Disturbance decoupling by state feedback and PD control law for systems with direct feedthrough matrices

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***Abstract*— In this paper new geometric necessary and sufficient conditions are provided for the solvability of the exact disturbance decoupling problem with a control input consisting of a static state feedback and a PD function of the signal to be rejected, with the requirement of internal stability of the closed loop. Differently from previous results presented in the literature on this issue, we will not restrict our attention to strictly proper systems. Indeed, both feedthrough matrices from the control and from the disturbance to the output will be considered to be possibly non-zero.**

I. Introduction and problem statement

In the last thirtyfive years, the so-called *geometric approach* to systems and control theory has provided valuable tools for the understanding of several system-theoretic properties of LTI models and for the solution of many control and estimation problems. The first and most famous problem that was studied and solved with geometric techniques by Basile and Marro in their first pioneering paper on the geometric approach [2] was the disturbance decoupling by state feedback.

Afterwards, many different versions of this problem have been investigated with geometric tools, e.g., the disturbance decoupling with stability of the closed-loop [16], the decoupling of measurable input functions [6] and the disturbance decoupling by dynamic output feedback [14].

The problem dealt with in this paper is the decoupling of an input signal by means of a control function involving a static state feedback and a feedforward PD function of the signal to be rejected, which can be precisely formulated as follows. Consider the LTI system

$$\begin{cases} \rho x(t) = A x(t) + B u(t) + H w(t), \\ y(t) = C x(t) + D u(t) + G w(t), \end{cases} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $H \in \mathbb{R}^{n\times r}$, $C \in \mathbb{R}^{p\times n}$, and the feedthrough matrices $D \in \mathbb{R}^{p\times m}$ and $G \in \mathbb{R}^{p\times r}$ are possibly non-zero. In what follows, whether the underlying system (1) evolves in continuous or discrete time is irrelevant and, accordingly, the time index set of any signal is denoted by $\mathbb{T}$, on the understanding that this represents either $\mathbb{R}^+$ in the continuous time or $\mathbb{N}$ in the discrete time. The symbol $\mathbb{C}_g$ denotes either the open left-half complex plane $\mathbb{C}^-$ in the continuous time or the open unit disc $\mathbb{C}^\circ$ in the discrete time. The operator ρ denotes either the time derivative in the continuous time, i.e., $\rho x(t) = \dot{x}(t)$, or the unit time shift in the discrete time, i.e., $\rho x(t) = x(t+1)$. In (1), for all $t \in \mathbb{T}$, $x(t) \in \mathbb{R}^n$ represents the state, $u(t) \in \mathbb{R}^m$ the control input and $w(t) \in \mathbb{R}^r$ the disturbance input to be decoupled from the output $y(t) \in \mathbb{R}^p$. From now on, we identify the *undisturbed system* characterized by the quadruple (A,B,C,D) with the symbol Σ, and the *disturbed system* characterized by the quadruple $(A,[B\ H],C,[D\ G])$ with the symbol Σ_d.

The purpose of this paper is finding necessary and sufficient conditions for the solution of the disturbance decoupling with internal stability of the closed-loop using a control law involving a state feedback and a PD function of the disturbance of a prescribed degree $q \in \mathbb{N}$, i.e., of the form

$$u(t) = F x(t) + \sum_{i=0}^{q} S_i \rho^i w(t), \tag{2}$$

where $F \in \mathbb{R}^{m\times n}$ and $S_i \in \mathbb{R}^{m\times r}$ for $i \in \{0,1,\ldots,q\}$, under the standing assumption that $\rho^i w(t)$ is bounded for all $i \in \{0,1,\ldots,q\}$. Our objective can be stated in more precise terms as follows.

Problem 1: Find necessary and sufficient conditions for the existence of a control law (2) such that:

1) for all initial conditions and all $w(t)$ the output $y(t)$ converges to zero as t approaches infinity;
2) the poles of the closed-loop are all stable, i.e., $\sigma(A+BF) \subset \mathbb{C}_g$.

In the discrete case, this problem is often referred to as the *previewed signal decoupling*, see e.g. [1]. A first solution to this problem with $D = 0_{p\times m}$ and $G = 0_{p\times r}$ was given by Willems in [13] in terms of the geometric notion of *almost invariance* (see also [12] for further details). In that paper, for the problem of the existence of an integer $q \in \mathbb{N}$ for which Problem 1 admits solutions the condition $\operatorname{im} H \subseteq \mathscr{V}^\star + \mathscr{S}^\star$ was proposed, where $\mathscr{V}^\star := \sup\{\mathscr{V} \subseteq \ker C \mid A\mathscr{V} \subseteq \mathscr{V} + \operatorname{im} B\}$ and $\mathscr{S}^\star := \inf\{\mathscr{S} \supseteq \operatorname{im} B \mid A(\mathscr{S} \cap \ker C) \subseteq \mathscr{S}\}$. Under the requirement of pole placement of the closed loop, such condition becomes $\operatorname{im} H \subseteq \mathscr{S}^\star$. The second fundamental contribution on this topic was given by Bonilla Estrada and Malabre in [7], by taking into account the more specific requirement of stability of the closed-loop system. In particular, in the case when D and G are both zero, a necessary and sufficient condition for the solvability of Problem 1 was presented, which was concisely expressed by means of the geometric inclusion $\operatorname{im} H \subseteq \mathscr{V}_g^\star + \mathscr{S}_{q+1}$, where $\mathscr{V}_g^\star := \sup\{\mathscr{V} \subseteq \ker C \mid \exists F \in \mathbb{R}^{m\times n} : (A+BF)\mathscr{V} \subseteq \mathscr{V}$ and[1]

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[1] If $A : \mathscr{X} \longrightarrow \mathscr{Y}$ and $\mathscr{J} \subseteq \mathscr{X}$, the restriction of the map A to $\mathscr{J}$ is denoted by $A|\mathscr{J}$. If $\mathscr{X} = \mathscr{Y}$ and $\mathscr{J}$ is A-invariant, the eigenvalues of A restricted to $\mathscr{J}$ are denoted by $\sigma(A|\mathscr{J})$.

$\sigma(A+BF\,|\,\mathscr{V})\subset\mathbb{C}_g\}$ and $\mathscr{S}_{q+1}$ is the $(q+1)$-th term of the sequence of subspaces $\mathscr{S}_0=\mathbf{0}_n$ and $\mathscr{S}_i=\operatorname{im}B+A(\mathscr{S}_{i-1}\cap\ker C)$ for $i\in\mathbb{N}\setminus\{0\}$.
An alternative solution to the same problem, (under the same standing assumption of zero feedthrough matrices) was presented by Barbagli, Marro and Prattichizzo in [1], relying on the geometric concept of self-bounded controlled invariance. By defining with $\Phi(\Sigma)$ the lattice of *self-bounded controlled invariant subspaces* of Σ contained in the null-space of C, i.e.,

$$\Phi(\Sigma):=\{\mathscr{V}\subseteq\ker C\,|\,A\mathscr{V}\subseteq\mathscr{V}+\operatorname{im}B\text{ and }\mathscr{V}\supseteq\mathscr{V}^\star\cap\operatorname{im}B\},$$

(see [3], [9] and [4, p.207] for details), the conditions of solvability of the disturbance decoupling problem with stability by PD action were expressed in terms of:

1) a geometric *structural* condition
$$\operatorname{im}H\subseteq\mathscr{V}^\star+\mathscr{S}_{q+1};\tag{3}$$
2) the so-called *stability condition*
$$\mathscr{V}_m:=\min\Phi(\Sigma_d)\quad\text{is internally stabilizable},\tag{4}$$
where $\mathscr{V}_m$ is the smallest element of the lattice $\Phi(\Sigma_d)$, this second condition meaning that a feedback matrix F exists such that $(A+BF)\,\mathscr{V}_m\subseteq\mathscr{V}_m$ and the eigenvalues of $A+BF$ restricted to $\mathscr{V}_m$ are stable, in symbols $\sigma(A+BF\,|\,\mathscr{V}_m)\subset\mathbb{C}_g$.

Checking the solvability of the decoupling problem through these conditions enables the burden deriving from the computation of eigenspaces in the determination of $\mathscr{V}_g^\star$ to be avoided, as pointed out in [5]. Indeed, $\mathscr{V}_m$ can be computed as the intersection $\mathscr{V}_m=\mathscr{V}^\star\cap\tilde{\mathscr{S}}$, where $\tilde{\mathscr{S}}$ is the limit of the sequence $\tilde{\mathscr{S}}_0=\mathbf{0}_n$ and $\tilde{\mathscr{S}}_i=\operatorname{im}[B\ H]+A(\tilde{\mathscr{S}}_{i-1}\cap\ker C)$, $i\in\mathbb{N}\setminus\{0\}$, which converges in at most n steps; hence, only the basic tools and algorithms of the geometric approach are necessary for the implementation of such solution. Moreover, the dimension of the resolvent subspace $\mathscr{V}_m$ is smaller than that of $\mathscr{V}_g^\star$ in the solution of [7], since in general $\mathscr{V}_m\subseteq\mathscr{V}_g$ holds. In the same paper, a hint on how to tackle this problem when the feedthrough matrices are not zero has been proposed, relying on the contrivance of solving a slightly modified decoupling problem, in which an integrator in the continuous time or a unit delay in the discrete time, described by

$$\begin{cases}\rho z(t)=y(t),\\ h(t)=z(t),\end{cases}$$

is inserted at the output of the original system and by solving the same decoupling problem with respect to the overall strictly proper system

$$\begin{cases}\rho s(t)=\widehat{A}s(t)+\widehat{B}u(t)+\widehat{H}w(t),\\ h(t)=\widehat{C}s(t),\end{cases}\tag{5}$$

where $s(t):=\begin{bmatrix}x(t)\\ z(t)\end{bmatrix}$, $\widehat{A}:=\begin{bmatrix}A & 0_{n\times p}\\ C & 0_{p\times p}\end{bmatrix}$, $\widehat{B}:=\begin{bmatrix}B\\ D\end{bmatrix}$, $\widehat{H}:=\begin{bmatrix}H\\ G\end{bmatrix}$, $\widehat{C}:=\begin{bmatrix}0_{p\times n} & I_{p\times p}\end{bmatrix}$. In fact, the following straightforward result holds.

Lemma 1: Problem 1 stated with respect to the non-strictly proper system Σ is equivalent to Problem 1 stated with respect to the strictly proper system $\widehat{\Sigma}:=\left(\widehat{A},\begin{bmatrix}\widehat{B} & \widehat{H}\end{bmatrix},\widehat{C}\right)$.
On the one hand, no further details were given in [1] concerning the structure of the solvability conditions in this case. On the other hand, the exploitation of this dummy integrator or delay unit, along with the material presented in [1], would lead to solvability conditions stated in terms of the fictitious variable $z(t)$. Hence, the purpose of this paper is to address this issue, by determining necessary and sufficient conditions for the solvability of Problem 1 expressed in terms of characteristic subspaces of the original system, and not of the extended system including the dummy unit (5). Hence, the main result of this paper is Theorem 1 in Section III, where, in the general case of D and G possibly non-zero, new necessary and sufficient conditions written in terms of the problem data are presented for the solvability of Problem 1, which cannot be trivially deducted from those expressed in terms of the matrices of the extended model $\widehat{\Sigma}$. It will be shown that the structural solvability condition proposed here encompasses those presented in [13], [7], [1] when the feedthrough matrices D and G are both zero.
Notice that the present is not the first attempt to extend geometric results and techniques to non-strictly proper system (see e.g. [8] and [10] for the model matching problem and for the disturbance decoupling with measurement feedback and internal stability of the closed loop, respectively). However, to date a similar extension to the decoupling with state-feedback and PD feedforward action has been neglected.

II. Geometric preliminaries

In this section some fundamental definitions and results of the geometric approach which are used in the sequel are recalled (for a detailed discussion on these topics we refer to [4], [11], [15]). First, recall that an *output-nulling subspace* $\mathscr{V}$ of $\Sigma=(A,B,C,D)$ is a subspace of $\mathbb{R}^n$ satisfying

$$\begin{bmatrix}A\\ C\end{bmatrix}\mathscr{V}\subseteq(\mathscr{V}\times\mathbf{0}_p)+\operatorname{im}\begin{bmatrix}B\\ D\end{bmatrix}.\tag{6}$$

The set of output-nulling subspaces $\mathscr{V}(\Sigma)$ of Σ is an upper semi-lattice with respect to subspace addition. Thus, the sum of all the output-nulling subspaces of Σ is the largest output-nulling subspace of Σ, and will be denoted by $\mathscr{V}^\star$. For the case where the feedthrough matrix D is zero, the output-nulling subspace coincides with the controlled invariant subspace contained in the null-space of matrix C introduced by Basile and Marro in [2]. In the following lemma, the most important properties of the output-nulling subspaces are presented.

Lemma 2: The following results hold:

- The subspace $\mathscr{V}$ is output-nulling for Σ if and only if a matrix $F\in\mathbb{R}^{m\times n}$ exists such that
$$(A+BF)\,\mathscr{V}\subseteq\mathscr{V}\subseteq\ker(C+DF).\tag{7}$$

- The sequence of subspaces $(\mathscr{V}_i)_{i\in\mathbb{N}}$ described by the recurrence

$$\begin{cases} \mathscr{V}_0 = \mathbb{R}^n, \\ \mathscr{V}_i = \begin{bmatrix} A \\ C \end{bmatrix}^{-1} \Big((\mathscr{V}_{i-1}\times \mathbf{0}_p) + \operatorname{im} \begin{bmatrix} B \\ D \end{bmatrix} \Big), \quad i>0, \end{cases} \tag{8}$$

is monotonically non-increasing. Moreover, there exists $k \le n-1$ such that $\mathscr{V}_{k+1}=\mathscr{V}_k$. For such k the identity $\mathscr{V}^\star=\mathscr{V}_k$ holds.

Any matrix F satisfying (7) will be referred to as a *friend* of the output-nulling subspace $\mathscr{V}$. We denote by $\mathfrak{F}_\Sigma(\mathscr{V})$ the set of friends of the output-nulling subspace $\mathscr{V}$. Let $F\in\mathfrak{F}_\Sigma(\mathscr{V})$: the *output-nulling reachability subspace* $\mathscr{R}_\mathscr{V}$ on $\mathscr{V}$ is the smallest $(A+BF)$-invariant subspace of $\mathbb{R}^n$ containing the subspace $\mathscr{V}\cap B\ker D$. We denote by $\mathscr{R}^\star$ the output-nulling reachability subspace on $\mathscr{V}^\star$, i.e., $\mathscr{R}^\star := <A+BF, \mathscr{V}^\star\cap B\ker D>$. For $F\in\mathfrak{F}_\Sigma(\mathscr{V})$, the eigenvalues of $(A+BF)$ restricted to $\mathscr{V}$, i.e. $\sigma(A+BF\,|\,\mathscr{V})$, can be split into two sets: the eigenvalues of $(A+BF\,|\,\mathscr{R}_\mathscr{V})$ are all freely assignable by a suitable choice of the friend F of $\mathscr{V}$ in $\mathfrak{F}_\Sigma(\mathscr{V})$ (provided that the eigenvalues to be assigned are mirrored with respect to the real axis). The eigenvalues $\sigma(A+BF \mid \mathscr{V}/\mathscr{R}_\mathscr{V})$ on the contrary do not depend on the choice of F; if $\sigma(A+BF \mid \mathscr{V}/\mathscr{R}_\mathscr{V})\subset\mathbb{C}_g$, the output-nulling $\mathscr{V}$ is said to be *internally stabilizable*. The dual concept is the input-containing subspace: a subspace $\mathscr{S}$ of $\mathbb{R}^n$ is said to be *input-containing* if

$$\begin{bmatrix} A & B \end{bmatrix} \Big((\mathscr{S}\times\mathbb{R}^m)\cap\ker\begin{bmatrix} C & D \end{bmatrix} \Big) \subseteq \mathscr{S}. \tag{9}$$

The set of all input-containing subspaces of Σ is a lower semi-lattice with respect to subspace intersection. The intersection of all input-containing subspaces of Σ is therefore the smallest input-containing subspace of Σ, and will be denoted by $\mathscr{S}^\star$. In the case when D is zero, $\mathscr{S}^\star$ reduces to the smallest (A,C)-conditioned invariant subspace containing the range of B, [2]. The counterpart of Lemma 1 for input-containing subspaces clearly holds; however, here we are only interested in the dual of property (8) in Lemma 2, which is as follows.

Lemma 3: The sequence of subspaces $(\mathscr{S}_i)_{i\in\mathbb{N}}$ described by the recurrence

$$\begin{cases} \mathscr{S}_0 = \mathbf{0}_n, \\ \mathscr{S}_i = \begin{bmatrix} A & B \end{bmatrix} \Big((\mathscr{S}_{i-1}\times\mathbb{R}^m)\cap\ker\begin{bmatrix} C & D \end{bmatrix} \Big), \quad i>0, \end{cases} \tag{10}$$

is monotonically non-decreasing. Moreover, there exists $k\le n-1$ such that $\mathscr{S}_{k+1}=\mathscr{S}_k$. For such k there holds $\mathscr{S}^\star=\mathscr{S}_k$.

III. Solution of Problem 1

The following lemma and corollary will be useful in the determination of a structural condition for the solution of Problem 1. As a second step, the stability requirement will be taken into account.

Lemma 4: Consider the sequence $(\mathscr{S}_i)_{i\in\mathbb{N}}$ given by (10) and the sequence $(\hat{\mathscr{S}}_i)_{i\in\mathbb{N}}$ described by

$$\begin{cases} \hat{\mathscr{S}}_0 = \mathbf{0}_{n+p}, \\ \hat{\mathscr{S}}_i = \operatorname{im}\widehat{B} + \widehat{A}(\hat{\mathscr{S}}_{i-1}\cap\ker\widehat{C}), \qquad i\in\mathbb{N}\setminus\{0\}, \end{cases} \tag{11}$$

where $\widehat{A},\widehat{B}$ and $\widehat{C}$ have been defined in Section I. For all $i\in\mathbb{N}$ there holds

$$\hat{\mathscr{S}}_i \cap \operatorname{im}\begin{bmatrix} I_n \\ 0_{p\times n} \end{bmatrix} = \mathscr{S}_i\times\mathbf{0}_p. \tag{12}$$

Proof: We proceed by induction. When $i=0$, the statement is clearly true. Let us now suppose that it holds for a given $i-1$, and let us prove the same fact for i. By definition of $\widehat{C}$ and by the inductive assumption it is found that

$$\begin{aligned} \hat{\mathscr{S}}_i &= \operatorname{im}\widehat{B}+\widehat{A}(\hat{\mathscr{S}}_{i-1}\cap\ker\widehat{C}) \\ &= \operatorname{im}\widehat{B}+\widehat{A}(\hat{\mathscr{S}}_{i-1}\cap\operatorname{im}\begin{bmatrix} I_n \\ 0_{p\times n}\end{bmatrix}) \\ &= \operatorname{im}\widehat{B}+\widehat{A}(\mathscr{S}_{i-1}\times\mathbf{0}_p) = \operatorname{im}\widehat{B}+\begin{bmatrix} A \\ C\end{bmatrix}\mathscr{S}_{i-1} \end{aligned}$$

Hence, we have shown that

$$\hat{\mathscr{S}}_i\cap\operatorname{im}\begin{bmatrix} I_n \\ 0_{p\times n}\end{bmatrix} = \Big(\operatorname{im}\widehat{B}+\begin{bmatrix} A \\ C\end{bmatrix}\mathscr{S}_{i-1}\Big)\cap\operatorname{im}\begin{bmatrix} I_n \\ 0_{p\times n}\end{bmatrix}.$$

Now, we want to prove that, given a matrix basis Z of $\mathscr{S}_{i-1}$, there holds

$$\begin{aligned} &\Big(\operatorname{im}\widehat{B}+\begin{bmatrix} A \\ C\end{bmatrix}\mathscr{S}_{i-1}\Big)\cap\operatorname{im}\begin{bmatrix} I_n \\ 0_{p\times n}\end{bmatrix} = \\ &\qquad = \begin{bmatrix} AZ & B\end{bmatrix}\ker\begin{bmatrix} CZ & D\end{bmatrix}\times\mathbf{0}_p. \end{aligned} \tag{13}$$

To this end, let $\begin{bmatrix} x \\ y\end{bmatrix}\in\Big(\operatorname{im}\widehat{B}+\begin{bmatrix} A \\ C\end{bmatrix}\mathscr{S}_{i-1}\Big)\cap\operatorname{im}\begin{bmatrix} I_n \\ 0_{p\times n}\end{bmatrix}$, such that $x\in\mathbb{R}^n$ and $y\in\mathbb{R}^p$. Hence, on the one hand, two vectors k_1 and k_2 exist such that $\begin{bmatrix} x \\ y\end{bmatrix} = \begin{bmatrix} A \\ C\end{bmatrix} Z k_1 + \begin{bmatrix} B \\ D\end{bmatrix} k_2$ and, on the other, $y=0$, the two leading to $\begin{bmatrix} CZ & D\end{bmatrix}\begin{bmatrix} k_1 \\ k_2\end{bmatrix}=0$. As a result, we find $x=\begin{bmatrix} AZ & B\end{bmatrix}\begin{bmatrix} k_1 \\ k_2\end{bmatrix}\in\begin{bmatrix} AZ & B\end{bmatrix}\ker\begin{bmatrix} CZ & D\end{bmatrix}$. It is easy to establish that these steps can be performed in the reversed order, hence (13) holds with the equality sign. The last fact to be proved is that

$$\mathscr{S}_i\times\mathbf{0}_p = \begin{bmatrix} AZ & B\end{bmatrix}\ker\begin{bmatrix} CZ & D\end{bmatrix}\times\mathbf{0}_p, \tag{14}$$

so that (13) and (14) yield (12). To this end, let $\xi\in\mathscr{S}_i\times\mathbf{0}_p = \begin{bmatrix} A & B\end{bmatrix}\Big((\mathscr{S}_{i-1}\times\mathbb{R}^m)\cap\ker\begin{bmatrix} C & D\end{bmatrix}\Big)$: a vector $\begin{bmatrix} l_1 \\ l_2\end{bmatrix}$ exists such that $\xi = A l_1 + B l_2$, where $\begin{bmatrix} l_1 \\ l_2\end{bmatrix}\in\mathscr{S}_{i-1}\times\mathbb{R}^m$ and $\begin{bmatrix} C & D\end{bmatrix}\begin{bmatrix} l_1 \\ l_2\end{bmatrix}=0$. Hence, since $l_1\in\mathscr{S}_{i-1}$, we can define $\tilde{l}_1$ such that $l_1 = Z\tilde{l}_1$. It follows that $\begin{bmatrix} \tilde{l}_1 \\ l_2\end{bmatrix}\in\ker\begin{bmatrix} CZ & D\end{bmatrix}$, so that $\xi = \begin{bmatrix} AZ & B\end{bmatrix}\begin{bmatrix} \tilde{l}_1 \\ l_2\end{bmatrix}\in\begin{bmatrix} AZ & B\end{bmatrix}\ker\begin{bmatrix} CZ & D\end{bmatrix}$. Again, by performing the same steps in the reversed order, (14) follows. This completes the proof. ■

Corollary 1: Consider the sequence $(\mathscr{S}_i)_{i\in\mathbb{N}}$ in (10) and the sequence $(\hat{\mathscr{S}}_i)_{i\in\mathbb{N}}$ in (11). Then

$$\hat{\mathscr{S}}_i = \text{im} \begin{bmatrix} B \\ D \end{bmatrix} + \begin{bmatrix} A \\ C \end{bmatrix} \mathscr{S}_{i-1} \qquad \forall\, i \in \mathbb{N}\setminus\{0\}. \qquad (15)$$

Proof: The proof follows straightforwardly from Lemma 4. Indeed,

$$\begin{aligned}\hat{\mathscr{S}}_i &= \text{im} \begin{bmatrix} B \\ D \end{bmatrix} + \begin{bmatrix} A & 0_{n\times p} \\ C & 0_{p\times p} \end{bmatrix} \left(\hat{\mathscr{S}}_{i-1} \cap \text{im} \begin{bmatrix} I_n \\ 0_{p\times n} \end{bmatrix}\right) \\ &= \text{im} \begin{bmatrix} B \\ D \end{bmatrix} + \begin{bmatrix} A & 0_{n\times p} \\ C & 0_{p\times p} \end{bmatrix} (\mathscr{S}_{i-1} \times \mathbf{0}_p) \\ &= \text{im} \begin{bmatrix} B \\ D \end{bmatrix} + \begin{bmatrix} A \\ C \end{bmatrix} \mathscr{S}_{i-1}.\end{aligned}$$

■

Now we introduce a lattice that will play a key role in the determination of a stability condition for the solvability of Problem 1. Let the set $\tilde{\Phi}(\Sigma)$ be defined as

$$\tilde{\Phi}(\Sigma) := \left\{ \mathscr{V} \in \mathscr{V}(\Sigma) \,\middle|\, \mathscr{V} \times \mathbf{0}_p \supseteq (\mathscr{V}^\star \times \mathbf{0}_p) \cap \text{im} \begin{bmatrix} B \\ D \end{bmatrix} \right\}.$$

The following preliminary result holds.

Lemma 5: The identities $\mathscr{V}(\widehat{\Sigma}) = \{\mathscr{V} \times \mathbf{0}_p \mid \mathscr{V} \in \mathscr{V}(\Sigma)\}$ and $\Phi(\widehat{\Sigma}) = \{\mathscr{V} \times \mathbf{0}_p \mid \mathscr{V} \in \tilde{\Phi}(\Sigma)\}$ hold.

Proof: It is first proved that $\mathscr{V}(\widehat{\Sigma}) \subseteq \{\mathscr{V} \times \mathbf{0}_p \mid \mathscr{V} \in \mathscr{V}(\Sigma)\}$. Let $\hat{\mathscr{V}} := \mathscr{V}_1 \times \mathscr{V}_2 \in \mathscr{V}(\widehat{\Sigma})$ be such that $\mathscr{V}_1$ and $\mathscr{V}_2$ are subspaces of $\mathbb{R}^n$ and $\mathbb{R}^p$, respectively. Since $\mathscr{V}_1 \times \mathscr{V}_2 \subseteq \ker \widehat{C} = \ker \begin{bmatrix} 0_{p\times n} & I_{p\times p} \end{bmatrix}$, it follows that $\mathscr{V}_2 = \mathbf{0}_p$. From $\widehat{A}(\mathscr{V}_1 \times \mathbf{0}_p) \subseteq (\mathscr{V}_1 \times \mathbf{0}_p) + \text{im}\,\widehat{B}$ it follows that $\begin{bmatrix} A \\ C \end{bmatrix} \mathscr{V}_1 \subseteq (\mathscr{V}_1 \times \mathbf{0}_p) + \text{im} \begin{bmatrix} B \\ D \end{bmatrix}$, so that $\mathscr{V}_1 \in \mathscr{V}(\Sigma)$. The opposite inclusion, i.e., $\mathscr{V}(\widehat{\Sigma}) \supseteq \{\mathscr{V} \times \mathbf{0}_p \mid \mathscr{V} \in \mathscr{V}(\Sigma)\}$, is now straightforward, and can be easily proved by following these steps backwards. As a consequence, it is now easily seen that $\hat{\mathscr{V}}^\star := \max \Phi(\widehat{\Sigma})$ can be written as $\mathscr{V}^\star \times \mathbf{0}_p$, where $\mathscr{V}^\star := \max \tilde{\Phi}(\Sigma)$.
Let us now prove that $\Phi(\widehat{\Sigma}) \subseteq \{\mathscr{V} \times \mathbf{0}_p \mid \mathscr{V} \in \tilde{\Phi}(\Sigma)\}$. Let $\hat{\mathscr{V}} = \mathscr{V}_1 \times \mathscr{V}_2$ be an element of $\Phi(\widehat{\Sigma})$. Since $\mathscr{V}_1 \times \mathscr{V}_2 \in \mathscr{V}(\widehat{\Sigma})$, as already seen $\mathscr{V}_2 = \mathbf{0}_p$ and $\mathscr{V}_1 \in \mathscr{V}(\Sigma)$. Now, since $\hat{\mathscr{V}}^\star = \mathscr{V}^\star \times \mathbf{0}_p$, from $\hat{\mathscr{V}} \supseteq \hat{\mathscr{V}}^\star \cap \text{im} \begin{bmatrix} B \\ D \end{bmatrix}$ it follows that $\mathscr{V}_1 \times \mathbf{0}_p \supseteq (\mathscr{V}^\star \times \mathbf{0}_p) \cap \text{im} \begin{bmatrix} B \\ D \end{bmatrix}$, so that $\mathscr{V}_1 \in \tilde{\Phi}(\Sigma)$. Again, the opposite inclusion $\Phi(\widehat{\Sigma}) \supseteq \{\mathscr{V} \times \mathbf{0}_p \mid \mathscr{V} \in \tilde{\Phi}(\Sigma)\}$ is now straightforward. ■

Corollary 2: The set $(\tilde{\Phi}(\Sigma), +, \cap; \subseteq)$ is a non-distributive modular lattice. It admits a maximum element, which is $\mathscr{V}^\star$, and a minimum element, which is $\mathscr{V}^\star \cap \mathscr{S}^\star$.

Proof: The proof is a direct consequence of Lemma 5, since $(\Phi(\widehat{\Sigma}), +, \cap; \subseteq)$ is a non-distributive modular lattice, [4, p.207]. Moreover, by Lemma 5 it follows that $\mathscr{V}^\star = \max \tilde{\Phi}(\Sigma)$ if and only if $\mathscr{V}^\star \times \mathbf{0}_p = \max \Phi(\widehat{\Sigma})$. We only have to show that $\min \Phi(\Sigma) = \mathscr{V}^\star \cap \mathscr{S}^\star$. It is a well-known fact that $\min \Phi(\widehat{\Sigma}) = \hat{\mathscr{V}}^\star \cap \hat{\mathscr{S}}^\star$, where $\hat{\mathscr{V}}^\star$ and $\hat{\mathscr{S}}^\star$ are respectively the largest controlled invariant subspace in $(\widehat{A}, \widehat{B})$ contained in the null space of $\widehat{C}$ and the smallest conditioned invariant subspace in $(\widehat{A}, \widehat{C})$ containing the image of $\widehat{B}$, see e.g. [4, Theorem 4.1.4]. By Lemma 4 it follows that

$$\begin{aligned}\min \Phi(\widehat{\Sigma}) &= (\mathscr{V}^\star \times \mathbf{0}_p) \cap \hat{\mathscr{S}}^\star \\ &= (\mathscr{V}^\star \times \mathbf{0}_p) \cap \text{im} \begin{bmatrix} I_n \\ 0_{p\times n} \end{bmatrix} \cap \hat{\mathscr{S}}^\star \\ &= (\mathscr{V}^\star \times \mathbf{0}_p) \cap (\mathscr{S}^\star \times \mathbf{0}_p) = (\mathscr{V}^\star \cap \mathscr{S}^\star) \times \mathbf{0}_p.\end{aligned}$$

As a consequence of Lemma 5, it is found that $\min \tilde{\Phi}(\Sigma)$ is indeed $\mathscr{V}^\star \cap \mathscr{S}^\star$. ■

Theorem 1: Let the pair (A, B) be stabilizable. Problem 1 is solvable if and only if

1) $\text{im} \begin{bmatrix} H \\ G \end{bmatrix} \subseteq (\mathscr{V}^\star \times \mathbf{0}_p) + \text{im} \begin{bmatrix} B \\ D \end{bmatrix} + \begin{bmatrix} A \\ C \end{bmatrix} \mathscr{S}_q$,
2) $\mathscr{V}_m := \min \tilde{\Phi}(\Sigma_d)$ is internally stabilizable.

Proof: By virtue of Lemma 1 and by taking [1, Theorem 3.2] into account, it is easily found that Problem 1 admits solutions if and only if $\text{im}\,\widehat{H} \subseteq \hat{\mathscr{V}}^\star + \hat{\mathscr{S}}_{q+1}$, where $\hat{\mathscr{V}}^\star = \max \mathscr{V}(\widehat{\Sigma})$ and $\hat{\mathscr{S}}_{q+1}$ is the $(q+1)$-th element of the sequence (11), and $\hat{\mathscr{V}}_m$ is internally stabilizable, where $\hat{\mathscr{V}}_m = \min \Phi(\widehat{\Sigma})$. By virtue of Corollary 1 the inclusion $\text{im}\,\widehat{H} \subseteq \hat{\mathscr{V}}^\star + \hat{\mathscr{S}}_{q+1}$ is equivalent to the structural condition *1)*. Hence, our aim now is to show that $\hat{\mathscr{V}}_m$ is internally stabilizable with respect to $\widehat{\Sigma}$ if and only if such is $\mathscr{V}_m$ with respect to system Σ. Let $\hat{\mathscr{V}}_m = \mathscr{V}_m \times \mathbf{0}_p$ be internally stabilizable with respect to $\widehat{\Sigma}$. Let $\widehat{F} = \begin{bmatrix} F_1 & F_2 \end{bmatrix} \in \mathfrak{F}_{\widehat{\Sigma}}(\mathscr{V}_m \times \mathbf{0}_p)$ be such that $\sigma(\widehat{A} + \widehat{B}\widehat{F} \,|\, \mathscr{V}_m \times \mathbf{0}_p) \subset \mathbb{C}_g$. The inclusion

$$\begin{aligned}(\widehat{A} + \widehat{B}\widehat{F})(\mathscr{V}_m \times \mathbf{0}_p) &= \begin{bmatrix} A + BF_1 & BF_2 \\ C + DF_1 & DF_2 \end{bmatrix} (\mathscr{V}_m \times \mathbf{0}_p) \\ &\subseteq \mathscr{V}_m \times \mathbf{0}_p,\end{aligned} \qquad (16)$$

can be written as $\begin{bmatrix} A + BF_1 & BF_2 \\ C + DF_1 & DF_2 \end{bmatrix} \begin{bmatrix} V \\ 0 \end{bmatrix} = \begin{bmatrix} V \\ 0 \end{bmatrix} X$, for a suitable matrix $X \in \mathbb{R}^{h\times h}$, where h is the dimension of $\mathscr{V}_m$ and where V is a basis matrix of $\mathscr{V}_m$, and the eigenvalues of X are the internal eigenvalues of $\mathscr{V}_m \times \mathbf{0}_p$ with respect to the feedback matrix $\widehat{F}$. Since $\widehat{F}$ is stabilizing for $\mathscr{V}_m \times \mathbf{0}_p$, it follows that $\sigma(X) = \sigma(\widehat{A} + \widehat{B}\widehat{F} \,|\, \mathscr{V}_m \times \mathbf{0}_p) \subset \mathbb{C}_g$. Thus, equation (16) yields $\begin{bmatrix} (A + BF_1)V \\ (C + DF_1)V \end{bmatrix} = \begin{bmatrix} VX \\ 0 \end{bmatrix}$, so that $F_1 \in \mathfrak{F}_\Sigma(\mathscr{V}_m)$ and $\sigma(A + BF_1 \,|\, \mathscr{V}_m) = \sigma(X) \subset \mathbb{C}_g$. As a result, $\mathscr{V}_m$ is stabilizable with respect to Σ.
These steps can be performed backwards. Indeed, if $\mathscr{V}_m$ is internally stabilizable with respect to Σ and we call F the friend of $\mathscr{V}_m$ such that $\sigma(A + BF \,|\, \mathscr{V}_m) \subset \mathbb{C}_g$, then $\hat{\mathscr{V}}_m = \mathscr{V}_m \times \mathbf{0}_p$ is stabilizable for $\widehat{\Sigma}$, since $\widehat{F} := \begin{bmatrix} F & 0_{m\times p} \end{bmatrix} \in \mathfrak{F}_{\widehat{\Sigma}}(\hat{\mathscr{V}}_m)$ and $\sigma(\widehat{A} + \widehat{B}\widehat{F} \,|\, \hat{\mathscr{V}}_m) \subset \mathbb{C}_g$. This completes the proof. ■

Remark 1: The stability condition *2)* in Theorem 1 is very easy to check from a computational point of view: indeed, if we denote by $\mathscr{V}_d^\star$ and by $\mathscr{S}_d^\star$ the largest output-nulling and the smallest input-containing subspaces of Σ_d, respectively, by (2) it is found that $\mathscr{V}_m = \min \tilde{\Phi}(\Sigma_d) = \mathscr{V}_d^\star \cap \mathscr{S}_d^\star$.

Remark 2: Notice that when the information available on the disturbance $w(t)$ consists of its sole current measure, i.e., when the control law is in the form $u(t) = F\,x(t) + S\,w(t)$, the structural condition *1)* in Theorem 1 reduces to the relation

$$\mathrm{im}\begin{bmatrix} H \\ G \end{bmatrix} \subseteq (\mathscr{V}^\star \times \mathbf{0}_p) + \mathrm{im}\begin{bmatrix} B \\ D \end{bmatrix},$$

in this case $\mathscr{S}_q = \mathscr{S}_0$ being the origin. Hence, as a particular case the presented approach solves the so-called *measurable signal decoupling problem*, [6], when the feedthough matrices are possibly non-zero.

It is also worth noticing that *1)* encompasses the classical condition (3) when both the feedthrough matrices D and G are zero. In fact, it can be proved that the inclusion

$$\mathrm{im}\begin{bmatrix} H \\ 0_{p\times r} \end{bmatrix} \subseteq (\mathscr{V}^\star \times \mathbf{0}_p) + \mathrm{im}\begin{bmatrix} B \\ 0_{p\times m} \end{bmatrix} + \mathrm{im}\begin{bmatrix} A \\ C \end{bmatrix}\mathscr{S}_q \quad (17)$$

is equivalent to $\mathrm{im}\,H \subseteq \mathscr{V}^\star + \mathscr{S}_{q+1}$. Suppose in fact that H satisfies (17), and let V and Z be basis matrices of $\mathscr{V}^\star$ and $\mathscr{S}_q$, respectively. Let n_v and n_s be the dimensions of $\mathscr{V}^\star$ and $\mathscr{S}_q$, respectively. It follows that three matrices $\Xi_1 \in \mathbb{R}^{n_v\times r}, \Xi_2 \in \mathbb{R}^{m\times r}$ and $\Xi_3 \in \mathbb{R}^{n_s\times r}$ exist such that

$$\begin{bmatrix} H \\ 0_{p\times r} \end{bmatrix} = \begin{bmatrix} V \\ 0_{p\times n_v} \end{bmatrix}\Xi_1 + \begin{bmatrix} B \\ 0_{p\times m} \end{bmatrix}\Xi_2 + \begin{bmatrix} A \\ C \end{bmatrix} Z\,\Xi_3.$$

It follows that $\mathrm{im}\,\Xi_3 \subseteq \ker(CZ)$, so that the former can be written as

$$\mathrm{im}\,H \subseteq \mathscr{V}^\star + \mathrm{im}\,B + AZ\ker(CZ). \quad (18)$$

On the other hand, it is not difficult to check that the subspace $\mathrm{im}\,B + AZ\ker(CZ)$ is indeed $\mathscr{S}_{q+1}$, since $AZ\ker(CZ) = A(\mathrm{im}\,Z \cap \ker C)$. In fact, $x \in \mathrm{im}\,Z \cap \ker C$ if and only if a vector h exists such that $x = Zh$ and $Cx = 0$, and the two together yield $CZh = 0$; this is equivalent to $x \in Z\ker(CZ)$. Therefore, (18) can be written as $\mathrm{im}\,H \subseteq \mathscr{V}^\star + A(\mathscr{S}_q \cap \ker C) + \mathrm{im}\,B = \mathscr{V}^\star + \mathscr{S}_{q+1}$ by (11) written with respect to Σ.

Remark 3: The solution presented for Problem 1 can be easily adapted to the case when the order q in (2) is no longer a prescribed integer but a design parameter. In that case, since the sequence (10) is monotonically non-decreasing, the structural solvability condition becomes

$$\mathrm{im}\begin{bmatrix} H \\ G \end{bmatrix} \subseteq (\mathscr{V}^\star \times \mathbf{0}_p) + \mathrm{im}\begin{bmatrix} B \\ D \end{bmatrix} + \begin{bmatrix} A \\ C \end{bmatrix}\mathscr{S}^\star. \quad (19)$$

Hence, (19) is the condition ensuring the existence of an integer q for which Problem 1 admits solutions.

IV. Conclusions

Geometric solvability conditions have been presented in this paper for the disturbance decoupling by state feedback and PD feedforward action, in the general case when the model of the plant is non-strictly proper, i.e., when the feedthrough matrices of the model may differ from the zero matrices. The approach suggested in [1] consisting of the insertion of a dummy unit at the output of the given system in order to obtain an overall strictly proper system has been exploited in this case only as an intermediate step to derive solvability conditions expressed in terms of the extended plant. Afterwards, new solvability conditions involving the characteristic subspaces of the original system have been obtained, which generalize the well-known conditions presented in [1]. A similar analysis leads to a generalization of the necessary and sufficient solvability condition of [7], yielding

$$\mathrm{im}\begin{bmatrix} H \\ G \end{bmatrix} \subseteq (\mathscr{V}_g^\star \times \mathbf{0}_p) + \mathrm{im}\begin{bmatrix} B \\ D \end{bmatrix} + \begin{bmatrix} A \\ C \end{bmatrix}\mathscr{S}_q$$

instead of *1)-2)* of Theorem 1, where $\mathscr{V}_g^\star$ is the *stabilizable output-nulling subspace*, see e.g. [11, p.171]. In order to determine the control law solving Problem 1, both techniques described in [7] and [1] may be easily adapted to this more general setting.

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