\\w(r_0,\theta,z,t)=-r_0Y'(z,t)\sin\theta\end{aligned}\right\}\tag{2}$$

Where prime holds for derivative with respect to "z". It is expected that deformation components in soil media to be single valued and periodic with period "2π" with respect to coordinate "θ". Therefore a Fourier series expansion with respect to this variable can be written for radial deformation component "$u(r,\theta,z,t)$" as follows:

$$u(r,\theta,z,t)=\sum_{m=1}^{\infty}[A_m(r,z,t)\sin(m\theta)+B_m(r,z,t)\cos(m\theta)]\tag{3}$$

Substituting Equation (3) in the first equation of Equation (2), multiplying by "$\cos(s\theta)d\theta$, $s=1,2,3,....$:" and integrating between 0 and 2π yields:

$$\left.\begin{aligned}&A_m(r_0,z,t)=\begin{cases}\pi Y(z,t) & for\ \ s=m=1\\0 & for\ \ s=m>1\end{cases}\\&B_m(r_0,z,t)=0\end{aligned}\right\}\tag{4}$$

In general, Equation (4) is only valid on the pile surface and Fourier series coefficients "A_m" and "B_m" may be non-zero elsewhere, provided that they have zeroes on $r=r_0$ for "$m>1$". However, for simplicity we assume the following expansions for displacement components to be valid everywhere including on pile surface. This assumption will obviously satisfy continuity conditions on the surface of the pile:

$$\left.\begin{aligned}u(r,\theta,z,t)=U(r,z,t)\sin\theta\\v(r,\theta,z,t)=V(r,z,t)\cos\theta\\w(r,\theta,z,t)=W(r,z,t)\sin\theta\end{aligned}\right\}\tag{5}$$

It is desirable to formulate the problem with respect to non dimensional radial and vertical coordinates "$\hat{r}=r/r_0$" and '"$\hat{z}=z/H$" respectively.

The media is assumed to be laid on a rigid boundary continuously and therefore deformations components on bottom should be zero. A sine Fourier series expansion of deformations in Equation (5) with respect to "$\hat{z}$" can satisfy bottom conditions:

$$\left.\begin{aligned}u(\hat{r},\theta,\hat{z},t)=e^{i\omega t}\sin\theta\sum_{n=1}^{\infty}U_n(\hat{r})\sin(k_n\hat{z})\\v(\hat{r},\theta,\hat{z},t)=e^{i\omega t}\cos\theta\sum_{n=1}^{\infty}V_n(\hat{r})\sin(k_n\hat{z})\\w(\hat{r},\theta,\hat{z},t)=e^{i\omega t}\sin\theta\sum_{n=1}^{\infty}W_n(\hat{r})\sin(h_n\hat{z})\end{aligned}\right\}\tag{6}$$

Where "h_n , k_n" are non dimensional factors which will be determined later.

Taking air pressure equal zero, all components of the stresses on the free surface (i.e. on "$\hat{z}=1$") should be zero. Shear stresses on free surface can be written as [e.g. 11]:

$$\tau_{rz} = \mu(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r})\bigg|_{\hat{z}=1} = \mu e^{i\omega t} \sin\theta \sum_{n=1}^{\infty}[\sin(h_n)W_n'(\hat{r}) + k_n \cos(k_n)(\frac{r_0}{H})U_n(\hat{r})] = 0 \quad (7)$$

$$\tau_{\theta z} = \mu(\frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta})\bigg|_{\hat{z}=1} = \mu e^{i\omega t} \cos\theta \sum_{n=1}^{\infty}[\sin(h_n)\frac{W_n(\hat{r})}{\hat{r}} + k_n \cos(k_n)(\frac{r_0}{H})V_n(\hat{r})] = 0 \quad (8)$$

In Equation (7) and Equation (8), prime sign holds for derivatives with respect to arguments "$\hat{r}$" and "μ" holds for shear modulus (together with "λ", Lamé constants of elasticity) of the soil media.

Normal stress "σ_z" should also vanish on the free surface and can be written as:

$$\sigma_z\big|_{\hat{z}=1} = \lambda\Delta + 2\mu\varepsilon_z\big|_{\hat{z}=1} = \mu e^{i\omega t} \sin\theta \sum_{n=1}^{\infty}\{\beta h_n \cos(h_n)(\frac{r_0}{H})W_n(\hat{r}) + \sin(k_n)[U'_n(\hat{r}) + \frac{U_n(\hat{r})}{\hat{r}} - \frac{V_n(\hat{r})}{\hat{r}}]\} = 0 \quad (9)$$

Where

$$\beta = \frac{\lambda + 2\mu}{\lambda} = \frac{1-\nu}{\nu} \quad (10)$$

Any solution to Equation (1) should satisfy Equations (7), (8) and (9) for all "$\hat{r}$" from pile's surface "$\hat{r}=1$" to infinity. It will be shown that this condition would pose unexpected changes to the solution.

3. Solution of Elasto–Dynamic Differential Equations

Substituting Equations (6) in Equations (1) reads:

$$\begin{aligned}&\sin(k_n z)\{\alpha\hat{r}^2U''_n(\hat{r}) + \alpha\hat{r}U'_n(\hat{r}) + [(\varpi^2 - (\frac{k_n r_0}{H})^2)\hat{r}^2 - (1+\alpha)]U_n(\hat{r}) + \\ &+ (1-\alpha)(\hat{r})V'_n(\hat{r}) + (1+\alpha)V_n(\hat{r})\} + \cos(h_n z)(\alpha-1)(\frac{h_n r_0}{H})\hat{r}^2W'(\hat{r}) = 0\end{aligned} \quad (11)$$

$$\begin{aligned}&\sin(k_n z)\{\hat{r}^2V''_n(\hat{r}) + \hat{r}V'_n(\hat{r}) + [(\varpi^2 - (\frac{k_n r_0}{H})^2)\hat{r}^2 - (1+\alpha)]V_n(\hat{r}) + \\ &+ (\alpha-1)(\hat{r})U'_n(\hat{r}) + (\alpha+1)U_n(\hat{r})\} + \cos(h_n z)(\alpha-1)(\frac{h_n r_0}{H})(\hat{r})W(\hat{r}) = 0\end{aligned} \quad (12)$$

$$\begin{aligned}&\sin(h_n z)\{\hat{r}^2W''_n(\hat{r}) + \hat{r}W'_n(\hat{r}) + [(\varpi^2 - (\sqrt{\alpha}\,\frac{h_n r_0}{H})^2)\hat{r}^2 - 1]W_n(\hat{r}) + \\ &+ (\alpha-1)(\frac{k_n r_0}{H})\hat{r}\cos(k_n z)[\hat{r}U'_n(\hat{r}) + U_n(\hat{r}) - V_n(\hat{r})] = 0\end{aligned} \quad (13)$$

Where

$$\alpha = \frac{(\lambda + 2\mu)}{\mu} = \frac{2(1-\nu)}{1-2\nu}$$

$$\nu = Poisson's \quad Ratio$$

$$V_s = \sqrt{\frac{\mu}{\rho}} \quad Shear\ Wave\ Velocity$$

$$\varpi = r_0\omega\sqrt{\frac{\rho}{\mu}} = \frac{r_0\omega}{V_s} \quad Non-\mathrm{dim}ensional \quad frequency$$

Since "$\sin$" and "$\cos$" never vanish simultaneously, their multipliers should be equated to zero, leading to the following six equations:

$$\alpha\hat{r}^2U''_n(\hat{r}) + \alpha\hat{r}U'_n(\hat{r}) + [(\varpi^2 - (\frac{k_n r_0}{H})^2)\hat{r}^2 - (1+\alpha)]U_n(\hat{r}) + (1-\alpha)\hat{r}V'_n(\hat{r}) + (1+\alpha)V_n(\hat{r}) = 0 \quad (14)$$

$$\hat{r}^2V''_n(\hat{r}) + \hat{r}V'_n(\hat{r}) + [(\varpi^2 - (\frac{k_n r_0}{H})^2)\hat{r}^2 - (1+\alpha)]V_n(\hat{r}) + (\alpha-1)(\hat{r})U'_n(\hat{r}) + (\alpha+1)U_n(\hat{r}) = 0 \quad (15)$$

$$\hat{r}^2W''_n(\hat{r}) + \hat{r}W'_n(\hat{r}) + [(\varpi^2 - (\sqrt{\alpha}\,\frac{h_n r_0}{H})^2)\hat{r}^2 - 1]W_n(\hat{r}) = 0 \quad (16)$$

$$(\alpha-1)(\frac{h_n r_0}{H})(\hat{r}^2)W'(\hat{r})=0 \tag{17}$$

$$(\alpha-1)(\frac{h_n r_0}{H})(\hat{r})W(\hat{r})=0 \tag{18}$$

$$(\alpha-1)(\frac{k_n r_0}{H})(\hat{r})[\hat{r}U'_n(\hat{r})+U_n(\hat{r})-V_n(\hat{r})]=0 \tag{19}$$

Starting from Equation (16), choice of

$$q_2^2=\varpi^2-(\sqrt{\alpha}\,\frac{h_n r_0}{H})^2 \tag{20}$$

turns it into a Bessel's differential equation with solutions of Bessel function of first and second kind. However, Bessel function of second kind is omitted from the solution due to its unboundedness when the argument approaches zero:

$$W_n(\hat{r})=C_n J_1(q_2\hat{r}) \tag{21}$$

In Equation (21), "C_n" is integration constant. Disregarding special case of "$\alpha=1$" (i.e. $\nu=1/3$) it can be seen that solution Equation (21) will not satisfy Equation (17) and Equation (18) for every "$\hat{r}$" unless "$q_2\hat{r}$" becomes zero. Since "$\hat{r}\geq 1$" it is concluded that "$q_2=0$" and therefore:

$$\left.\begin{aligned}&W_n(\hat{r})=C_n J_1(q_2\hat{r})=0\\&W'_n(\hat{r})=C_n[q_2 J_0(q_2\hat{r})-\frac{1}{\hat{r}}J_1(q_2\hat{r})]=0\end{aligned}\right\} \tag{22}$$

Equation (14) can be simplified by substituting for "$V_n(\hat{r})$"from Equation (19):

$$\hat{r}^2U''_n(\hat{r})+3\hat{r}U'_n(\hat{r})+[(\varpi^2-(\frac{k_n r_0}{H})^2)\hat{r}^2]U_n(\hat{r})=0 \tag{23}$$

Equation (23) can be simplified by setting "$U_n(\hat{r})=\hat{r}^{-1}g_n(\hat{r})$":

$$\hat{r}^2g''_n(\hat{r})+\hat{r}g'_n(\hat{r})+(q_1^2\hat{r}^2-1)g_n(\hat{r})=0 \tag{24}$$

Where

$$q_1^2=\varpi^2-(\frac{k_n r_0}{H})^2 \tag{25}$$

Since Equation (24) is in standard form of Bessel's differential equation, solution for "U_n" can be established in this basis. Similar to Equation (21), Bessel function of the second kind is omitted from the solution:

$$U_n(\hat{r})=A_n\hat{r}^{-1}J_1(q_1\hat{r}) \tag{26}$$

Substituting this result in Equation (19) reads:

$$V(\hat{r})=A_n J'_1(q_1\hat{r})=A_n[q_1 J_0(q_1\hat{r})-\frac{1}{\hat{r}}J_1(q_1\hat{r})] \tag{27}$$

Solutions Equation (26) and Equation (27) must satisfy Equation (15), which after substitution it concludes:

$$-2A_n[J_1(q_1\hat{r})-\frac{1+\alpha}{\hat{r}}J_1(q_1\hat{r})]=0 \tag{28}$$

Equation (28) is valid for every "r" only if "$q_1=0$" . Moreover, the above solutions should satisfy free surface boundary conditions. Substituting Equations (21) (26) and (27) in Equations (7), (8) and (9) reads:

$$\sin(h_n)q_2C_nJ_1'(q_2\hat{r})+k_n\cos(k_n)(\frac{r_0}{H})A_n\hat{r}^{-1}J_1(q_1\hat{r})=0 \tag{29}$$

$$\sin(h_n)\frac{C_nJ_1'(q_2\hat{r})}{\hat{r}}+k_n\cos(k_n)(\frac{r_0}{H})A_nJ'_1(q_1\hat{r})=0 \tag{30}$$

$$\beta h_n\cos(h_n)(\frac{r_0}{H})C_nJ_1'(q_2\hat{r})=0 \tag{31}$$

It can be seen that Equations (29) (30) and (31) are satisfied for every "r" provided that:

$$q_1=q_2=0 \tag{32}$$

It is concluded from Equation (32) that:

$$\overline{\omega}_n = \frac{k_n r_0}{H} = \sqrt{\alpha}\,\frac{h_n r_0}{H} \tag{33}$$

And therefore

$$\omega_n = \frac{k_n V_s}{H} = \sqrt{\alpha}\,\frac{h_n V_s}{H} \Rightarrow k_n = \sqrt{\alpha} h_n = h_n \sqrt{\frac{1-\nu}{2(1-2\nu)}} \tag{34}$$

Solutions (21), (26) and (27) may seem trivial in combination with Equation (32). However, by considering Equation (32) as a limiting condition, new equation can be established by substituting *"$q_2=0$"* in Equation (14):

$$\hat{r}^2 W''_n(\hat{r}) + \hat{r} W'_n(\hat{r}) - W_n(\hat{r}) = 0 \Rightarrow W_n(\hat{r}) = E_n \hat{r}^{-1} + F_n \hat{r} \tag{36}$$

Constant "F_n" must be zero as "$W(r)$" should be bounded for large "r".

Substituting *"$q_1=0$"* in combined Equations (15) and (19) gives:

$$\hat{r}^2 U''_n(\hat{r}) + 3\hat{r} U'_n(\hat{r}) = 0 \Rightarrow U_n(\hat{r}) = G_n r^{-2} + D_n \tag{37}$$

Constant *"D_n"* denotes a rigid body motion in radial direction for the entire media. Since such a motion cannot be physically justified, this constant will be taken zero. Consequently "V_n" can be determined from Equation (19) as:

$$V_n(\hat{r}) = -G_n r^{-2} \tag{38}$$

It can be investigated by direct substitution that solutions (37) and (38) for "$U_n(r)$" and *"$V_n(r)$"* respectively, will satisfy Equation (15). To investigate if obtained solutions are satisfying free surface boundary conditions, Equations (36), (37) and (38) are substituted in Equations (6) , (7) and (8):

$$\tau_{rz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)\Big|_{\hat{z}=1} = \hat{r}^{-2}\mu e^{i\omega t}\sin\theta \sum_{n=1}^{\infty}\left[-E_n \sin(h_n) + k_n \cos(k_n)\left(\frac{r_0}{H}\right)G_n\right] = 0 \tag{39}$$

$$\tau_{\theta z} = \mu\left(\frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta}\right)\Big|_{\hat{z}=1} = \hat{r}^{2}\mu e^{i\omega t}\cos\theta \sum_{n=1}^{\infty}\left[E_n \sin(h_n) - k_n \cos(k_n)\left(\frac{r_0}{H}\right)G_n\right] = 0 \tag{40}$$

$$\sigma_z\big|_{\hat{z}=1} = \lambda\Delta + 2\mu\varepsilon_z\big|_{\hat{z}=1} = \hat{r}^{2}\mu e^{i\omega t}\sin\theta \sum_{n=1}^{\infty} E_n \beta h_n \cos(h_n)\left(\frac{r_0}{H}\right) = 0 \tag{41}$$

To satisfy Equation (41), it is necessary that:

$$\cos(h_n) = 0 \Rightarrow h_n = (2n-1)\frac{\pi}{2} \tag{42}$$

While Equations (39) and (40) lead to a relationship between integration constants:

$$E_n = (-1)^{n+1} k_n \cos(k_n)\left(\frac{r_0}{H}\right)G_n \tag{43}$$

An important conclusion is made by combining Equations (42) and (34), obtaining resonant frequencies of lateral vibrations in the media:

$$\omega_{pn} = \sqrt{\alpha}\,\frac{V_s}{H}(2n-1)\frac{\pi}{2} \Rightarrow T_1 = \frac{4H}{V_s}\sqrt{\frac{1-2\nu}{2(1-\nu)}} = \frac{4H}{V_p} \tag{44}$$

Where $V_p = \sqrt{(\lambda + 2\mu)/\rho}$ is propagation speed of P-waves [12]. Another possibility to satisfy free surface boundary conditions Equations (39) to (41) is to choose *"$E_n=0$"* and *"$\cos(k_n)=0$"* which concludes;

$$W_n(\hat{r}) = 0$$

$$k_n = (2n-1)\frac{\pi}{2} \Rightarrow \omega_n = \frac{V_s}{H}(2n-1)\frac{\pi}{2} \Rightarrow T_1 = \frac{4H}{V_s} \tag{45}$$

Relationship (45) is comparable with vertical resonant period of a limited thickness-half infinite elastic layer [13]:

$$T_1 = 4H/V_s \tag{46}$$

Nogami and Novak [3] used the assumption "*$W(r)=0$*" and proposed a solution to elasto-dynamic equations of motion, which did not satisfy free surface un-stressed conditions. Unpublished works of the authors on Nogami's and Novak's [3] approach to reach a solution which satisfies free surface boundary conditions resulted in similar deformation field as described in Equations (37) and (38) together with frequencies in Equation (44). The present solution, however, is more general as no initial assumption is made for zero vertical deformation. Frequencies in Equation (44) are comparable to resonant frequencies of vertical vibration of layer as described by Dobry and Oweis et al. [13].

Frequencies Equations (44) and (45) clearly describe two distinct kinds of waves propagating in soil media, i.e. pressure and shear waves respectively. It is also concluded that radiated shear waves are accompanied with zero vertical deformation in the media. In other words generation of shear waves is only related to lateral deformations of pile. It can be also seen that deformation field in soil media has decay characteristics of inverse powers of distance from the pile's axis. Deformation field corresponding to shear waves and pressure waves are hereby given, respectively:

$$\left.\begin{aligned} u_s(\hat{r},\theta,\hat{z},t) &= \hat{r}^{-2}\sin\theta\sum_{n=1}^{\infty} G_n \exp(i\omega_{sn}t)\sin(\eta_n\hat{z}) \\ v_s(\hat{r},\theta,\hat{z},t) &= -\hat{r}^{-2}\cos\theta\sum_{n=1}^{\infty} G_n \exp(i\omega_{sn}t)\sin(\eta_n\hat{z}) \\ w_s(\hat{r},\theta,\hat{z},t) &= 0 \end{aligned}\right\} \quad (47)$$

$$\left.\begin{aligned} u_p(\hat{r},\theta,\hat{z},t) &= \hat{r}^{-2}\sin\theta\sum_{n=1}^{\infty} G_n \exp(i\omega_{pn}t)\sin(\sqrt{\alpha}\eta_n\hat{z}) \\ v_p(\hat{r},\theta,\hat{z},t) &= -\hat{r}^{-2}\sin\theta\sum_{n=1}^{\infty} G_n \exp(i\omega_{pn}t)\sin(\sqrt{\alpha}\eta_n\hat{z}) \\ w_p(\hat{r},\theta,\hat{z},t) &= (\frac{r_0}{H})\hat{r}^{-1}\sin\theta\sum_{n=1}^{\infty} G_n(-1)^{n+1}\sqrt{\alpha}\eta_n\cos(\sqrt{\alpha}\eta_n)\exp(i\omega_{pn}t)\sin(\eta_n\hat{z}) \end{aligned}\right\} \quad (48)$$

Complete deformation field can be considered as a linear combination of Equations (47) and (48). Introducing new coefficients "P_n" and "S_n":

$$\left.\begin{aligned} u(\hat{r},\theta,\hat{z},t) &= \hat{r}^{-2}\sin\theta\sum_{n=1}^{\infty}[P_n\exp(i\omega_{pn}t)\sin(\sqrt{\alpha}\eta_n\hat{z}) + S_n\exp(i\omega_{sn}t)\sin(\eta_n\hat{z})] \\ v(\hat{r},\theta,\hat{z},t) &= -\hat{r}^{-2}\cos\theta\sum_{n=1}^{\infty}[P_n\exp(i\omega_{pn}t)\sin(\sqrt{\alpha}\eta_n\hat{z}) + S_n\exp(i\omega_{sn}t)\sin(\eta_n\hat{z})] \\ w(\hat{r},\theta,\hat{z},t) &= (\frac{r_0}{H})\hat{r}^{-1}\sin\theta\sum_{n=1}^{\infty} P_n(-1)^{n+1}\sqrt{\alpha}\eta_n\cos(\sqrt{\alpha}\eta_n)\exp(i\omega_{pn}t)\sin(\eta_n\hat{z}) \end{aligned}\right\} \quad (49)$$

Where

$$\eta_n = (2n-1)\frac{\pi}{2}, \quad \omega_{sn} = \frac{V_s}{H}(2n-1)\frac{\pi}{2}, \quad \omega_{pn} = \frac{V_p}{H}(2n-1)\frac{\pi}{2}$$

The displacement field Equation (49) should satisfy continuity boundary conditions Equation (2). Substituting the third of Equation (49) in the relevant equation of Equation (2) gives:

$$(\frac{r_0}{H})\sin\theta\sum_{n=1}^{\infty} P_n(-1)^{n+1}\sqrt{\alpha}\eta_n\cos(\sqrt{\alpha}\eta_n)\exp(i\omega_{pn}t)\sin(\eta_n\hat{z}) = -r_0\sin\theta Y'(z,t) \quad (50)$$

To determine *"P_n"*, both sides of Equation (50) is multiplied by "$\exp(-i\omega_{pm}t)\sin(\eta_m\hat{z})d\hat{z}dt$" and integrated over "$0\le\hat{z}\le1$" and "$0\le t\le 2\pi/\omega_{pm}$". Here *"m"* is an arbitrary integer. It is known that for "*m≠n*" the integral $\int_0^1\sin(\eta_m\hat{z})\sin(\eta_n\hat{z})d\hat{z} = 0$. Only for "*m=n*" this integral has the value of "1/2". Hence the unknown coefficient *"G_n"* is determined as follows:

$$P_n = \frac{\omega_{pn}(-1)^n}{\pi\sqrt{\alpha}\eta_n\cos(\sqrt{\alpha}\eta_n)}\int_0^{2\pi/\omega_{pn}}\int_0^1 \exp(-i\omega_{pn}t)\frac{\partial Y(\hat{z},t)}{\partial\hat{z}}\sin(\eta_n\hat{z})d\hat{z}dt \quad (51)$$

From the first of the Equation (2), shear waves modal amplitude is derived:

$$\begin{aligned} S_n &= \frac{\omega_{sn}}{\pi}\{\int_0^{2\pi/\omega_{pn}}\int_0^1 e^{-i\omega_{sn}t}Y(\hat{z},t)\sin(\eta_n\hat{z})d\hat{z}dt - \sum_{m=1}^{\infty}P_m\int_0^{2\pi/\omega_{sn}}\int_0^1 e^{i(\omega_{pm}-\omega_{sm})t}\sin(\sqrt{\alpha}\eta_m\hat{z})\sin(\eta_m\hat{z})d\hat{z}dt\} \\ &= \frac{\omega_{sn}}{\pi}\{\int_0^{2\pi/\omega_{pn}}\int_0^1 e^{-i\omega_{sn}t}Y(\hat{z},t)\sin(\eta_n\hat{z})d\hat{z}dt + i\sum_{m=1}^{\infty}P_m(-1)^m\frac{\sqrt{\alpha}\cos(\sqrt{\alpha}\eta_m)(e^{i2\pi\sqrt{\alpha}}-1)}{\omega_{sm}\eta_m(\alpha-1)(\sqrt{\alpha}-1)}\} \end{aligned} \quad (52)$$

It can be seen from the above formulation that generation of pressure waves have two major restriction. The first restriction is material compressibility which is determined by Poisson's ratio. According to Equation (42) the higher the Poisson's ratio of the media the smaller the frequency of P-waves becomes. The second restriction is the amount of vertical deformation that will be induced in the media by lateral movements of the pile. According to the third of Equations (2), magnitude of vertical deformations is related to the slope of the pile axis while vibrating laterally. Magnitude of this slope might be very small compared to the lateral deformations of the pile. Moreover, in many practical cases some degree of slippage between the pile's surface and the media takes place due to partial discontinuities which may occur due to very soft pile's surface or a lack of bounding between the pile's skin and the media. This effect will further reduce the amplitude of P-waves. On the other hand, expression (52) guarantees existence of shear waves with presence of lateral deformation of pile. The lower the amplitude of the P-waves, the higher the amplitude of the shear waves becomes. Another conclusion from Equations (51) and (52) is that both shear waves and pressure waves are complex valued and therefore they have a phase delay compared to the motions of the pile.

4. Effect of Material Damping in Soil Media

Some media like soil show damping characteristics in form of viscous or hysteresis damping. In case of presence of hysteresis damping, Lamé constants can be written in the form of complex valued parameters with imaginary part showing damping characteristics of shear and compressive waves [e.g. 3], i.e. $\lambda^*=\lambda(1+i2D_v)$ and $\mu^*=\mu(1+i2D_s)$, where "D_v" and "D_s" are hysteresis damping ratios associated with volumetric and shear strains respectively. Therefore factors "α", "β" and "ϖ" will be represented by complex valued parameters as follows:

$$\left.\begin{aligned}
&\alpha=\frac{\lambda^*+2\mu^*}{\mu^*}=\frac{1}{1+4D_s^2}\left[\frac{2\nu}{1-2\nu}(1+4D_sD_v)+2(1+4D_s^2)+i\frac{4\nu}{1-2\nu}(D_v-D_s)\right]\\
&\beta=\frac{\lambda^*+2\mu^*}{\lambda^*}=\frac{1}{1+4D_v^2}\left[\frac{1-2\nu}{\nu}(1+4D_sD_v)+1+4D_v^2+i\frac{2(1-2\nu)}{\nu}(D_s-D_v)\right]\\
&\varpi^2=r_0^2\frac{\rho}{\mu^*}\omega^2=\left(\frac{r_0\omega}{V_s}\right)^2\frac{1-i2D_s}{1+4D_s^2}\quad\Rightarrow\quad\varpi=\left(\frac{r_0\omega}{V_s}\right)(a-ib)\\
&a=\sqrt{\frac{1+\sqrt{1+4D_s^2}}{1+4D_s^2}}\quad,\ b=\frac{\sqrt{2}D_s}{\sqrt{(1+4D_s^2)(1+\sqrt{1+4D_s^2})}}
\end{aligned}\right\}\tag{53}$$

Vibration frequencies for a media with hysteresis damping can be obtained by substituting Equation (53) in Equations (44) and (45):

$$\left.\begin{aligned}
&\omega_{sm}=\left(\frac{V_s}{H}\right)(2n-1)(a-ib)\frac{\pi}{2},\ n=1,2,...\\
&\omega_{pn}=\left(\frac{V_p}{H}\right)(2n-1)(a-ib)\frac{\pi}{2},\ n=1,2,...
\end{aligned}\right\}\tag{54}$$

5. Example

A rigid pile is pinned on its tip to the rigid bed, undergoes a known harmonic deformation on its top $\delta(t)=\delta_0\exp(i\omega_0 t)$. Deformation function "$Y(z,t)$" and its derivative can be written as:

$$\begin{cases}Y(z,t)=\dfrac{\delta_0}{H}ze^{i\omega_0 t}=\delta_0\hat{z}e^{i\omega_0 t}\\[2mm]\dfrac{\partial Y(\hat{z},\hat{t})}{\partial\hat{z}}=\delta_0 e^{i\omega_0 t}\end{cases}\tag{55}$$

After substituting (55) in (51) and some simplification the pressure wave's amplitude "P_n" is obtained:

$$P_n=\frac{i(-1)^{n+1}[e^{i2\pi\omega_0/\omega_{pn}}-1]}{\pi\eta_n^2\cos(\sqrt{\alpha}\eta_n)(\omega_0/\omega_{pn}-1)}\delta_0\tag{56}$$

Similarly by substituting (55) in (52), the shear waves amplitude can be written as:

$$S_n=\delta_0\left\{\frac{\eta_n}{\pi^2}\sum_{m=1}^{\infty}\frac{(e^{i2\pi\omega_0/\omega_{pm}}-1)(e^{i2\pi\sqrt{\alpha}}-1)}{\eta_m^4(\omega_0/\omega_{pm}-1)}\frac{\sqrt{\alpha}}{(\alpha-1)(\sqrt{\alpha}-1)}+i\frac{(-1)^{n+1}[e^{i2\pi\omega_0/\omega_{sn}}-1]}{\pi\eta_n^2(\omega_0/\omega_{sn}-1)}\right\}\tag{57}$$

It should be noted that "P_n" approaches a finite limit as vibration frequency of the pile approaches pressure wave frequency. Amplitude of shear waves also approaches finite limits when frequency of vibration of the pile approaches both pressure and shear waves frequencies.

Figure 1 shows variation of amplitudes of pressure and shear waves radiated from a rigid pile in an elastic homogeneous media. Figure 1(a) show maximum pressure wave amplitude for frequency of vibration equal to the pressure wave frequency and zero amplitudes for higher integer multiples. However, according to Figure 1(b), absolute magnitude of the shear waves never vanishes for any frequency of the pile.

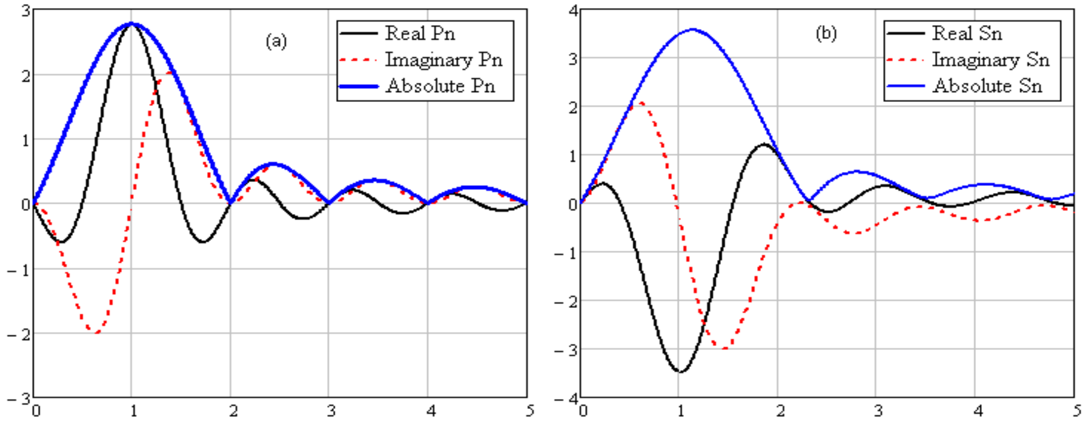

Figure1. Amplitude of (a) pressure waves (b) shear waves radiated from a laterally vibrating rigid cylinder (amplitudes are given per unit lateral deformation of pile top)

6. Summary and Conclusion

A generalized solution to steady state problem of radiated waves from a laterally vibrating pile in elastic semi infinite media with finite thickness is provided. The solution is such that shear waves and pressure waves can be clearly distinguished and their amplitudes determined.

Application of un-stressed (zero traction) free surface conditions lead to determination of certain set of frequencies to which the waves are propagated. These set of frequencies serve as a Fourier expansion basis for radiated waves with arbitrary frequencies and amplitudes.

It is shown that radiation of pressure waves depend on the vertical component of deformations in the media, which in turn relates to the magnitude of slope of the pile's axis in its lateral deflections and the degree of connectivity between the pile and the media. It is argued that if such connectivity is violated due to extra slippery surface of pile or lack of bound between soil and pile surface, formation of pressure waves becomes restricted.

Presence of shear waves is only depend on the lateral movements of the pile and is not affected by the magnitude of vertical component of deformations in soil. It is therefore concluded that shear waves will be radiated by a laterally vibrating pile even where there is complete slippage between pile skin and soil.

Modal amplitudes of both forms of waves are uniquely determined by the continuity conditions between pile and the media at the pile's surface. A simple example of a rigid pile pinned to the bottom with a known harmonic lateral deflection on its top is used to show the real and imaginary parts of waves' amplitudes. It is seen that resonance conditions may occur if frequency of vibration of the pile is equal to pressure or shear waves' first frequencies. However, such resonance conditions will be accompanied with finite amplitudes even in the absence of any kind of material damping due to the presence of the so called radiation damping.

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