School of Mathematics and Statistics

Extremal Problems and Designs on Finite Sets

This thesis is presented as part of the requirements for the award of the Degree of Doctorate of Philosophy of the Curtin University of Technology.

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Declaration

I hereby declare that the work herein is the work of myself alone except where specifically acknowledged. The work has not been submitted previously, in whole or in part, in respect of any other academic award.

(Dated & Signed)

Ian Thomas Roberts

Publications

The following publications by the author are related to the work in this thesis.

Lieby P., Roberts I.T. *Antichains and Completely Separating Systems.* Submitted.

Ramsay C., Roberts I.T., *Minimal Completely Separating Systems of Sets*, Australasian Journal of Combinatorics, **13**, 129–150, 1996.

Ramsay C., Roberts I.T., Ruskey F. *Separation of Elements in Finite Sets*, Proceedings of AWOCA, NTU, 1994.

Ramsay C., Roberts I.T., Ruskey F. *Completely Separating Systems of k-sets*, Discrete Mathematics, **183**, 265-275, 1998.

Roberts I.T. *The Union-Closed Sets Conjecture*, School of Mathematics and Statistics Technical Report, Curtin University, 2/92, 1992.

Roberts I.T. *Minimal* (n) *and* (n, h, k) *Completely Separating Systems.* Submitted.

Abstract

This thesis considers three related structures on finite sets and outstanding conjectures on two of them. Several new problems and conjectures are stated.

A union-closed collection of sets is a collection of sets which contains the union of each pair of sets in the collection. A completely separating system of sets is a collection of sets in which for each pair of elements of the universal set, there exists a set in the collection which contains the first element but not the second, and another set which contains the second element but not the first. An antichain (Sperner Family) is a collection of distinct sets in which no set is a subset of another set in the collection. The size of an antichain is the number of sets in the collection. The volume of an antichain is the sum of the cardinalities of the sets in the collection. A flat antichain is an antichain in which the difference in cardinality between any two sets in the antichain is at most one.

The two outstanding conjectures considered are:
The union-closed sets conjecture - In any union-closed collection of non-empty sets there is an element of the universal set in at least half of the sets in the collection;
The flat antichain conjecture - Given an antichain with size s and volume V, there is a flat antichain with the same size and volume.

Union-closed collections are considered in two ways. Improvements are made to the previously known bounds concerning the minimum size of a counterexample to the union-closed sets conjecture. Results are derived on the minimum size of a union-closed collection generated by a given number of k-sets. An ordering on sets is described, called order R, and it is conjectured that choosing a collection of m k-sets in order R will always minimise the size of the union-closed collection generated by m k-sets.

Several variants on completely separating systems of sets are considered. A determination is made of the minimum size of such collections, subject to various constraints on the collections. In particular, for each n and k, exact values or bounds are determined for the minimum size of completely separating systems on a n-set in which each set has cardinality k.

Antichains are considered in their relationship to completely separating systems and the flat antichain conjecture is shown to be true in certain cases.

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Preface and Acknowledgement

As with any mathematical work, there are many results that have been enhanced by discussions, comments and criticisms of fellow mathematicians. Subject to this caveat and unless otherwise stated, the work in this thesis is the result of several years of work by the author.

The chapters where the new results are not totally attributable to the author are Chapters 3, 4, 5 and 7. Lemma 3.9 is due to Jamie Simpson. Some of the smaller results in Chapters 4 were derived by Colin Ramsay or Frank Ruskey whilst working together with the author on new results for completely separating systems. The passing of time, and the true nature of collaboration, has meant that it is no longer easy to discern the parts best attributable to each of us. However, the proofs of the main result in Chapter 4 which is Theorem 4.2 (except for the case when $r = 0$), and the proof of the main result in Chapter 5, which is Theorem 5.1, are solely the work of the author. In Chapter 7 the proof of Theorem 6.4 is due to Paulette Lieby. All other results are due to the author unless otherwise indicated within the thesis.

Of course many of the proofs throughout this thesis have been improved with critical comments from others. Jamie Simpson must be thanked for the clarity and precision which he so often helped to achieve.

Colin Ramsay, Paulette Lieby and Brendan McKay have helped in the computer generation and checking of completely separating systems and/or tables for the size of minimal completely separating systems, some of which appear in this thesis.

In general the production of this thesis has been greatly enhanced by many people and I wish to thank them all. Mathematically these people include Jamie Simpson, Paulette Lieby, Colin Ramsay, Douglas Rogers, Peter Pleasants, Mike Houle, Frank Ruskey and Brendan McKay. Personally they include my wife Clare and two children Joanna and Fiona, Judy, Tom and Genny Simpson and Peter Eades. Both Curtin University of Technology and the Northern Territory University have provided support for this thesis.

The thesis is dedicated to my parents Tom and Mavis Roberts, my aunts Maisie Downes and Anne Sharrock and my grandmother Eva May Tither.

Chapter 1

Introduction

1.1 Overview

This thesis considers a number of related problems on finite sets. In broad terms the problems are to do with union-closed collections of sets, completely separating systems and antichains.

The first problem considered is the Union-Closed Sets Conjecture. The conjecture is that in any union-closed collection of non-empty sets on a finite set there is an element in at least half of the sets in the collection. The actual history of this conjecture is not entirely clear although it is stated in [25] that Frankl first stated the conjecture.

There has been little progress made on the proof of the conjecture. The majority of published papers have considered the conjecture from the point of view of assuming a minimum-size counterexample, with some appropriate definition of minimum-size, and then increasing the lower bound on certain variables associated with a minimum-size counterexample, if one exists. Chapter 2 contributes to this approach.

Consideration of this approach led to the following question: Given a choice of m k-sets on an n-set S, how does one choose the collection of k-sets, called the generating set, to minimise the number of sets in the union-closure. The conjectured answer to this question can be seen as a union-closed counterpart of the Kruskal-Katona Theorem. It is dealt with in Chapter 3. That chapter also applies the likely answer to the question to the union-closed sets conjecture.

Whilst considering the minimisation question above, the following question was also posed: Given that a union-closed collection generated by a collection of k-sets contains all of the $(n-1)$-sets on an n-set, what properties must the collection of k-sets possess? The answer is that the generating set must be a completely separating system.

Separating systems and completely separating systems have been considered by various authors and some results on the minimum number of sets required to separate or completely separate n elements have been obtained by them. A study of minimum-size completely separating systems is included here in Chapters 4, 5, 7, 8 and 9.

Completely separating systems are related to antichains through a duality relationship. The study of completely separating systems has led to some new results on antichains. The new results on antichains and the applications to completely separating systems are included in Chapters 6 and 7.

It is common in mathematics for similar concepts to be dealt with in the language of different but related theories. This often reflects different historical motivations for the consideration of the ideas. The problem which motivated this thesis, the union-closed sets conjecture, has its natural interpretation within the theory of the combinatorics of finite sets. Given the way that the problems concerning completely separating systems arose, it was natural to consider them as examples of combinatorial designs rather than by their dual formulation as antichains or within the related but distinct context of hypergraphs. It is beyond the scope of this thesis to translate the results included here into all of the various contexts in which they have relevance. However, this would be an interesting future project.

Open problems and some ideas for future work are included throughout the thesis. The thesis also includes an appendix containing some sample minimum completely separating systems.

Most of the notation and definitions used in more than one chapter are given in the next section. Notations and definitions used in one chapter only are introduced within that chapter.

1.2 Sets and Orderings

The n-set $\{1, 2, \ldots, n\}$ is denoted by $[n]$. Unless otherwise stated, all sets in this thesis are taken to be subsets of $[n]$. The power set of a set A is denoted by $P(A)$. The **set difference** of two sets A and B is $A - B = \{a \in A : a \notin B\}$. The **symmetric difference** of two sets $A, B \subseteq [n]$ is $A \triangle B = (A \cap B') \cup (A' \cap B)$ where $A' = [n] - A$ is the **complement** of the set A in $[n]$. If $\mathcal{A}$ is a collection of subsets of $[n]$, then a **dominates** b in $\mathcal{A}$ if for each $A \in \mathcal{A}$, $b \in A$ only if $a \in A$.

Where convenient the braces are left out when denoting sets. For example the collection $\{\{1, 2\}, \{3, 4\}\}$ may be written $\{12, 34\}$. $\lfloor . \rfloor$ and $\lceil . \rceil$ are used to denote the floor and ceiling functions respectively. By convention, $\binom{x}{y} = 0$ if $y < 0$ or $y > x$.

A relation, denoted by $\leq$, on a collection $\mathcal{C}$ of subsets of $[n]$ is said to be an **order** relation if it satisfies (i)–(iii) below. If the relation also satisfies (iv) the order is said to be **total**.

(i) reflexive: $\forall A \in \mathcal{C}, A \leq A$

(ii) antisymmetric: If $A, B \in \mathcal{C}, A \leq B$ and $B \leq A$ then $A = B$

(iii) transitive: If $A, B, C \in \mathcal{C}, A \leq B, B \leq C$ then $A \leq C$

(iv) total: $\forall A, B \in \mathcal{C}$ then either $A \leq B$ or $B \leq A$.

There are several natural orderings that can be defined on collections $\mathcal{C}$ of subsets of $[n]$: the **lexicographic order** on $\mathcal{C}$ is a total order relation defined by $A \leq_L B$ if the smallest element of $A \triangle B$ is in A, or if $A = B$; the **anti-lexicographic order** on $\mathcal{C}$ is a total order relation defined by $A \leq_A B$ if the largest element of $A \triangle B$ is in A, or if $A = B$; the **squashed order** or **colex order** on $\mathcal{C}$ is a total order relation defined by $A \leq_S B$ if the largest element of $A \triangle B$ is in B, or if $A = B$. For a given $[n]$, the squashed order on all k-subsets of $[n]$ is the reverse of the anti-lexicographic order.

When analysing problems involving squashed order it is sometimes convenient to use the l-binomial representation of a positive integer m. This is defined as: Given positive integers l and m, the **l-binomial representation** of m is

$$m = \binom{a_l}{l} + \binom{a_{l-1}}{l-1} + \ldots + \binom{a_t}{t}.$$

where $a_l > a_{l-1} > \ldots > a_t \geq t \geq 1$. Theorem 7.2.1 of [2] shows that this representation is unique for each l and m. Chapter 7 of [2] provides some detail of its use in considering collections of sets in squashed order.

Two new but natural orderings are defined as follows: **order T** on $\mathcal{C}$ is a total order relation defined by $A \leq_T B$ if the largest element in A is not in B or if A and B have the same largest element and the smallest element of $A \triangle B$ is in B, or if $A = B$; **order R** on $\mathcal{C}$ is a total order relation defined by $A \leq_R B$ if the largest element of $A \cup B$ is in B only, or if A and B have the same largest element and the smallest element of $A \triangle B$ is in A, or if $A = B$.

Order T is the reverse of order R on $[n]$. Further, if $\mathcal{C}$ consists of the sub-collection of k-sets then order T is the reverse of the lexicographic ordering on $\mathcal{C}$. Also, if the element which is contained in each set in $\mathcal{C}$ is ignored, then order T is equivalent to the squashed order. The importance of order R is expressed in Chapter 3. The five orderings defined above are illustrated below for the 3-subsets of [5].

Lexicographic order	Anti-lexicographic order	Squashed order	Order T	Order R
123	345	123	345	123
124	245	124	245	124
125	145	134	235	134
134	235	234	145	234
135	135	125	135	125
145	125	135	125	135
234	234	235	234	145
235	134	145	134	235
245	124	245	124	245
345	123	345	123	345

Notice that in order R all subsets of $\{1,2,3\}$ are included before subsets containing 4, all of which precede subsets containing 5. Consider order R on the six sets which contain 5. Then these six 3-sets occur in lexicographic order.

A collection of sets $\mathcal{A}$ is called a **union-closed** collection if $A \cup B \in \mathcal{A}$ whenever $A, B \in \mathcal{A}$. A collection of sets $\mathcal{A}$ is said to be **generated** by a collection of non-empty sets $\mathcal{B}$ if $\mathcal{A}$ is the smallest union-closed collection containing $\mathcal{B}$. For example, $\mathcal{A} = \{12, 13, 24, 123, 124, 1234\}$ is generated by the set $\mathcal{B} = \{12, 13, 24\}$.

A set A is said to be **allowed** in a union-closed collection $\mathcal{A}$ if $\mathcal{A} \cup \{A\}$ is union-closed. A collection of sets $\mathcal{C}$ is said to be **allowed** in a union-closed collection $\mathcal{A}$ if $\mathcal{A} \cup \mathcal{C}$ is union-closed. If $\mathcal{A}_1$ is not a union-closed collection, A is said to be **allowed** in $\mathcal{A}_1$ if $A \in \mathcal{A}_1$.

1.3 Sets and Antichains

The **volume** of a collection of sets $\mathcal{C}$ is $V = V(\mathcal{C}) = \sum_{A \in \mathcal{C}} |A|$. An **antichain** on $[n]$ is a collection $\mathcal{A}$ of subsets of $[n]$ such that for any distinct $A, B \in \mathcal{A}$, $A \not\subset B$. The **average set size** of a collection of sets $\mathcal{C}$ is $\overline{\mathcal{C}} = \frac{V(\mathcal{C})}{|\mathcal{C}|}$. Two antichains $\mathcal{A}, \mathcal{A}'$ are said to be **equivalent** if $|\mathcal{A}| = |\mathcal{A}'|$ and $\overline{\mathcal{A}} = \overline{\mathcal{A}'}$. Note that the meaning of the previous definition would be maintained if the condition that $\overline{\mathcal{A}} = \overline{\mathcal{A}'}$ is replaced by the condition that $V(\mathcal{A}) = V(\mathcal{A}')$.

A **flat antichain** is an antichain $\mathcal{A}$ such that $|A| = \lfloor \overline{\mathcal{A}} \rfloor$ or $\lceil \overline{\mathcal{A}} \rceil$ for each $A \in \mathcal{A}$. When an antichain $\mathcal{A}$ is replaced by a flat antichain $\mathcal{A}'$ which is equivalent to $\mathcal{A}$ then $\mathcal{A}$ is said to be **flattened.**

Example 1.1. $\mathcal{A} = \{1234, 125, 135, 235, 45, 16, 26, 36, 46, 56\}$ is an antichain with flat equivalent antichain $\mathcal{A}' = \{123, 124, 134, 234, 125, 35, 45, 16, 26, 36\}$.

The **shadow** of a collection of sets $\mathcal{A}$, with each set $A \in \mathcal{A}$ of cardinality k, is $\triangle\mathcal{A} = \{B : |B| = k-1, B \subset C \text{ for some } C \in \mathcal{A}\}$. The **shade** of a collection of sets $\mathcal{A}$, with each set $A \in \mathcal{A}$ of cardinality k, is $\nabla\mathcal{A} = \{B : |B| = k+1, C \subset B$ for some $C \in \mathcal{A}\}$. The **shadow at level** h of a collection of sets $\mathcal{A}$, with each set $A \in \mathcal{A}$ of cardinality at least $h+1$, is $\triangle^{(h)}\mathcal{A} = \{B : |B| = h, B \subset C$ for some $C \in \mathcal{A}\}$. The **shade at level** h of a collection of sets $\mathcal{A}$, with each set $A \in \mathcal{A}$ of cardinality at most $h-1$, is $\nabla^{(h)}\mathcal{A} = \{B : |B| = h, C \subset B \text{ for some } C \in \mathcal{A}\}$.

Let $\mathcal{A}$ be an antichain with p_i sets of size i. (The p_i are called the **parameters** of the antichain). Let $\mathcal{A}_i = \{A \in \mathcal{A} : |A| > i\}$. $\mathcal{A}$ is called a **squashed antichain** if for each i, $\triangle^{(i)}\mathcal{A}_i$ precedes the sets in $\mathcal{A}$ of size i in squashed order and the sets in $\triangle^{(i)}\mathcal{A}_i$, together with the sets in $\mathcal{A}$ of size i, constitute an initial segment of the sets of size i in squashed order. Hence if A, B are sets in a squashed antichain $\mathcal{A}$ and $|A| > |B|$ then A precedes B in $\mathcal{A}$. The collections $\mathcal{A}$ and $\mathcal{A}'$ in Example 1.1 have been written as squashed antichains. Note that it is known (see [2]) that any antichain has an equivalent squashed antichain. This result is often used in this thesis to replace an antichain by a squashed antichain equivalent to it.

Let $\mathcal{A}$ be a squashed antichain and $C \subset [n]$. The notation $C <_s \mathcal{A}$ is used to denote the fact that $C <_s A$ for all $A \in \mathcal{A}$. Similarly $C >_s \mathcal{A}$ is used to denote the fact that $A <_s C$ for all $A \in \mathcal{A}$. The **new shadow** of $\mathcal{A}$ is $\triangle_N \mathcal{A} = \{B \in \triangle\mathcal{A} : B \notin \triangle C \text{ for all } C <_s \mathcal{A}\}$. The **new shade** of $\mathcal{A}$ is $\nabla_N \mathcal{A} = \{B \in \nabla\mathcal{A} : B \notin \nabla C \text{ for all } C >_s \mathcal{A}\}$. The **new shadow at level** h of $\mathcal{A}$, with each set in $\mathcal{A}$ of cardinality at least k, is defined for $h < k$ to be $\triangle_N^{(h)} \mathcal{A} = \{B \in \triangle^{(h)}\mathcal{A} : B \notin \triangle^{(h)} C \text{ for all } C <_s \mathcal{A}\}$. The **new shade at level** h of $\mathcal{A}$, with each set in $\mathcal{A}$ of cardinality at most k, is defined for $h > k$ to be $\nabla_N^{(h)} \mathcal{A} = \{B \in \nabla^{(h)}\mathcal{A} : B \notin \nabla^{(h)} C \text{ for all } C >_s \mathcal{A}\}$.

$F_k(m)$ is used to denote the first m k-sets in the squashed ordering of sets. The squashed ordering is such that $F_k(m)$ is independent of the universal set. Note that if a collection of sets $\mathcal{A}$ consists of the initial segment in a squashed antichain then $\triangle_N \mathcal{A} = \triangle\mathcal{A}$.

1.4 Collections of Sets and Completely Separating Systems

Let $\mathcal{C} = \{A_1, \ldots, A_m\}$ be a collection of subsets of $[n]$. The **complementary collection** $\mathcal{C}'$ of $\mathcal{C}$ is $\mathcal{C}' = \{A' = [n] - A : A \in \mathcal{C}\}$. If $|A_i| \geq |A_j|$ for $i < j$ then the **cardinality sequence** of $\mathcal{C}$ is the non-increasing sequence $|A_1|, \ldots, |A_m|$. A collection of sets $\mathcal{C}$ is said to be a **representation** of a cardinality sequence S if $\mathcal{C}$ has cardinality sequence S. A collection of sets $\mathcal{B}$ is said to be **isomorphic** to the collection $\mathcal{C}$ (written $\mathcal{B} \equiv \mathcal{C}$) if $\mathcal{B}$ can be obtained from $\mathcal{C}$ by relabelling the elements of $[n]$. $\mathcal{C}$ is said to be **unique** if all other collections on $[n]$ with cardinality sequence $|A_1|, \ldots, |A_m|$ are isomorphic to $\mathcal{C}$. $\mathcal{C}$ is said to be written in **standard form** if the set of largest cardinality in $\mathcal{C}$ is an m-set and the first set in $\mathcal{C}$ is the set $[m]$.

An element $a \in [n]$ is said to be **separated** from $b \in [n]$ in $\mathcal{C}$ if there is a set in $\mathcal{C}$ which contains a but not b. A **separating system (or SS)** $\mathcal{C}$ on $[n]$ is a collection of subsets of $[n]$ in which for any pair of elements $a, b \in [n]$ either a is separated from b or b is separated from a. A **completely separating system (or CSS)** $\mathcal{C}$ on $[n]$ is a collection of subsets of $[n]$ in which each element occurs at least once and in which no element dominates another element. That is, for each $a, b \in [n]$, a is separated from b and b is separated from a in $\mathcal{C}$.

Observe that if $\mathcal{C}$ is a CSS on $[n]$ then the complementary collection $\mathcal{C}'$ is also a CSS. Also observe that any CSS is a SS, but not vice-versa. For example, in $\{\{1,2\},\{1,3\}\}$, 1 is separated from 2 by $\{1,3\}$, but 2 is not separated from 1. The collection $\{\{1,2\},\{1,3\}\}$ is a SS but not a CSS.

Let h and k be positive integers. A CSS on $[n]$ without restrictions on the size of the sets in the collection is said to be a **(n)CSS**. If $h \le |A| \le k$ for all $A \in \mathcal{C}$ then $\mathcal{C}$ is said to be a **(n,h,k)CSS**. If $h = k$ then $\mathcal{C}$ is said to be a **(n,k)CSS**. Note that if $h > 1$ then every element of $[n]$ must occur at least twice in a (n,h,k)CSS to ensure that the elements of any set in the CSS are completely separated from one another.

For n, h, k fixed positive integers **R(n), R(n,h,k), R(n,k)** are defined as follows: $R(n) = \min\{|\mathcal{C}| : \mathcal{C} \text{ is a } (n)CSS\}$; $R(n,h,k) = \min\{|\mathcal{C}| : \mathcal{C} \text{ is a } (n,h,k)CSS\}$; and $R(n,k) = \min\{|\mathcal{C}| : \mathcal{C} \text{ is a } (n,k)CSS\}$.

A (n)CSS for which $|\mathcal{C}| = R(n)$ is a **minimal** (n)CSS. A (n,h,k)CSS for which $|\mathcal{C}| = R(n,h,k)$ is a **minimal** (n,h,k)CSS. A (n,k)CSS for which $|\mathcal{C}| = R(n,k)$ is a **minimal** (n,k)CSS. A (n,h,k)CSS $\mathcal{C}$ is said to be **strongly minimal** when $\mathcal{C}$ is minimal and $\mathcal{C}$ also has minimum volume compared to all other minimal (n,h,k)CSSs.

Note that if $\mathcal{C}$ is a minimal (n, h, k)CSS then a set $A \in \mathcal{C}$ cannot occur more than once in $\mathcal{C}$ as the removal of one occurrence of A would produce a smaller CSS. Also note that any collection of k-subsets of $[n]$ that is a superset of a (n, k)CSS is also a (n, k)CSS.

Example 1.2. $R(6,3) = 4$ since $\mathcal{C} = \{\{1,2,3\}, \{1,4,5\}, \{2,4,6\}, \{3,5,6\}\}$ is a $(6,3)$CSS and no fewer number of 3-sets will completely separate 6 elements.

Let $\mathcal{C}$ be a collection of k-subsets of $[n]$. Let $\mathcal{C}_x$ denote $\{A \in \mathcal{C} : x \in A\}$. Let $\mathcal{C}_{\overline{x}}$ denote $\{A \in \mathcal{C} : x \notin A\}$. Observe that $\mathcal{C}$ is a (n, k)CSS if and only if $\bigcap\{A \in \mathcal{C}_x\} = \{x\}$, for all $x \in [n]$ if and only if $\bigcup\{A \notin \mathcal{C}_x\} = [n] - \{x\}$, for all $x \in [n]$.

A p-**element** in a collection of sets is an element which occurs in exactly p sets in the collection.

It is often convenient to represent the elements of a completely separating system on an n-set by elements of $[n]$ for the 2-elements and by alphabetic characters for the 3-elements. In this case lower and upper case letters may be used when the number of 3-elements exceed 26. If 4-elements are to be used, numeric elements of $[n]$ will be used to denote them.

It is also convenient to represent the sets in a CSS as rows in an array with the elements of the array being the elements of the universal set. In some of these representations spaces are left in some rows to help highlight the structures of the CSS represented.

An alternative representation of a CSS is in the form of an incidence matrix. In this case, a CSS $\mathcal{C}$ on $[n]$ with $|\mathcal{C}| = m$ is represented by an $m \times n$ matrix M with the rows representing the sets in $\mathcal{C}$ and the columns representing the elements of $[n]$. The element $m_{ij} \in M$ is set equal to 1 if $j \in A_i \in \mathcal{C}$ and 0 otherwise. Although this representation is useful especially for computer manipulation, it is not the best representation for many of the arguments used in this thesis.

Suppose $\mathcal{C}$ is a minimal (n,k)CSS. Then the **excess** is $E = kR(n,k) - 2n$. Note that when $kR(n,k) \leq 3n$, E is the maximum number of elements of $[n]$ which can occur in more than two sets in $\mathcal{C}$.

1.5 Antichains and Completely Separating Systems

In [5] Cai made the following definition. Let $\mathcal{A} = \{A_1, \ldots, A_m\}$ be a collection of subsets of $[n]$. The **dual** $\mathcal{A}^*$ of $\mathcal{A}$ is the collection $\mathcal{A}^* = \{X_1, \ldots, X_n\}$ of subsets of $[m]$ given by $X_j = \{i : j \in A_i\}$. In the case when $\mathcal{A}$ is a CSS then $\mathcal{A}$ has an associated incidence matrix M (as defined above for the CSS $\mathcal{C}$ but with the role of the rows and columns reversed). Then the columns of M represent the sets in $\mathcal{A}^*$ and M is an incidence matrix for $\mathcal{A}^*$ with $m_{ij} = 1$ if $i \in X_j$.

Note that for any collection $\mathcal{A}$, $V(\mathcal{A}) = V(\mathcal{A}^*)$ since

$$V(\mathcal{A}) = \sum_{A \in \mathcal{A}} |A| = \sum_{i=1}^{m} |A| = \sum_{j=1}^{n} |\{i : j \in A_i\}| = \sum_{j=1}^{n} |X_j| = V(\mathcal{A}^*).$$

A CSS $\mathcal{C}$ on $[n]$ is said to be **fair** if there is a positive integer x such that each element of $[n]$ occurs in exactly x or $x+1$ sets in $\mathcal{C}$. It will be shown later that the dual of a fair CSS is a flat antichain and vice versa.

The notation $a|b$ is used to denote that a divides b. Further notation and definitions will be introduced as required in later chapters.

Chapter 2

The Union-Closed Sets Conjecture

2.1 Introduction

The union-closed sets conjecture asserts that in any union-closed collection of non-empty subsets of a finite set there is an element which occurs in at least half of the sets in the collection. Several papers by other authors have provided information concerning bounds on the size of a possible counterexample to this conjecture. This chapter improves the lower bounds on several key variables for a minimum size counterexample.

Many, but not all, of the results included here were obtained independently by LoFaro [10]. His results appear in [11] whereas most of the results included here first appeared in [30].

2.1.1 Background

According to Renaud [25] the conjecture was originally proposed in 1979 by Frankl. The problem has an intersection-closed sets form but most of the published work deals with the union-closed form. Some of the published work does not allow the empty set in the collections and some do allow it. The convention adopted here is to exclude the empty set. The theory in this thesis is correct even if the empty set is included in the collections.

The papers appearing so far have tended to use a variety of notation, so one has to be careful when reading each of the references. Some of the early definitions and results below are from various papers and they have been summarised in a notation which is consistent within this chapter.

The formal statement of the union-closed sets conjecture is:

Conjecture 2.1 (The Union-Closed Sets Conjecture or UCS).
Let $\mathcal{A} = \{A_1, \ldots, A_m\}$ be a non-empty union-closed collection of non-empty sets on $[n]$. Then there exists an element of $[n]$ which belongs to at least $\lceil \frac{m}{2} \rceil$ of the sets in $\mathcal{A}$.

Henceforth, within this chapter, it is assumed that $\mathcal{A}$ is a union-closed collection of sets which is a minimum size counterexample to UCS. That is, if $\mathcal{C}$ is the collection of all union-closed collections of sets which do not satisfy UCS then $\mathcal{A}$ satisfies $|\mathcal{A}| = min_{C \in \mathcal{C}}\{|C|\}$ and $|\bigcup_{A_i \in \mathcal{A}} A_i|$ is minimum over all counterexamples with $|\mathcal{A}|$ sets. In [33], Renaud and Sarvate showed that $|\mathcal{A}|$ is odd.

2.1.2 Definitions

For the remainder of this chapter, it is assumed that $\bigcup \mathcal{A} = [n]$, with $|\mathcal{A}| = 2N+1$. Note that no pair of elements can be mutually dominant in $\mathcal{A}$.

Define

$$\mathcal{A}_x = \{A \in \mathcal{A} : x \in A\};$$

$$C_a = \bigcup\{A \in \mathcal{A} : a \notin A\};$$

$$\mathcal{C} = \{C_a : a \in [n]\};$$

$$\mathcal{D} = \mathcal{A} - [n] - \mathcal{C};$$

$$\mathcal{D}_a = \{A \in \mathcal{D} : a \in A\};$$

$$\mathcal{A}'_x = \mathcal{A} - \mathcal{A}_x;$$

$$\mathcal{D}'_x = \mathcal{D} - \mathcal{D}_x.$$

Note that $[n] \in \mathcal{A}$, $C_a \in \mathcal{A}$ for each $a \in A$ and that $b \notin C_a$ if and only if a dominates b.

The elements of $[n]$ which occur most often in $\mathcal{A}$ belong to the set H where

$$H = \{x \in [n] : X \text{ is an element of } N \text{ sets in } \mathcal{A}\}.$$

Note 2.1. (i) The fact that H is non-empty and has at least 3 elements is shown in [19].
(ii) It is easily shown that each set in $\mathcal{A}$ contains at least three elements(see [33]).

By convention x, y, z will denote elements of H while a, b, c denote elements which may not be in H. A **basis** set is a set $A \in \mathcal{A}$ which is not the union of other sets in A. $\mathcal{B}$ will denote the set of **basis sets** in $\mathcal{A}$. Note that a collection remains union-closed when one or more of its basis sets are removed.

In [34], Savarte and Renaud and showed that $n \geq 7$ and in [10], Lo Faro showed that $|\mathcal{A}| \geq 25$ and $|\mathcal{A}| \geq 2n + 1$. Poonen [20] has shown, by methods different from those of this chapter, that $|\mathcal{A}| \geq 29$ and $n \geq 8$.

In this chapter it will be shown that $|\mathcal{A}| \geq 41$, $|\mathcal{A}| \geq 4n - 1$ and $n \geq 8$, thus confirming Poonen's result on n.

2.2 Concepts

There are several concepts which have arisen in the papers which have dealt with the conjecture. A collection of results on these concepts are presented in this section.

2.2.1 Results Concerning C_a

Lemma 2.1. *In a minimum size counterexample to UCS*

(i) $H \subseteq C_a$ *for each* $a \notin H$*;*

(ii) $H - \{x\} \subseteq C_x$ *for each* $x \in H$*;*

(iii) $C_a \notin \mathcal{B}$ *for any* $a \in [n]$*;*

(iv) $C_a \neq B \cup C$ *for any* $a \notin H$*,* $B, C \in \mathcal{B}$*;*

(v) $|C_a| \geq 5$ *for all* $a \notin H$*;*

(vi) For each distinct pair $a, b \in [n]$*,* $C_a \neq C_b$*;*

(vii) For each distinct pair $a, b \in [n]$*, if* $|C_a| \leq |C_b|$ *then* $a \in C_b$*;*

(viii) For each distinct pair $a, b \in [n]$*, if* $|C_a| = |C_b|$ *then* $a \in C_b$ *and* $b \in C_a$*.*

Proof. (i) The elements in H occur most often in a minimum size counterexample to UCS. Therefore, as no pair of elements can be mutually dominant, if $x \in H$, $a \in [n]$ and $a \neq x$, then there exists an $A \in \mathcal{A}$ such that $x \in A$, $a \notin A$. So $x \in C_a$.

(ii) This follows from a similar argument to (i).

(iii) Suppose C_a is a basis set. First suppose that $a \notin H$. Then, by (i), $\mathcal{A} - C_a$ is a union-closed collection of $2N$ sets in which each element occurs less than N times and so is a counterexample to the conjecture. Now suppose that $a \in H$. Form a union-closed collection $\mathcal{A} - C_a - B$ where B is any basis set which contains a. Each element of H occurs in $C_a \cup B$, so occurs in $\mathcal{A} - C_a - B$ $N-1$ times. Thus $\mathcal{A} - C_a - B$ contains $2N-1$ sets in which no element occurs at least N times. Both cases contradict the minimality of $|\mathcal{A}|$.

(iv) If $C_a = B \cup C, a \notin H$ and $B, C \in \mathcal{B}$, then the removal of B and C provides a smaller counterexample as in the proof of (iii).

(v) By Note 2.1(ii), $|A| \geq 3$ for all $A \in \mathcal{A}$, all 3-sets in $\mathcal{A}$ are basis sets. Further, all 4-sets in $\mathcal{A}$ are basis sets or the union of two basis sets. Applying (iii) or (iv) as appropriate gives the required result.

(vi) This is Corollary 3 of [10]. It is easily shown that $C_a = C_b$ implies that a and b are mutually dominant which is impossible.

(vii) By (vi), $C_a \neq C_b$ for each $a, b \in [n]$, $a \neq b$. Therefore, if $|C_a| \leq |C_b|$ there exists $c \in C_b$, $c \notin C_a$. As $c \notin C_a$, a dominates c. Thus, as $c \in C_b$, $a \in C_b$ as required.

(viii) This follows from (vii). □

Theorem 2.1. *In all union-closed collections of sets, if $a, b \in [n]$ and $C_a \not\subseteq C_b$ then $b \in C_a$.*

Proof. By the assumption there exists $c \in C_a$, $c \notin C_b$. Therefore b dominates c and so $b \in C_a$. □

2.2.2 Results Concerning H

Lemma 2.2. *In a minimum size counterexample to UCS the following statements hold.*

(i) $|H| \geq 3$.

(ii) $H \not\subseteq B \cup C$ *for any* B, $C \in \mathcal{B}$. *In particular,* $H \not\subseteq B$ *for any basis set* B.

(iii) For each pair x, $y \in H$ *there exists* B, $C \in \mathcal{B}$ *such that* $x \in B$, $x \notin C$, $y \notin B$, $y \in C$.

(iv) For each $x \in H$ *there exists at least 2 distinct sets* B, $C \in \mathcal{B}$ *with* $x \in B$ *and* $x \in C$.

Proof. (i) This is Note 2.1(i).

(ii) If $H \subseteq B \cup C$ then the removal of B and C from $\mathcal{A}$ gives a smaller counterexample to UCS.

(iii) In a minimum size counterexample to UCS no two elements can occur in exactly the same sets. If (iii) was not true then either x would occur more often then y or vice-versa. This is not possible for x, $y \in H$.

(iv) Assume B is the only basis set containing x. As $|B| \geq 3$ there exists an $a \in B$, $a \neq x$. Then a occurs in at least as many sets as x, but in no more as $x \in H$. Thus a and x are mutually dominant in $\mathcal{A}$, which is a contradiction. □

2.2.3 Results Concerning $\mathcal{B}$

Lemma 2.3. *In a minimum size counterexample to UCS*

$|B| \leq n - 2$ *for each* $B \in \mathcal{B}$.

Proof. Let B be a basis set. Then $|B| \neq n$ else there is an element which occurs only in B and the removal of B would give a smaller counterexample to UCS. If $|B| = n - 1$ then $B = [n] - \{a\}$ for some $a \in [n]$. There must be a basis set C, $a \in C$. Then $H \subseteq B \cup C = [n]$ which contradicts Lemma 2.2(ii). Hence $|B| \leq n - 2$. □

2.2.4 General Results

Theorem 2.2. *In a minimum size counterexample to UCS*

$$|\mathcal{A}| \le 2\left(2^p - \frac{p^2+p}{2}\right) - 3$$

where $p = min_{a\in[n]}|C_a|$.

Proof. For all $a \in [n]$

$$|\mathcal{A}'_a| \ge \frac{|\mathcal{A}|+1}{2}. \tag{2.1}$$

All sets have cardinality at least 3, thus

$$\begin{aligned} |\mathcal{A}'_a| &\le 2^p - \binom{p}{2} - \binom{p}{1} - 1 \\ &= 2^p - \frac{p^2+p}{2} - 1. \end{aligned} \tag{2.2}$$

The theorem follows from (2.1) and (2.2). □

Corollary 2.1.
If $|\mathcal{A}| \ge 33$ *then* $|C_a| \ge 6$ *for all* $a \in [n]$.
If $|\mathcal{A}| \ge 85$ *then* $|C_a| \ge 7$ *for all* $a \in [n]$.

Proof. Immediate from the theorem and the fact that $|\mathcal{A}|$ is odd. □

2.3 Improved Bounds on the Validity of the Conjecture

In this section several bounds on variables involved with the conjecture are improved. Recall that

$$\mathcal{D} = \mathcal{A} - [n] - \mathcal{C},$$

and $\mathcal{D}_x$ is the collection of sets in $\mathcal{D}$ which contain x.

Lemma 2.4. $|D_x| = N - n$ *for each* $x \in H$.

Proof. By Lemmas 2.1.(i) and 2.1(ii), $x \in H$ occurs in all sets C_b with $b \neq x$. There are $n-1$ such sets in $\mathcal{A}$. Further $x \in [n]$ which is a member of $\mathcal{A}$. Thus x belongs to n sets not included in $\mathcal{D}$, but x occurs in N sets in $\mathcal{A}$. Therefore, for $x \in H$, x occurs in $(N-n)$ sets in $\mathcal{D}$ so $|\mathcal{D}_x| = N - n$. □

Theorem 2.3. *In a minimum size counterexample to UCS each* $x \in H$ *occurs in more sets in* $\mathcal{D}$ *than does any* $a \notin H$.

Proof. Suppose $a \in [n]$ is such that $|\{A \in \mathcal{D} : a \in A\}|$ is maximal. Then, by Lemma 2.4, $|\{A \in \mathcal{D} : a \in A\}| \geq N - n$.

If there exists $b \neq a$ such that $a \notin C_b$ then b dominates a. Thus there exists a basis set B with $b \in B$, $a \notin B$ and so b occurs more often than a in $\mathcal{D}$. This is a contradiction.

If $a \in C_b$ for all $b \neq a$, $b \in [n]$, then a occurs in n sets of $\mathcal{A} - \mathcal{D}$. If a occurs in $N-n$ sets of $\mathcal{D}$ then it occurs in at least $(N-n)+n = N$ sets in $\mathcal{A}$, which means that $a \in H$. □

In [10] it was shown that a minimum size counterexample to the conjecture has $|\mathcal{A}| \geq 2n+1$. This is improved in the following theorem.

Theorem 2.4. *In a minimum size counterexample to UCS*

$$|\mathcal{A}| \geq 4n - 1.$$

Proof. Consider $\mathcal{A}_x$, $\mathcal{A}'_x$ for some $x \in H$. Then $|\mathcal{A}_x| = N$, $|\mathcal{A}'_x| = N+1$, $|\mathcal{D}_x| = N - n$. $\mathcal{A}'_x$ is union-closed so there exists an element a in at least half of the sets in $\mathcal{A}'_x$. By Lemma 2.4 and Theorem 2.3, a occurs in at most $(N-n)$ sets in $\mathcal{D}$. Also $a \in C_x = \bigcup \mathcal{A}'_x$. By Lemma 2.1(i) $C_y \notin \mathcal{A}'_x$ for $x \neq y$. Therefore

a is in at most $(N - n + 1)$ sets in $\mathcal{A}'_x$. Hence

$$|\mathcal{A}'_x| \leq 2(N - n + 1).$$

That is,

$$N + 1 \leq 2(N - n + 1),$$
$$2n - 1 \leq N.$$

So $|\mathcal{A}| = 2N + 1 \geq 4n - 1$. □

For the rest of this chapter assume $|\mathcal{A}| = 4n + 2r + 1$ with $r \geq -1$ an integer. That is $N = 2n + r$. A simple but useful result is the following theorem.

Theorem 2.5. *In a minimum size counterexample to UCS*

(i) There are at least n sets which contain neither a nor b for any $a, b \in [n]$.

(ii) Each $a \in [n]$ is in less then half of the sets in $\mathcal{D}$.

Proof. As $|\mathcal{A}| = 4n+2r+1$ and $\mathcal{D} = \mathcal{A} - [n] - \mathcal{C}$, $|\mathcal{D}| = 3n+2r$. By Lemma 2.4, each $x \in H$ is in exactly $N - n = n + r$ sets in $\mathcal{D}$. By Theorem 2.3 each $a \in [n]$ occurs in no more than $n + r$ sets in $\mathcal{D}$. Part (ii) follows from this. For part (i) note that each of a and b occur in at most $n + r$ sets in $\mathcal{D}$. Since $\mathcal{D}$ contains $3n + r$ sets there will be at least n sets in $\mathcal{D}$ containing neither a nor b. □

Note that the collection of sets with neither a nor b in the proof of Theorem 2.5 is union-closed.

Corollary 2.2.

$$\overline{\mathcal{D}} < \frac{n}{2}.$$

Proof. This follows directly from Theorem 2.5(ii). □

Corollary 2.3. *In a minimum size counterexample to UCS*

$$9n + 6r \leq V(\mathcal{D}) \leq n(n+r) = n(N-n).$$

Proof. As $|\mathcal{A}| = 4n + 2r + 1$, $N = 2n + r$. As in the proof of Theorem 2.5, each $a \in [n]$ is in no more than $n+r$ sets in $\mathcal{D}$. Therefore $V(\mathcal{D}) \leq n(n+r) = n(N-n)$. As each set in $\mathcal{A}$ is of cardinality at least 3, $V(\mathcal{D}) \geq 3(3n + 2r)$. □

Corollary 2.4. *In a minimum size counterexample to UCS, if* $|\mathcal{A}| = 4n-1$ *then* $n \geq 10$.

Proof. If $|\mathcal{A}| = 4n - 1$ then $r = -1$. Corollary 2.3 implies that $9n - 6 \leq n^2 - n$. That is,

$$n^2 - 10n + 6 \geq 0.$$

Thus $n > 9$ or $n \leq 0$. As n is a positive integer, $n \geq 10$. □

Corollary 2.5. *In a minimum size counterexample to UCS*

(i) If $n = 7$, $|\mathcal{A}| \geq 57$.

(ii) If $n = 8$, $|\mathcal{A}| \geq 41$.

Proof. Corollary 2.3 implies that $r \geq n\frac{9-n}{n-6}$, for $n > 6$. Substituting for n and using $|\mathcal{A}| = 4n + 2r + 1$ gives the required inequalities. □

The following result is an improvement on Lemma 2.1(v). Note that Theorems 2 and 3 of [33] show that UCS is satisfied if $n \leq 6$, so this is assumed for the remainder of this chapter.

Corollary 2.6. *In a minimum size counterexample to UCS,* $|C_a| \geq 6$ *for all* $a \in [n]$.

Proof. It can be assumed that $n \geq 7$, so Corollary 2.5 and Theorem 2.4 ensure that $|\mathcal{A}| \geq 35$. Then $|C_a| \geq 6$ by Corollary 2.1. $\square$

The following theorem presents a different proof of the result of Poonen [20].

Theorem 2.6. *In a minimum size counterexample to UCS, $n \geq 8$.*

Proof. It can be assumed that $n \geq 7$. Assume $n = 7$. By Corollary 2.5, if $n = 7$ then $|\mathcal{A}| \geq 57$.

There are at most $\binom{7}{3} = 35$ 3-sets in $\mathcal{A}$. By Corollary 2.6, $|C_a| \geq 6$ for each $a \in [n]$. Thus $\mathcal{A}$ contains at most 35 3-sets and $\mathcal{A}$ also contains 7 6-sets, and 1 7-set. Then $\mathcal{A}$ contains at least $|\mathcal{A}| - 43$ other sets of size at least 4. Therefore

$$V(\mathcal{A}) \geq 35 \times 3 + (|\mathcal{A}| - 43) \times 4 + 7 \times 6 + 7 = 4|\mathcal{A}| - 18.$$

As each element can occur at most N times in $\mathcal{A}$, $V(\mathcal{A}) \leq 7N = 7\frac{|\mathcal{A}|-1}{2}$. Combining these inequalities gives $\frac{7|\mathcal{A}|-7}{2} \geq 4|\mathcal{A}| - 18$ which does not hold when $|\mathcal{A}| \geq 57$. Therefore $n \neq 7$ and the theorem follows. $\square$

Theorem 2.7. *In a minimum size counterexample to the union-closed sets conjecture $|\mathcal{A}| \geq 41$.*

Proof. By Theorem 2.4 and Corollaries 2.4 and 2.5 only the cases $|\mathcal{A}| = 37, 39$ for $n = 9$ and $|\mathcal{A}| = 39$ for $n = 10$ need to be checked. Assume $|\mathcal{A}| = 39$ and $n = 10$. Then $r = -1$ and $|\mathcal{D}| = 28$. For $x \in H$, $\mathcal{D}$ may be partitioned into the two collections $\mathcal{D}_x$ and $\mathcal{D}'_x$, where $\mathcal{D}_x$ consists of those sets in $\mathcal{D}$ which contain x, and $\mathcal{D}'_x = \mathcal{D} - \mathcal{D}_x$. Then $|\mathcal{D}_x| = 9$ and $|\mathcal{D}'_x| = 19$. It will be shown that this implies that $V(\mathcal{D}) > 90$, which is a contradiction of Corollary 2.3, which gives the bound $84 \leq V(\mathcal{D}) \leq 90$.

As $|\mathcal{D}| = 28$, $V(\mathcal{D}) \leq 90$ and $|A| \geq 3$ for all $A \in \mathcal{A}$, $\mathcal{D}$ must contain at least 22 3-sets and at most 6 sets of size greater than 3. Thus $\mathcal{D}'_x$ must contain at least $|\mathcal{D}'_x| - 6 = 13$ 3-sets. By the pigeon-hole principle and as $|\bigcup \mathcal{D}'_x| \leq 9$, an element $a \in [10]$ must occur in at least $\lceil \frac{13 \times 3}{9} \rceil = 5$ of these 3-sets. By the pigeon-hole principle, another element b must occur in at least two of these 3-sets. If b occurs in at least three of these sets then it can be assumed that the sets include $\{a, b, 1\}, \{a, b, 2\}, \{a, b, 3\}$ and $\{a, c, d\}$ where c or d may or may not be b. In any case, by Corollary 2.6 $|C_a| \geq 6$ for all $a \in [10]$, so the union-closure of this collection contains enough 4-sets or 5-sets to ensure that $V(\mathcal{D}) > 90$, which is a contradiction. The same contradiction can be seen to arise if every b occurs in at most two of the five 3-sets containing a, by considering the collection $\{a, b, 1\}, \{a, b, 2\}, \{a, c, d\}$ and $\{a, e, f\}$ where $c, d, e, f \neq b$. Thus $|\mathcal{A}| \neq 39$. The cases for $n = 9$ can be argued in a similar manner. □

This last proof would be more elegant if there were results on the minimum number of sets in a union-closed collection generated by a given number of k-sets. This issue is considered in Chapter 3.

Chapter 3

Union-Closed Collections of Sets

3.1 Introduction

Chapter 2 closed with a slight improvement on the minimum number of sets that must occur in a minimum-size counterexample to UCS. The argument required particular case arguments about the minimum size of a union-closed collection generated by a given number of sets of a given size. It is the aim of this chapter to seek a more general solution to this type of minimisation problem. Order R, which is defined in Chapter 1, is shown to minimise the number of sets in a union-closed collection generated by a given number of k-sets in many cases. It is conjectured in Conjecture 3.1 that order R provides a general solution to this problem although a proof of this has not been found for all cases.

One of several approaches tried to prove the conjecture concerning order R is included in this chapter. The method has been only partially successful as there is one case for which it has not been proved. However, this approach appears to be the most promising and so it is presented in full with the incomplete case shown. Hopefully, this case will be completed at some future date. Applications of the truth of the conjecture are included in this chapter along with some properties of order R.

Conjecture 3.1. *For given m and k, let C denote the set of union-closed collections of sets generated by m k-sets. Choosing the first m k-sets in order R simultaneously achieves $min_{\mathcal{A}\in C}|\mathcal{A}|$ and $min_{\mathcal{A}\in C}V(\mathcal{A})$.*

This conjecture can be seen as a union-closed counterpart of the Kruskal-Katona Theorem (see Theorem 3.1) for subsets and its various extensions. The problem for unions is more complicated than that for subsets and it requires care in its statement. In fact several sub-problems naturally arise. Consider the following example.

Example 3.1. To minimise the number of subsets of two 2-sets there is a unique solution up to labelling of elements. This is to choose the 2-sets to be $\{1, 2\}$ and $\{1, 3\}$ rather than $\{1, 2\}$ and $\{3, 4\}$.

To minimise the cardinality of the union-closure of two 2-sets, choosing the 2-sets to be either $\{1, 2\}$ and $\{3, 4\}$, or $\{1, 2\}$ and $\{2, 3\}$, are both valid solutions. In this case there is an argument that the second solution should be regarded as being superior to the first as it has fewer elements in its universal set and it has a smaller volume union-closure.

This leads to variations on the problem such as the following. One could require that it is necessary to minimise the cardinality of the union-closure and then minimise the volume of the union-closure. One could require that a minimum-size universal set is chosen and then require that the minimum possible number of sets is included. In both of these cases the second solution above is the only solution, up to labelling of elements.

3.2 Basic Results

The following classical theorems are required. Recall that $F_k(p)$ denotes the collection of the first p k-sets in squashed order.

Theorem 3.1 (Kruskal 1963, Katona 1966, see [2]). *Let $\mathcal{B}$ be a collection of k-subsets of n. Then*

$$|\triangle\mathcal{B}| \geq |\triangle F_k(|\mathcal{B}|)|$$

Theorem 3.1 means that the collection of the first m k-sets in squashed order has the smallest possible shadow of any collection of m k-sets. Note that squashed order automatically implies the use of a minimum size universal set.

Theorem 3.2 (Clements 1974, see [2]). *For positive integers a, b,*

$$|\triangle F_k(a+b)| \leq |\triangle F_k(a)| + |\triangle F_k(b)|.$$

The following technical lemma is necessary.

Lemma 3.1. *For fixed n, k and r with $k < n-1, r < n$,*

$$\sum_{i=1}^{r} \binom{n-i-1}{k-i+1} = \binom{n-1}{k} - \binom{n-r-1}{k-r}. \tag{3.1}$$

Proof. (3.1) is true for $r = 1$. Assume (3.1) is true for $r-1$, $r \geq 2$. Then for $r \leq k$

$$\begin{aligned}\sum_{i=1}^{r} \binom{n-i-1}{k-i+1} &= \left[\binom{n-1}{k} - \binom{n-r}{k-r+1}\right] + \binom{n-r-1}{k-r+1} \\ &= \binom{n-1}{k} - \binom{n-r-1}{k-r}\end{aligned}$$

as required.

For $r = k+1$, the left-hand side of (3.1) is

$$\sum_{i=1}^{k} \binom{n-i-1}{k-i+1} + \binom{n-k-2}{0} = \binom{n-1}{k} - \binom{n-k-1}{0} + 1$$
$$= \binom{n-1}{k}$$

by (3.1). For $r > k+1$, the right-hand side of (3.1) is $\binom{n-1}{k}$.

Thus the lemma is true for all $r < n$. □

3.3 The Cardinality of a Union-Closed Collection of Sets

The following definitions are applicable in this section only.

Definition 3.1. For a union-closed collection of sets $\mathcal{A}$ with $\bigcup \mathcal{A} = [n]$, let $\mathcal{A}_k = \{A \in \mathcal{A} : |A| = k\}$ and $\mathcal{B}_k = \{B \subset [n], B \notin \mathcal{A} : |B| = k\}$.

That is, $\mathcal{A}_k, \mathcal{B}_k$ respectively denote the collection of k-subsets of $[n]$ which are in $\mathcal{A}$ and the collection of k-subsets of $[n]$ which are not in $\mathcal{A}$. Clearly $|\mathcal{B}_k| = \binom{n}{k} - |\mathcal{A}_k|$.

Definition 3.2. Let $\mathcal{A}'$ denote the union-closed collection on $[n]$ which contains the first m_k k-sets in order R and the maximum possible number m_{k-1} of $(k-1)$-subsets in order R which are allowed in $\mathcal{A}$. Let $\mathcal{A}'_k = \{A \in \mathcal{A}' : |A| = k\}$ and $\mathcal{B}'_k = \{B \subset [n], B \notin \mathcal{A}' : |B| = k\}$.

Note 3.1. The sets in $\mathcal{B}'_k$ can be thought of as being consecutive to the sets in $\mathcal{A}'_k$ with respect to order R. Hence, reversing the order, they are in order T.

3.3.1 Statement of Problems

For fixed positive integers m_k and k let $\mathcal{C}$ be the collection of all possible union-closed collections of sets generated by m_k k-sets. For $\mathcal{A} \in \mathcal{C}$ let $\mathcal{B}$ be the collection of k-sets which generates $\mathcal{A}$. Let $[n] = \bigcup \mathcal{A}$ and throughout this section assume a minimum-size universal set is always used. That is, assume $\binom{n-1}{k} < m_k \leq \binom{n}{k}$. Given m_k and k, the following problems are defined.

Problem 3.1. Choose $\mathcal{B}$ to achieve $\min_{\mathcal{A}\in\mathcal{C}} |\mathcal{A}|$.

Problem 3.2. Choose $\mathcal{B}$ to achieve $\min_{\mathcal{A}\in\mathcal{C}} V(\mathcal{A})$.

Problem 3.3. If it exists, find a simultaneous solution of Problems 3.1 and 3.2.

It will be shown that using order R on k-sets solves Problems 3.1, 3.2 and 3.3 for some values of k, m_k and n. The uniqueness of the solution to these problems is proved in certain cases. The conjectured solution to Problem 3.3 has been given earlier as Conjecture 3.1.

The following lemmas are useful in considering these problems.

Lemma 3.2. *Let $\mathcal{A}_k$ be a collection of m_k k-sets on $[n]$ with $\binom{n-1}{k} < m_k \leq \binom{n}{k}$. Let $\mathcal{B}_k$ be the collection of k-sets on $[n]$ which are not in $\mathcal{A}_k$. Assume the elements of the sets in $\mathcal{A}_k$ are labelled in decreasing order of the number of times they occur in k-sets in $\mathcal{A}_k$ and hence in increasing number of times they occur in $\mathcal{B}_k$.*

Let $\mathcal{A}_{k,\bar{n}} = \{A \in \mathcal{A}_k : n \notin A\}$. Then $|\mathcal{A}_{k,\bar{n}}| > \binom{n-2}{k}$.

Proof. Note that n occurs in more sets in $\mathcal{B}_k$ than does any other element of $[n]$. Let z_i denote the number of k-sets in $\mathcal{A}_k$ which contain $i \in [n]$ and let $Z = \sum_{i=1}^{n-1} z_i$. Thus Z counts the number of occurrences of elements other than n in the sets in $\mathcal{A}_k$. The proof is by contradiction.

Assume $|\mathcal{A}_{k,\bar{n}}| \leq \binom{n-2}{k}$. Then, as $m_k > \binom{n-1}{k}$,

$$z_n > \binom{n-1}{k} - \binom{n-2}{k} = \binom{n-2}{k-1}. \tag{3.2}$$

Hence, as n is assumed to be the element which occurs least often in sets in $\mathcal{A}_k$, $z_i \geq z_n$ for each $i \in [n]$. Thus

$$Z \geq (n-1)z_n. \tag{3.3}$$

As $|\mathcal{A}_{k,\bar{n}}| \leq \binom{n-2}{k}$,

$$Z \leq k\binom{n-2}{k} + (k-1)z_n. \tag{3.4}$$

By (3.3) and (3.4), $(n-1)z_n \leq k\binom{n-2}{k} + (k-1)z_n$, so $(n-k)z_n \leq k\binom{n-2}{k}$.

By (3.2) this means that $(n-k)\binom{n-2}{k-1} < k\binom{n-2}{k}$.

This simplifies to $n-k < n-k-1$ which gives the necessary contradiction to prove the claim. $\square$

Lemma 3.3. *If C is the collection of all union-closed collections of sets on $[n]$ generated by m_k k-sets, $k < n-1$, $\binom{n-1}{k} < m_k \leq \binom{n}{k}$, then for each $\mathcal{A} \in C$,*
(i) $|\bigcup \mathcal{A}| = n$,
(ii) $\mathcal{A}$ contains at least 2 $(n-1)$-sets.

Proof. (i) Since $m_k > \binom{n-1}{k}$, each element of $[n]$ appears in at least one k-set in $\mathcal{A}$. Hence $\bigcup \mathcal{A} = [n]$.
(ii) Assume $\mathcal{A}$ contains at most one $(n-1)$-set. Then there are at least $(n-1)$ $(n-1)$-sets not in $\mathcal{A}$. A contradiction will be obtained by finding bounds on a given sum in two different ways.

Let $\mathcal{A}_{k,\bar{i}}$ be the collection of k-sets in $\mathcal{A}$ which do not contain i, $1 \leq i \leq n$. Then $|\mathcal{A}_{k,\bar{i}}| \leq \binom{n-1}{k}$. Suppose $[n] - \{j\} \notin \mathcal{A}$. Then j dominates some other element of

$[n]$ so $|\mathcal{A}_{k,\bar{j}}| \leq \binom{n-2}{k}$. Hence, if $\mathcal{A}$ contains at most one $(n-1)$-set

$$\sum |\mathcal{A}_{k,\bar{i}}| \leq (n-1)\binom{n-2}{k} + \binom{n-1}{k} = (n-k)\binom{n-1}{k}. \tag{3.5}$$

On the other hand, each k-set occurs in exactly $(n-k)$ of the $\mathcal{A}_{k,\bar{i}}s$. Therefore

$$\sum |\mathcal{A}_{k,\bar{i}}| = (n-k)m_k > (n-k)\binom{n-1}{k} \tag{3.6}$$

as $\binom{n-1}{k} < m_k \leq \binom{n}{k}$ for all $k < n-1$. Inequalities(3.5) and (3.6) give the required contradiction. □

Lemma 3.3 implies that in seeking a solution to Problem 3.1 one need only consider union-closed collections on $[n]$ which contain at least 2 $(n-1)$-sets.

3.3.2 Unions, $(n-1)$-sets and $(k-1)$-sets

This and the following subsections address Problems 3.1 to 3.3 which are considered from the following perspective. If any union-closed collection containing m_j j-sets is allowed to contain at most m_k k-sets, $k < j$, then a union-closed collection containing more than m_k k-sets must contain more than m_j j-sets. Thus, given m_j, the maximum possible value of m_k is desired. The first theorem concerns the case $j = n-1$.

Theorem 3.3. *Let $\mathcal{C}$ be the collection of all union-closed collections of sets on $[n]$ containing at least 2 $(n-1)$-sets. For each $\mathcal{A} \in \mathcal{C}$ assume $|\mathcal{A}_{n-1}| = n-r$, where $0 \leq r < n-1$. Then for $0 < k < n$*
(i) $|\mathcal{A}_k| \leq \binom{n-1}{k} + \binom{n-r-1}{k-r-1}$,
(ii) up to labelling of elements there is a unique choice of $\mathcal{A}$ such that $|\mathcal{A}_k| = \binom{n-1}{k} + \binom{n-r-1}{k-r-1}$. This unique choice of $\mathcal{A}$ has the k-sets in $\mathcal{A}$ in order R.

Proof. (i) If $k = n-1$, $r = 0$ or $|\bigcup \mathcal{A}| < n$ then the result is trivial. Assume that $r > 0, k < n-1$, $|\bigcup \mathcal{A}| = n$ and $[n] - \{i\} \notin \mathcal{A}$ for $1 \leq i \leq r$.

Consider the properties that $\mathcal{A}$ must possess if

$$|\mathcal{A}_k| = \max_{\mathcal{D}\in\mathcal{C}}\{|\mathcal{D}_k| : |\mathcal{D}_{n-1}| = n-r, 0 \le r < n-1, 0 < k < n\}.$$

As $[n]-\{i\} \notin \mathcal{A}$ for each $i \in \{1,\ldots,r\}$ there is an $x_i \in [n]-\{i\}$ such that i dominates x_i in $\mathcal{A}$. Otherwise

$$[n]-\{i\} = \bigcup_{i\notin A_i\in\mathcal{A}} A_i \in \mathcal{A}.$$

The aim is to determine an upper bound on the number of subsets of $[n]-\{1\},\ldots,[n]-\{r\}$ which may be in $\mathcal{A}$. As 1 dominates x_1 there exist at most $\binom{n-2}{k}$ k-subsets of $[n]-\{1\}$ in $\mathcal{A}$. As 2 dominates x_2 there exist at most $\binom{n-3}{k-1}$ k-subsets of $[n]-\{2\}$ in $\mathcal{A}$ which have not already been counted as subsets in $\mathcal{A}$ of $[n]-\{1\}$.

Generally, there are at most $\binom{n-i-1}{k-i-1}$ k-subsets of $[n]-\{i\}$ in $\mathcal{A}$ and which have not been counted as subsets in $\mathcal{A}$ of $[n]-\{1\},\ldots,[n]-\{i-1\}, 1 \le i \le r$. Hence the number of k-subsets of $\mathcal{A}$ which do not contain all the elements $1,2,\ldots,r$ is bounded above by

$$\sum_{i=1}^{r}\binom{n-i-1}{k-i+1} = \binom{n-1}{k} - \binom{n-r-1}{k-r}$$

by Lemma 3.1. Further, there are at most $\binom{n-r}{k-r}$ k-subsets in $\mathcal{A}$ which contain each of $1,\ldots,r$. Therefore the upper bound on $|\mathcal{A}_k|$ is

$$\binom{n-1}{k} - \binom{n-r-1}{k-r} + \binom{n-r}{k-r} = \binom{n-1}{k} + \binom{n-r-1}{k-r-1}.$$

(ii) The bound in (i) can be attained by choosing $\mathcal{A}$ such that in the proof of (i) $x_1 = x_2 = \cdots = x_r = x$ where x is any element of $\{r+1,\ldots,n\}$. One may assume $x = n$. In this case, the k-sets which occur in $\mathcal{A}$ consist of all k-subsets of $[n]-\{n\}$ plus all k-subsets of n which contain each of the elements $1,\ldots,r,n$. This means that the k-sets in $\mathcal{A}$ occur in order R.

To show that the bound is not attained if $x_i \notin \{r+1, \ldots, n\}$ note that the order of counting subsets in the proof of (i) can be permuted into any order of $\{1, 2, \ldots, r\}$. Hence the x_i corresponding to $[n] - \{i\}$ must be an element of $\{r+1, \ldots, n\}$.

Finally the bound is not attained if there exists $x_i \neq x_j$. To see this assume that 1 dominates x_1 and 2 dominates x_2, $x_2 \neq x_1$. Then the bound of $\binom{n-2}{k}$ on subsets of $[n] - \{1\}$ which may be included in $\mathcal{A}$ will not be attained, as the subsets of $[n] - \{1\}$ which contain x_2 and not 2 cannot be included in $\mathcal{A}$. This completes the proof. □

Corollary 3.1. *Let C be the collection of all union-closed collections of sets on $[n]$ containing at least 2 $(n-1)$-sets. Assume $\mathcal{A} \in C$ with $|\bigcup \mathcal{A}| = n$ achieves the bound in Theorem 3.3. Then the k-sets not in $\mathcal{A}, 0 < k < n$, are exactly the k-subsets of $[n]$ which contain each of the elements $\{n-r+1, \ldots, n\}$.*

Proof. This result is an immediate consequence of the construction of the unique collection $\mathcal{A} \in C$ which achieves the bound in Theorem 3.3. □

Let $m_{n,k,r} = \binom{n-1}{k} + \binom{n-r-1}{k-r-1}$ for $1 \leq k \leq n-1$.

Corollary 3.2. *Let C be the collection of all union-closed collections of sets $\mathcal{A}$ on $[n]$ with $|\mathcal{A}_{n-1}| = t$, $2 \leq t \leq n$. Then for each t, there is a chain of integers $m_{n,n-1,t}, \ldots, m_{n,j,t}, \ldots, m_{n,k,t}$, $k \leq j \leq n-1$, such that for each j:*
(i) no more than $m_{n,j,t}$ j-sets are allowed in $\mathcal{A}$;
(ii) the bound in (i) is attained if and only if the j-sets in $\mathcal{A}$ occur in order R.

Proof. This follows immediately from Theorem 3.3. □

Corollary 3.3. *Let C be the collection of all union-closed collections of sets $\mathcal{A}$ on $[n]$ generated by $m_{n,k,t}$ k-sets with $m_{n,k,t}$ assuming the values in Corollary 3.2. Assume $\mathcal{A}$ and $\mathcal{A}'$ are distinct members of C with $\mathcal{A}'$ generated by $m_{n,k,t}$ k-sets in order R. Then* *(i) $\mathcal{A}'$ contains exactly $m_{n,j,t}$ j-sets for each $j \geq k$;*

(ii) $|\mathcal{A}'| < |\mathcal{A}|$ for any $\mathcal{A} \in \mathcal{C}$ not isomorphic to $\mathcal{A}'$; and

(iii) $V(\mathcal{A}') < V(\mathcal{A})$ for any $\mathcal{A} \in \mathcal{C}$ not isomorphic to $\mathcal{A}'$.

Proof. (i) follows from the definition of $m_{n,k,t}$ and Corollary 3.2. (ii) and (iii) follow from (i) and the uniqueness of the choice of sets in Corollary 3.2(ii). □

This means that up to labelling of elements, order R uniquely minimises the cardinality and volume of a union-closed collection of sets on $[n]$ generated by $m_{n,k,t}$ k-sets whenever $m_{n,k,t}$ is as defined in Corollary 3.2(ii).

3.3.3 Unions, k-sets and $(k-1)$-sets

This section provides one framework to attempt to show that a generating collection of m k-sets chosen in order R always solves Problem 3.1. The approach uses a method similar to that used to prove Theorem 3.3. A proof that order R solves Problem 3.1 is provided for a subclass of union-closed collections. This is done in Theorem 3.4. There is one subclass for which this approach has not been proved but an outline of this case and the problem with trying to include this class is given. If the result for this latter class could be finalised, then it would be shown that order R on a collection of k-sets always provides a solution to Problem 3.1 when a minimum size universal set is assumed.

Note 3.2. Note that Corollary 3.2 can be used to define a rooted tree structure. In this structure the internal vertices are labelled by the maximum number of k-sets allowed in $\mathcal{A}$ for each value of $m_{n,n-1,t}$ and $k < n-1$. To be precise, using the notation in Corollary 3.2, the tree has a root vertex of degree $n-1$ which can be labelled m_n and with all other internal vertices having degree 2. The vertices in branch t, $2 \leq t \leq n$ are labelled by the set $\{m_{n,n-1,t}, \ldots, m_{n,j,t}, \ldots, m_{n,k,t}\}$ with $m_{n,k,t}$ being the label of the leaf vertex.

The need for Theorem 3.4 to solve Problems 3.1 to 3.3 rests on the fact that the tree structure defined above has $t < \binom{n}{k}$ vertices at level k, $k < n-1$. It is desired to 'fill' the tree until it has $\binom{n}{k} - \binom{n-1}{k}$ k-sets at level k so that the union-closure of any number of m_k k-sets can be considered for $\binom{n-1}{k} < m_k \leq \binom{n}{k}$.

Consider all choices of union-closed collections of sets on $[n]$ and containing m_k k-sets, with $\binom{n-1}{k} < m_k \leq \binom{n}{k}$. The aim is to show that the collection where the k-sets are in order R achieves the maximum possible number of $(k-1)$-subsets of the k-sets that can be included in the union-closed collection, without adding any more j-sets to the collection, for any $j \geq k$. This is achieved in Theorem 3.4 for Classes 1 and 2 in Definition 3.3.

Definition 3.3. Let $\mathcal{D}$ be the collection of union-closed collections of sets such that for k, m_k and n given, each $\mathcal{A} \in \mathcal{D}$ is generated by a collection of k-sets with $\bigcup \mathcal{A} = [n]$ and $|\mathcal{A}_k| = m_k$ with $\binom{n-1}{k} < m_k \leq \binom{n}{k}$. For each $\mathcal{A} \in \mathcal{D}$ let $\mathcal{A}_{k-1}$ denote a non-empty collection of $(k-1)$-sets which are allowed in $\mathcal{A}$. Let $\mathcal{U} = \{\mathcal{A} \cup \mathcal{A}_{k-1} : \mathcal{A} \in \mathcal{D}\}$.
Class 1 consists of all $(\mathcal{A} \cup \mathcal{A}_{k-1}) \in \mathcal{U}$ where there is an element of $[n]$, say n, which occurs in every k-set not in $\mathcal{A}$ and in every $(k-1)$-set not in $\mathcal{A}_{k-1}$.
Class 2 consists of all $(\mathcal{A} \cup \mathcal{A}_{k-1}) \in \mathcal{U}$ where there is an element of $[n]$, say n, which occurs in every k-set not in $\mathcal{A}$ but not in every $(k-1)$-set not in $\mathcal{A}_{k-1}$.
Class 3 consists of all $(\mathcal{A} \cup \mathcal{A}_{k-1}) \in \mathcal{U}$ where there is no element of $[n]$ which occurs in every k-set not in $\mathcal{A}$.

Let m_{k-1} be as in Definition 3.2 and to simplify notation in the proof of the next theorem the collection $(\mathcal{A} \cup \mathcal{A}_{k-1}) \in \mathcal{U}$ will be referred to simply as $\mathcal{A}$.

Theorem 3.4. *Assume Definition 3.3. Let C be the union of the collections of sets in Class 1 and Class 2. Then $max_{\mathcal{A} \in C}|\mathcal{A}_{k-1}|$ is achieved when $\mathcal{A}$ contains m_k k-sets and m_{k-1} $(k-1)$-sets with both collections in order R.*

Definition 3.4. If $\mathcal{A} \in \mathcal{C}$ achieves the maximum in the statement of the theorem then $\mathcal{A}$ is said to be **best**.

Proof. Assume Definition 3.1. Assume $\mathcal{A} \in \mathcal{C}$ and $\mathcal{A}$ contains m_h $(k-1)$-sets. It is necessary to show that $m_h \leq m_{k-1}$. Note that for each $B \in \mathcal{B}_k$, at most one $(k-1)$-subset of B is allowed in $\mathcal{A}$, as $\mathcal{A}$ is union-closed and $B \notin \mathcal{A}$. Hence there are at least $(k-1)$ $(k-1)$-subsets of B not in $\mathcal{A}$. Hence there exists $x \in B$ such that every $(k-1)$-subset of B containing x is not allowed in $\mathcal{A}$.

Assume the elements of the sets in $\mathcal{A}$ are labelled in increasing order of the number of times they occur in sets in $\mathcal{B}_k$ so that n occurs in the greatest number of sets in $\mathcal{B}_k$. There are two cases corresponding to the two classes in $\mathcal{C}$.

Case 1 (corresponding to Class 1). Assume that $n \in B$ for each $B \in \mathcal{B}_k$ and for each B, no subset of B containing n is allowed in $\mathcal{A}$. Then, for each $B \in \mathcal{B}_k$, at most one $(k-1)$-subset which does not contain n is allowed in $\mathcal{A}$.

Consider $\mathcal{A}'$ and $\mathcal{B}'_k$ as in Definition 3.2. There are $|\mathcal{B}'_k| = |\mathcal{B}_k|$ k-sets not in $\mathcal{A}'$ and these can be arranged in order T (see Note 3.1). Recall that the k-sets in order T are effectively $(k-1)$-sets in squashed order with n attached to each set. Hence, by the Kruskal-Katona Theorem (Theorem 3.1), this ordering minimises the number of $(k-1)$-subsets of $|\mathcal{B}_k|$ k-sets, given that n occurs in each subset. For each $B \in \mathcal{B}_k$, this arrangement also allows in $\mathcal{A}$ exactly one $(k-1)$-subset of B which does not contain n. Further, if $B_1, B_2 \subset [n], B_1 \neq B_2, B_1, B_2 \notin \mathcal{B}_k$, then the $(k-1)$-subsets allowed in $\mathcal{A}$ in the previous sentence are distinct.

Hence $\mathcal{A}'$, which is union-closed, contains at least as many $(k-1)$-sets as $\mathcal{A}$. That is, $m_{k-1} \geq m_h$ as required, and $\mathcal{A}'$ is best.

Case 2 (corresponding to Class 2). Assume n is an element of each set in $\mathcal{B}_k$ but there are sets in $\mathcal{B}_k$ which have a $(k-1)$-subset which contains n and is allowed in $\mathcal{A}$. Note that all the k-sets without n are in $\mathcal{A}_k$. Partition the sets in

$\mathcal{B}_k$ into two parts $\mathcal{P}_1$ and $\mathcal{P}_2$ where $\mathcal{P}_1 = \{A \in \mathcal{B}_k : \forall B \in \{B : B \subset A, |B| = k-1, n \in B\}, B \notin \mathcal{A}\}$ and $\mathcal{P}_2 = \{A \in \mathcal{B}_k : \exists B \in \{B : B \subset A, |B| = k-1, n \in B\}$ such that $B \in \mathcal{A}\}$.

Consider the collection $\mathcal{A}_1$ which compares with $\mathcal{A}$ as follows.
If $A = \{a_1, \ldots, a_{k-1}, n\} \in \mathcal{P}_2$ and $A - \{a_i\} \in \mathcal{A}$, then $A - \{n\} \in \mathcal{A}_1$ and $A - \{a_i\} \notin \mathcal{A}_1$. That is, the $(k-1)$-subsets of sets in $\mathcal{P}_2$ allowed in $\mathcal{A}$ are swapped in $\mathcal{A}_1$ with $(k-1)$-subsets which do not contain n and which are not allowed in $\mathcal{A}$. Here there are exactly $|\mathcal{P}_2|$ distinct $(k-1)$-sets on $\{1, \ldots, n-1\}$ not allowed in $\mathcal{A}$ but in $\mathcal{A}_1$, and at most $|\mathcal{P}_2|$ distinct $(k-1)$-sets containing n which are allowed in $\mathcal{A}$ and not in $\mathcal{A}_1$. Note that $\mathcal{A}_1$ may not be union-closed but that it contains at least as many $(k-1)$-sets as $\mathcal{A}$.

Now compare $\mathcal{A}_1$ with $\mathcal{A}'$. To compare the number of $(k-1)$-sets in these two collections it is sufficient to compare the number of $(k-1)$-sets not in each collection. As n is contained in each set not in $\mathcal{A}_1$ or $\mathcal{A}'$ this is done as in (i). Thus $m_{k-1} \geq m_h$ as required, so $\mathcal{A}'$ is best. □

3.3.4 A Problem

It is desirable to prove that order R is best when considering collections in Class 3. This has been attempted based upon the approach above. The approach runs into difficulties and several variations have been tried to deal with these. Generally speaking, the approach seems to require some sort of shifting argument, perhaps not dissimilar to the argument used in Anderson[2] to prove the Kruskal-Katona Theorem. Here the required shifting argument is more difficult and a valid argument has not been found. A quasi-proof is provided to give an idea of the approach. After the quasi-proof, the part which makes the quasi-proof incomplete as a proof is stated. All definitions and ideas used here relate to Definitions 3.1 and 3.2 and Theorem 3.4.

Proof. (Quasi-proof)

Consider the collection of sets in Class 3. That is, assume there are sets in $\mathcal{B}_k$ which do not contain n. Partition the k-sets in $\mathcal{B}_k$ into t parts $\mathcal{Q}_n, \ldots, \mathcal{Q}_{n-t+1}$ so that $A \in \mathcal{Q}_i$ if and only if i is the largest element of A.

Replace each $\mathcal{Q}_i$ by the same number of k-sets in order T on $\bigcup \mathcal{Q}_i$ to obtain a new collection $\mathcal{Q}'_i$, with i taken as the largest element in the sets in $\mathcal{Q}'_i$. Note that $\mathcal{Q}'_i \cap \mathcal{Q}'_j = \emptyset$ whenever $i \neq j$. Let $\mathcal{A}'$ and $\mathcal{B}'_k$ be as in Definition 3.2 and note that $|\mathcal{B}_k| = |\mathcal{B}'_k|$. Let

$$\mathcal{P}_i = \{A \notin \mathcal{A} : |A| = k+1, A \subset B \text{ for some } B \in \mathcal{Q}_i\},$$

$$\mathcal{P}'_i = \{A \notin \mathcal{A} : |A| = k+1, A \subset B \text{ for some } B \in \mathcal{Q}'_i\},$$

$$\mathcal{P}' = \{A \notin \mathcal{A}' : |A| = k+1, A \subset B \text{ for some } B \in \mathcal{B}'_k\}.$$

Note the following.

1. $|\mathcal{P}_i| - |\mathcal{Q}_i|$ is a lower bound on the number of $(k-1)$-subsets of members of $\mathcal{B}_k$, containing i as the largest element, which are not allowed in $\mathcal{A}$.
2. By the argument in the proof of Case 1 of Theorem 3.4 applied to each $\mathcal{Q}_i$, $|\mathcal{P}'_i| - |\mathcal{Q}'_i| \leq |\mathcal{P}_i| - |\mathcal{Q}_i|$.
3. $|\mathcal{P}'| - |\mathcal{B}_k|$ is the minimum number of $(k-1)$-subsets of k-sets in $\mathcal{B}_k$ which are not allowed in $\mathcal{A}'$.

By applying Clement's result on the subadditivity of the squashed order (Theorem 3.2), $|\mathcal{P}'| - |\mathcal{B}_k| \leq \sum_{i=n-t+1}^{n}(|\mathcal{P}'_i| - |\mathcal{Q}'_i|) \leq \sum_{i=n-t+1}^{n}(|\mathcal{P}_i| - |\mathcal{Q}_i|)$. Hence $\mathcal{A}'$ is best in this case. □

The missing part causing this derivation to be only a quasi-proof occurs at note 2. The crux of the problem is that each Q_i cannot be treated as being independent in the sense that there may be the same sets excluded from $\mathcal{A}$ by sets in different Q_i. Thus there is no guarantee that the shifting argument used will minimise the number of excluded sets as some extra excluded sets may be induced by the shifting.

A simple example helps to illustrate the immediate problem. Assume $n = 8$ and $k = 5$. Assume Q_8 includes the set $\{4,5,6,7,8\}$, Q_7 includes the set $\{3,4,5,6,7\}$, P_8 includes the sets $\{4567, 4568, 4578, 4678\}$ and P_7 includes the sets $\{3457, 3467, 3567, 4567\}$. Assume that the set $\{5,6,7,8\}$ is allowed in the union-closed collection. Then the transformation from Q_8 to Q_8' means that the set $\{5,6,7,8\}$ is now not allowed in the collection but the set $\{4,5,6,7\}$ is also still not allowed because it is in P_7'. Thus, for this intermediate stage, the transformation may increase the number of sets not allowed in the union-closed collection. This situation has not been resolved.

3.3.5 The Final Step?

If Theorem 3.4 can be proved for Class 3 then the following results are valid.

Corollary 3.4. *Assume Theorem 3.4 is true when Class 3 is included in the collection C in Theorem 3.4. Assume $\mathcal{A}$ is a union-closed collection of sets containing m_{k-1} $(k-1)$-sets with $\binom{n-1}{k-1} < m_{k-1} \leq \binom{n}{k-1}$. Assume the following bounds occur for any union-closed collection $\mathcal{A}'$ whose k-sets occur in order R: if there are m k-sets in $\mathcal{A}'$ then there are at most h_1 $(k-1)$-sets in $\mathcal{A}'$; if there are $m+1$ k-sets in $\mathcal{A}'$ then there are at most h_2 $(k-1)$-sets in $\mathcal{A}'$, $h_1 \leq h_2$.*
1. If $h_1 < m_{k-1} \leq h_2$ then $\mathcal{A}$ contains at least $m+1$ k-sets. Further, if the $(k-1)$-sets in $\mathcal{A}$ occur in order R then $\mathcal{A}$ contains exactly $m+1$ k-sets.
2. If $h_1 = m_{k-1} = h_2$ then $\mathcal{A}$ contains at least m k-sets. Further, if the $(k-1)$-sets in $\mathcal{A}$ occur in order R then $\mathcal{A}$ contains exactly m k-sets.

As a complement to Corollary 3.2, the truth of Theorem 3.4 for all classes would yield

Corollary 3.5. *Let C be the collection of all union-closed collections of sets with*

$|\bigcup \mathcal{A}| = n$ *and with* m_k *k-sets,* $\binom{n-1}{k} < m_k \leq \binom{n}{k}$. *Assume Theorem 3.4 is true when Class 3 is included in the collection* $\mathcal{C}$ *in Theorem 3.4. Then for each* m_k*, there is a chain of integers* $m_{k-1}, \ldots, m_{k-j}, \ldots, m_1$ *such that for each* j*,* $1 \leq j \leq k-1$*,*

(i) no more than m_{k-j} $(k-j)$*-sets are allowed in* $\mathcal{A}$*,*

(ii) the bound in (i) is attained if the $(k-j)$*-sets and k-sets in* $\mathcal{A}$ *occur in order* R *for each* j*.*

Proof. This follows immediately from Theorem 3.4. □

This Corollary will be used in the next section to solve Problems 3.1, 3.2 and 3.3 subject to the assumption that Theorem 3.4 is true when the collections in Class 3 are included in the collection $\mathcal{C}$ in Theorem 3.4.

3.4 More on Order R

Some of the properties of collections of sets in order R are examined in this section and one of these is applied to Problems 3.1, 3.2 and 3.3.

Theorem 3.5. *Let* $\mathcal{A}$ *be a union-closed collection of sets generated by the collection* $\mathcal{B}$ *of* m *k-sets in order* R*,* $\binom{n-1}{k} < m \leq \binom{n}{k}$*, Let* $\mathcal{A}'$ *be the union-closed collection of sets generated by the collection* $\mathcal{B}'$ *of* $m+1$ *k-sets in order* R*.*

Assume $m < \binom{n}{k}$ *and* $\mathcal{B}' - \mathcal{B} = \{A\}$ *where* $A = \{x_1, \ldots, x_{k-1}, n\}$*. Then*

(i) $\mathcal{A}'$ *contains* $\binom{n-x_{k-1}-1}{j-k}$ *more j-sets than* $\mathcal{A}$*, for each* $j \geq k$*,*

(ii) $|\mathcal{A}'| = |\mathcal{A}| + 2^{n-x_{k-1}-1}$*.*

Assume $m = \binom{n}{k}$*. Then*

(iii) $\mathcal{A}'$ *contains* $\binom{n+1-k}{j-k}$ *more j-sets than* $\mathcal{A}$*, for each* $j \geq k$*,*

(iv) $|\mathcal{A}'| = |\mathcal{A}| + 2^{n+1-k}$*.*

Proof. Let $k \leq j \leq n$ and assume $m < \binom{n}{k}$ for each j satisfying $k \leq j < n$, and A contains n. Both $\mathcal{A}$ and $\mathcal{A}'$ contain all j-sets on $[n-1]$ for each $j < n$. Hence all supersets of A on $[n]$ are included in $\mathcal{A}'$. As the sets in $\mathcal{B}$ occur in order T, each k-set in $\mathcal{A}$ which contains n also contains an $x \notin A$, $x < x_{k-1}$. Hence no set in $\mathcal{A}$ is a set which contains n and simultaneously contains $x_1, \ldots, x_{k-1}$ and no other element less than x_{k-1}.

Consider the collection of j-sets in $\mathcal{A}'$ which are not in $\mathcal{A}$. The collection must consist of all j-sets of the form $A \cup C$ where C is a $(j-k)$-subset of $\{x_{k-1}+1, \ldots, n-1\}$. Part (i) follows from this observation as there are $\binom{n-x_{k-1}-1}{j-k}$ such j-sets. Part (ii) follows from (i) as $\sum_{j=k}^{n-x_{k-1}-1} \binom{n-x_{k-1}-1}{j-k} = 2^{n-x_{k-1}-1}$.

Let $k \leq j \leq n$ and assume $m = \binom{n}{k}$. Then $\mathcal{A}$ contains all k-sets on $[n]$ and $\mathcal{A}'$ contains all k-sets on $[n]$ plus the k-set $\{1, \ldots, k-1, n+1\}$. No set in $\mathcal{A}$ contains the element $n+1$. Therefore the difference between $\mathcal{A}$ and $\mathcal{A}'$ is that only $\mathcal{A}'$ contains all the j-supersets of $\{1, \ldots, k-1, n+1\}$ on $[n+1]$. There are exactly $\binom{n+1-k}{j-k}$ such supersets for each j and this gives (iii). (iv) follows as $\sum_{j=k}^{n+1-k} \binom{n+1-k}{j-k} = 2^{n+1-k}$. □

Corollary 3.6. *Let $\mathcal{A}$ be a union-closed collection of sets generated by the collection $\mathcal{B}$ of m k-sets in order R, $\binom{n-1}{k} < m \leq \binom{n}{k}$. Assume Theorem 3.4 is true when Class 3 is included in the collection $\mathcal{C}$ in Theorem 3.4. Then the collection of j-sets in $\mathcal{A}$ occur in order R for each $j \geq k$.*

Proof. All j-sets on $[n-1]$ are included in $\mathcal{A}$. By the proof of Theorem 3.5, the j-sets which contain n appear in lexicographic order so the j-sets in $\mathcal{A}$ occur in order R. □

An interesting characteristic of collections in order R is provided in Theorem 3.6 and Corollary 3.7.

Theorem 3.6. *Let $\mathcal{B}$ denote a collection of m k-sets in order R, $\binom{n-1}{k} < m \leq \binom{n}{k}$. For some $x \in [n]$, let $\mathcal{B}_x = \{B \in \mathcal{B} : x \in B\}$. Then $\mathcal{B} - \mathcal{B}_x$ is isomorphic to a collection of sets on $[n-1]$ in order R.*

Proof. The result is immediate if $x = n$ since then $\mathcal{B} - \mathcal{B}_x = P([n-1])$. Assume $x < n$. In this case form a new collection of sets $\mathcal{B}'$ from $\mathcal{B} - \mathcal{B}_x$ by relabelling each y as $y - 1$ for each y with $x < y \leq n$. $\mathcal{B}'$ includes all k-sets on $[n-2]$. The sets in $\mathcal{B}'$ which contain $n-1$ appear in lexicographic order on $[n-1]$. Therefore $\mathcal{B}'$ consists of the first $m - |\mathcal{B}_x|$ k-sets in order R. □

The next Corollary generalises Theorem 3.6 to the union-closed collection generated by $\mathcal{B}$ in Theorem 3.6.

Corollary 3.7. *Let $\mathcal{A}$ be a union-closed collection generated by m k-sets in order R, $\binom{n-1}{k} < m \leq \binom{n}{k}$. Let $\mathcal{A}_x = \{A \in \mathcal{A} : x \in A\}$. Then $\mathcal{A} - \mathcal{A}_x$ is equivalent to a union-closed collection of sets on $[n-1]$ generated by a collection of k-sets in order R.*

Proof. This follows from Corollary 3.6 and Theorem 3.6. □

The next theorem provides the solution to Problems 3.1, 3.2 and 3.3 if Theorem 3.4 is true for all classes.

Theorem 3.7. *Let C be the collection of all union-closed collections of k-sets on $[n]$ generated by m_k k-sets, $\binom{n-1}{k} < m_k \leq \binom{n}{k}$. Let $\mathcal{A}' \in C$ be the union-closed collection generated by m_k k-sets in order R. Assume Theorem 3.4 is true when Class 3 is included in the collection C in Theorem 3.4. Then*

(i) $\mathcal{A}'$ will achieve $min_{\mathcal{A} \in C}|\mathcal{A}_j|$ for each $j > k$.

(ii) The j-sets in $\mathcal{A}'$ occur in order R for each $j > k$.

(iii) $min_{\mathcal{A} \in C}|\mathcal{A}|$ is achieved by $\mathcal{A}'$.

(iv) $V(\mathcal{A}') = min_{\mathcal{A} \in C} V(\mathcal{A})$.

Proof. (i) follows from Corollary 3.5.

(ii) follows from Corollary 3.6.

(iii) follows from (i).

(iv) follows from (i) since $\mathcal{A}'$ contains the minimum possible number of j-sets for each $j \geq k$. □

3.5 A Formula for Union-Closed Collection of Sets

It is a relatively easy matter to calculate the size of a union-closed collection generated by a collection of m k-sets in order R. This is shown in this section.

Theorem 3.8. *Let $\mathcal{A}$ be a union-closed collection of sets generated by a collection $\mathcal{B}$ of m k-sets in order R with $\binom{n-1}{k} < m \leq \binom{n}{k}$. Let $x = \binom{n}{k} - m$. Assume the $(k-1)$-binomial representation of x is $\sum_{i=0}^{t} \binom{a_i}{k-1-i}$, $t < k-1$.*
Then
(i) The last element of $\mathcal{B}$ is the $(x+1)$ k-set in order T.
(ii) $|\mathcal{A}| = 2^n - \sum_{i=0}^{k-1} \binom{n}{i} - \sum_{i=0}^{t} \left(2^{a_i} - \sum_{l=0}^{k-2-i} \binom{a_i}{l}\right)$.

Proof. Note that there are $\sum_{i=0}^{k-1} \binom{n}{i}$ subsets of $[n]$ of size less than k and none of these sets are in $\mathcal{A}$. If $m = \binom{n}{k}$ then $x = 0$ and the theorem is true in this case. Assume $x > 0$.

(i) The k-sets in $\mathcal{A}$ occur in order R and there are x k-sets not in $\mathcal{B}$. As the reverse of order R is order T, the last set in $\mathcal{B}$ must be the $(x+1)$ k-set in order T.

(ii) $|\mathcal{A}|$ can be determined by considering the j-sets which are in $\mathcal{A}$ for each $j \geq k$. As $\mathcal{B}$ contains all k-sets on $[n-1]$, $\mathcal{A}$ contains all j-subsets of $[n-1]$. Therefore, if the set $A = \{a_1, \ldots, a_{k-1}, n\} \in \mathcal{A}$ then all supersets of A are in $\mathcal{A}$.

Conversely, assume $A = \{a_1, \ldots, a_{k-1}, n\} \notin \mathcal{A}$. It is claimed that the only supersets of A which are in $\mathcal{A}$ are exactly those which are also supersets of $B \in \mathcal{B}$ where $B <_R A (\equiv B >_T A)$ and $n \in B$. To see that this claim is true note that any j-set in $\mathcal{A}$ is the union of at least two k-sets in $\mathcal{B}$ and n is contained in any superset of A. Thus if $C \in \mathcal{A}$ is a superset of $A \notin \mathcal{B}$ then C must be a superset of at least one set $D \in \mathcal{B}$ with $n \in D$. As noted above, if $D \in \mathcal{B}$ then all supersets of D are in $\mathcal{A}$, so all sets which are simultaneously supersets of A and D are in $\mathcal{A}$.

Thus the subsets of $[n]$ which are not in $\mathcal{A}$ can be partitioned into two parts. The first part consists of the j-subsets of $[n]$ for $j \leq k-1$. There are $\sum_{i=0}^{k-1} \binom{n}{i}$ such sets. The second part consists of the collection of j-sets, $j \geq k$, which are not supersets of any set in $\mathcal{B}$. As n is in each set in this part, the number of sets in this part corresponds to the number of supersets of the first x $(k-1)$-sets in squashed order which are not supersets of any $(k-1)$-set after the xth set. The number of such supersets is determined below. Note that true squashed order (that is assuming the alphabet is in lexicographic order) is used in the determination to simplify the analysis. This does not affect the numeric answer.

Consider the terms in the $(k-1)$-binomial expansion of x. The term $\binom{a_0}{k-1}$ is the number of $(k-1)$-subsets of $A_0 = \{1, \ldots, a_0\}$. There are $2^{a_0} - \sum_{l=0}^{k-2} \binom{a_0}{l}$ subsets of A_0 of size at least $k-1$. None of these sets are supersets of later sets in squashed order. Therefore this is also the number of supersets of the first $\binom{a_0}{k-1}$ k-sets in order T which are not in $\mathcal{A}$ for $j \geq k$.

The term $\binom{a_1}{k-2}$ is the number of $(k-1)$-sets formed by adjoining $a_0 + 1$ to the $(k-2)$-subsets of $A_1 = \{1, \ldots, a_1\}$. There are $2^{a_1} - \sum_{l=0}^{k-3} \binom{a_1}{l}$ subsets of A_1 with $a_0 + 1$ appended and of size at least $k-1$.

The ith term $\binom{a_{i-1}}{k-i}$ is the number of $(k-1)$-sets which are formed by adjoining $\{a_{i-1} + 1, \ldots, a_0 + 1\}$ to the $(k-i)$-subsets of $A_{i-1} = \{1, \ldots, a_{i-1}\}$. There are

$2^{a_{i-1}} - \sum_{l=0}^{k-i-1} \binom{a_{i-1}}{l}$ subsets of A_{i-1} with the elements $a_{i-1}+1, \ldots, a_0+1$ appended and of size at least $k-1$.

Thus there are $\sum_{i=0}^{t} \left(2^{a_i} - \sum_{l=0}^{k-2-i} \binom{a_i}{l}\right)$ sets in the second part. The value of $|\mathcal{A}|$ follows from removing the sets in each part from the collection of subsets on $[n]$. □

In Theorem 3.8 the last element of $\mathcal{B}$ is the pth k-set in order T where $p = x+1$. It is shown in Lemma 3.4 and Note 3.3 how to determine this set, based upon the $(k-1)$-binomial expansion of p. Lemma 3.4 and the subsequent example is adapted from Anderson [2] (pages 115-117).

Lemma 3.4. *Assume p has the $(k-1)$-binomial representation $\binom{a_{k-1}}{k-1} + \binom{a_{k-2}}{k-2} + \cdots + \binom{a_t}{t}$. Then the pth $(k-1)$-set in squashed order is the set* $\{a_k, a_{k-2}+1, \ldots, a_{t+1}+1; a_t, a_t-1, \ldots, a_t-t+1\}$.

Example 3.2. Let $k-1=3$. As $9 = \binom{4}{3} + \binom{3}{2} + \binom{2}{1}$ the 9th 3-set in squashed order is the set $\{5,4,2\}$. As $5 = \binom{4}{3} + \binom{2}{2}$ the 5th 3-set in squashed order is the set $\{5,2,1\}$. As $4 = \binom{4}{3}$ the 4th 3-set in squashed order is the set $\{4,3,2\}$.

Note 3.3. To find the pth k-set M in order T, the pth $(k-1)$-set on $[n-1]$ in squashed order is found using Lemma 3.4. The alphabet is then reversed and the element n is appended to M. To be more precise, let $A = \{a_1, \ldots, a_{k-1}\}$ be the pth $(k-1)$-set in squashed order on $[n-1]$. When the alphabet is reversed on $[n-1]$ then the pth $(k-1)$-set in squashed order is the set $\{n-a_1+1, \ldots, n-a_{k-1}+1\}$. Then M is the set $\{n-a_1+1, \ldots, n-a_{k-1}+1, n\}$. The last element of $\mathcal{B}$ in Theorem 3.8 is M. This is illustrated in the following example.

Example 3.3. Let $n=6$, $k=4$ and $m=11$ in Theorem 3.8. Then $x = \binom{n}{k} - m = 4$ so that $m \neq x+1 = 5$. The set $\{1,2,5\}$ is the 5th 3-set on $[5]$. If the alphabet is reversed on $[5]$ then the 5th 3-set is the set $\{1,4,5\}$. Thus the 11th 4-set in order R is the set $\{1,4,5,6\}$.

The next example illustrates the calculation of $|\mathcal{A}|$ in Theorem 3.8.

Example 3.4. 1. Assume $m = 52$ and $k = 3$ so that $n = 8$, $x = \binom{8}{3} - 52 = 4 = \binom{3}{2} + \binom{1}{1}$ and $k - 1 = 2$. The four 3-sets not in $\mathcal{A}$ are the sets 876, 875, 865, 874. The sets 876, 875, 865 correspond to the term $\binom{3}{2}$ in the 2-binomial representation of x. As the sets $876, 875, 865 \notin \mathcal{A}$ all the supersets of these containing no element less than 5 are not in $\mathcal{A}$. Similarly, the set $874 \notin \mathcal{A}$, but the sets 864, 854 are each in $\mathcal{A}$ and so all proper supersets of the set 874, and only containing elements greater than or equal to 4 are in $\mathcal{A}$. All j-sets, $j < 3$, are not in $\mathcal{A}$.
Thus $|\mathcal{A}| = 2^8 - \sum_{i=0}^{2} \binom{8}{i} - \left(\left(2^3 - \binom{3}{0} - \binom{3}{1}\right) + \left(2^1 - \binom{1}{0}\right)\right) = 214$.

2. Assume $m = 55$ and $k = 4$ so that $n = 8$, $x = \binom{8}{4} - 55 = 15 = \binom{5}{3} + \binom{3}{2} + \binom{2}{1}$ and $k - 1 = 3$. Then
$|\mathcal{A}| = 2^8 - \sum_{i=0}^{3} \binom{8}{i} - \left(\left(2^5 - \binom{5}{0} - \binom{5}{1} - \binom{5}{2}\right) + \left(2^3 - \binom{3}{0} - \binom{3}{1}\right) + \left(2^2 - \binom{2}{0}\right)\right) = 140$.

Note 3.4. As with many formulas in mathematics there is often a quick way of manually generating the values of the formula when one wants to consider a few special cases. For example, it may be quicker to write out the terms of Pascal's triangle up to and including the nth row than it is to calculate each of the binomial coefficients $\binom{n}{k}$ directly. The analogous situation occurs in the case of order R when one wants to consider just a few values. This is illustrated in the following example where Theorem 3.8 is applied rather than Theorem 3.5. The key point in the example is that when the next set in a generating set in order R is $A = \{a_1, \ldots, a_r, n\}$ then A adds exactly 2^{n-a_r-1} sets to the union-closure when it is included. This means that knowing the form of the next set in order R allows incremental additions to the size of the union-closure to be written very quickly with little calculation.

Example 3.5. If $n = 8$ and $k = 4$ then the additional number of sets in the union-closure generated by m 4-sets, $\binom{7}{4} < m \leq \binom{7}{4} + 15$ can be easily derived by consideration of Note 3.4. For $m = 36, \ldots, 50$ the respective mth 4-sets in order R are: 1238, 1248, 1258, 1268, 1278, 1348, 1358, 1368, 1378, 1458, 1468, 1478, 1568, 1578, 1678. These respectively give the following incremental additions to the size of the union-closed collection of sets as they are included in order R: $16, 8, 4, 2, 1, 8, 4, 2, 1, 4, 2, 1, 2, 1, 1$. Here, for example, the 16 corresponds to the additional sets in the union-closure when the set $\{1, 2, 3, 8\}$ is included, and the second 8 corresponds to the additional sets in the union-closure when the set $\{1, 3, 4, 8\}$ is included.

3.6 The Volume of a Union-Closed Collection of Sets

Theorem 3.7 provides a solution to the problem of minimising the size and volume of a union-closed collection generated by m_k k-sets when a minimum-size universal set is assumed. In this section the restriction on the size of the universal set is removed and the problem of achieving minimum volume is considered in this context. To formalise this, let $\mathcal{C}$ be the collection of union-closed collection of sets generated by m_k k-sets, with no restriction on the size of the universal set $[n]$. This section considers some of the properties that a union-closed collection of sets $\mathcal{A}$ must possess if it is generated by m_k k-sets and $V(\mathcal{A}) = min_{\mathcal{A} \in \mathcal{C}}\{V(\mathcal{A})\}$. The following definitions are used in this section only.

Definition 3.5. Let $\mathcal{C}$ denote the collection of all union-closed collections of sets generated by m_k k-sets.

Definition 3.6. A **reduced** system of sets $\mathcal{A}$ is a non-empty collection of sets such that:

(i) No two elements of $\bigcup \mathcal{A}$ are mutually dominant.

(ii) $\emptyset \notin \mathcal{A}$.

(iii) If $\bigcap \mathcal{A} = \{a\}$ then $\{a\} \in \mathcal{A}$.

Note that (i) implies that $|\bigcap \mathcal{A}| \leq 1$.

The following theorem provides some necessary characteristics of a minimum volume union-closed collection generated by a given number of k-sets.

Theorem 3.9. *Let $\mathcal{C}$ be as in Definition 3.5. If $\mathcal{A} \in \mathcal{C}$ and $\mathcal{A}$ achieves* $\min_{\mathcal{D}\in\mathcal{C}} V(\mathcal{D})$ *then*

(i) $\mathcal{A}$ is a reduced system of sets,

(ii) 'dominates' is an order relation on $\mathcal{A}$,

(iii) if $|\bigcup \mathcal{A}| = t$ then $\mathcal{A}$ contains at least 2 $(t-1)$-sets.

For clarity of argument, the proof of Theorem 3.9 is presented as a sequence of lemmas (Lemmas 3.5 to 3.8).

Lemma 3.5. *Let $\mathcal{C}$ be as in Definition 3.5. If $\mathcal{A} \in \mathcal{C}$ satisfies $V(\mathcal{A}) = \min_{\mathcal{D}\in\mathcal{C}} V(\mathcal{D})$ then $\mathcal{A}$ is a reduced system of sets.*

Proof. Assume $\mathcal{A}$ contains a pair of mutually dominant elements 1, 2. Then each occurrence of 1 and 2 together can be replaced by 1 to form a new collection $\mathcal{A}^* \in \mathcal{C}$, with $V(\mathcal{A}^*) < V(\mathcal{A})$. By the definitions of $\mathcal{A}$ and $\mathcal{C}$, $\emptyset \notin \mathcal{A}$. If $\bigcap \mathcal{A} = \{a\}$ and $\{a\} \notin \mathcal{A}$ then the removal of a from each set in $\mathcal{A}$ forms a new collection $\mathcal{A}^* \in \mathcal{C}$ with $V(\mathcal{A}^*) < V(\mathcal{A})$. Hence the lemma is proved. □

Lemma 3.6. *Let $\mathcal{C}$ be as in Definition 3.5. If $\mathcal{A} \in \mathcal{C}$ satisfies $V(\mathcal{A}) = \min_{\mathcal{D}\in\mathcal{C}} V(\mathcal{D})$ then dominates is an order relation on $\bigcup \mathcal{A}$ in $\mathcal{A}$.*

Proof. Clearly dominates is reflexive and transitive. If $\mathcal{A} \in \mathcal{C}$ satisfies $V(\mathcal{A}) = \min_{\mathcal{D}\in\mathcal{C}} V(\mathcal{D})$ then by Lemma 3.5 $\mathcal{A}$ is a reduced system of sets and therefore dominates is antisymmetric. □

Lemma 3.7. *Let $\mathcal{C}$ be as in Definition 3.5. Let $\mathcal{A}$ be an element of $\mathcal{C}$. Assume T is the largest set in $\mathcal{A}$ and $|T| = t$. Assume that a next largest set in $\mathcal{A}$ has cardinality $t - r, r > 1$. Then $\mathcal{A}$ does not achieve $\min_{\mathcal{D} \in \mathcal{C}} V(\mathcal{D})$.*

Proof. Without loss of generality assume that $T = [t]$ and suppose that a next largest set is $X = \{1, 2, \ldots, t-r\}$. Let $\mathcal{A}_t'$ be the collection of sets in $\mathcal{A}$ which do not contain the element t. Then $X \subset \bigcup \mathcal{A}_t'$ and $t \notin \mathcal{A}_t'$ so $\bigcup \mathcal{A}_t' \neq T$. It follows that $\bigcup \mathcal{A}_t' = X$. Similarly $\bigcup \mathcal{A}_{t-1}' = X$ as $r > 1$. This implies that t and $t-1$ are mutually dominant. Thus $\mathcal{A}$ is not a reduced system of sets and by Lemma 3.5 $\mathcal{A}$ does not achieve $\min_{\mathcal{D} \in \mathcal{C}} V(\mathcal{D})$. □

Lemma 3.8. *Let $\mathcal{C}$ be as in Definition 3.5. If $\mathcal{A}$ satisfies $V(\mathcal{A}) = \min_{\mathcal{D} \in \mathcal{C}} V(\mathcal{D})$ and $|\bigcup \mathcal{A}| = t$ then there are at least 2 $(t-1)$-sets in $\mathcal{A}$.*

Proof. By Lemma 3.7 there is at least one $(t-1)$-set in $\mathcal{A}$. Assume that there is exactly one $(t-1)$-set in $\mathcal{A}$. Let this set be $T - \{t\}$ with $T = \{1, 2, \ldots, t\}$. As $T - \{t\} \in \mathcal{A}$, t dominates no other element of T. As $T - \{i\} \notin \mathcal{A}$ for each $i \neq t$, each element i of $T - \{t\}$ must dominate another element of T.

Note that not all i can dominate t as then t would occur in only one set in $\mathcal{A}$. This would necessarily be the set T. However $\mathcal{A}$ is generated by m_k k-sets, so every element of T must occur in one of the generating sets as well as in T.

Assume some $x_1 \in M$ does not dominate t. Then there is an $x_2 \in M - \{t\}$ which x_1 dominates. In turn x_2 dominates some distinct $x_3 \in M$. By Lemma 3.5, $\mathcal{A}$ is a reduced system of sets. Hence $x_3 \neq t$ or x_1. Continuing in this manner we find each $x_i \in M - \{t\}$ dominates some x_{i+1} with $x_{i+1} \neq x_1, x_2, \ldots, x_i, t$. At some stage a contradiction will arise in this process as follows. Either there must occur an x_i which is mutually dominant with some $x_j \neq x_i$, thus contradicting Lemma 3.5, or there must occur an x_i that dominates t, thus implying that x_1 dominates t. This completes the proof of Lemma 3.8 and Theorem 3.9. □

3.7 $k \leq 3$

It has not been proved that order R is always best for a union-closed collection generated by m k-sets although it is suspected that order R is always best. Some further evidence is presented in this section that order R provides a solution to Problem 3.3.

For $k = 1$ all orderings are equivalent and so the size of any union-closed collection generated by m 1-sets will be the same.

Lemma 3.9 (Simpson [35]). *If $\mathcal{B}$ is a union-closed collection of m 2-subsets of $[n]$ with $\binom{n-1}{2} < m \leq \binom{n}{2}$ then the union-closure of $\mathcal{B}$ contains at least $2^n - 2^x - n$ sets where $x = \binom{n}{2} - m$. This bound is attained if $\mathcal{B}$ consists of the first m 2-sets in order R.*

Proof. Let $\mathcal{B}^*$ be the collection of 2-subsets of $[n]$ which are not in $\mathcal{B}$. Then

$$|\mathcal{B}^*| = \binom{n}{2} - m = x. \tag{3.7}$$

Let $\mathcal{C}$ be the union-closure of $\mathcal{B}$ and $\mathcal{C}^*$ be the union-closure of $\mathcal{B}^*$. It is shown by induction that every k-subset of $[n]$, $k \geq 2$, belongs to either $\mathcal{C}$ or $\mathcal{C}^*$. This clearly holds for $k = 2$. Assume it holds for $k - 1 \geq 2$ and consider an arbitrary k-subset of $[n]$. This has k $(k-1)$-subsets each of which belongs to $\mathcal{C}$ or $\mathcal{C}^*$. Since $k \geq 3$ at least two of these belong to $\mathcal{C}$ or at least two belong to $\mathcal{C}^*$. Since the union of these two $(k-1)$-sets is the k-set, the k-set belongs to either $\mathcal{C}$ or $\mathcal{C}^*$. Thus $|\mathcal{C}| + |\mathcal{C}^*| \geq |\{A \subset [n] : |A| \geq 2\}| = 2^n - \binom{n}{1} - \binom{n}{0}$. Hence

$$|\mathcal{C}| \geq 2^n - n - 1 - |\mathcal{C}^*|. \tag{3.8}$$

Now $|\mathcal{C}^*|$ cannot be greater than the number of non-empty subcollections of $\mathcal{B}^*$. Thus, applying 3.7, $|\mathcal{C}^*| \leq 2^{|\mathcal{B}^*|} - 1 = 2^x - 1$. Substituting in 3.8 gives the required bound. This is the number of sets in the union-closure of the first m 2-sets in order R by Theorem 3.8. □

Note that for $k = 2$ order R and squashed order are the same, so this theorem does not distinguish between the two orders.

Now consider union-closed collections generated by m 3-sets with $5 \leq m \leq 20$ and with a minimum size universal set assumed. The following table shows the size of the union-closed collection in the cases when the m 3-sets are chosen in three common orders, namely lexicographic order, squashed order and order R respectively.

m	5	6	7	8	9	10	11	12	13	14	15	66	17	18	19	20
Lex.	10	11	12	14	15	16	27	29	33	34	36	37	38	40	41	42
Sq.	9	11	13	14	15	16	24	28	32	34	36	37	38	40	41	42
R	9	11	12	14	15	16	24	28	30	31	35	37	38	40	41	42

Clearly, choosing order R generates the smallest size union-closed collection for each value of m in the table. Note that the table shows that there are cases when order R is not unique in achieving the minimum cardinality and minimum volume of a union-closed collection. For example, when $m = 6$ the collections in order R and lexicographic order are not isomorphic and yet both collections generated have the same cardinality and volume. To see this let $\mathcal{A} = \{123, 124, 125, 134, 135, 145\}$ and $\mathcal{A}' = \{123, 124, 134, 234, 125, 135\}$. $\mathcal{A}$ contains the first 6 3-sets in lexicographic order. $\mathcal{A}'$ contains the first 6 3-sets in order R on [5]. The sets in $\mathcal{A}$ are neither in order R nor isomorphic to a collection in order R. The 4-sets and 5-sets in $\mathcal{A}$ are 1234, 1235, 1245, 1345, 12345. The 4-sets and 5-sets in $\mathcal{A}'$ are 1234, 1235, 1245, 1345, 12345. Thus $|\mathcal{A}| = |\mathcal{A}'|$ and $V(\mathcal{A}) = V(\mathcal{A}')$.

3.8 An Application of Order R to UCS

Evidence has been provided which suggests that order R minimises both the cardinality and the volume of the sets in union-closed collections generated by a given number of k-sets, assuming a minimum-size universal set. As yet a complete proof of this conjecture has not been determined.

In this section improvements are made to the known lower bounds on a minimum-size counterexample to the union-closed sets conjecture under the assumption that Conjecture 3.1 is true for all values of m and k.

In Chapter 2 it is shown that if $\mathcal{A}$ is a minimum size counterexample to UCS then $n \geq 8$ and $|\mathcal{A}| \geq 41$. This bound on $|\mathcal{A}|$ is improved in this section. Symbols used in this section, other than m and k, are as defined in Chapter 2. In particular $\mathcal{A}$ stands for a minimum size counterexample to UCS.

Theorem 3.10. *If Conjecture 3.1 is true for all values of m and k then in a minimum size counterexample to UCS:*
(i) If $n = 8$, $|\mathcal{A}| \geq 95$;
(ii) If $n = 9$, $|\mathcal{A}| \geq 51$;
(iii) If $n = 10$, $|\mathcal{A}| \geq 43$.

Proof. Assume Conjecture 3.1 is true for all values of m and k. By Theorem 2.7 it can be assumed that $|\mathcal{A}| \geq 41$.
(i) Assume $n = 8$. Recall that $|A| \geq 3$ for all $A \in \mathcal{A}$ and $|\mathcal{A}| = 4n + 2r + 1 = 33 + 2r$. Further

$$|\mathcal{D}| = 3n + 2r = 24 + 2r \tag{3.9}$$

by the definition of $\mathcal{D}$, (see Section 2.1.2).

$$V(\mathcal{D}) \leq n(n + r) = 64 + 8r \tag{3.10}$$

by Corollary 2.3.

Now suppose that $\mathcal{D}$ contains d 3-sets and $|\mathcal{D}| - d$ sets of cardinality greater than 3. Then $V(\mathcal{D}) \geq 3d + 4(|\mathcal{D}| - d) = 4|\mathcal{D}| - d$. Thus $d \geq 4|\mathcal{D}| - V(\mathcal{D}) \geq 4(24+2r)-(64+8r) = 32$. Thus $\mathcal{A}$ contains at least 32 3-sets. If Conjecture 3.1 is true then the union-closure of 32 3-sets contains at least 95 sets by Theorem 3.8. Thus $|\mathcal{A}| \geq 95$.

A similar argument to that used in (i) proves (ii) and (iii). For (ii), with $r = 2, \ldots, 6$ $\mathcal{A}$ must contain at least 21 3-sets. If Conjecture 3.1 is true then the union-closure of 21 3-sets contains at least 58 sets by Theorem 3.8. Thus $|\mathcal{A}| \geq 58$. Note that if $r \geq 7$ so that $|\mathcal{A}| \geq 51$ then no contradiction arises from this argument alone.

For (iii), with $r = 0$, $\mathcal{A}$ must contain at least 20 3-sets. If Conjecture 3.1 is true then the union-closure of 20 3-sets contains at least 42 sets by Theorem 3.8. Thus $|\mathcal{A}| \geq 42$. As noted in Chapter 2, $|\mathcal{A}|$ is odd, so $|\mathcal{A}| \geq 43$. □

Corollary 3.8. *In a minimum size counterexample to UCS* $|\mathcal{A}| \geq 43$.

Proof. The Corollary follows from Theorem 3.10 and Theorem 2.4. □

These new results on a minimum size counterexample to UCS are limited. It seems that an approach to UCS is needed which is different from the minimum size counterexample approach.

3.9 Comments

It seems highly desirable to determine if a collection of k-sets chosen in order R does always achieve a minimum size union-closed collection. It is hoped that the partial solutions and approaches in this chapter can be used as building blocks for determining the validity of Conjecture 3.1 for all values of m and k.

It should be noted that considerable time has been spent in researching the validity of Conjecture 3.1. One aspect of the problem on which considerable time was spent, and which has not been dealt with in this chapter, is to show that the minimum size union-closed collection generated by m k-sets will always be achieved on a minimum size universal set. This particular aspect of Conjecture 3.1 is not considered in this thesis.

Another interesting point to note concerns the relationship (via de Morgan's Laws) between union-closed collections and intersection-closed collections. It might be conjectured that if it is always known how to achieve the minimum size union-closed collection generated by m k-sets then it will also be known how to minimise the size of the intersection-closed collections generated by a collection of m $(n-k)$-sets. This conjecture is not necessarily true for two reasons.

First, it is conjectured that choosing a generating collection of m k-sets from a minimum size universal set minimises the union-closure of any collection of m k-sets. The corresponding result is certainly not true for intersection-closed collections, as the intersection-closure of m k-sets is minimised when all of the k-sets are disjoint.

This observation leads to the second reason. Assume that a minimum size union-closed collection $\mathcal{C}$ generated by m k-sets is chosen with $\binom{n-1}{k} \leq m \leq \binom{n}{k}$. Consider the complementary collection $\mathcal{C}'$ generated by a collection of m $(n-k)$-sets defined by $\mathcal{C}' = \{A \subset [n] : A' \in \mathcal{C}\}$. Then it is not always the case that $\binom{n-1}{n-k} \leq m \leq \binom{n}{n-k}$ so that a minimum size universal set is not always being used for the given number of $(n-k)$-sets.

The example below illustrates these ideas. That is, the example illustrates the point that knowing how to choose m k-sets to generate a minimum size union-closed collection on a minimum size universal set does not automatically provide a solution to the problem of choosing m $(n-k)$-sets to generate a minimum size intersection-closed collection on a minimum size universal set.

Example 3.6. The problem of minimising the size of an intersection-closed collection generated by a collection of five 2-sets is considered in two different ways.

Consider the intersection-closed collection generated by five 2-sets on a minimum-size universal set. There are six 2-sets on $[4]$, so one can consider the intersection-closed collection $\mathcal{E}$ generated by the set $\mathcal{D} = \{12, 13, 23, 14, 24\}$.
Here $\mathcal{E} = \{\emptyset, 1, 2, 3, 4, 12, 13, 23, 14, 24\}$ Thus $|\mathcal{E}| = 10$.

Now assume that $n = 5$, $m = 5$ and $k = 3$ in the discussion on the second reason above. Then a minimum size union-closed collection is generated by the sets in $\mathcal{A} = \{123, 124, 134, 234, 125\}$. Consider the complementary collection $\mathcal{A}' = \{54, 53, 52, 51, 43\}$. Here $\bigcup \mathcal{A}' = [5]$ and the intersection-closure of $\mathcal{A}'$ is the set $\mathcal{B} = \{\emptyset, 3, 4, 5, 34, 15, 25, 35, 45\}$ with $|\mathcal{B}| = 9$.

Clearly $|\mathcal{B}| < |\mathcal{E}|$ so $\mathcal{A}'$ generates a smaller intersection-closed collection than does $\mathcal{D}$. However, $\mathcal{A}'$ is not defined on a minimum size universal set given that $|\mathcal{A}'| = 5$.

Thus $\mathcal{A}$ minimises the size of the union-closed collection generated by five 3-sets but the complementary collection $\mathcal{A}'$ is not a valid solution to the problem of minimising the size of an intersection-closed collection generated by five 2-sets if the condition that a smallest size universal set is assumed when defining the generating set for the intersection-closed collection.

The remainder of the thesis deals with completely separating systems and antichains.

Chapter 4

(n, k)Completely Separating Systems, Part 1

4.1 Introduction

The major part of the remainder of this thesis deals with completely separating systems. All necessary definitions are included in Chapter 1.

A question which arose whilst considering union-closed collections was stated in Section 1.1 and it is restated here in a more general form. Let $\mathcal{B}$ be a collection of subsets of $[n]$. What properties must $\mathcal{B}$ possess if its union-closure contains all of the $(n-1)$-subsets of $[n]$? The answer is that $\mathcal{B}$ must be a completely separating system. More generally it can be asserted that $\mathcal{B}$ being a completely separating system is both necessary and sufficient to ensure that all of the $(n-1)$-sets are contained in the union-closure of $\mathcal{B}$. It can also be asserted that $\mathcal{B}$ being a completely separating system is both necessary and sufficient to ensure that all of the 1-subsets of $[n]$ are contained in the intersection-closure of $\mathcal{B}$. For those with an interest in finite topolgies this means that a completely separating system is the subbasis for the T_1 (discrete) topology on a finite set. Similarly, a separating system is always a subbasis for a T_0 topology on a finite set.

The purpose of this chapter is to present some new results about minimum size completely separating systems. Most of the results in this chapter have already appeared in [23] or [24]. The connections between completely separating systems, separating systems and finite topologies will be considered outside this thesis. The generic problem considered in this chapter and several later chapters is to find completely separating systems of various types and which have the smallest size in terms of containing the least number of sets for each type. The chapter begins with a brief mention of some of the historical developments in the study of completely separating systems.

In 1961 Rényi [26] raised the problem of finding minimum separating systems in the context of solving certain problems in information theory. Subsequently, several variants have been treated in the literature. (See, for example, [4, 5, 6, 8, 12, 36, 39].) Completely separating systems were introduced by Dickson [8]. Dickson [8] showed that $R(n) \sim \log_2 n$. Spencer [36] obtained the sharper result in Lemma 4.1 by exploiting the duality of CSSs and antichains and by applying the well-known Sperner's theorem which is stated next.

Theorem 4.1 (Sperner). *The maximum size of an antichain on an r-set is* $\binom{r}{\lfloor r/2 \rfloor}$.

Lemma 4.1 (Spencer).

$$R(n) = min\{r : \binom{r}{\lfloor r/2 \rfloor} \geq n\}. \tag{4.1}$$

Proof. Let $\mathcal{C} = \{A_1, \dots, A_r\}$ be a completely separating system on $[n]$. The dual of $\mathcal{C}$ is an antichain consisting of n sets on $[r]$. By Theorem 4.1 such an antichain exists if and only if $n \leq \binom{r}{\lfloor r/2 \rfloor}$. □

Cai [5] has shown that

Lemma 4.2.

$$R(n, 1, k) = \lceil 2n/k \rceil, \quad \textit{if } n > k^2/2 \geq 2.$$

It is obvious that $R(n) \leq R(n,1,k) \leq R(n,2,k) \leq \cdots \leq R(n,k-1,k) \leq R(n,k)$. Combining this observation with Lemma 4.1 the following lower bound on $R(n,k)$ is obtained.

Lemma 4.3. $R(n,k) \geq min\{r : \binom{r}{\lfloor r/2 \rfloor} \geq n\}$.

4.2 First Results

The following lemma formalises a result on CSSs which has already been informally stated in Section 1.4.

Lemma 4.4. *If $h > 1$ and $\mathcal{C}$ is a (n,h,k)CSS then evey element of $[n]$ occurs at least twice in $\mathcal{C}$. In particular if $k > 1$ and $\mathcal{C}$ is a (n,k)CSS then evey element of $[n]$ occurs at least twice in $\mathcal{C}$.*

Proof. It is clear that each $a \in [n]$ must occur at least once in $\mathcal{C}$, say in the set A. As $h > 1$, a must reoccur at least once more in $\mathcal{C}$ so that it is completely separated from the other elements of A. □

Lemma 4.5. *For $1 \leq h \leq k < n$, $R(n,h,k) = R(n,n-k,n-h)$.*
In particular $R(n,k) = R(n,n-k)$.

Proof. Let $\mathcal{C}$ be a minimal (n,h,k)CSS with $|\mathcal{C}| = R$. The complementary CSS $\mathcal{C}'$ contains R sets with $n-k \leq |B| \leq n-h$ for all $B \in \mathcal{C}'$. Thus $R(n,n-k,n-h) \leq R$. If $R(n,n-k,n-h) = T < R$ then there exists a $(n,n-k,n-h)$CSS $\mathcal{D}$ in T sets. Then $\mathcal{D}'$ is a (n,h,k)CSS in T sets which contradicts the minimality of $\mathcal{C}$. Therefore $R(n,n-k,n-h) = R$. Letting $h = k$ provides the particular case for $R(n,k)$. □

The importance of this result lies in the fact that one normally need only consider values of $k \leq n/2$ when trying to determine the size of minimal (n,k)CSSs for $1 \leq k \leq n-1$.

Lemma 4.6. *For all* $2 \leq k < n$,

$$R(n,k) \geq \left\lceil \frac{2n}{k} \right\rceil . \tag{4.2}$$

Proof. For $k > 1$, every element of $[n]$ must appear in at least two k-sets of any (n,k)CSS by Lemma 4.4. Thus $k \times R(n,k) \geq 2n$, from which (4.2) follows. □

A trivial upper bound on $R(n,k)$ is

$$R(n,k) \leq n, \tag{4.3}$$

which follows from consideration of the set system

$$\{\{1,\ldots,k\},\{2,\ldots,k+1\},\ldots,\{n,1,\ldots,k-1\}\}.$$

The following lemma gives an improvement on Lemma 4.6 for some values of k and n.

Lemma 4.7. *For* $2 \leq k < n$,

$$R(n,k) \geq \left\lceil \frac{n}{k} \left\lceil \frac{n-1}{n-k} \right\rceil \right\rceil . \tag{4.4}$$

Proof. For $k \geq 2$, every element a of $[n]$, in a (n,k)CSS, must occur without each of the other $n-1$ elements of $[n]$. Each time a occurs in a k-subset, it is separated from $n-k$ elements of $[n]$. Therefore a must occur in at least $\lceil \frac{n-1}{n-k} \rceil$ sets. This value replaces the constant 2 in Lemma 4.6. □

Note that (4.4) is also trivially valid for $k = 1$ (see Lemma 4.9). To use Lemma 4.7 for $k' < n/2$, Lemma 4.5 and then Lemma 4.7 for $k = n - k'$ can be used. This provides a better bound than Lemma 4.6 only for those k near $n/2$. For example, if $n = 16, k' = 7$ then $R(n,k') \geq \lceil 2n/k' \rceil = 5$ by Lemma 4.6. Lemma 4.7 gives $R(16,9) \geq 6$, so $R(16,7) \geq 6$ by Lemma 4.5.

Lemma 4.8. *For all n and k, $R(n,k) \geq R(k) + 1$.*

Proof. Let $\mathcal{C}$ be a minimal (n,k)CSS and $A \in \mathcal{C}$. The k elements of A need to be completely separated in $\mathcal{C} - A$. Thus $\mathcal{C} - A$ contains at least $R(k)$ sets, so $\mathcal{C}$ contains at least $R(k) + 1$ sets. □

Lemma 4.9. *For $n \geq 2$, $R(n,1) = n$. For $n \geq 3$, $R(n,2) = n$.*

Proof. That $R(n,1) \leq n$ follows from (4.3). That $R(n,1) \geq n$ follows from the observation that each element of $[n]$ must appear in at least one set. Now consider the case when $k = 2$. That $R(n,2) \leq n$ follows from (4.3). For the lower bound use Lemma 4.6. □

Note that Lemmas 4.5 and 4.9 immediately give $R(n, n-1) = n$ for $n \geq 2$ and $R(n, n-2) = n$ for $n \geq 3$.

For a collection of k-sets $\mathcal{C}$ on the set $[n]$, let $\mathcal{C}[x] = \{A \in \mathcal{C} : x \in A\}$. Recall that $\mathcal{C}$ is a (n,k)CSS if and only if $\bigcap\{A \in \mathcal{C}[x]\} = \{x\}$, for all $x \in [n]$. In particular, if for all $x \in [n]$ there are two sets $A, B \in \mathcal{C}$ such that $A \cap B = \{x\}$ then $\mathcal{C}$ is a (n,k)CSS. This criterion is used in the proof of the next theorem which ensures $R(n,k)$ is known for a given k for all but a finite number of values of n.

Theorem 4.2. *If $n \geq k(k-1)$, $k \geq 3$ then*

$$R(n,k) = \left\lceil \frac{2n}{k} \right\rceil.$$

Proof. Let $n = kp + r$ where $0 \leq r < k$. Then $p = \lfloor \frac{n}{k} \rfloor \geq k - 1$. The strategy is to create a (n,k)CSS so that kp elements appear in exactly two sets, and the remaining elements appear at least twice. The cases $p \geq k$ and $p = k - 1$ require slightly different constructions, and each of these cases is broken into subcases depending on whether $r = 0$, $0 < r \leq k/2$, or $r > k/2$. Let M be a p by k array whose jth column consists of the integers $1 + (j-1)p, 2 + (j-1)p, \ldots, jp$ in that order.

Assume $p \geq k$. Form one set system $\mathcal{R} = \{R_1, R_2, \ldots, R_p\}$ by taking the rows of M, and another $\mathcal{C} = \{C_0, C_1, \ldots, C_{p-1}\}$ by successively stripping k elements from the columns of M from top-to-bottom and left-to-right. To be more precise, define

$$C_j = \{1 + jk, 2 + jk, \ldots, k + jk\} \quad \text{for } 0 \leq j \leq p - 1 \text{ and}$$

$$R_i = \{i, i + p, \ldots, i + (k-1)p\} \quad \text{for } 1 \leq i \leq p.$$

Note that $\mathcal{C}$ is a partition of $[kp]$, as is $\mathcal{R}$. Furthermore $|R_i \cap C_j| \leq 1$ for all i and j. Hence $\mathcal{C} \cup \mathcal{R}$ is a (kp, k)CSS. Thus if $r = 0$ the result follows.

Now assume that $r > 0$. To get a (n, k)CSS the remaining elements must be taken into account. Assume they are the elements of E, where $E = [n] - [kp]$. Take D to be the diagonal elements of M. That is,

$$D = \{1, 2 + p, \ldots, k + (k-1)p\} = \{i + (i-1)p : 1 \leq i \leq k\}.$$

Now modify the sets in $\mathcal{R}$ by removing an element of D (if possible) from each set R_i and replacing it with an element of E so that each element of E gets used, as close as possible, the same number of times. To be unambiguous, let $\mathcal{R}' = \{R'_1, R'_2, \ldots, R'_p\}$ where

$$R'_i = \begin{cases} (R_i - \{i + (i-1)p\}) \cup \{kp + 1 + ((i-1) \bmod r)\} & \text{if } 1 \leq i \leq k, \\ R_i & \text{if } k < i \leq p. \end{cases}$$

There are two subcases, $r \leq k/2$ and $r > k/2$.

(i) If $r \leq k/2$ then $\mathcal{S} = \mathcal{R}' \cup \mathcal{R} \cup \{D\}$ is a (n, k)CSS of size $2p+1$. To show that $\mathcal{S}$ is a CSS, observe that for each element $x \in [n]$, there are two sets containing x whose intersection is $\{x\}$. If $x \in D$ then the two sets are D and the set C_j that contains x (recall that $\mathcal{C}$ is a partition of $[kp]$). If $x \in [kp] - D$, then use the two sets C_j and R'_i that contain x, as in the $r = 0$ case. If $x \in [n] - [kp]$, then the two sets are R'_{x-kp} and R'_{x-kp+r}.

(ii) If $r > k/2$, then there are some elements of E that have occurred only once in $\mathcal{S}$, so another set is formed containing these singletons. The elements $1 + kp, 2 + kp, \ldots, k - r + kp$ have all appeared twice, the elements $1 + k - r + kp, 2 + k - r + kp, \ldots, n$ have appeared only once. Define X to be the singletons together with the first $2(k - r)$ elements of D. That is,

$$\begin{aligned} X &= \{1 + k - r + kp, 2 + k - r + kp, \ldots, n\} \\ &\quad \cup \{1, 2 + p, \ldots, r + (2(k - r) - 1)p\}. \end{aligned} \tag{4.5}$$

Now $\mathcal{S} \cup \{X\}$ is a (n, k)CSS of size $2p+2$. This follows from the same arguments given in the $r \leq k/2$ case, except for those elements $x \in X - D$ (that is the singletons). Those elements are dealt with by noting that $X \cap R'_{x-kp} = \{x\}$.

Assume $p = k - 1$. Let N be a k by k array constructed from M by adding an extra bottom row identical to the first row of M but rotated left one position. Now form a set system $\mathcal{R} = \{R_2, R_3, \ldots, R_{k-1}\}$ by taking all rows of N except the first and last, and another set system $\mathcal{C} = \{C_0, C_1, \ldots, C_{k-1}\}$ consisting of the columns of N. To be more precise, define

$$C_j = \{1 + j(k-1), 2 + j(k-1), \ldots, k + j(k-1)\} \quad \text{for } 0 \leq j \leq k-1 \text{ and}$$

$$R_i = \{i, i + k - 1, \ldots, i + (k-1)(k-1)\} \quad \text{for } 1 \leq i \leq k-1.$$

The element $k + (k-1)^2$ of C_{k-1} is taken to be 1. The set R_1 will be used only when $r > 0$.

Note that $C_j \cap C_{j+1}$ (index addition taken mod k) is $\{k + j(k-1)\}$ and that otherwise the intersection of two sets in $\mathcal{C}$ is empty. The union of all sets C_j is $[pk]$. The elements $k + j(k-1)$ do not occur in $\mathcal{R}$. In fact, $\mathcal{R}$ forms a partition of $[n] - \{k + j(k-1) : 0 \leq j \leq k-1\}$. Since $|R_i \cap C_j| \leq 1$ for all i and j, the union $\mathcal{C} \cup \mathcal{R}$ is a (pk, k)CSS. Thus if $r = 0$ the result follows.

Now assume that $r > 0$. To obtain a (n, k)CSS the remaining elements must be taken into account. Assume that they are the elements of E, where $E = [n] - [kp]$. Again, there are two subcases, $r \leq k/2$ and $r > k/2$.

(i) If $r \leq k/2$ then set, for $0 \leq j \leq k-1$,

$$C'_j = (C_j - \{k + j(k-1)\}) \cup \{1 + kp + (j \bmod r)\}.$$

Now the intersection of two sets in $\mathcal{C}' = \{C'_0, C'_1, \ldots, C'_{k-1}\}$ is either empty or is an element of E. It is easy to verify that the set system $\mathcal{S} = \mathcal{C}' \cup \mathcal{R} \cup R_1$ is a (n, k)CSS of size $2k - 1$, as required.

(ii) If $r > k/2$ then define X as in (4.5). The set system $\mathcal{S} \cup \{X\}$ is a (n, k)CSS of size $2k$, as required. The verification is left to the reader. □

4.3 Particular Values of $R(n, k)$

Lemma 4.10. *For $k = 3$, $R(4, 3) = 4$, $R(5, 3) = 5$, and $R(n, 3) = \lceil 2n/3 \rceil$ if $n \geq 6$.*

Proof. Note that $R(4, 3) = R(4, 1) = 4$ and $R(5, 3) = R(5, 2) = 5$, using Lemmas 4.5 and 4.9. For $n \geq 6$, the result follows from Theorem 4.2. □

Lemma 4.11. *For $k = 4$, $R(5, 4) = 5$, $R(6, 4) = 6$, $R(7, 4) = 5$, $R(8, 4) = 5$, $R(9, 4) = 6$, $R(10, 4) = 5$, $R(11, 4) = 6$ and $R(n, 4) = \lceil n/2 \rceil$ if $n \geq 12$.*

Proof. The values for $R(5,4)$, $R(6,4)$, and $R(7,4)$ follow from Lemma 4.5 and the earlier results for $k = 1,2,3$. By Lemma 4.1 $R(4) = 4$. By Lemma 4.8 $R(8,4) \geq 5$. To see that $R(8,4) = 5$ consider the set system
$\mathcal{C} = \{1234, 1678, 2578, 3568, 4567\}$.

By Lemma 4.6, $R(9,4) \geq 5$. It is shown that $R(9,4) \neq 5$ as follows. Assume that the set $A = \{1,2,3,4\}$ occurs in a $(9,4)$CSS $\mathcal{C}$ containing 5 sets. The elements 1,2,3,4 need to be completely separated from one another in the four sets other than A. It can be seen that this cannot be done with each other set in $\mathcal{C}$ containing exactly one element of A by noting that $R(5,3) = 5$, so that the elements 5,6,7,8,9 cannot be completely separated from one another in these four sets. Thus $\mathcal{C}$ contains a set other than A which contains more than one element of A.

Note that $V(\mathcal{C}) = 20$ and by Lemma 4.4 each element of [9] must occur at least twice in $\mathcal{C}$. Thus no more than two elements of $[n]$ can occur more than twice in $\mathcal{C}$ and this means that no more than two elements of A can occur together in a set other than A. Thus it may be assumed the set 1256 occurs. Then, to separate 1 and 2, 1 must occur in one set and 2 in another. Similarly, to separate 5 and 6, 5 must occur in one set and 6 in another. Hence, one of the pairs 15, 16, 25, 26 occurs again. Assume it is the pair 15. To separate these, 5 must occur in another set without 1. This means that each of 1,2,5 occur in at least 3 sets which contradicts the fact that only two such elements can exist. Hence, $R(9,4) \neq 5$. To see that $R(9,4) = 6$, consider the set system {1234, 1235, 1467, 2568, 3789, 4569}.

That $R(10,4) = 5$ follows from Lemma 4.6 and the set system {1234, 1567, 2589, 368A, 479A}.

That $R(11,4) = 6$ follows from Lemma 4.6 and the set system {1234, 4567, 789A, B368, B259, 156A}.

For $k \geq 12$, the result follows from Theorem 4.2. □

Lemma 4.12. *$R(n, 5) = 6$ for $n = 6, 8, 9$. $R(7, 5) = 7$.*

Proof. The stated values of $R(n,k)$ follow from the values of $R(n, n-k)$ as stated in previous lemmas and by applying Lemma 4.5. □

4.4 More Results on $R(n, k)$

Theorem 4.2 says that the lower bound on $R(n,k)$ can be achieved for sufficiently large n compared to k. The following lemma says that it is not possible to achieve the bound on $R(n,k)$ in Lemma 4.6 for sufficiently small n compared to k in the case when $k \mid 2n$. That is, at least one element of $[n]$ must be used more than twice in a CSS in this case.

Lemma 4.13. *If $n < \binom{k+1}{2}$, $1 < k < n$, and $k \mid 2n$, then $R(n,k) > \lceil 2n/k \rceil$.*

Proof. As $k \mid 2n$, $\lceil 2n/k \rceil = 2n/k$. If $R(n,k) \leq 2n/k$ then $R(n,k) = 2n/k$ by Lemma 4.6. Suppose that $R(n,k) = \frac{2n}{k}$. Thus each element of $[n]$ occurs exactly twice in a CSS. Without loss of generality it can be assumed that $\{1, \ldots, k\}$ is a member of a (n,k)CSS. To completely separate these elements, using each only once more, each element of $[k]$ must appear in a set which contains no other element of $[k]$ and so there are at least $k+1$ sets in the (n,k)CSS. As each element of $[n]$ occurs twice, $2n \geq k(k+1)$ which contradicts the hypothesis of the lemma and the result follows. □

The next two lemmas show how one can use a (n,k)CSS to construct CSSs for larger values of n and k.

Lemma 4.14.
If $R(n,k) \leq k+1$, $1 < k < n$, then $R(n+k+1, k+1) \leq k+2$.

Proof. Let $\mathcal{A} = \{A_1, A_2, \dots, A_{k+1}\}$ be a (n,k)CSS in $(k+1)$ sets. Note that a (n,k)CSS can be extended as necessary, whilst maintaining the complete separation property, by adding arbitrary k-subsets of $[n]$. Consider the set system $\mathcal{C} = \{C_0, C_1, \dots, C_{k+1}\}$ where $C_0 = \{n+1, n+2, \dots, n+k+1\}$ and $C_i = A_i \cup \{n+i\}$ for $i = 1, 2, \dots, k+1$. It is easy to verify that $\mathcal{C}$ is a $(n+k+1, k+1)$CSS in $k+2$ sets. □

Lemma 4.15. *If $R(n,k) \geq k+1$, $1 < k < n$, then $R(n+k+1, k+1) \leq 1 + R(n,k)$.*

Note 4.1. CSSs which illustrate the construction described in the proof of this lemma appear in Subsection 5.2.3.

Proof. Set $R = R(n,k)$ and let $\{A_1, \dots, A_R\}$ be a minimal $(n,k)CSS$. A $(n+k+1, k+1)$CSS containing $R+1$ sets will be constructed. Consider the collection of sets $\mathcal{B} = \{B_0, \dots, B_R\}$ defined by:

(1) $B_0 = \{n+1, \dots, n+k+1\}$;

(2) For $i = 1, \dots, k+1$ set $B_i = A_i \cup \{n+i\}$;

(3) For $i = k+2, \dots, R$ set $B_i = A_i \cup \{b_i\}$ where b_i is an element of B_0 chosen inductively as follows.

For each $i \geq k+2$ let $C_i = \{x \in A_i : x$ has appeared at least once in $B_1, \dots B_{i-1}\}$. For $x \in C_i$ let $B_{i(x)}$ be the first member of $B_1, \dots, B_{i-1}$ which contains x. Set $q(x) = B_0 \cap B_{i(x)}$. Note that $B_0 \cap B_{i(x)}$ contains a single element so $q(x)$ is well defined. Now set $Q(i) = \{q(x) : x \in C_i\}$. Note that $Q(i)$ is a proper subset of B_0 since $|Q(i)| \leq |C_i| \leq |A_i| = k$, whereas $|B_0| = k+1$. Finally choose b_i to be an element of $B_0 - Q(i)$.

It is now shown that $\mathcal{B}$ is a $(n+k+1, k+1)$CSS. Each set in $\mathcal{B}$ has cardinality $k+1$ so this part of the definition is satisfied. It is necessary to show that each pair of elements are separated from one another.

Consider each pair of elements $a, b \in [n+k+1]$. The elements of $[n]$ are completely separated by the sets $B_1, \dots, B_R$ as $\{A_1, \dots, A_R\}$ is a CSS on $[n]$ and $b_1, \dots, b_R \notin [n]$. If $a, b \in \{n+1, \dots, n+k+1\}$ then each belongs to a unique set among $B_1, \dots, B_{k+1}$. These sets are distinct and so completely separate a and b.

If $a \leq n$ and $b \geq n+1$ then b appears in B_0 without a. It is necessary to also find a set which contains a but not b. It is known that a occurs in at least two of the sets $A_1, \dots, A_R$. Suppose that the first two sets in which a occurs are A_i and A_j. Then a also occurs in both $B_i \cup \{b_i\}$ and $B_j = A_j \cup \{b_j\}$. The method of choosing b_i and b_j ensures that $b_i \neq b_j$ and so at least one of b_i and b_j is not b. The associated set separates a from b. □

Lemma 4.16. *Let $p + q = n$. Then $R(n,k) \leq R(p,k) + R(q,k)$.*

Proof. A (p,k)CSS $\mathcal{P}$ on $[p]$ and a (q,k)CSS $\mathcal{Q}$ on $[n] - [p]$ may be used to create a (n,k)CSS $\mathcal{C}$ by taking $\mathcal{C} = \mathcal{P} \cup \mathcal{Q}$. □

Lemma 4.17. *For $n \geq 1$, $R(4n, 2n) \leq 2 + R(2n, n)$.*

Proof. Assume that $\mathcal{C} = \{S_1, S_2, \dots, S_m\}$ is a $(2n,n)$CSS where $m = R(2n,n)$. Construct $\mathcal{A} = \{A_1, A_2, \dots, A_m, A_{m+1}, A_{m+2}\}$ as follows. If $i \in [m]$ then $A_i = S_i \cup (S_i + 2n)$. (Here $(S_i + 2n)$ denotes the set defined by adding the number $2n$ to each element of S_i). The other two sets are formed by taking $A_{m+1} = [2n]$ and $A_{m+2} = [2n] + 2n$. It is now argued that $\mathcal{A}$ has the desired properties . Assume $p, q \in [2n]$. If S_i separates p from q then A_i also separates p from q. Assume $p, q \in [2n] + 2n$. If S_i separates $p - 2n$ from $q - 2n$ then A_i separates p from q. Assume $p \in [2n]$ and $q \in [2n] + 2n$. Then A_{m+1} separates p from q. Assume $p \in [2n] + 2n$ and $q \in [2n]$. Then A_{m+2} separates p from q. Thus, in all cases p is separated from q, and so $\mathcal{A}$ is a $(4n, 2n)$CSS of size $2 + R(2n,n)$. □

As a consequence it can be shown, for example, that $R(12,6) \leq 6$, $R(16,8) \leq 7$, $R(20,10) \leq 8$, $R(24,12) \leq 8$ and $R(28,14) \leq 10$.

Corollary 4.1. $R(2^m, 2^{m-1}) \leq 2m$.

Proof. The proof is by induction, the base case being $R(2,1) = 2$ and the inductive step following from Lemma 4.17. □

Note that the bound of the previous lemma is not tight since, for example, $R(8,4) = 5 < 6$. However, it does suggest logarithmic growth in $R(n,k)$ for $k = n/2$. The final lemma provides a bound on the rate of growth of $R(n,k)$ with respect to n, for a fixed k.

Lemma 4.18. *(i) If $n \geq 2k-2$, $1 < k < n$, then $R(n+1,k) \leq 2 + R(n,k)$,*
(ii) If $n \geq 2k-3$, $1 < k < n$, then $R(n+2,k) \leq 3 + R(n,k)$,
(iii) If $n \geq 3k-6$, $1 < k < n$, then $R(n+3,k) \leq 3 + R(n,k)$.

Proof. Begin with a minimal (n,k)CSS $\mathcal{C}$.
(i) Append to $\mathcal{C}$ the sets $\{1,\ldots,k-1,n+1\}$ and $\{k,\ldots,2k-2,n+1\}$.
(ii) Append to $\mathcal{C}$ the sets $\{1,\ldots,k-1,n+1\}$, $\{1,\ldots,k-1,n+2\}$ and $\{k,\ldots,2k-3,n+1,n+2\}$.
(iii) Append to $\mathcal{C}$ the sets $\{1,\ldots,k-2,n+1,n+2\}$, $\{k-1,\ldots,2k-4,n+2,n+3\}$ and $\{2k-3,\ldots,3k-6,n+3,n+1\}$. □

The results in this chapter provide a partial solution of the $R(n,k)$ problem. Together with the results in Chapter 5, $R(n,k)$ is fully determined for $k \leq 6$ and for many other values of n for $k > 6$. These values appear in Table 5.5 at the end of Chapter 5.

Chapter 5

(n, k)Completely Separating Systems, Part 2

5.1 Introduction

It is the purpose of this chapter to derive more results about minimal completely separating systems. The results build on those obtained in Chapter 4 and most of the results also appear in [22].

In Chapter 4 it is shown that $R(n,k) \geq \lceil 2n/k \rceil$, $k > 1$, and that this bound can be achieved for $n \geq k(k-1)$. This result is extended here to show that $R(n,k) = \lceil 2n/k \rceil$ for $n > k^2/2$, except when $n = \binom{k+1}{2} - 1$. The solution to the $R(n,k)$ problem is also extended to include all $n > \binom{k}{2} - \frac{k}{3}$. The proofs of these results are constructive, thus sample minimal CSSs are included for each value of n and k. Constructive examples and values for $R(n, 2, k)$ are also included in the proofs.

The results in this chapter allow the complete solution of the $R(n, k)$ problem for the $k \leq 6$ and $k \geq n - 6$ cases and they allow the determination of the value of $R(n,k)$ in many other cases. A table of known values or known bounds for $R(n,k)$, based upon the results in Chapter 4 and this chapter, appears as Table 5.5 at the end of this chapter. Throughout this chapter it is assumed that $k > 1$, so that every element of $[n]$ must occur at least twice in any (n,k)CSS.

5.2 Main Result

The main result of this chapter is the following theorem.

Theorem 5.1. *For $k < n$:*
(i) If $n \geq \binom{k+1}{2}$, $k \geq 2$, then $R(n,k) = \lceil 2n/k \rceil$;
(ii) If $n = \binom{k+1}{2} - 1$, $k \geq 3$, then $R(n,k) = k+2 = \lceil 2n/k \rceil + 1$;
(iii) If $k^2/2 \leq n \leq \binom{k+1}{2} - 2$, $k \geq 5$, then $R(n,k) = k+1$ with $R(n,k) = \lceil 2n/k \rceil$ except for $n = k^2/2$;
(iv) If $\binom{k}{2} \leq n < k^2/2$, $k \geq 5$, then $R(n,k) = k+1 > \lceil 2n/k \rceil$.

Theorem 5.1 is a compilation of several lemmas and theorems. These results, and their proofs, are presented in Sections 5.2.1–5.2.5. Section 5.3 presents an alternative construction for parts (iii) & (iv) of Theorem 5.1, using Lemma 4.14.

An immediate outcome of Theorem 5.1 is the determination of $R(n,5)$ for all values of n. To see this, recall that $R(n,5)$ was determined in Lemma 4.12 for $n \leq 9$. Theorem 5.1 yields the following Corollary for $k = 5$.

Corollary 5.1. $R(n,5) = 6$ *for* $10 \leq n \leq 13$, $R(14,5) = 7$ *and* $R(n,5) = \lceil 2n/5 \rceil$ *for* $n \geq 15$.

5.2.1 Bounds on n for $R(n,k) \leq k$

In this section some useful results concerning minimal (n,k)CSSs are given as well as showing that if $n > \binom{k}{2} - k/3$, then $R(n,k) > k$. This lower bound on $R(n,k)$ is used in the proof of Theorem 5.1(iv) and is also an improvement on the lower bound $\lceil 2n/k \rceil$ for $n < k^2/2$.

Theorem 5.2. *Let $\mathcal{C}$ be a $(n,k)CSS$ with $|\mathcal{C}| \leq k$. Then each set in $\mathcal{C}$ contains at most $|\mathcal{C}| - 5$ 2-elements.*

Proof. Suppose that $\mathcal{A} = \{A_1, \ldots, A_{|\mathcal{C}|}\}$ is a $(n,k)CSS$ with $|\mathcal{C}| \leq k$ in which one set, say A_1 contains $s \geq |\mathcal{C}| - 4$ 2-elements, say $1, \ldots, s$. A_1 will contain $k-s$ other elements, say $x_1, \ldots, x_{k-s}$. The elements $1, \ldots, s$ must occur in s distinct sets other than A_1 to be completely separated from one another. Assume that i occurs in A_{i+1} for $i = 1, \ldots, s$. It is not possible for any of $x_1, \ldots, x_{k-s}$ to belong to any of $A_2, \ldots, A_{s+1}$, for if $x_j \in A_i, i \in [2, s+1]$, then x_j and $i-1$ would not be completely separated. Thus $x_1, \ldots, x_{k-s}$ need to be separated by the $|\mathcal{C}| - s - 1$ sets $A_{s+2}, \ldots, A_{|\mathcal{C}|}$. Note that $|\mathcal{C}| - s - 1 \leq 3$ by assumption and $R(m) = m$ for $m \leq 3$ by Lemma 4.1. This means that the collection $\{A_{s+2}, \ldots, A_{\mathcal{C}}\}$ can completely separate at most $|\mathcal{C}| - s - 1$ elements, so $k - s \leq |\mathcal{C}| - s - 1$. This implies that $k \leq |\mathcal{C}| - 1$, contradicting the hypothesis that $|\mathcal{C}| \leq k$. Thus no set can contain more than $|\mathcal{C}| - 5$ 2-elements. □

An immediate corollary of this theorem is the following result.

Corollary 5.2. *Assume $R(n,k) = R \leq k$, $k \geq 6$. Then at most $\frac{R(k-5)}{2}$ elements of $[n]$ occur exactly twice in a minimal $(n,k)CSS$.*

Proof. By Theorem 5.2, a minimal (n,k)CSS contains at most $(R-5)$ 2-elements in each of its sets. Therefore there are at most $\frac{R(R-5)}{2} \leq \frac{R(k-5)}{2}$ 2-elements in any minimal (n,k)CSS for $k \geq 6$. □

Lemma 5.1. *If $R(n,k) = R \leq k$, $k \geq 6$, then $n \leq R(3k-5)/6 \leq \binom{k}{2} - k/3$.*

Proof. Let $\mathcal{C}$ be a minimal (n,k)CSS with $k \geq 6$ and $R(n,k) \leq k$. By Corollary 5.2 at most $\frac{R(k-5)}{2}$ elements of $[n]$ occur twice in $\mathcal{C}$. Each of the other elements occur at least 3 times. Thus the volume of $\mathcal{C}$ is at least $\frac{2R(k-5)}{2} + 3(n - \frac{R(k-5)}{2}) = 3n - \frac{R(k-5)}{2}$. However, the volume of $\mathcal{C}$ is kR. Thus $kR \geq 3n - \frac{R(k-5)}{2}$ which implies that $n \leq \frac{R(3k-5)}{6}$. The second inequality follows upon substitution of k for R and rearrangement of the expression. □

Incorporating the results for the $k \leq 5$ cases and applying Lemma 5.1 yields

Lemma 5.2. *If $n > \binom{k}{2} - k/3$, then $R(n,k) > k$.*

5.2.2 Proof of Theorem 5.1.(i)

The first step is to show that the lower bound of $\lceil 2n/k \rceil$ can be achieved for $n \geq \binom{k+1}{2}$. Example constructions of minimum CSSs for $n \geq \binom{k+1}{2}$ appear at the end of this subsection. Once the method of construction is known it is straightforward to obtain a CSS for each value of n and k, $n \geq \binom{k+1}{2}$.

The proof that the construction works is not simple. This section describes the construction and, through a series of lemmas and notes, proves the validity of the construction for all n and k satisfying $n \geq \binom{k+1}{2}$.

In the process, constructions for minimal $(n,2,k)$CSSs are provided for each $n \geq \binom{k+1}{2}$. In this case the constructions also provide strongly minimal $(n,2,k)$CSSs, that is minimum volume minimal $(n,2,k)$CSSs. This result appears as Theorem 5.4. The initial step in the proof of Theorem 5.5 is given below as Construction M.

Construction M.
Assume $n \geq \binom{k+1}{2}$ and let $R = \lceil 2n/k \rceil$. An $R \times k$ array M will be constructed

where the R rows of M form a (n,k)CSS. Let m_{ij} denote the element of M in row i column j. Initialise all elements of M to zero. Note that with the given assumptions, $R = k+1$ only when $n = \binom{k+1}{2}$, and $R \geq k+2$ otherwise.

For each m, in lexicographic order, include m in turn in the two positions of M defined by:

$$\min_j \min_i \{m_{ij} : m_{ij} = 0\},$$

$$\min_i \min_j \{m_{ij} : m_{ij} = 0\}.$$

That is, m is placed in the first row of M containing 0, in the first 0-valued place in that row. m is then also placed in the first column of M containing 0, in the first 0-valued place in that column. Clearly M is sufficiently large to do this. This concludes Construction M.

Now consider the special case when $n = \binom{k+1}{2}$, $k > 1$.

Theorem 5.3. *If $n = \binom{k+1}{2}$, $k > 1$, then $R(n,k) = \lceil 2n/k \rceil = k+1$.*

Proof. Using Construction M note that $m_{ij} = m_{j+1,i}$ for all $1 \leq i \leq j \leq k$, and that there are no 0-valued elements left in M. Hence each element of $[n]$ appears in exactly two positions in M and, given any pair of elements (m_{ij}, m_{il}) occurring in the same row i of M, m_{ij} and m_{il} occur once more in two different rows. Hence, by Lemma 4.4, the rows of M form a minimal (n,k)CSS. □

Corollary 5.3. *If $n = \binom{k+1}{2}$, $k > 1$, then $R(n,2,k) = k+1$.*

Proof. In the minimal (n,k)CSS used in the proof of the theorem, each element appears twice. Hence, this example is a minimal $(n,2,k)$CSS and the result follows. □

For the remainder of this subsection assume $n > \binom{k+1}{2}$. The following notes and technical lemmas concern Construction M after the above replacement of $2n$

0-valued elements has occurred.

Note 5.1. The elements of M in a given row may be partitioned into 3 parts, allowing for some of these parts to be empty. For a given row r let H or H_r denote the set of consecutive integers in row r which are occurring for the first time in M using Construction M. Let D or D_r denote the set of integers in row r which are the second occurrence of those integers in M using Construction M. Let B or B_r denote the 0-valued elements or row r. Let h_r denote the least element of H_r.

Note 5.2. $|B| = 0$ for any row above a row which contains a non-empty set H. This follows immediately from the construction. Hence there is in M at most one row with both $|H|$, $|B| > 0$.

Note 5.3. $|H_r| \leq |H_{r-1}|$ for all $r > 1$. This follows immediately from the construction.

Lemma 5.3. *For* $n > \binom{k+1}{2}$*:*
(i) in row r, and in terms of column order, all elements of D_r *occur before all elements of* H_r *which occur before all elements of* B_r*;*
(ii) $|H_R| = 0$.

Proof. (i) For any row r and any $m \in D_r$ with $m \in H_s$ for some $s < r$, it must be that m occurs in row r before any element of H_r.

Assume $r < R$. In Construction M, if an element h of H_r is included in M at m_{rj}, then $m_{r+1,j}$ is 0 at this stage. Therefore, the latest occurrence of the second occurrence of h in M is in column j. Hence no second occurrence of an element of H_r can occur in row r after H_r. Hence all elements of D_r occur before all elements of H_r. It is clear that no 0-valued element can occur in row r before a non-zero element. The case $r = R$ is dealt with in (ii).

(ii) Assume $|H_R| > 0$. By Note 5.2, $|B_{R-1}| = 0$. By Note 5.3, $|H_{R-1}| \geq |H_R|$. By Construction M, if h_{R-1} is $m_{R-1,j}$ then $m_{R-1,j-1}$ is non-zero and equal to an element of D_{R-1}. Hence the first column in row R where an element of H_{R-1} can be placed the second time is at least column $j-1$. Therefore the elements of H_{R-1}, when they have been placed for the second time, leave at most one 0-valued element in row R of M. This contradicts the choice of the size for M as it then leaves insufficient places in M for the insertion of elements of H_R in at least 2 positions in M. It follows that $|H_R| = 0$ and that part (i) of the lemma is true for $r = R$. □

Note 5.4. Hence, in Construction M, the rows of M in numeric order can be partitioned into non-empty collections with parts $\mathcal{H}$, $\mathcal{DH}$, $\mathcal{DHB}$, $\mathcal{DB}$ or $\mathcal{H}$, $\mathcal{DH}$, $\mathcal{DB}$ where, for example, the collection $\mathcal{H}$ represents the set of rows in M with $|H| > 0$ and $|D| = |B| = 0$. $\mathcal{DH}$ represents the set of rows of M with $|D|, |H| > 0$ and $|B| = 0$.

The next lemma is pivotal to the proof of Theorem 5.1. It shows that in each row i of M there are at least $|H_i|$ rows below row i.

Lemma 5.4. *For $n > \binom{k+1}{2}$, in each row i of M, $|H_i| \leq R - i$.*

Note 5.5. This lemma implies that if h_i is in column j then h_{i+1} is in column j or $j+1$.

Proof of Lemma 5.4. The proof involves four claims.

Claim 5.1. *The first row of M with $|H_r| > R - r$, if such a row exists, is not the first row i of M for which h_i is m_{ij} with $j < i$.*

Proof of Claim 5.1. Note that M has a sequence of rows for which $h_i = m_{ij}$ with $i = j$. Let r be the first row for which $j < i$. By Note 5.3 $|H_r| \leq |H_{r-1}|$ for all r. As $R > k+1$, row $r-1$ has at least $|H_{r-1}| + 1$ rows below it in M. Therefore row r has at least $|H_r|$ rows below it. □

Claim 5.2. *The first row of M with $|H_r| > R - r$, if such a row exists, is not a row t in M with D_t, H_t, B_t each non-empty.*

Proof of Claim 5.2. Assume row t has each of D, H and B non-empty and suppose that row t is the first for which $|H_t| > R - t$. By Note 5.3 $H_t \leq |H_{t-1}|$. By Note 5.2 $|B_{t-1}| = 0$. By assumption $|B_t| > 0$. Therefore $|H_t| < |H_{t-1}|$. Hence, as row $t-1$ has at least $|H_{t-1}|$ rows below it, row t has at least $|H_t|$ rows below it. □

Claim 5.3. *Each row r with $|D_r|, |H_r| > 0$ and $|B_r| = 0$ has at least $|H_r|$ rows below it in M.*

Proof of Claim 5.3. Assume row r is the first row of M with $|H_r| > R - r$, $|D_r| > 0$, $|B_r| = 0$. Assume $h_r = m_{rj}$.

The following assumptions may be made:

(1) $|H_r| > 1$. This follows from the assumption that $|H_r| > 0$ and $|H_r| \neq 1$ as $r \neq R$ by Lemma 5.3.2. Hence there is at least one row below row r in M.

(2) $m_{r,j-1}$ is equal to an element of H_i for some $i \leq r - 1$. This follows from the construction and the assumed position of h_r in M.

(3) $|H_r| = |H_{r-1}|$. This follows from Notes 5.2 & 5.3, Lemma 5.3 and the choice of r.

This means that row $r-1$ has exactly $|H_{r-1}|$ rows below it. Thus $m_{r,j-1}$ must be equal to an element of H_{r-1} and $m_{r-1,j-1}$ must be equal to an element of H_{r-2}. Therefore the first occurrence of an element of H_{r-1} below row $r-1$ is at $m_{r,j-1}$. That is, h_{r-1} occurs at $m_{r,j-1}$ for the second time.

(4) h_{r-2} occurs at $m_{r-2,j-1}$. To see this, suppose h_{r-2} is at $m_{r-2,l}$. If $l > j$, this contradicts Note 5.3. If $l = j$, then by the above discussion, the element at

$m_{r-1,j-1}$ is an element of H_{r-2} while the element at $m_{r-2,j-1}$ is an element of H_{r-3}. This implies that $|H_{r-2}| = 1$, contradicting assumption (1) and Note 5.1. If it is assumed that $l < j-1$, then $|H_{r-2}| > R-(r-2)$, contradicting the choice of r. Thus, h_{r-2} is at $m_{r-2,j-1}$.

The proof of Claim 5.3 needs Claim 5.4.

Claim 5.4. *With the assumptions stated immediately above for each $i < r$, row i has exactly $|H_i|$ rows below it in M.*

Claim 5.4 immediately leads to a contradiction as $|H_1| = k$ and $R > k+1$. That is, row 1 has more than k rows below it. Thus Claim 5.3 is proved, once Claim 5.4 is proved. □

Proof of Claim 5.4. Use induction on i. The claim is true for $i = r-1$ by assumption (3). Assume that the claim is true for all i, $p \leq i \leq r-1$. Assume h_{p+1} is at $m_{p+1,q}$.

It is clear that $m_{p+1,q-1}$ is not equal to an element of H_{p+1}, and as H_p has exactly $|H_p|$ rows below it, $m_{p+1,q-1}$ must be equal to an element of H_p. Further, $m_{p,q-2}$ is not an element of H_p. Assume $|H_{p-1}| = |H_p|$. Then $m_{p-1,q-2}$ is not an element of H_{p-1}.

Therefore the first ocurrence of an element of H_{p-1} below row $p-1$ is in column $q-2$ at or below row p. As $|H_{p-1}| = |H_p|$ and $m_{p,q-1}$ is an element of H_p, the first occurrence of an element of H_p below row p is at $m_{R,q-2}$ or at $m_{p+1,q-1}$.

If the first occurrence of an element of H_p below row p is at $m_{R,q-2}$ then the first occurrence of an element of H_{p+1} is at $m_{R,q-1}$. As each row i, $p \leq i \leq r-1$, has exactly $|H_i|$ rows below it, h_{r-1} is at $m_{R,j-2}$. This contradicts Assumption (3).

If the first occurrence of an element of H_p below row p is at $m_{p+1,q-1}$, then the first occurrence of h_{r-1} is at m_{rj}. Again this contradicts Assumption (3).

Thus $|H_{p-1}| > |H_p|$. If $|H_{p-1}| > |H_p| + 1$ then H_{p-1} has less than $|H_{p-1}|$ rows below it, contradicting the choice of r. Hence $|H_{p-1}| = |H_p| + 1$ and Claim 5.4 is proved. □

This completes the proof of Lemma 5.4. □

Theorem 5.4. *For $n > \binom{k+1}{2}$ the rows of M in the Construction M, ignoring the 0-valued elements, form a minimal $(n, 2, k)$CSS.*

Proof. Consider the rows of M as being sets consisting of the non-zero valued elements in the rows. Each element of $[n]$ occurs exactly twice in M, and $R = \lceil \frac{2n}{k} \rceil$.

By taking $n = pk + r$, $0 \leq r < k$, it is easily seen that the number of 0-valued elements of M are at most $Rk - 2n \leq k - 2$. Therefore M has no singleton set, and it can be concluded that $R(n, 2, k) \geq \lceil \frac{2n}{k} \rceil$. It needs to be shown that the rows of M, ignoring the 0-valued elements, form a completely separating system. For any D_i, each element of D_i is separated from each element of H_i as each element of D_i occurs above row i and no element of H_i occurs above row i.

By Lemma 5.4 the elements of D_i will each occur in different rows above row i as members of different H sets. By Lemma 5.4 the elements of H_i appear in different rows of M below row i. Therefore each of these are separated from one another. The elements of H_i are also separated from all elements of D_i as no element of D_i appears below row i in M. Thus the rows of M from a completely separating system. □

Lemma 5.5. *Assume $n > \binom{k+1}{2}$.*

(i) Each row of M containing a non-empty H has the least element of H in position m_{ij} where $j \leq i$.

(ii) Let $t \leq k$ be the last row in M with $|H| \neq 0$. Then the least element of H_t is in position m_{tj} where $j < t$.

Proof. (i) Note that by Lemma 5.4, if h_{r-1} is $m_{r-1,j}$ then h_r is m_{rj} or $m_{r,j+1}$. Hence as h_1 is in column 1 the result is true for all subsequent rows.

(ii) Assume the conditions of Lemma 5.5.2. By part (i), $j \leq t$. To show that $j < t$, assume $j = t$. Then by Note 5.5 and Lemma 5.5.1 $|H_r| = k - r + 1$ for all $r \leq t$. Therefore $n = \sum_{i=1}^{t} |H_i| \leq \sum_{i=1}^{k} i = \binom{k+1}{2}$. This contradicts $n > \binom{k+1}{2}$. □

Lemma 5.6. *Let* $n > \binom{k+1}{2}$. *Let row* t *be the last row* t *of* M *with* $|H_t| > 0$. *Then* $|D_t| \leq t - 2$.

Proof. Let row t be as in the statement of the lemma. Lemma 5.4 ensures that the second occurence of elements in a given H do not occur in adjacent positions in the same row. Hence if h_i is at m_{ij} then h_{i+1} is at $m_{i+1,j}$ or at $m_{i+1,j+1}$. This, together with Lemma 5.5.2, implies that $|D_t| \leq t - 2$ when $t \leq k$. If $t > k$ then as $|D_t| \leq k - 1$, $|D_t| \leq t - 2$. □

Lemma 5.7. *If* $n > \binom{k+1}{2}$ *and* $k|2n$ *then the rows of* M *form a minimal* (n, k)*CSS.*

Proof. If $k|2n$ then M has no 0-valued elements at this stage. Hence, as shown in Theorem 5.4, the rows of M form an appropriate completely separating system.

□

Lemma 5.8. *If* $n > \binom{k+1}{2}$ *and* $k \nmid 2n$ *then the 0-valued elements in* M *can be replaced by elements of* $[n]$ *to form a minimal* (n, k)*CSS.*

Proof. The 0-valued elements of M need to be replaced whilst ensuring that the complete separation property is maintained. This is done in numeric order of the rows. Consider two cases, with t defined as in Lemma 5.6 to be the last row of M with $|H_t| > 0$.

(a) Assume that row t of M has $|H_t| \neq 0$ and $|B_t| \neq 0$. By Note 5.2 there is at most one of these rows. Each element of D_t occurs in exactly one row above row

t. No element of H_t occurs in a row above t. By Lemma 5.6 there are at least $|D_t| + 1$ rows above row t. Hence there is a row r above row t which contains no element of row t. The elements of row r will be used to fill B_t.

It must be ensured that the elements of row r used in B_t are already separated from the elements of row t. Note that at this stage any two elements of row r are already completely separated in M.

The elements of D_t appear in exactly $|D_t|$ rows of M above row t. Hence they occur with at most $|D_t|$ different elements of row r. These elements of row r cannot be used to place in B_t as this would destroy the complete separation property.

It is necessary to ensure that the elements of H_t are separated in a row below row t from the set of elements of row r used to replace B_t. To do this note that the elements of H_t occur in exactly $|H_t|$ different rows below row t. Hence at most $|H_t|$ elements of row r occur with elements of H_t in these lower rows.

Thus there are at least $k - |D_t| - |H_t| = |B_t|$ elements of row r that can be used to replace the 0-valued elements in row t, whilst maintaining the complete separation property.

(b) Consider any row s of M with $|H_s| = 0$ and $|B_s| > 0$. Then $s > t$ by Note 5.2. By Lemmas 5.4 and 5.6, $|D_s| \leq |D_t| + 1 \leq t - 1$. Then if $|B_t| = 0$ there is at least one row r at or above row t which contains no element of row s. Each element of row s occurs in at most one row at or above row t and hence with at most $|D_s|$ elements of row r. Hence there are at least $k - |D_s| = |B_s|$ elements of row r which can be used to replace the elements of $|B_s|$ whilst maintaining the complete separation property.

Note that $|D_s| = |D_t| + 1$ if and only if D_s contains an element of H_t. Therefore, if $|B_t| > 0$ and D_s contains no element h of H_t, then $|D_s| \leq |D_t|$. If $|B_t| > 0$ and D_s

contains an element h of H_t then $|D_s - \{h\}| \leq |D_t|$. Note that in this case h does not occur above row t in M. Therefore, in either case, by applying Lemma 5.6, there is at least one row r above row t which contains no elements of D_s. Each element of row D_s occurs in at most one row above row t and hence with at most $|D_s|$ elements of row r. Hence there are at least $k - |D_s| = |B_s|$ elements of row r which can be used to replace the elements of $|B_s|$ whilst maintaining the complete separation property. This completes the proof of the lemma. □

Theorem 5.5. *If* $n \geq \binom{k+1}{2}$, $k > 1$, *then* $R(n,k) = \lceil 2n/k \rceil$.

Proof. Combine Theorem 5.3 and Lemmas 5.7 and 5.8. □

As an example of this construction, consider the following three arrays, the rows of which are minimal CSSs for the $(10,4)$, $(13,4)$ and $(16,5)$CSS cases. The elements used to fill the 0-valued positions of M have been offset for clarity.

1	2	3	4
1	5	6	7
2	5	8	9
3	6	8	10
4	7	9	10

,

1	2	3	4	5
1	5	6	7	
2	7	8	9	
3	8	10	11	
4	9	12	13	
5	10	12		2
6	11	13		2

,

1	2	3	4	5	
1	6	7	8	9	
2	7	10	11	12	
3	8	11	13	14	
4	9	13	15	16	
5	10	14	16		6
6	11	15		1	3

Note that this construction is not fair in general. In the second example above, 2 occurs four times and there is no other possible choice with the given construction.

5.2.3 Proof of Theorem 5.1.(ii)

In [23] it was noted that $R(n,k)$ is not monotonic in n. For example $R(4,3) = 4$, $R(5,3) = 5$ and $R(6,3) = 4$. In that paper it was asked whether or not the values of $R(n,k)$ are monotonic with n, for fixed $k \neq 4,5$ and $n \geq 2k$. This question is answered in the negative, by showing that the lower bound of $\lceil 2n/k \rceil = k+1$

cannot be achieved for $n = \binom{k+1}{2} - 1$, whilst it is achieved for $n = \binom{k+1}{2} - 2$ and, as seen above, for $n = \binom{k+1}{2}$.

Lemma 5.9. *If $k \geq 3$ then $R(\binom{k+1}{2} - 1, k) = k + 2 = \lceil 2n/k \rceil + 1$.*

Proof. Let $n = \binom{k+1}{2} - 1$. The proof is in two parts. Firstly, induction is used to show that $R(n, k) > k + 1$. Then the construction method in Lemma 4.15 can be applied to find a (n, k)CSS in $k + 2$ sets.

The base case in the induction is the result that $R(5, 3) = R(5, 2) = 5$. Note that, if $k \geq 3$, then the lower bound $\lceil 2n/k \rceil$ is $k + 1$.

Assume that the lemma holds for all values of $k \leq k'$, for some $k' \geq 3$. Let $k = k' + 1$ and assume that $R(n, k) = k + 1$, with $n = \frac{k(k+1)}{2} - 1$. If this is the case, then one element of $[n]$ occurs 4 times or two elements of $[n]$ occur 3 times in the CSS, with all other elements occuring exactly twice. That is, the excess is 2. It can be assumed, without loss of generality, that $\{1, \ldots, k\}$ occurs in the CSS. There are three cases for the other k sets in the CSS.

(a) If each of $1, \ldots, k$ occurs only once more then, to be separated, they must occur singly in the remaining sets. Thus, the k $(k-1)$-sets formed by removing each of $1, \ldots, k$ from these sets must form a $(n - k, k - 1)$CSS in k sets. That is, a $\binom{k}{2} - 1, k - 1)$CSS. This is not possible by the induction hypothesis.

(b) Assume that 1 occurs two or three times and each of $2, \ldots, k$ occur once in the remaining k sets. To separate $2, \ldots, k$ with only one occurence of each, they must appear in $k - 1$ separate sets of the k available. Since the 1's cannot be in the same set, at least one of $2, \ldots, k$ must occur with 1 and thus cannot be separated from it.

(c) Assume that 1 and 2 each occur twice and each of $3, \ldots, k$ occur once in the remaining k sets. If 1 or 2 occur singly in the remaining two sets then at least one of $3, \ldots, k$ must occur with 1 or 2 and thus cannot be separated from it. If 1 and

2 occur together in each of the remaining two sets then they are not separated from each other. If 1 and 2 occur in the $k-2$ sets with $3, \ldots, k$ then at least one of $3, \ldots, k$ is not separated from at least one of 1 and 2.

Thus, in all cases there is a contradiction, so $R(n,k) > k+1$. That $R(n,k) = k+2$ follows by applying Lemma 4.15 with the base case being $n = 5$ and $k = 3$. □

As an example of the construction provided by Lemma 4.15 for the values of n and k in Lemma 5.9, consider the following four arrays, the rows of which are minimal CSSs for the $k = 3, 4, 5, 6$ cases.

$$\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 1 \\ 5 & 1 & 2 \end{array}, \quad \begin{array}{cccc} 1 & 2 & 3 & 6 \\ 2 & 3 & 4 & 7 \\ 3 & 4 & 5 & 8 \\ 4 & 5 & 1 & 9 \\ 5 & 1 & 2 & 9 \\ 6 & 7 & 8 & 9 \end{array}, \quad \begin{array}{ccccc} 1 & 2 & 3 & 6 & 10 \\ 2 & 3 & 4 & 7 & 11 \\ 3 & 4 & 5 & 8 & 12 \\ 4 & 5 & 1 & 9 & 13 \\ 5 & 1 & 2 & 9 & 14 \\ 6 & 7 & 8 & 9 & 14 \\ 10 & 11 & 12 & 13 & 14 \end{array}$$

$$\begin{array}{cccccc} 1 & 2 & 3 & 6 & 10 & 15 \\ 2 & 3 & 4 & 7 & 11 & 16 \\ 3 & 4 & 5 & 8 & 12 & 17 \\ 4 & 5 & 1 & 9 & 13 & 18 \\ 5 & 1 & 2 & 9 & 14 & 19 \\ 6 & 7 & 8 & 9 & 14 & 20 \\ 10 & 11 & 12 & 13 & 14 & 20 \\ 15 & 16 & 17 & 18 & 19 & 20 \end{array}$$

5.2.4 Proof of Theorem 5.1.(iii)

Lemma 5.10. *If* $k^2/2 \le n \le \binom{k+1}{2} - 2$, $k > 1$, *then:*
(i) $R(n, 2, k) \ge k+1$;
(ii) $R(n, 1, k) \ge k+1$.

Proof. (i) Assume the conditions of the lemma. Assume $R(n,2,k) \leq k$. In any $(n,2,k)$CSS each element must occur in at least two different sets. Therefore, the number of sets in a minimum $(n,2,k)$CSS is at least $2n/k$. For $n > k^2/2$ this means that $R(n,k) > k$.

If $n = k^2/2$ then $2n/k = k$ so at least k k-sets are required in a CSS. If $R(n,2,k) = k$ each element must occur in exactly two sets in a $R(n,2,k)$CSS. Assume $[k]$ is one of the sets in the minimal CSS. Then, as each element of $[k]$ must occur exactly once more without the other elements of $[k]$, at least k more sets are required in the CSS. Hence the minimum CSS requires at least $k+1$ sets.

(ii) Assume the conditions of the lemma and that, for some n, k, there exists a $(n,1,k)$CSS $\mathcal{C}$ with $|\mathcal{C}| \leq k$. By part (i), $\mathcal{C}$ must contain a singleton set and therefore $\sum_{A \in \mathcal{C}} |A| \leq k(k-1)+1 = k^2-(k-1)$. As $2n \geq k^2$, there must be at least $k-1$ elements of $[n]$ which occur in only one set in $\mathcal{C}$ and hence must occur in singletons sets in $\mathcal{C}$. As there are at least $k-1$ such singleton sets and $|\mathcal{C}| \leq k$, it is clearly impossible for $\mathcal{C}$ to completely separate the remaining elements of $[n]$. □

Theorem 5.6. *If $k^2/2 \leq n \leq \binom{k+1}{2} - 2$, $k > 1$, then $R(n,k) = k+1$.*

Proof. Assume the conditions of the theorem. The theorem is vacuously true for $k \leq 3$ so assume $k \geq 4$. By Lemma 5.2 $R(n,k) > k$ if $n \geq k^2/2$. An $R \times k = (k+1) \times k$ array M will be constructed such that its rows form a (n,k)CSS. Note that the excess E (see Section 1.5) has $4 \leq E \leq k$ and E is always even.

The initial step is to use Construction M as given in Section 5.2.2. This provides a fair $(n,2,k)$CSS and a fair $(n,1,k)$CSS. The fairness and the complete separation property of the system is clear.

Once Construction M has been completed consider the position of the 0-valued elements remaining in array M. Consideration of Construction M, with the given size of array M, easily leads to the truth of the following statement for all values of E. With the possible exception of one column, each column in M which now contains a 0-valued element contains at least three 0-valued elements. The notation h_t, H_t, B_t as defined in the proof of Theorem 5.1(i) is used where appropriate in the remainder of this proof. Assume h_t occurs at m_{tt}. Define the subarray A of M by $A = \{m_{ij} \in M : i, j \geq t\}$ if $|B_t| > 0$ and $A = \{m_{ij} \in M : i, j > t\}$ if $|B_t| = 0$. Assume A is a $r \times (r-1)$ array with column vectors A_i, $i = 1, \ldots, r-1$. There are four cases to consider.

Case 1. If row t contains no 0-valued elements each A_j contains at least three 0-valued elements. Then, in row order, replace the first 0-valued element in A_j by $m_{k+1,j}$ and for each remaining 0-valued element in row i and in A_j replace it by $m_{i-1,j}$. Given that Construction M forms a fair $(n, 2, k)$CSS it is easy to see that in this case the rows of M form a fair (n, k)CSS.

For the remaining cases assume row t contains some 0-valued elements.

Case 2. If A is a 3×2 array set $m_{k+1,t} = m_{tt}$. The remaining columns of A can be dealt with as in case 1.

Case 3. If A is a 4×3 array and $|H_t| = 2$ set $m_{k+1,t} = m_{t,t}$. The remaining columns of A can be dealt with as in case 1.

Case 4. If A is an $r \times (r-1)$ array with $r > 3$, set $m_{ii} = m_{it}$ for $t < i \leq k$, set $m_{k+1,k+1} = m_{tt}$ and then set $m_{it} = 0$ for $i > t$. Now each column of M which contains a 0-valued element contains at least three such elements. The remaining 0-valued elements can be replaced in a similar way to that outlined in case 1. Let a_{ij} denote the element in row i column j of A with $i = t, \ldots, k+1$ and $j = 1, \ldots, r-1$. For each column $A_j = \{a_{ij}\}$ form the column $C_j = \{a_{ij}\}$ of elements in column j of M. In row order, replace the first 0-valued element in

A_j by the last element of C_j in the same row as a 0-valued element of A_j. For each a_{ij}, $i > 1$, simultaneously set $a_{ij} = c_{mj}$ where $m < i$ is the row index of the first 0-valued element immediately above a_{ij} in A_j.

By the nature of Construction M, to show that the rows of M now form a fair (n, k)CSS, it is only necessary to consider the elements of the array A. All elements of $[n]$ not in A are completely separated from elements of A in some row above row t. The elements of different A_j, other than elements of H_t, are completely separated from one another in rows above row t. Elements of the same A_j are completely separated from one another in the corresponding sets A_j and C_j.

The elements of H_t are completely separated from one another in A. They are completely separated from elements of the A_j by occurring in row t or, in the case of h_t when $|B_t| = 0$, by occurring three times in A.

The fairness of the system is clear as every element occurs either two or three times in M. This completes the proof. □

As an example of this construction, consider the following four arrays, the rows of which are minimal CSSs for the $k = 10$, $n = 50, 51, 52$ & 53 cases. These illustrate, respectively, cases 4, 3, 1 & 2 of the proof.

1	2	3	4	5	6	7	8	9	10	
1	11	12	13	14	15	16	17	18	19	
2	11	20	21	22	23	24	25	26	27	
3	12	20	28	29	30	31	32	33	34	
4	13	21	28	35	36	37	38	39	40	
5	14	22	29	35	41	42	43	44	45	
6	15	23	30	36	41	46	47	48	49	
7	16	24	31	37	42	46		50	19	26
8	17	25	32	38	43	47		10	50	24
9	18	26	33	39	44	48		8	16	25
10	19	27	34	40	45	49		9	18	50

1	2	3	4	5	6	7	8	9	10	
1	11	12	13	14	15	16	17	18	19	
2	11	20	21	22	23	24	25	26	27	
3	12	20	28	29	30	31	32	33	34	
4	13	21	28	35	36	37	38	39	40	
5	14	22	29	35	41	42	43	44	45	
6	15	23	30	36	41	46	47	48	49	
7	16	24	31	37	42	46		50	51	19
8	17	25	32	38	43	47		50	10	16
9	18	26	33	39	44	48		51	8	17
10	19	27	34	40	45	49		50	9	18

1	2	3	4	5	6	7	8	9	10	
1	11	12	13	14	15	16	17	18	19	
2	11	20	21	22	23	24	25	26	27	
3	12	20	28	29	30	31	32	33	34	
4	13	21	28	35	36	37	38	39	40	
5	14	22	29	35	41	42	43	44	45	
6	15	23	30	36	41	46	47	48	49	
7	16	24	31	37	42	46	50	51	52	
8	17	25	32	38	43	47	50		10	19
9	18	26	33	39	44	48	51		8	17
10	19	27	34	40	45	49	52		9	18

1	2	3	4	5	6	7	8	9	10	
1	11	12	13	14	15	16	17	18	19	
2	11	20	21	22	23	24	25	26	27	
3	12	20	28	29	30	31	32	33	34	
4	13	21	28	35	36	37	38	39	40	
5	14	22	29	35	41	42	43	44	45	
6	15	23	30	36	41	46	47	48	49	
7	16	24	31	37	42	46	50	51	52	
8	17	25	32	38	43	47	50		53	10
9	18	26	33	39	44	48	51		53	8
10	19	27	34	40	45	49	52		53	9

Corollary 5.4. *If $k^2/2 \le n \le \binom{k+1}{2} - 2$ then minimum $(n,2,k)$CSSs and minimum $(n,1,k)$CSSs contain $k+1$ sets. In each case fair minimal CSSs exist.*

Proof. Lemma 5.10 ensures that $R(n,1,k) > k$ and $R(n,2,k) > k$. Construction M in Section 5.2.2 provides a fair $(n,1,k)$CSS and a fair $(n,2,k)$CSS using $k+1$ sets. □

5.2.5 Proof of Theorem 5.1.(iv)

Lemma 5.11. *If $\binom{k}{2} \leq n < k^2/2$, $k \geq 5$, then $R(n,k) = k+1(> \lceil 2n/k \rceil)$ and a fair CSS exists in each case.*

Proof. Assume the conditions of the theorem. Then $R(n,k) > k$ by Lemma 5.2.

Let $R = k+1$. An $R \times k$ array M will be constructed with the R row vectors of M forming a (n,k)CSS. Initialise all elements of M to zero. Let m_{ij} denote the element of M in row i, column j.

Partition M into four parts defined by:

$$A = \{m_{1j} : 1 \leq j \leq k\};$$

$$B = \{m_{ij} : 2 \leq j \leq k+1, 1 \leq j \leq 2\};$$

$$C = \{m_{ij} : 3 \leq i \leq k, 3 \leq j \leq k\};$$

$$D = \{m_{k+1,j} : 3 \leq j \leq k\}.$$

The elements of these parts are now defined for various cases.

Case 1. Assume $n = \binom{k}{2}$.

For part A, set $m_{1j} = j$ for $1 \leq j \leq k$.

For part B, set $m_{k+1,2} = 1$ and for $i \neq k+1$, $j \neq 2$ set $m_{1j} = i+j-2$ for $1 \leq i \leq 2$ and $2 \leq j \leq k+1$.

For part C, use the Construction M on the set $\{k+1, \ldots, n\}$. At this stage C has exactly two 0-valued elements at $m_{k-1,k}$ and $m_{k,k}$. Set $m_{k-1,k} = m_{k-2,3}$ and

$m_{kk} = m_{k-1,3}$.

Part D is filled by setting $_{mk+1,j} = m_{jj}$, $3 \leq j \leq k-1$ and $m_{k+1,k} = m_{k,3}$.

The elements of A are completely separated in B and are completely separated in A from all other elements of $[n]$. It can be noted that the element n first occurs exactly at $m_{k-3,k}$ and hence all elements of $[n]$ occur in a row above row $k-1$. The fact that the elements of C are completely separated from one another is then easily seen as a feature of the construction with the only special cases being the separation of $m_{k,3}$ from $m_{k-1,3}$ at $m_{k+1,k}$ and the separation of $n-1$ from $m_{k-3,3}$ at $m_{k+1,k-1}$. The choice of elements of D other than $m_{k+1,k}$ ensures the separation of the elements of C from the elements of B. Hence the row vectors of M form a fair CSS on $[n]$.

Case 2. Assume $n = \binom{k}{2} + 1$.

The construction in this case is the same as for case 1, with one modification. For this case set $m_{i,k} = n$ for $k-1 \leq i \leq k+1$. It is easier than in case 1 to see that M is a fair CSS in this case.

Case 3. Assume $\binom{k}{2} + 1 < n \leq k^2/2 - 1$.

For this case note that, with the assumed bounds on n, there are less than $k/2$ elements of $[n]$ greater than $\binom{k}{2}$. For ease of notation, assume there are r elements greater than $\binom{k}{2}$, denoted by $d_1, \ldots, d_r$.

To construct a fair CSS on $[n]$ now proceed as for case 2. Then the remaining elements not yet included in M are used to replace elements of M as follows: For $2 \leq i \leq r$, set $m_{i,2} = d_i$ and $m_{k+1,i+1} = d_i$. Set $m_{k+1,2} = m_{r+1,1}$.

Given that $k \geq 6$ and $r < k/2$ it is a small matter to check that the rows of M form a fair CSS on $[n]$. To check that complete separation is maintained note that the elements replaced in D are the ones which now occur in the same row as only one element of $[k]$. Hence they no longer need to be repeated in D to separate them from the elements of $[k]$. The change in value of $m_{k+1,2}$ is important to ensure that r is separated from $r+1$. It is clear that the elements $d_1, \ldots, d_r$ are completely separated from one another with the construction.

Hence the theorem is proved. □

As an example of this construction, consider the following four arrays, the rows of which are minimal CSSs for the $k=7$, $n = 21, 22, 23$ & 24 cases.

1	2	3	4	5	6	7
1	2	8	9	10	11	12
2	3	8	13	14	15	16
3	4	9	13	17	18	19
4	5	10	14	17	20	21
5	6	11	15	18	20	10
6	7	12	16	19	21	11
7	1	8	13	17	20	12

,

1	2	3	4	5	6	7
1	2	8	9	10	11	12
2	3	8	13	14	15	16
3	4	9	13	17	18	19
4	5	10	14	17	20	21
5	6	11	15	18	20	22
6	7	12	16	19	21	22
7	1	8	13	17	20	22

1	2	3	4	5	6	7
1	23	8	9	10	11	12
2	3	8	13	14	15	16
3	4	9	13	17	18	19
4	5	10	14	17	20	21
5	6	11	15	18	20	22
6	7	12	16	19	21	22
7	2	23	13	17	20	22

,

1	2	3	4	5	6	7
1	23	8	9	10	11	12
2	24	8	13	14	15	16
3	4	9	13	17	18	19
4	5	10	14	17	20	21
5	6	11	15	18	20	22
6	7	12	16	19	21	22
7	2	23	24	17	20	22

5.3 An Alternative Construction

Corollary 5.5 proves an alternative method of constructing a fair (n,k)CSS in $k+1$ sets, for $\binom{k}{2} \leq n \leq \binom{k+1}{2} - 2$, based on Lemma 4.14. Note that $\left(\binom{k+1}{2} - 2\right) - \binom{k}{2} = k-2$. The construction is by induction, with the base cases being the construction of Lemma 5.11 (for the $n = \binom{k}{2}$ case only) and some known $(n,5)$CSSs.

The inductive step is based on the observations that, if $\binom{k}{2} < n$ then $\binom{k-1}{2} \le n-k$, and if $n \le \binom{k+1}{2} - 2$ then $n - k \le \binom{k}{2} - 2$. Thus, the construction of Lemma 4.14 can be used for the inductive step.

Corollary 5.5. *If* $\binom{k}{2} \le n \le \binom{k+1}{2} - 2$, $k \ge 5$, *then* $R(n,k) \le k+1$ *and a fair* (n,k)*CSS in* $k+1$ *sets exists.*

Proof. Lemma 5.11 establishes the result for the case $n = \binom{k}{2}$. The collections {129AB, 13678, 24578, 345AB, 5689B, 4679A}, {12345, 16ABC, 26789, 389BC, 479AC, 578AB}, and {12345, 16789, 26BCD, 37ACD, 48ABD, 59ABC} establish the result for the remaining cases when $k = 5$. Together, these form the basis for the induction, which is on k.

Assume that the corollary is true for some $k' \ge 5$ and consider the case $k = k'+1$. If $n = \binom{k}{2}$, the result follows from the base case. If $\binom{k}{2} < n \le \binom{k+1}{2} - 2$ then $\binom{k-1}{2} \le n - k \le \binom{k}{2} - 2$. By the induction hypothesis the result is true for $k - 1 = k'$ and Lemma 4.14 can be used to construct a CSS of k-sets in $k+1$ sets, from the CSS of k'-sets in $k' + 1 = k$ sets. Thus $R(n,k) \le k+1$.

Note that the construction of Lemma 4.14 adds the new elements exactly twice. Thus, if the original CSS is fair, with all elements occuring two or three times, then the new CSS is fair. Thus the inductively constructed CSSs are fair and the result follows. □

As an example of the construction of Lemma 4.14, consider the following six arrays, the rows of which form CSSs. The first example in each row are the base cases for $R(13,5)$, $R(15,6)$ and $R(21,7)$ respectively. The second example in each row, for $R(19,6)$, $R(22,7)$ and $R(29,8)$, are built from the first example in the same row, via the construction in Lemma 4.14.

$$
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
1 & 6 & 7 & 8 & 9 \\
2 & 6 & 11 & 12 & 13 \\
3 & 7 & 10 & 12 & 13 \\
4 & 8 & 10 & 11 & 13 \\
5 & 9 & 10 & 11 & 12
\end{array}
,\quad
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 14 \\
1 & 6 & 7 & 8 & 9 & 15 \\
2 & 6 & 11 & 12 & 13 & 16 \\
3 & 7 & 10 & 12 & 13 & 17 \\
4 & 8 & 10 & 11 & 13 & 18 \\
5 & 9 & 10 & 11 & 12 & 19 \\
14 & 15 & 16 & 17 & 18 & 19
\end{array}
$$

$$
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 2 & 7 & 8 & 9 & 10 \\
2 & 3 & 7 & 11 & 12 & 13 \\
3 & 4 & 8 & 11 & 14 & 15 \\
4 & 5 & 9 & 12 & 14 & 8 \\
5 & 6 & 10 & 13 & 15 & 9 \\
6 & 1 & 7 & 11 & 14 & 10
\end{array}
,\quad
\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 16 \\
1 & 2 & 7 & 8 & 9 & 10 & 17 \\
2 & 3 & 7 & 11 & 12 & 13 & 18 \\
3 & 4 & 8 & 11 & 14 & 15 & 19 \\
4 & 5 & 9 & 12 & 14 & 8 & 20 \\
5 & 6 & 10 & 13 & 15 & 9 & 21 \\
6 & 1 & 7 & 11 & 14 & 10 & 22 \\
16 & 17 & 18 & 19 & 20 & 21 & 22
\end{array}
$$

$$
\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 2 & 8 & 9 & 10 & 11 & 12 \\
2 & 3 & 8 & 13 & 14 & 15 & 16 \\
3 & 4 & 9 & 13 & 17 & 18 & 19 \\
4 & 5 & 10 & 14 & 17 & 20 & 21 \\
5 & 6 & 11 & 15 & 18 & 20 & 10 \\
6 & 7 & 12 & 16 & 19 & 21 & 11 \\
7 & 1 & 8 & 13 & 17 & 20 & 12
\end{array}
,\quad
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 22 \\
1 & 2 & 8 & 9 & 10 & 11 & 12 & 23 \\
2 & 3 & 8 & 13 & 14 & 15 & 16 & 24 \\
3 & 4 & 9 & 13 & 17 & 18 & 19 & 25 \\
4 & 5 & 10 & 14 & 17 & 20 & 21 & 26 \\
5 & 6 & 11 & 15 & 18 & 20 & 10 & 27 \\
6 & 7 & 12 & 16 & 19 & 21 & 11 & 28 \\
7 & 1 & 8 & 13 & 17 & 20 & 12 & 29 \\
22 & 23 & 24 & 25 & 26 & 27 & 28 & 29
\end{array}
$$

5.4 $k \leq 6$

Theorem 5.1, together with the result that $R(n,1) = R(n,2) = n$ and Lemma 4.5, provides a complete solution to the $R(n,k)$ problem for all $k \leq 5$. Using the results in Chapter 4, the only remaining unknown case for $k = 6$ is $R(13,6)$. This is now determined.

Lemma 5.12. $R(13,6) = 7$.

Proof. By Lemma 4.3, $R(13,6) \geq 6$. That $R(13,6) \leq 7$ follows from consideration of the collection {12345D, 12345C, 16789A, 2678BD, 369ABC, 479BCD, 58ABCD}. To prove that $R(13,6) \neq 6$, assume $\mathcal{C}$ is a $(13,6)$CSS with $|\mathcal{C}| = 6$.

Here the excess $E = k|\mathcal{C}| - 2n = 10$ so there are at least three elements, say 1, 2 and 3, which occur exactly twice in sets in $\mathcal{C}$. If i of these occur in one set A, then to obtain complete separation there are i other sets which contain exactly one of these elements of A. This leaves $6 - i$ elements in A to be separated in less than $6 - i$ sets. This is impossible for $i > 1$ by Lemma 4.1.

Assume all sets of $\mathcal{C}$ contain exactly one element which occurs exactly twice in sets in $\mathcal{C}$. Assume 1 occurs in sets A and B. Let the other sets in $\mathcal{C}$ be C, D, E and F. It can be assumed that $2 \in C, D$ and $3 \in E, F$.

A contains exactly five elements which occur more than twice in $\mathcal{C}$. None of these can occur in B to ensure that 1 is separated from each of the other elements of A. It is shown later (see Theorem 8.2) that there is exactly one way, up to labelling of elements, to separate five elements in no more than four sets, namely {456, 478, 57, 68}. These four sets must be subsets of C, D, E and F. For all possible arrangements of these sets it is now easy to check that either 2 or 3 cannot now be separated from at least one element of A other than 1. Hence $R(13,6) \neq 6$, and the result follows. □

5.5 Comments

Combining the results in this chapter with those in Chapter 4, the exact values or bounds on the values of $R(n,k)$ are obtained and these are shown in Table 5.5.

To read the table, note the following details. The lower and upper bounds are shown when exact values are unknown. For example the bounds on $R(15,7)$ are shown as $6 - 9$ meaning that $6 \leq R(15,7) \leq 9$. The terms written in the form [.] are the diagonal elements in the table. That is, they are in the positions in the table in which $n = 2k$. By Lemma 4.5 the values in one row, to the right of the main diagonal, can be obtained from the values in the same row to the left of the main diagonal.

An open question is whether or not fair minimal (n,k)CSSs exist for all values of n and k. In particular, do fair minimal (n,k)CSSs always exist whenever $n \geq \binom{k+1}{2}$?

It appears to be increasingly difficult to calculate $R(n,k)$ as n approaches $2k$ from above. Chapters 7 and 9 will consider this problem further. However, those chapters require several diversions from the (n,k)CSS problem. These diversions appear as Chapters 6 and 8.

	k													
n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
2	[2]													
3	3	3												
4	4	[4]	4											
5	5	5	5	5										
6	6	6	[4]	6	6									
7	7	7	5	5	7	7								
8	8	8	6	[5]	6	8	8							
9	9	9	6	6	6	6	9	9						
10	10	10	7	5	[6]	5	7	10	10					
11	11	11	8	6	6	6	6	8	11	11				
12	12	12	8	6	6	[6]	6	6	8	12	12			
13	13	13	9	7	6	7	7	6	7	9	13	13		
14	14	14	10	7	7	7	[6–8]	7	7	7	10	14	14	
15	15	15	10	8	6	7	6–9	6–9	7	6	8	10	15	15
16	16	16	11	8	7	7	6–8	[6–7]	6–8	7	7	8	11	16
17	17	17	12	9	7	7	7–8	6–9	6–9	7–8	7	7	9	12
18	18	18	12	9	8	7	7–8	6–9	[6–10]	6–9	7–8	7	8	9
19	19	19	13	10	8	7	8	7–9	6–11	6–11	7–9	8	7	8
20	20	20	14	10	8	8	8	7–9	7–10	[6–8]	7–10	7–9	8	8
21	21	21	14	11	9	7	8	7–9	7–10	7–10	7–10	7–10	7–9	8
22	22	22	15	11	9	8	8	7–9	7–10	7–11	[7–12]	7–11	7–10	7–9
23	23	23	16	12	10	8	8	8–9	7–10	7–11	7–13	7–13	7–11	7–10
24	24	24	16	12	10	8	8	8–9	7–10	7–11	7–12	[7–8]	7–12	7–11
25	25	25	17	13	10	9	8	8–9	7–10	7–11	7–12	7–10	7–10	7–12
26	26	26	18	13	11	9	8	9	8–10	7–11	7–12	7–11	[7–12]	7–11
27	27	27	18	14	11	9	9	9	8–10	7–11	7–12	7–13	7–13	7–13
28	28	28	19	14	12	10	8	9	8–10	7–11	7–12	7–13	7–14	[7–10]
29	29	29	20	15	12	10	9	9	8–10	7–13	7–12	7–13	7–14	7–12
30	30	30	20	15	12	10	9	9	9–10	8–11	7–12	7–13	7–14	7–13
31	31	31	21	16	13	11	9	9	9–10	8–11	7–12	7–13	7–14	7–15
32	32	32	22	16	13	11	10	9	9–10	8–11	7–12	7–13	7–14	7–15
33	33	33	22	17	14	11	10	9	9–10	8–11	8–12	7–13	7–14	7–15
34	34	34	23	17	14	12	10	9	10	9–11	8–12	7–13	7–14	7–15
35	35	35	24	18	14	12	10	10	10	9–11	8–12	7–14	7–14	7–15
36	36	36	24	18	15	12	11	9	10	9–11	8–12	8–13	8–14	8–15
37	37	37	25	19	15	13	11	10	10	9–11	8–12	8–13	8–14	8–15
38	38	38	26	19	16	13	11	10	10	10–11	9–12	8–13	8–14	8–15
39	39	39	26	20	16	13	12	10	10	10–11	9–12	8–13	8–14	8–15
40	40	40	27	20	16	14	12	10	10	10–11	9–14	8–13	8–14	8–15
41	41	41	28	21	17	14	12	11	10	10–11	9–12	8–13	8–14	8–15
42	42	42	28	21	17	14	12	11	10	11	9–12	9–13	8–14	8–15
43	43	43	29	22	18	15	13	11	10	11	10–12	9–13	8–14	8–15
44	44	44	30	22	18	15	13	11	11	11	10–12	9–13	8–14	8–15
45	45	45	30	23	18	15	13	12	10	11	10–12	9–13	8–14	8–15
46	46	46	31	23	19	16	14	12	11	11	10–12	9–13	9–14	8–15
47	47	47	32	24	19	16	14	12	11	11	11–12	10–13	9–14	8–15
48	48	48	32	24	20	16	14	12	11	11	11–12	10–13	9–16	8–15
49	49	49	33	25	20	17	14	13	11	11	11–12	10–13	9–14	8–15
50	50	50	34	25	20	17	15	13	12	11	11–12	10–13	9–14	9–15
51	51	51	34	26	21	17	15	13	12	11	11–12	10–13	9–14	9–15
52	52	52	35	26	21	18	15	13	12	11	12	11–15	10–14	9–15
53	53	53	36	27	22	18	16	14	12	11	12	11–13	10–14	9–15
54	54	54	36	27	22	18	16	14	12	12	12	11–13	10–14	9–15
55	55	55	37	28	22	19	16	14	13	11	12	11–13	10–14	9–15
56	56	56	38	28	23	19	16	14	13	12	12	11–13	10–14	10–15
57	57	57	38	29	23	19	17	15	13	12	12	12–13	11–14	10–15
58	58	58	39	29	24	20	17	15	13	12	12	12–13	11–14	10–15
59	59	59	40	30	24	20	17	15	14	12	12	12–13	11–14	10–15
60	60	60	40	30	24	20	18	15	14	12	12	12–13	11–14	10–15
61	61	61	41	31	25	21	18	16	14	13	12	12–13	11–14	10–15
62	62	62	42	31	25	21	18	16	14	13	12	12–13	11–14	11–17
63	63	63	42	32	26	21	18	16	14	13	12	13	12–14	11–15
64	64	64	43	32	26	22	19	16	15	13	12	13	12–14	11–15
65	65	65	44	33	26	22	19	17	15	13	13	13	12–16	11–15
66	66	66	44	33	27	22	19	17	15	14	12	13	12–14	11–15
67	67	67	45	34	27	23	20	17	15	14	13	13	12–14	11–15
68	68	68	46	34	28	23	20	17	16	14	13	13	12–14	12–15
69	69	69	46	35	28	23	20	18	16	14	13	13	13–14	12–15
70	70	70	47	35	28	24	20	18	16	14	13	13	13–14	12–15

Table 5.1: Known bounds on $R(n,k)$ for $2 \leq n \leq 70$ and $1 \leq k \leq \min\{14, n-1\}$.

Chapter 6

The Flat Antichain Conjecture

6.1 Introduction

The following conjecture was first stated by Lieby [13]. It arose whilst considering the $R(n,k)$ problem.

Conjecture 6.1. ***Flat Antichain Conjecture (FAC)***
Given any antichain $\mathcal{A}$ on $[n]$, there is an equivalent flat antichain $\mathcal{A}'$ on $[n]$.

The conjecture is intrinsically interesting and its validity would have numerous important consequences in Sperner Theory. The conjecture has been difficult to validate except in some very restricted cases. The range of known cases for which it is valid is increased in this chapter. These cases are sufficient to allow further advances in the determination of $R(n,k)$. These are dealt with in Chapters 7 and 9.

Throughout this chapter assume that $\mathcal{A}$ is an antichain on $[n]$ with average set size $\overline{\mathcal{A}}$ with $x \leq \overline{\mathcal{A}} < x+1$ for some $x \in \boldsymbol{Z}$.

Note 6.1. If $\mathcal{A}$ is a flat antichain on $[n]$ then the complementary antichain $\mathcal{A}'$ is also flat. In general if the antichain $\mathcal{A}$ has $h \leq |A| \leq k$ for all $A \in \mathcal{A}$ then $\mathcal{A}'$ has $n-k \leq |A| \leq n-h$ for each $A \in \mathcal{A}'$.

The following result is well-known (see [2]).

Lemma 6.1. *(Sperner)*
For any collection $\mathcal{B}$ of k-sets on $[n]$, $\nabla\mathcal{B} \geq \frac{n-k}{k+1}|\mathcal{B}|$.

The following theorem is a consequence of Theorem 3.1 (see Corollary 7.3.4 in [2]) in a form which is useful for this chapter.

Theorem 6.1. *To minimise the cardinality of the shadow at level h of a collection of m k-sets on $[n]$, $k > h$, choose the k-sets in squashed order.*

The following theorem can be found in [7] where it is stated in terms of multisets.

Theorem 6.2. *(Clements)*
To minimise the cardinality of the new shadow at level h over all possible collections of m k-sets on $[n]$, $k > h$, choose the k-sets to be the last m k-sets in squashed order on $[n]$.
To minimise the cardinality of the new shade at level h over all possible collections of m k-sets on $[n]$, $h > k$, choose the k-sets to be the first m k-sets in squashed order on $[n]$.

The following important result can be found in [2].

Theorem 6.3. *(Clements, Daykin, Godfrey, Hilton)*
Let $p_0, \ldots, p_n$ be non-negative integers, and let k and g respectively be the smallest and largest values of i for which $p_i \neq 0$. Define integers $n_k, \ldots, n_g$ by $n_g = 0$, $n_{g-1} = |\triangle F_g(p_g)|, n_{g-2} = |\triangle F_{g-1}(p_{g-1} + n_{g-1})|, \ldots, n_k = |\triangle F_{k+1}(p_{k+1} + n_{k+1}|$. Then, provided that $p_i + n_i \leq \binom{n}{i}$ for each $i = k, \ldots, g$ there exists an antichain $\mathcal{A}^*$ with parameters p_i such that, for each i, the sets in $\mathcal{A}^*$ of size i, together with

the sets of size i contained in larger members of $\mathcal{A}^*$, constitute an initial segment of the sets of size i in the squashed order.

Theorem 6.3 implies that for any antichain $\mathcal{A}$ there is a squashed antichain $\mathcal{A}^*$ with the same parameters as $\mathcal{A}$. Hence, to consider whether or not an antichain can be flattened, it suffices to consider only squashed antichains. Thus, in all cases below, it is appropriate and useful to assume that each antichain $\mathcal{A}$ is a squashed antichain. Detailed discussion on antichains can be found in [2], [3] or [9]. The following theorem is found in [17].

Theorem 6.4. *(Maire)*
FAC is true for all antichains $\mathcal{A}$ with $\overline{\mathcal{A}}$ an integer.

A proof of this theorem, which is simpler than Maire's proof, can be found in Chapter 7.

6.2 Main Results

Amongst other things, it will be shown in this section that FAC is true for all antichains $\mathcal{A}$ with $\overline{\mathcal{A}} \leq 3$. In the remainder of this chapter, it is assumed for integers a_i, a_j that $a_i < a_j$ whenever $i < j$. Note that braces are sometimes left out when denoting sets to aid readability. Hence $\{s\}$ may be written as s when no confusion arises.

Theorem 6.5. *FAC is true for all antichains $\mathcal{A}$ with $1 \leq \overline{\mathcal{A}} \leq 2$.*

Proof. The theorem is trivially true if $\mathcal{A}$ is an antichain with $\overline{\mathcal{A}} = 1$. It is also true if $\overline{\mathcal{A}} = 2$ by Theorem 6.4.
Assume $\mathcal{A}$ is an antichain with $1 < \overline{\mathcal{A}} < 2$. If $\mathcal{A}$ is not a flat antichain, then it contains sets of cardinality greater than two, but does not contain sets of

cardinality less than one. Assume the 1-sets in $\mathcal{A}$ are $s, s+1, ..., n$. A flat antichain $\mathcal{A}'$ equivalent to $\mathcal{A}$ can be constructed as follows.

Initialise $\mathcal{A}' = \mathcal{A}$. Assume that the last set in squashed order in $\mathcal{A}$ with cardinality greater than 2 is the p-set $\{a_1, ..., a_{p-1}, a_p\}$. Replace this set in $\mathcal{A}'$ by the 2-set $\{1, s\}$. As $\overline{\mathcal{A}} < 2$ there are at least $(p+1)$ 1-sets in $\mathcal{A}$. Replace the first $(p-2)$ 1-sets in $\mathcal{A}'$ by the first $(p-2)$ 2-sets in squashed order after $\{1, s\}$, namely $\{2, s\}, \{3, s\}, \ldots, \{p-1, s\}$. This is possible since $s > a_p \geq p$. Noting that no superset of any of the sets $s, s+1, ..., s+p-3$ can be in $\mathcal{A}'$ before the changes, it is clear that the changes can be done whilst maintaining the antichain property for $\mathcal{A}'$.

If there are sets in $\mathcal{A}'$ with cardinality greater than two, replace $\mathcal{A}'$ by a squashed antichain equivalent to $\mathcal{A}'$. This is possible by Theorem 6.3. Label this new antichain $\mathcal{A}'$ and repeat the procedure above from the point when $\mathcal{A}'$ is initialised. Eventually $\mathcal{A}'$ will be a flat antichain equivalent to $\mathcal{A}$. □

Theorem 6.6. *FAC is true for all antichains $\mathcal{A}$ with $|A| = 1$ for some $A \in \mathcal{A}$ if FAC is true for all antichains $\mathcal{A}'$ with* $min\{|A| : A \in \mathcal{A}'\} > 1$.

Proof. By Theorems 6.4 and 6.5 FAC is true for antichains $\mathcal{A}$ with $\overline{\mathcal{A}} \leq 2$. Assume $\mathcal{A}$ is a squashed antichain on $[n]$ which contains at least one singleton set and with $\overline{\mathcal{A}} > 2$. Assume the singleton sets in $\mathcal{A}$ are $s, s+1, ..., n$. Assume that the last set in squashed order with cardinality greater than $\overline{\mathcal{A}}$ is the p-set $\{a_1, ..., a_{p-1}, a_p\}$. $\mathcal{A}$ will be modified as shown below until it is transformed into an equivalent antichain which contains no singleton sets.

Replace $\{a_1, ..., a_{p-1}, a_p\}$ by the $(p-1)$-set $\{a_1, ..., a_{p-2}, s\}$. Replace the first singleton set $\{s\}$ by the 2-set $\{s-1, s\}$. As $\mathcal{A}$ is an antichain, $a_p < s$ so $a_{p-2} < s-1$. Then it is clear that the replacements preserve the antichain property for the collection of sets.

For this new collection, order the sets with cardinality greater than 1 in squashed order, and repeat the replacement process above. Eventually all 1-sets will be removed from $\mathcal{A}$ and the transformed $\mathcal{A}$ will be an antichain equivalent to the original antichain. Thus, $\mathcal{A}$ can be flattened if and only if the transformed version of $\mathcal{A}$ can be flattened. □

Corollary 6.1. *FAC is true for all antichains $\mathcal{A}$ with $|A| = n - 1$ for some $A \in \mathcal{A}$ if FAC is true for all antichains $\mathcal{A}'$ with $max\{|A| : A \in \mathcal{A}'\} < n - 1$.*

Proof. This follows from Theorem 6.6 and Note 6.1. □

The following result follows directly from Theorem 6.6 and its proof and Corollary 6.1.

Corollary 6.2. *If there exists a collection $\mathcal{A}$ containing a singleton set and which is a counterexample to FAC, then there exists a collection $\mathcal{A}'$ equivalent to $\mathcal{A}$ which is also a counterexample and for which $min\{|A| : A \in \mathcal{A}'\} > 1$ and $max\{|A| : A \in \mathcal{A}'\} \leq max\{|A| : A \in \mathcal{A}\}$.*
For fixed n, if there exists a collection $\mathcal{A}$ containing an $(n-1)$-set and which is a counterexample to FAC, then there exists a collection $\mathcal{A}'$ equivalent to $\mathcal{A}$ which is also a counterexample and for which $max\{|A| : A \in \mathcal{A}'\} < n - 1$ and $min\{|A| : A \in \mathcal{A}'\} \geq min\{|A| : A \in \mathcal{A}\}$.

The next lemmas are required for Theorem 6.7.

Lemma 6.2. *Let $\mathcal{A}$ denote the collection of the first t 2-sets in squashed order. If $t = \binom{a}{2} + b$ with $a > b \geq 0$ then $|\nabla_N \mathcal{A}| = \binom{a}{3} + \binom{b}{2}$.*

Proof. $\mathcal{A}$ consists of the union of the collection of the $\binom{a}{2}$ 2-subsets of $[a]$ and the collection of 2-sets $\{\{i, a+1\} : i = 1, \ldots, b\}$. If $\{x, y, z\}$ is in $\nabla_N \mathcal{A}$ then either $\{x, y, z\} \subseteq [a]$ or $z = a + 1$ and $\{x, y\} \subseteq [b]$. The number of such sets is $\binom{a}{3} + \binom{b}{2}$ as required. □

Lemma 6.3. *Let $\mathcal{A}$ denote the collection of the first t 2-sets in squashed order. Then $|\nabla_N \mathcal{A}| \geq 2t$ for $t \geq 33$.*

Proof. Write $t = \binom{a}{2} + b$ where $a > b \geq 0$. Then a is the greatest integer such that $t \geq \binom{a}{2}$. Thus

$$a = \lfloor \frac{1+\sqrt{1+8t}}{2} \rfloor$$

and

$$b = t - \binom{a}{2}.$$

By Lemma 6.2 $|\nabla_N \mathcal{A}| = \binom{a}{3} + \binom{a}{2}$. Now

$$\binom{a}{3} \geq \frac{c(c-1)(c-2)}{3!}$$

where

$$c = \frac{1+\sqrt{1+8t}}{2} - 1$$

and

$$\binom{b}{2} \geq \frac{d(d-1)}{2!}$$

where

$$d = t - \frac{1+\sqrt{1+8t}}{2}.$$

Then

$$\begin{aligned}\binom{a}{3} + \binom{b}{2} &\geq \frac{c(c-1)(c-2)}{3!} + \frac{d(d-1)}{2!} \\ &= \frac{t+3}{6}\sqrt{1+8t} - \frac{1+3t}{2} \\ &> 2t\end{aligned}$$

for $t \geq 50$.

Thus the statement of the lemma holds for $t \geq 50$. The truth of the lemma for $33 \leq t \leq 49$ can be established by direct calculation. □

Lemma 6.4. *Let $\mathcal{B}$ denote a collection of t consecutive 2-sets in squashed order. Then:*

(i) $|\nabla_N \mathcal{B}| \geq 2t$ for $t \geq 33$;

(ii) if $t \geq 3$ and $\mathcal{B}$ does not contain any of the first four 2-sets in squashed order then $|\nabla_N \mathcal{B}| \geq t$;

(iii) if $t \geq 10$ and $\mathcal{B}$ does not contain any of the first ten 2-sets in squashed order then $|\nabla_N \mathcal{B}| \geq 2t$.

Proof. Observe that Theorem 6.2 implies that over all possible collections of t 2-sets, the collection consisting of the first t 2-sets in squashed order has the smallest possible number of sets in its new shade. Then (i) follows from Lemma 6.3.

(ii) Let $\mathcal{C} = \{12, 13, 23, 14\}$ be the first 4 2-sets in squashed order. Then $|\mathcal{B} \cup \mathcal{C}| = t + 4 \geq 7$ and $|\nabla_N \mathcal{B}| = |\nabla_N(\mathcal{B} \cup \mathcal{C})| - |\nabla_N \mathcal{C}| = |\nabla_N(\mathcal{B} \cup \mathcal{C})| - 1$. By Lemma 6.2, $|\nabla_N(\mathcal{B} \cup \mathcal{C})| \geq t + 1$ for $t + 4 \geq 7$, so $|\nabla_N \mathcal{B}| \geq t$.

(iii) By (i), only $10 \leq t \leq 32$ needs to be considered. Let $\mathcal{C}$ be the first 10 2-sets in squashed order. Then $|\mathcal{B} \cup \mathcal{C}| = t + 10$ and $|\nabla_N \mathcal{C}| = 10$ so that $|\nabla_N \mathcal{B}| = |\nabla_N(\mathcal{B} \cup \mathcal{C})| - |\nabla_N \mathcal{C}| = |\nabla_N(\mathcal{B} \cup \mathcal{C})| - 10$. For $t \geq 23$, $|\mathcal{B} \cup \mathcal{C}| \geq 33$ so that $|\nabla_N(\mathcal{B} \cup \mathcal{C})| \geq 2t + 20$ by Lemma 6.3. Thus $|\nabla_N \mathcal{B}| \geq 2t$ in these cases.

For $10 \leq t \leq 22$ it can be checked by direct calculation that $|\nabla_N \mathcal{D}| \geq 2t - 10$ where $\mathcal{D}$ is the collection of the first $t + 10$ 2-sets in squashed order. Thus, as $|\nabla_N \mathcal{C}| = 10$ and by Theorem 6.2, $|\nabla_N \mathcal{B}| \geq 2t$. □

Lemma 6.5. *Let $\mathcal{C}$ be the initial segment of a squashed antichain with $|\mathcal{C}| = r$ and with $|C| \geq 4$ for all $C \in \mathcal{C}$. Then,*

(i) $|\triangle_N^{(3)} \mathcal{C}| \geq 4$ for all $r \geq 1$.

If each set in $\mathcal{C}$ has cardinality 4 then

(ii) $|\triangle_N^{(3)} \mathcal{C}| \geq r$ for $r \leq 10$

(iii) $|\triangle_N^{(2)} \mathcal{C}| \geq 10$ for $r \geq 3$.

If $\mathcal{C}$ contains a set of cardinality at least five then

(iv) $|\triangle_N^{(3)}\mathcal{C}| \geq 10$ *for all* r

(v) $|\triangle_N^{(2)}\mathcal{C}| \geq 10$ *for all* r.

Proof. Recall that $\triangle_N^{(3)}\mathcal{C} = \triangle^{(3)}\mathcal{C}$ and $\triangle_N^{(2)}\mathcal{C} = \triangle^{(2)}\mathcal{C}$ as $\mathcal{C}$ is an initial segment of a squashed antichain.

(i) $|\triangle_N^{(3)}\mathcal{C}| = |\triangle^{(3)}\mathcal{C}| \geq |\triangle^{(3)}[k]|$ where $[k]$ is the first set in $\mathcal{C}$, with $k \geq 4$. Since $|\triangle^{(3)}[k]| = \binom{k}{3} \geq 4$, (i) holds for all r.

(ii) can be easily checked by calculating $|\triangle_N^{(3)}\mathcal{C}|$ for $|\mathcal{C}|$ taking values $1, 2 \ldots, 10$.

(iii) $|\triangle_N^{(2)}\mathcal{C}|$ is clearly non-decreasing with $|\mathcal{C}|$. When $|\mathcal{C}| = 3$, that is $\mathcal{C} = \{1234, 1235, 1245\}$, then $|\triangle_N^{(2)}\mathcal{C}| = |\{12, 13, 23, 14, 24, 34, 15, 25, 35, 45\}| = 10$. Part (iii) follows.

(iv) As in part (i), $|\triangle_N^{(3)}\mathcal{C}| \geq \binom{k}{3}$ where k is the cardinality of the first set in $\mathcal{C}$. Since $k \geq 5$ the result follows.

(v)By reasoning similar to that used in part (iv), $|\triangle_N^{(2)}\mathcal{C}| \geq \binom{k}{2}$ which is at least 10. □

Theorem 6.7. *FAC is true for all antichains $\mathcal{A}$ with $2 < \overline{\mathcal{A}} \leq 3$.*

Proof. The theorem is true for $\overline{\mathcal{A}} = 3$ by Theorem 6.4. Assume $\mathcal{A}$ is an antichain with $2 < \overline{\mathcal{A}} < 3$. By Theorem 6.6 it can be assumed that $|A| \geq 2$ for all $A \in \mathcal{A}$. Let $\mathcal{C} = \{A \in \mathcal{A} : |A| > 3\}$, $\mathcal{B} = \{A \in \mathcal{A} : |A| = 2\}$ and set $r = |\mathcal{C}|$. If $|\mathcal{C}| = 0$ the antichain is already flattened so it can be assumed that $r \geq 1$. Let

$$\begin{aligned} t &= V(\mathcal{C}) - 3|\mathcal{C}| \\ &\geq 4|\mathcal{C}| - 3|\mathcal{C}| \\ &= r. \end{aligned}$$

Let $\mathcal{F}$ be the first t 2-sets in $\mathcal{B}$ in squashed order. A sufficient condition to allow $\mathcal{A}$ to be flattened is that all of the sets in $\mathcal{F}$ and $\mathcal{C}$ can be replaced by 3-sets to form a new antichain $\mathcal{A}'$. For then $|\mathcal{A}'| = |\mathcal{A}|$ and

$$\begin{aligned} V(\mathcal{A}') &= V(\mathcal{A}) - V(\mathcal{C}) + 3r + 3t - 2|\mathcal{F}| \\ &= V(\mathcal{A}) - t + 3t - 2t \\ &= V(\mathcal{A}). \end{aligned}$$

As $\mathcal{A}$ is an antichain, $\triangle_N^{(3)}\mathcal{C}$ and $\nabla_N\mathcal{F}$ contain neither 3-sets in $\mathcal{A}$ nor contain any 2-subsets which come after $\mathcal{F}$ in $\mathcal{A}$. Therefore, if $|\triangle_N^{(3)}\mathcal{C}| + |\nabla_N\mathcal{F}| \geq r + t$, $\mathcal{A}$ can be flattened by replacing the sets in $\mathcal{B}$ and $\mathcal{C}$ by distinct sets in $\triangle_N^{(3)}\mathcal{C} \cup \nabla_N\mathcal{F}$. Note that as $r + t \leq 2t$ the condition is met if $|\nabla_N\mathcal{F}| \geq 2t$. There are three cases to consider.

If $t = 1$ or 2 then by Lemma 6.5(i) $|\triangle_N^{(3)}\mathcal{C}| \geq 4 \geq r + t$ and $\mathcal{A}$ can be flattened.

If $3 \leq t \leq 10$ then $\mathcal{C}$ contains at least three 4-sets or a set of cardinality at least 5. By applying Lemma 6.5(iii) or (v), $|\triangle_N^{(2)}\mathcal{C}| \geq 10$. Hence $\mathcal{F}$ does not contain any of the first 10 2-sets in squashed order and so $|\nabla_N\mathcal{F}| \geq t$ by Lemma 6.4(ii). Further, by Lemma 6.5(ii), $|\triangle_N^{(3)}\mathcal{C}| \geq r$. Thus $|\triangle_N^{(3)}\mathcal{C}| + |\nabla_N\mathcal{F}| \geq r + t$ and $\mathcal{A}$ can be flattened.

If $t \geq 11$ then $\mathcal{C}$ contains at least 11 4-sets or a set of cardinality at least 5. Then again $\mathcal{F}$ does not contain any of the first 10 2-sets in squashed order. Thus, by Lemma 6.4(iii), $|\nabla_N\mathcal{F}| \geq 2t \geq r + t$ and $\mathcal{A}$ can be flattened. □

Corollary 6.3. *FAC is true for all antichains $\mathcal{A}$ on $[n]$ with $n - 3 \leq \overline{\mathcal{A}} \leq n$.*

Proof. By Theorems 6.5 and 6.7 FAC is true for all antichains $\mathcal{A}$ when $\overline{\mathcal{A}} \leq 3$. Applying Note 6.1 gives the result. □

Corollary 6.4. *FAC is true on $[n]$ for $n \leq 6$.*

Proof. This follows from Corollary 6.3 and Theorems 6.5 and 6.7. □

The following technical lemma is required for Theorem 6.8.

Lemma 6.6. *Let $\mathcal{A}$ be a squashed antichain on $[n]$ containing b 2-sets, $b > 0$, and with $|A| \geq 2$ for all $A \in \mathcal{A}$.*
(i) Assume that $n = 7$ or 8 and that $\mathcal{A}$ also contains more than b 4-sets or a set of cardinality at least 5. Then $|\nabla\mathcal{B}| \geq 2b + n - 4$.
(ii) If $n \geq 9$ then $|\nabla\mathcal{B}| \geq 2b + n - 4$.

Proof. It can be assumed that the collection $\mathcal{B}$ of 2-sets in $\mathcal{A}$ consists of the last b 2-sets in squashed order on $[n]$.
(i) This follows from a direct calculation of $|\nabla\mathcal{B}|$ for each possible value of $|\mathcal{B}|$ with the given assumptions on $\mathcal{A}$.
(ii) By Lemma 6.1, $|\nabla\mathcal{B}| \geq \frac{n-2}{3}b$. This is at least $2b + n - 4$ when $n = 9$ and $b \geq 15$, and also when $n \geq 10$ and $b \geq 9$. Assume $n = 9$ and $1 \leq b \leq 15$ or $n \geq 10$ and $1 \leq b \leq 9$. The result can be checked easily by exhaustion for $n = 9$ and 10. The result follows for $n > 10$ and $b < 9$ from the fact that $|\nabla\mathcal{B}|$ is a strictly increasing function of n for $n \geq 10$ and for the assumed values of b. □

Note 6.2. Assume $\mathcal{A}$ is a squashed antichain and P is the last set in $\mathcal{A}$ with cardinality $p > r$. Let $\mathcal{B} = \{A \subset P : |A| = r\}$. $\mathcal{B}$ is called the r-shadow of P. Then any r-set in $\mathcal{A}$ that comes after the r-shadow of P in the squashed ordering cannot be a subset of P or any set which precedes P in $\mathcal{A}$.

Theorem 6.8. *FAC is true for all antichains $\mathcal{A}$ with $\overline{\mathcal{A}} > 3$ and with $|A| \leq 2$ for some $A \in \mathcal{A}$ if FAC is true for all antichains $\mathcal{A}'$ with $min\{|A| : A \in \mathcal{A}'\} > 2$.*

Proof. It is sufficient to show that if $\mathcal{A}$ is an antichain with $\overline{\mathcal{A}} > 3$ and $|A| \leq 2$ for some $A \in \mathcal{A}$ then there is an equivalent antichain $\mathcal{A}'$ with $|A| > 2$ for all $A \in \mathcal{A}'$. By Theorem 6.6 the antichains which contain 1-sets need not be considered. By Corollary 6.4 it can be assumed that $n \geq 7$. Thus it can be assumed that $\mathcal{A}$ is a squashed antichain on $[n]$, $n \geq 7$, containing $b(\geq 1)$ 2-sets, no 1-sets and with $\overline{\mathcal{A}} > 3$. Note that $\mathcal{A}$ can be modified, if necessary, so that the 2-sets in $\mathcal{A}$ consist of the last b 2-sets in squashed order on $[n]$. Call this collection $\mathcal{B}$. Assume this is done. Note that as $\overline{\mathcal{A}} > 3$, $\mathcal{A}$ must contain more than b 4-sets or contain sets of cardinality greater than 4, so Lemma 6.6 is applicable in all cases considered here. Let $\mathcal{C}$ be the collection of sets in $\mathcal{A}$ with cardinality greater than 3.

An antichain $\mathcal{A}'$ equivalent to $\mathcal{A}$ with $|A| > 2$ for all $A \in \mathcal{A}$ is created iteratively as follows. Form a new antichain $\mathcal{A}_0$ by replacing the b 2-sets in $\mathcal{A}$ by b 3-sets in $\nabla\mathcal{B}$. This can be done, as by Lemma 6.6 $|\nabla\mathcal{B}| \geq 2b + n - 4$ for $b \geq 1$, and thus for all b $|\nabla\mathcal{B}| > 2b$. Note that

$$V(\mathcal{A}_0) = V(\mathcal{A}) + b \tag{6.1}$$

Label the sets in $\mathcal{C}$ in the reverse of squashed order as P_i for $i = 1, \ldots, |\mathcal{C}|$. The sets in $\mathcal{C}$ are considered one at a time in the ordering imposed above. The aim is to construct a sequence of antichains $\mathcal{A}_1, \mathcal{A}_2, \ldots$ such that $|\mathcal{A}_{i-1}| = |\mathcal{A}_i|, V(\mathcal{A}_{i-1}) > V(\mathcal{A}_i)$ until the last antichain $\mathcal{A}'$ is equivalent to $\mathcal{A}$ but with $|A| > 2$ for all $A \in \mathcal{A}'$. The construction is done as follows.

Whilst $|P_i| - 3 \leq V(\mathcal{A}_{i-1}) - V(\mathcal{A})$ replace P_i by a 3-set which is in $\nabla\mathcal{B}$ and not in $\mathcal{A}_{i-1}$ to create $\mathcal{A}_i$. This is possible for each iteration as $|\nabla\mathcal{B}| > 2b$ and there will be at most b iterations. This means that $V(\mathcal{A}_i) = V(\mathcal{A}_{i-1}) - |P_i| + 3$ and so $V(\mathcal{A}_i) = V(\mathcal{A}_0) - (|P_1| - 3) - (|P_2| - 3) - \ldots - (|P_i| - 3) \leq V(\mathcal{A}_0) - i$. Thus, by 6.1,

$$V(\mathcal{A}_i) \leq V(\mathcal{A}) + b - i. \tag{6.2}$$

If, at some stage, $\mathcal{A}_i$ is equivalent to $\mathcal{A}$, set $\mathcal{A}' = \mathcal{A}_i$ and the result is achieved. Otherwise, at some stage t, $V(\mathcal{A}_t) > V(\mathcal{A})$, so that by 6.2,

$$t < b \tag{6.3}$$

but $|P_{t+1}| - 3 > V(\mathcal{A}_t) - V(\mathcal{A})$. This means that P_{t+1} cannot be replaced by a 3-set without creating a new antichain with volume smaller than the volume of

$\mathcal{A}$. If this happens, form a new antichain $\mathcal{A}^*$ which consists of the sets in $\mathcal{A}_t$ but with all 3-sets in $\mathcal{A}_t$ being replaced by the last 3-sets in squashed order on $[n]$. Call this collection of 3-sets $\mathcal{D}$. Let $P = P_{t+1}$. Note that $|P| > 4$ so that P is the last set in squashed order in $\mathcal{A}^*$ with cardinality greater than 4 and $\mathcal{A}^*$ contains no 4-sets. Let $P = \{a_1, \dots, a_p\}$ and note that $p < n$ as $\mathcal{A}$ is defined on $[n]$ and $b \geq 1$. Define $r = V(\mathcal{A}^*) - V(\mathcal{A})$. Then r is the amount by which the volume of $\mathcal{A}^*$ has to be reduced for its volume to be the same as the volume of $\mathcal{A}$. Note that $|\mathcal{A}^*| = |\mathcal{A}_t| = |\mathcal{A}|$. There are four cases to consider.

Before considering these cases note that: there were at least $|\nabla\mathcal{B}|$ 3-sets which were not subsets of any set in $\mathcal{A}$; there were b of these used when $\mathcal{A}_0$ was constructed; there were t of these used in the construction of $\mathcal{A}_1, \dots, \mathcal{A}_t$. Hence the difference between the number of 3-sets in $\mathcal{A}^*$ and $\mathcal{A}$ is $b + t$. As $t < b$ and by Lemma 6.6, $|\nabla\mathcal{B}| - b - t \geq 2b + n - 4 - b - t \geq n - 3$ for $b > 1$. If $b = 1$, $t = 0$ so $|\nabla\mathcal{B}| - b - t = n - 3$. Set $g = |\nabla\mathcal{B}| - b - t$. Thus there are at least $g(\geq n - 3)$ 3-sets which are not members of $\mathcal{A}^*$ nor in the 3-shadow of any set in $\mathcal{A}^*$. These 3-sets lie in the interval (in the sense of squashed order) between the 3-shadow of P and the 3-sets in $\mathcal{D}$. Call this interval **the gap**.

Case 1: Assume that $a_{p-2} + 1 < a_{p-1} < a_p < n$. Then the next 3-set in squashed order after the 3-shadow of P is $\{a_{p-2} + 1, a_{p-1}, a_p\}$. As $p \geq 5$, $a_{p-2} \geq 3$ so $a_{p-2} + 1 \geq 4$. Combining this with the facts that $g \geq n - 3$ and $a_{p-1} < n - 1$, it can be seen that if $a_{p-1} + 1 < a_p$ then the the set $\{1, a_{p-1} + 1, a_p\}$ is contained in the gap. If $a_{p-1} + 1 \neq a_p$ the first 3-set after the gap comes after $\{1, a_{p-1} + 1, a_p\}$. If $a_{p-1} + 1 = a_p$ then the first 3-set after the gap comes after $\{1, 2, a_p + 1\}$. Replace P by the $(p - r)$-set $Q = \{1, \dots, p - r - 3, a_{p-2} + 1, a_{p-1}, a_p\}$. Then Q comes immediately after the r-shadow of P and the 3-shadow of Q does not include any 3-sets after the gap. Thus the new collection is an antichain equivalent to $\mathcal{A}$.

Case 2: Assume that $a_{p-2}+1 = a_{p-1}$ and $a_{p-1}+1 < a_p < n$. Then the next 3-set in squashed order after the 3-shadow of P is $\{1, a_{p-1}+1, a_p\}$.

As $g \geq n-3$ and $a_{p-1}+1 \leq n-2$, the first possible 3-set in squashed order after the gap is not before the set containing a_p+1 if $a_{p-1}+2 = a_p$, or containing both $a_{p-1}+2$ and a_p otherwise. Replace P by the $(p-r)$-set $Q = \{1, \ldots, p-r-2, a_{p-1}+1, a_p\}$. Then Q comes after the $(p-r)$-shadow of P and the 3-shadow of Q does not include any 3-sets after the gap. Thus this new collection is an antichain equivalent to $\mathcal{A}$.

Case 3: Assume that $a_{p-2}+1 = a_{p-1}$ and $a_{p-1}+1 = a_p < n$. Then the next 3-set in squashed order after the 3-shadow of P is $\{1, 2, a_p+1\}$.

Assume $|\mathcal{D}| \leq p-5$. Then all sets in $\mathcal{D}$ contain the elements $n-1$ and n. Replace P by the $(p-r)$-set $Q = \{1, \ldots, p-r-1, n\}$. As $p-r-1 < n-1$, Q is not a superset of any set in $\mathcal{D}$. As $n \in Q$, Q is not a subset of any set in $\mathcal{A}^* - \mathcal{D}$. Hence this new collection is an antichain.

Assume $|\mathcal{D}| > p-5$. Then it may not be possible to replace P by a $(p-r)$-set whose 3-shadow contains no 3-set after the gap so another transformation is used.

Claim: If $|\mathcal{D}| > p-5$ then the antichain $\mathcal{A}'$ created by the following changes to $\mathcal{A}^*$ provides an antichain equivalent to $\mathcal{A}$. Replace P by the 4-set $Q = \{1, 2, 3, a_p+1\}$ and replace the first $(p-r-4)$ 3-sets in $\mathcal{A}^*$ by the next $(p-r-4)$ 4-sets after Q in squashed order. (Note that this is possible as $|\mathcal{D}| > p-5$. Also note that each of these $(p-r-4)$ 4-sets contain a_p+1 as its largest element as $a_p+1 > p$).

Proof of claim: Let the collection of 4-sets in $\mathcal{A}'$ be $\mathcal{G} = \{\{1, 2, 3, a_p+1\}, \{1, 2, 4, a_p+1\}, \ldots, \{b_1, b_2, b_3, a_p+1\}\}$. Clearly $b_3 \geq 4$. Define the **new gap** to be the collection of 3-sets that come after the 3-shadow of P and before the remaining 3-sets in $\mathcal{A}'$. The size of the new gap is $g' = g+(p-r-4)$.

Assume $b_3 = 4$. Then $2 \leq |\mathcal{G}| \leq 4$. As $g \geq n-3 \geq 4$, $g' \geq 5$, 6 or 7 for $|\mathcal{G}| = 2, 3$ or 4 respectively. However $|\triangle_N(\mathcal{G})|$ is 5,6 or 6 respectively. Therefore $\mathcal{A}'$ is an antichain.

Assume $b_3 \geq 5$. Then it must be the case that at least $\binom{b_3-1}{3}$ 3-sets have been replaced by 4-sets, so $p - r - 4 \geq \binom{b_3-1}{3}$ and $g \geq n - 3 > p - r - 2 \geq \binom{b_3-1}{3} + 2$. Thus $g' \geq 2\binom{b_3-1}{3} + 2 \geq \binom{b_3}{2} \geq |\triangle_N(\mathcal{G})|$. Hence $\mathcal{A}'$ is an antichain.

Case 4: Assume that $P = \{a_1, \ldots, a_{p-1}, n\}$. Note that $a_{p-1} \neq n-1$ as $\{n-1, n\} \in \mathcal{B}$. The next 3-set in squashed order after the 3-shadow of P is either $A = \{a_{p-2}+1, a_{p-1}, n\}$ or $B = \{1, a_{p-1}+1, n\}$. Assume that $b = 1$. Then, as $\mathcal{A}$ is a squashed antichain, if $a_{p-1} = n-2$ then $\mathcal{A}$ cannot contain a 3-set and $\mathcal{A}^*$ contains only the 3-set $\{n-2, n-1, n\}$. Also, if $a_{p-2} = n-3$ then the next 3-set after the 3-shadow of P is B. Otherwise, with $a_{p-1} = n-2$ it is A.

Assume that $b > 1$. Then $\mathcal{B}$ also contains the sets $\{n-2, n\}$. Therefore P cannot contain $n-2$ so $a_{p-1} < n-2$ and $a_{p-2} < n-3$.

Form a new antichain $\mathcal{A}'$ from $\mathcal{A}^*$ as follows. If the first 3-set after the 3-shadow of P is A, replace P by the $(p-r)$-set $Q = \{1, \ldots, p-r-3, a_{p-2}+1, a_{p-1}, n\}$. If the first 3-set after the 3-shadow of P is B, replace P by the $(p-r)$-set $Q = \{1, \ldots, p-r-2, a_{p-1}+1, n\}$.

It is now shown that $\mathcal{A}'$ is an antichain. Assume that $b = 1$. If $a_{p-1} = n-2$ and $a_{p-2} = n-3$ then $p-r-2 < n-2$, so that $\{n-2, n-1, n\}$ is not in the 3-shadow of Q. If $a_{p-1} = n-2$ and $a_{p-2} < n-3$ then there are at most $(n-3)$ 3-sets with $n-2$ and n as the two largest elements. For all other cases $a_{p-1} < n-2$ so there are at most $(n-3)$ 3-sets with either a_{p-1} and n as the two largest elements. In each situation, the 3-shadow of Q does not contain any 3-sets after the gap as the gap has size at least $n-3$. Hence $\mathcal{A}'$ is an antichain equivalent to $\mathcal{A}$.

Assume that $b > 1$. Then there are at most $(n-3)$ 3-sets with a_{p-1} and n or with $a_{p-1}+1$ and n as the two largest elements. In either case the 3-shadow of Q does not contain any 3-sets after the gap. Thus $\mathcal{A}'$ is an antichain equivalent to $\mathcal{A}$. This completes the proof. □

Corollary 6.5. *FAC is true for all antichains* $\mathcal{A}$ *with* $|A| = n-2$ *for some* $A \in \mathcal{A}$ *if FAC is true for all antichains* $\mathcal{A}'$ *with* $max\{|A| : A \in \mathcal{A}'\} < n-2$.

Proof. This follows from Theorem 6.8 and Note 6.1. □

The following result follows directly from Corollary 6.1, Corollary 6.5 and Theorem 6.8 and its proof.

Corollary 6.6. *If there exists a collection* $\mathcal{A}$ *containing sets of cardinality 1 or 2, and which is a counterexample to FAC, then there exists a collection* $\mathcal{A}'$ *equivalent to* $\mathcal{A}$ *which is also a counterexample and for which* $min\{|A| : A \in \mathcal{A}'\} > 2$ *and* $max\{|A| : A \in \mathcal{A}'\} \leq max\{|A| : A \in \mathcal{A}\}$.
For fixed n, if there exists a collection $\mathcal{A}$ *containing sets of cardinality at least (n-2) and which is a counterexample to FAC, then there exists a collection* $\mathcal{A}'$ *equivalent to* $\mathcal{A}$ *which is also a counterexample and for which* $max\{|A| : A \in \mathcal{A}'\} < n-2$ *and* $min\{|A| : A \in \mathcal{A}'\} \geq min\{|A| : A \in \mathcal{A}\}$.

Corollary 6.7. *FAC is true on* $[n]$ *for* $n \leq 8$.

Proof. The cases when $n \leq 6$ appear as Corollary 6.4. The case when $n = 7$ follows from Corollaries 6.5 and 6.6. Assume $n = 8$. By Corollaries 6.2 and 6.6 it is only necessary to consider antichains $\mathcal{A}$ with $3 \leq |A| \leq 5$ for each $A \in \mathcal{A}$. Further, by complements and Theorem 6.4, it can be assumed that $3 < \overline{\mathcal{A}} < 4$. Let C denote the collection of 5-sets in $\mathcal{A}$. Let $\mathcal{B}$ denote the first $|\mathcal{C}|$ 3-sets in squashed order in $\mathcal{A}$. To show that there is a flat equivalent $\mathcal{A}'$ of $\mathcal{A}$, $|\mathcal{C}|$ 5-sets and the first $|\mathcal{C}|$ 3-sets in $\mathcal{A}$ will be replaced by 4-sets and at least one 3-set will still occur in $\mathcal{A}'$. It can be checked that for $\mathcal{A}$ to be an antichain with at least $|\mathcal{C}|+1$ 3-sets, then $|\mathcal{C}| \leq 20$. It can also be checked that $|\Delta\mathcal{C}| \geq 2|\mathcal{C}|$ for $|\mathcal{C}| \leq 17$ and thus $\mathcal{A}$ can be flattened in these cases. Assume $18 \leq |\mathcal{C}| \leq 20$. As $|\mathcal{C}|$ 3-sets are required to be replaced by $|\mathcal{C}|$ 4-sets it is easy to check that $|\Delta\mathcal{C}|+|\nabla_N\mathcal{B}| > 2|\mathcal{C}|$ so that $\mathcal{A}$ can be flattened. This completes the proof. □

Note that Corollary 6.7 can be improved with individual case arguments so that it is true at least for $n \geq 10$. These arguments are not included here.

6.3 Comments

The results in this chapter show that FAC is true whenever the antichain has average set size at most 3 or when the universal set has cardinality at most 8. Theorem 6.8 shows that if one is trying to prove FAC then it is sufficient to prove FAC is true for antichains in which each set has size no less than 3.

Although these results are limited in scope, they do represent an improvement on known results on FAC and they provide a possible strategy for proving FAC in general. Whilst it is desirable to generalise the arguments here, it may be quite difficult, especially to prove a generalised form of Theorem 6.8. Despite the lack of a more general result, the fact that FAC is true when the average set size is no more than three is important in applications. In fact the motivation for this chapter was to achieve the result that FAC is true for $\overline{\mathcal{A}} \leq 3$ so that FAC can be applied to the $R(n,k)$ problem. This application is considered in Chapter 7.

It should be noted that at the time of completing this thesis, Lieby [15] has a long draft proof that FAC is true for all antichains on 3 consecutive levels. That is, FAC is true for antichains on $[n]$ with only 3 non-zero parameters p_{k-1}, p_k, p_{k+1} for some $2 < k < n-1$. The fact that this is the strongest "known" result concerning the validity of FAC gives some indication that proving its validity in all cases is a very challenging problem. Of course, if Lieby's proof is correct, then the 3-level result can be applied to prove Corollary 6.7 and to aid in increasing the upper bound on the values of n for which FAC is known to be true.

Chapter 7

Antichains and Completely Separating Systems

7.1 Introduction

The results in Chapter 6 are considered in this chapter in terms of their relationship to (n)CSSs and (n,k)CSSs. This chapter also relies on some classical results from Sperner Theory and results in Chapters 4 and 5. The results in this chapter are used in Chapters 8 and 9.

The volume of a collection of sets is very important for the arguments in this and subsequent chapters. The use of arguments which specifically use volume provides a seemingly new but important approach to considering problems in Sperner Theory and designs for Completely Separating Systems.

The following theorem can be found in [7] where it is stated in terms of multisets.

Theorem 7.1. *Assume n, $p \in Z^+$ are fixed. Let $k \in Z^+$ and $k \leq \frac{n}{2}$ be such* *that* $\binom{n}{k} < p \leq \binom{n}{k+1}$. *Let* e' *denote the smallest of the solutions* e *of*

$$|\triangle F_{k+1}(e)| + (p-e) = \binom{n}{k}. \tag{7.1}$$

Then

$$min\ V(\mathcal{A}) = e'(k+1) + (p-e')k$$

where the minimum is taken over all antichains on $[n]$ *of size* p.

Note 7.1. To see why a solution to (7.1) always exists, let e be such that

$$|\triangle F_{k+1}(e)| + (p-e) \leq \binom{n}{k}. \tag{7.2}$$

Note that such an e always exists as $e = p$ is a solution of (7.2). Consider the antichain $\mathcal{A}$ of p sets formed by taking the first e $(k+1)$-sets in squashed order and the last $(p-e)$ k-sets in squashed order. Such an antichain exists by (7.2). If equality is not achieved in (7.1) there exists at least one k-set A such that $A \notin \triangle F_{k+1}(e)$ and $A \notin \mathcal{A}$. Then a new antichain $\mathcal{A}^*$ can be formed by taking the first $(e-1)$ $(k+1)$-sets in squashed order and the last $(p-e+1)$ k-sets in squashed order. If $|\triangle F_{k+1}(e-1)| + (p-e+1) < \binom{n}{k}$ the process is repeated with $\mathcal{A}^*$ replacing $\mathcal{A}$. As $p > \binom{n}{k}$ it follows that there always exists a solution which achieves equality in (7.1).

The minimum in the theorem is attained by an antichain consisting of e' $(k+1)$-sets and $(p-e')$ k-sets. Thus the theorem says that over all antichains of size p on $[n]$ there is always a minimum volume antichain which is flat.

As stated in Chapter 6 it is known that for each antichain on $[n]$ with p_k k-sets, $k = 1, \ldots, n$, there is a corresponding squashed antichain on $[n]$ with p_k k-sets. Hence, to consider whether or not an antichain can be flattened, it suffices to consider only squashed antichains. Thus, in all cases below, it is appropriate and useful to assume that each antichain $\mathcal{A}$ is a squashed antichain. Theorem 7.1 has various important consequences. For example, it can be used to prove Theorem 6.4 as shown below.

Theorem 7.2 (Restatement of Theorem 6.4). *FAC is true for all antichains $\mathcal{A}$ with $\overline{\mathcal{A}}$ an integer.*

Alternate proof of Theorem 6.4. Let $\mathcal{A}$ be an antichain on $[n]$ with $\overline{\mathcal{A}} = k$, $k \in Z^+$. Without loss of generality it can be assumed that $k \leq \frac{n}{2}$, for if $k > \frac{n}{2}$, then one can consider the antichain $\mathcal{A}' = \{A : ([n] - A) \in \mathcal{A}\}$. Let $\mathcal{C}$ be the minimum volume squashed and flat antichain on $[n]$ with $|\mathcal{C}| = |\mathcal{A}|$. $\mathcal{C}$ exists by Theorem 7.1. Further, $|\mathcal{C}| \leq \binom{n}{k}$ by Theorem 7.1 since $\overline{\mathcal{C}} \leq \overline{\mathcal{A}} = k$ and $k \leq \frac{n}{2}$. Hence, as $|\mathcal{C}| = |\mathcal{A}|$, $\mathcal{A}$ can be flattened by replacing it by an antichain of k-subsets of $[n]$. □

7.2 Completely Separating Systems

The original motivation for considering FAC is that the truth of FAC would aid in the determination of minimum size (n,k)CSSs. The relationship between antichains and CSSs is examined in the next section. This section contains some necessary results.

Recall that Spencer (see Lemma 4.1) has shown that

$$R(n) = \min\{m : \binom{m}{\lfloor \frac{m}{2} \rfloor} \geq n\}$$

so that $R(n)$ is known for all n.

In the process of proving this result Spencer stated the following theorem, which is restated here using the definition of the dual of a collection of sets.

Theorem 7.3. *If $\mathcal{A}$ is a CSS then its dual $\mathcal{A}^*$ is an antichain and vice versa.*

Spencer used a description of CSSs and antichains involving arrays with all elements being 0 or 1. This can be viewed as a basis for a computationally inefficient method for finding minimal CSSs by searching through the set of arrays of appropriate size in which all entries are 0 or 1. Spencer did not explicitly construct any minimal CSSs although his proof provides sufficient information to guide the reader who wishes to do this. A more efficient way of constructing minimal CSSs is considered later in this chapter.

Note 7.2. If $\mathcal{A}$ is a CSS and $\mathcal{A}^*$ is its dual then $V(\mathcal{A}) = V(\mathcal{A}^*)$.

Recall that a CSS $\mathcal{C}$ on $[n]$ is said to be **fair** if there is a positive integer k such that each element of $[n]$ occurs in exactly k or $k+1$ sets in $\mathcal{C}$.

Note 7.3. The dual of a fair CSS is a flat antichain and vice versa.

The following result is a corollary of Theorem 7.1.

Corollary 7.1. *For any n, there is always a fair minimum volume minimal $(n)CSS$.*

Proof. Let $\mathcal{C}$ be a minimum volume minimal (n)CSS and $\mathcal{A}$ its dual. Let $|\mathcal{C}| = m$. Then $\mathcal{A}$ is an antichain of n subsets of $[m]$. By Note 7.2, $V(\mathcal{C}) = V(\mathcal{A})$ so $\mathcal{A}$ is a minimum volume antichain of size n on $[m]$. By Theorem 7.1, either $\mathcal{A}$ is flat or $\mathcal{A}$ can be flattened to an antichain with the same volume and number of sets. In either case, by Note 7.3, there is a dual minimum volume minimal (n)CSS which is fair. □

Note 7.4. It is unknown if in general all minimum volume minimal (n)CSSs are fair. It is shown in Chapter 8 that all minimum volume minimal (n)CSSs are fair for $n \leq 10$.

Let n be given. Note that the dual of a minimal (n)CSS is an antichain of size n on $[R(n)]$ and vice versa. The corollary implies that searching through the collection of minimum volume antichains on $[R(n)]$ will provide the duals of minimum volume minimal (n)CSSs. This search may not provide all examples of minimal (n)CSSs as not all minimal CSSs have minimum volume. If FAC is true, then for each dual antichain on $[R(n)]$ of a minimal (n)CSS, there will always be a flat antichain with the same cardinality and volume. Each antichain which can be flattened to this one will be the dual of a minimal (n)CSS. Hence, if one knew the number of antichains on $[R(n)]$ which can be flattened to a given antichain, then one can calculate the exact number of minimal (n)CSSs. Unfortunately, these values are not known in general, and it is a problem worthy of further consideration. The cases when $n \leq 10$ are fully considered in Chapter 8.

The next section explores the use of antichains in the study of CSSs.

7.3 Antichains and Minimal (n)CSSs

The construction of fair minimal (n)CSS of minimum(maximum) volume using antichains is considered here. Recall that by the result of Spencer (Lemma 4.1), $R(n)$ is known for all n. Recall that if $\mathcal{C}$ is a collection of sets on $[n]$ then the **complementary collection $\mathcal{C}'$** is $\mathcal{C}' = \{A' = [n] - A, A \in \mathcal{C}\}$.

Note 7.5. 1. If $\mathcal{C}$ in the previous definition is a minimum volume CSS then $\mathcal{C}'$ is a maximum volume CSS.

2. If $\mathcal{C}$ is a flat antichain then $\mathcal{C}'$ is a flat antichain.

3. If $\mathcal{C}$ is a fair CSS then $\mathcal{C}'$ is a fair CSS.

Lemma 7.1. *Let $\mathcal{D}$ denote the collection of non-equivalent minimal (n)CSSs with $|\mathcal{C}| = R$ for each $\mathcal{C} \in \mathcal{D}$. Assume $\mathcal{B}, \mathcal{C} \in \mathcal{D}$ with $V(\mathcal{B}) = min_{\mathcal{A}\in\mathcal{D}}(V(\mathcal{A}))$ and $V(\mathcal{C}) = max_{\mathcal{A}\in\mathcal{D}}(V(\mathcal{A}))$. Then $V(\mathcal{B}) + V(\mathcal{C}) = Rn$.*

Proof. Let $\mathcal{B}'$ and $\mathcal{C}'$ be the complementary collections of $\mathcal{B}$ and $\mathcal{C}$ respectively. Note that $V(\mathcal{B}) + V(\mathcal{B}') = V(\mathcal{C}) + V(\mathcal{C}') = Rn$. The assumptions concerning the minimum and maximum volumes of $\mathcal{B}$ and $\mathcal{C}$ imply that $V(\mathcal{B}') \leq V(\mathcal{C})$ and

$V(\mathcal{C}') \geq V(\mathcal{B})$. Combining these inequalities with the previous equation completes the proof. □

Theorem 7.4. *Let $\mathcal{A}$ be an antichain of n sets on $[R]$. Let $\mathcal{A}^*$ be the dual of $\mathcal{A}$. Assume $\mathcal{A}$ and its complementary collection $\mathcal{A}'$ are ordered so that $A_i \in \mathcal{A}$ and $A_i' \in \mathcal{A}'$ are in the same relative position in the ordering. Assume that the ith element of $[n]$ corresponds to the ith set in $\mathcal{A}$ and $\mathcal{A}'$. Then $(\mathcal{A}^*)' = (\mathcal{A}')^*$.*

Proof. The dual of $\mathcal{A}$ is the CSS $\mathcal{A}^*$ on $[n]$ with $|\mathcal{A}^*| = R$. By definition $|(\mathcal{A}^*)'| = |(\mathcal{A}')^*|$. Hence the proof is complete if it is shown that each set in $(\mathcal{A}^*)'$ is a set in $(\mathcal{A}')^*$. Without loss of generality assume $A \in \mathcal{A}$. Then $A' = [n] - A \in \mathcal{A}'$. $A \in \mathcal{A}$ means that some element of A, say p, is in exactly k sets of $\mathcal{A}^*$, say $\{A_i : i \leq k\}$. Hence p is in exactly $(n-k)$ sets of $(\mathcal{A}^*)'$, namely $\{A_i : i > k\}$. $A' \in \mathcal{A}'$ means that the element p is in exactly $(n-k)$ sets of $(\mathcal{A}')^*$, namely $\{A_i : i > k\}$. This proves the theorem. □

It is possible to construct a minimal (n)CSS which is both fair and of minimum volume as follows. Determine $R(n)$. Construct a flat, minimum volume squashed antichain $\mathcal{A}$ on $[R(n)]$ of size n as in Theorem 7.1. The dual of this antichain $\mathcal{A}^*$ is a fair minimal (n)CSS of minimum volume. By Lemma 7.1 $(\mathcal{A}^*)'$ is a fair maximal $(n)CSS$ of maximum volume.

The ideas above are illustrated with the following example.

Example 7.1. Consider minimum volume minimal (22)CSSs. Note that $R(22) = 7$. Clearly there is no antichain on a 7-set containing a 1-set and 22 sets in total, or containing 22 2-sets. A flat antichain exists containing 22 3-sets. The dual CSS of this antichain is a minimal fair (22)CSS with volume 66.

If a smaller volume minimal (22)CSS exists then the dual antichain contains some 2-sets so one tries to construct a flat squashed antichain consisting of 2-sets and 3-sets. It is required to construct a flat antichain, from the left (squashed order) for 3-sets, and from the right (reverse of squashed order) for 2-sets, which includes the maximum number of 2-sets whilst allowing the inclusion of sufficient 3-sets to obtain a total of 22 sets.

The antichain obtained is

$\mathcal{A} = \{123, 124, 134, 234, 125, 135, 235, 145, 245, 345, 126, 136, 236, 146, 246, 56, 17, 27, 37, 47, 57, 67\}$. This has volume 59. The corresponding fair minimal (n)CSS on the 22-set $\{1, \dots, 9, A \dots, M\}$ is $\mathcal{C} = \{123568BCEH, 124579BDFI, 13467ACDJ, 23489AEFK, 56789AGL, ABCDEFM, HIJKLM\}$.

As no squashed flat antichain with more 2-sets can be constructed, the dual of this antichain is a minimum volume minimal (22)CSS. The complement of this CSS is a maximum volume fair minimal (n)CSS which is the dual of $\mathcal{A}'$.

Note 7.6. There are smaller volume (22)CSSs but they are not minimal. For example, there is an antichain of 22 sets on an 8-set with volume 43. This antichain contains 21 2-sets and one 1-set. The dual (22)CSS contains 8 sets so it is not minimal. In the extreme case there is a (22)CSS with volume 22. This CSS consists of singleton sets only.

The minimum and maximum volumes of a minimal (n)CSS for $n \leq 70$ are given explicitly in Lemma 7.3. One of the reasons to state Lemma 7.3 is to illustrate the pattern in growth in the volume of minimum volume minimal (n)CSSs. Bounds on this growth are given in Lemma 7.2.

Lemma 7.2. *Let C be the collection of minimal (n)CSSs for a given n. Then*
$n(p-1) < min_{A \in C} V(A) \leq np$
where p is the least integer for which $n \leq \binom{R}{p}$.

Proof. By Theorem 7.1 there is always a minimum volume minimal (n)CSS whose dual is a flat antichain on $[R(n)]$. As p is the least integer for which $n \leq \binom{R}{p}$, $p \leq \binom{R}{\lfloor \frac{R}{2} \rfloor}$ and it is easy to see that the dual flat antichain must contain sets of size greater than $p-1$. Clearly there are n sets of size p on $R(n)$. The inequalities follow from these observations. □

Lemma 7.3. *For $n \leq 70$ the minimum volume of a minimal (n)CSS is given by the value $min(V)$ in the tables below. The corresponding maximum volume is given by the equality $max(V) = Rn - min(V)$ for each n.*

1. For $n \leq 4$, $R(n) = n$.
For $5 \leq n \leq 6$, $R(n) = 4$.
For $7 \leq n \leq 10$, $R(n) = 5$.
For $n = 11$ to 20, $R(n) = 6$.

n	$1-4$	$5-6$	7	$8-10$	11	$12-15$	$16-20$
$min(V)$	n	$2n$	13	$2n$	21	$2n$	$3n$

2. For $n = 21$ to 35, $R(n) = 7$.

n	21	$22-23$	$24-26$	$27-28$	$29-31$	$32-35$
$min(V)$	$2n$	$3n-7$	$3n-6$	$3n-2$	$3n-1$	$3n$

3. For $n = 36$ to 70, $R(n) = 8$.

n	$36-38$	$39-42$	$43-44$	$45-47$	$48-56$
$min(V)$	$3n-8$	$3n-7$	$3n-3$	$3n-2$	$3n$

n	$57-58$	$59-61$	$62-63$	$64-66$	$67-70$
$min(V)$	$4n-7$	$4n-6$	$4n-2$	$4n-1$	$4n$

Proof. The values of $R(n)$ are given by Lemma 4.1. The values for $min(V)$ are determined by finding the duals of minimum volume flat antichains on $[R(n)]$ of size n. The process to obtain these minimum volume flat antichains is explained before Example 7.1. The values of $max(V)$ follow from Lemma 7.1. □

7.4 Antichains and (n, k)CSSs

The use of antichains in searching for minimal (n, k)CSSs is not as straightforward as for minimal (n)CSSs. The main reason is that the value k is a constraint on the number of times an element can occur in a dual antichain. An (n,k)CSS has the property that each element of $[n]$ must occur exactly the same number of times in the dual antichain. In many cases this means that a squashed antichain will not provide a minimal (n,k)CSS as the element 1 occurs most often in a squashed antichain and the element n occurs the least often.

Example 7.2. By Theorem 5.1 $R(13,4) = 7$. A fair minimal (13,4)CSS exists as shown by the collection $\mathcal{C} = \{1247, 1258, 139A, 236B, 456C, 789D, ABCD\}$ with dual antichain $\mathcal{A} = \{123, 124, 34, 15, 25, 45, 16, 26, 36, 37, 47, 57, 67\}$. The dual antichain of a minimal $(13,4)$CSS cannot be a squashed antichain as such an antichain involves breaking the restriction imposed by k.

Despite the restriction imposed by k, there are some benefits in using the antichain approach for minimal (n,k)CSS problems. In [22] it is conjectured that a fair minimal (n,k)CSS exists for all n and k. If this is true for (n,k)CSS and FAC is true, flat antichains are a useful aid in determining values of $R(n,k)$. For example, if FAC is true and fair minimal (n,k)CSSs always exist, then it is only necessary to search through flat antichains with appropriate parameters to find a corresponding minimal CSS. Without FAC, an exhaustive search for minimal CSSs with given constraints can require considerable amounts of calculations. This is true, for example, if one searches through the collection of all possible antichains with appropriate parameters.

Even if fair minimal (n,k)CSSs do not always exist, FAC can be used to assert that $R(n,k) > R$ for some R if there does not exist a flat antichain on $[R]$ containing n sets. This approach has been used by Ramsay in [21] where the lower bounds on $R(n,k)$ in Chapter 5 were improved in several cases by assuming that FAC is true. The fact that FAC has been verified for $\overline{\mathcal{A}} \leq 3$ in Chapter 6 allows the calculations by Ramsay to be made certain in some cases. This is now stated explicitly for $k \leq 10$.

The unknown values of $R(n,k)$ for $k \leq 10$ in Chapter 5 are as follows: $k = 7$ for $n = 14$ to 18; $k = 8$ for $n = 15$ to 25; $k = 9$ for $n = 16$ to 33; $k = 10$ for $n = 17$ to 41. All remaining values of $R(n,k)$ for $k \leq 10$ were fully determined in Chapters 4 and 5. For some of the unknown cases above, an application of FAC allows improvements to the lower bounds on $R(n,k)$ found in Chapter 5. These improvements on the lower bound on $R(n,k)$ are:

$k = 7$: $R(n,7) \geq 7$ for $n = 15, 16$.

$k = 8$: $R(n,8) \geq 7$ for $n = 15, 17, 18, 19$; $R(n,8) \geq 8$ for $n = 21, 22$.

$k = 9$: $R(n,9) \geq 7$ for $n = 16, 17, 19, 20$; $R(n,9) \geq 8$ for $n = 22, 23, 24, 25$; $R(29,9) \geq 9$.

$k = 10$: $R(n,10) \geq 6$ for $n = 18, 19$; $R(n,10) \geq 8$ for $n = 26, 27, 28, 29$; $R(n,10) \geq 9$ for $n = 32, 33$; $R(n,10) \geq 10$ for $n = 37$.

For most of the new bounds above, sample CSSs which achieve the stated lower bound have been constructed. Thus the actual value of $R(n,k)$ in each of these cases is now known. The exceptions are for $k = 10$ and $n = 18, 19, 29$. Recall that in Chapter 4 it is shown $R(n,k) = R(n, n-k)$. This allows the values of $R(18,8)$ and $R(19,8)$ to be used to determine the values of $R(18,10)$ and $R(19,10)$. The fact that $R(29,10) = 9$ is shown in Chapter 9.

Note 7.7. An example minimal $(32,10)$CSS is included below because of the difficulty of finding one, either by hand or computer search. Examples of several of the remaining minimal (n,k)CSSs can be found in the appendix.

In the example $(32,10)$CSS shown below each row of the array represents a set in the CSS. Alphabetic and numeric characters are used to represent the 32 elements.

Extra spaces are left in rows of the array to help clarify the structure of the CSS.

1	2	3	4		a	b	c	d	e	f		
1	5	9		t	u	v		l	m		p	q
2	6	10		t	u	j		l	n		p	r
3	7	11		t	v	j		m	o		q	s
4	8	12		u	v	j		n	o		r	s
a	b	c		5	6	7	8		g	h	i	
a	d	e		9	10	11	12		g	h	k	
b	d	f		g	i	k		p	q	r	s	
c	e	f		h	i	k		l	m	n	o	

Example CSSs which achieve the lower bound on $R(n,k)$ in Chapter 5 have been constructed for several of the other unknown values in Chapter 5 for $k \leq 10$. The actual values of $R(n,k)$ in these cases are:

$k = 7$: $R(14,7) = 6$.

$k = 8$: $R(16,8) = 6$; $R(n,8) = 8$ for $n = 23, 24$.

$k = 9$: $R(18,9) = 6$; $R(21,9) = 7$; $R(26,9) = 8$; $R(n,9) = 9$ for $n = 30, 31, 32$.

$k = 10$: $R(17,10) = 7$; $R(20,10) = 6$; $R(n,10) = 7$ for $n = 21, 22, 23$; $R(34,10) = 9$; $R(n,10) = 10$ for $n = 38, 39, 40, 41$.

7.5 Comments about (n,k)CSSs

The results in this chapter leave the following values of $R(n,k)$ undetermined for $k \leq 10$: $k = 7$ for $n = 17, 18$; $k = 8$ for $n = 20, 25$; $k = 9$ for $n = 27, 28, 33$; $k = 10$ for $n = 24, 25, 29, 30, 31, 35, 36$. These are determined in Chapter 9 where a table of values of $R(n,k)$ is included for $k \leq 10$ and $n \leq 41$.

In Chapter 9 structural constraints on (n,k)CSSs are introduced for various cases. These decrease the reliance on brute force searches, which cannot be exhaustive because of the problem size. These structural constraints allowed the sample $(32,9)$CSS in 9 sets to be made, whilst the original application of the search program mentioned below did not find a $(32,9)$CSS in 9 sets.

Computationally, any results which limit the search space for minimal (n,k)CSSs are useful. The complexity of determining minimal CSSs is partially reflected in Cai [6]. Cai defines a CSS on a graph G to be a family F of collections of vertex sets in G which completely separate the adjacent vertices of G. Cai shows that the problem of determining the minimum size of a CSS on G is NP-complete.

The application of the truth of FAC for $\overline{\mathcal{A}} \leq 3$ to improving lower bounds on $R(n,k)$ for $k \geq 11$ is to be included in [31]. That paper will also include the results of computer searches by McKay [18] for minimal (n,k)CSSs, using antichains. In many cases this search produces an example CSS of minimum size. Otherwise it shows that a dual antichain with appropriate parameters has not been found. As the searches are not exhaustive, this leads to no definite conclusion.

7.6 A Use of CSSs in Sperner Theory

It is important to realise that the consideration of the duality of antichains and CSSs has benefits flowing in both directions in the study of these structures. It has been shown above how antichains can be used to find minimum size CSSs or to improve the known lower bounds on minimum size CSSs. Knowledge of the size of minimum CSSs can solve certain natural questions involving antichains. One such question is the following:

For what values of k, m and n does there exist an antichain of size n on $[m]$ in which each element of $[m]$ occurs k times? (An antichain which satisifies this condition is said to be *equitable*.)

The answer to this question is quite simple based upon the work in this thesis as stated in the following results.

Theorem 7.5. *Let C be a minimal (n,k)CSS with $R(n,k) = m$. Then there exists an antichain of size n on $[m]$ in which each element of $[m]$ occurs k times.*

Proof. The dual of $\mathcal{C}$ is an antichain of size n on $[m]$ in which each element of $[m]$ occurs k times. □

Corollary 7.2. *Assume that* $R(n,k) = m$. *Then*
(i) there does not exist an antichain of size n *on* $[p]$ *in which each element of* $[p]$ *occurs* k *times for each* $p < m$;
(ii) there exists an antichain of size n *on* $[p]$ *in which each element of* $[p]$ *occurs* k *times for each* $p \geq m$.

Proof. (i) If there exists a p for which (i) is not true, then the dual of the appropriate antichain is a (n,k)CSS with cardinality less than m. This is a contradiction.
(ii) By Theorem 7.5 there exists an antichain $\mathcal{A}$ of size n on $[m]$ in which each element of $[m]$ occurs k times. As $n > k$, the elements $m+1, \ldots, p$ can be added to k of the sets in $\mathcal{A}$ to form an antichain satisfying the required conditions. □

Example 7.3. Consider the following question: Does there exist an antichain of size 18 on [7] in which each element of [7] occurs exactly 7 times? If such an antichain exists then it has volume 49. It is stated earlier that $R(18,7) \geq 7$ so such an antichain may exist. There are many antichains of size 18 on [7] with this volume including the squashed flat antichain
$\{123, 124, 134, 234, 125, 135, 235, 145, 245, 345, 126, 136, 236, 17, 27, 37, 47, 57\}$.
This antichain does not include each element exactly 7 times and it is not immediately clear if there exists an antichain of size 18 on [7] in which each element of [7] occurs exactly 7 times. However, an answer to the question can be found by applying Corollary 7.2 and the fact that $R(18,7) = 8$ (shown in Lemma 9.2). This proves that such an antichain does not exist.

Chapter 8

Minimal (n) and (n, h, k) Completely Separating Systems

The main aims of this chapter are to catalogue all non-isomorphic ways of achieving $R(n)$ for $n \leq 10$ and to determine $R(n, h, k)$ for $1 \leq h < k$ with $n \leq 10$ or with n large compared to k. A motivation for creating the catalogue is that it aids in increasing the known values of $R(n, k)$ and this is done in Chapter 9. The catalogue also makes it relatively easy to determine $R(n, h, k)$ for $n \leq 10$. Of course the catalogue is of interest in its own right within the context of combinatorial designs.

Note that in any minimal CSS no set will contain more than $n - 1$ elements for $n > 1$ and no set will be duplicated. Values of $R(n)$ for all n have been determined by Spencer [36] as stated in Lemma 4.1, using the duality of CSSs and antichains.

8.1 Basic Results on Minimal (n) and (n, h, k) CSSs.

Designing an (n)CSS which achieves $R(n)$ is not difficult using the duality of CSSs and antichains. The process is described in Section 7.3. Determining the number of non-equivalent (n)CSS which achieve $R(n)$ for general n is computationally difficult. Some basic results to aid in this determination are included in this section.

Recall the result of Cai (Lemma 4.2) that $R(n, 1, k) = \lceil \frac{2n}{k} \rceil$ for all $n > \frac{k^2}{2} \geq 2$.

The following Lemmas are easy to obtain. It is clear that

$$R(n, a, b) \geq R(n, h, k) \tag{8.1}$$

whenever $a \geq h$ and $b \leq k$. Note that $R(n) = R(n, 1, n)$ and that $R(n, k) = R(n, k, k)$ so

Lemma 8.1. *For all n, h and k, $R(n) \leq R(n, h, k) \leq R(n, k)$.*

Lemma 8.2. *For $n \geq 3$ there is a minimal $(n, 1, k)$CSS with at most two singleton sets contained in it.*

Proof. Assume C is a minimal $(n, 1, k)$CSS containing at least three singleton sets $\{a\}$, $\{b\}$ and $\{c\}$. These can be replaced by the three 2-sets $\{a, b\}$, $\{a, c\}$ and $\{b, c\}$ to maintain the CSS property and not increase the number of sets. □

Lemma 8.3. *For all n and k, $R(n + 1, 1, k) \geq R(n, 1, k)$.*

Proof. Given any CSS $\mathcal{C}$ which achieves $R(n + 1, 1, k)$, remove from $\mathcal{C}$ all occurrences of the element $n + 1$. The remaining n elements are then still completely separated in no more than $|\mathcal{C}|$ sets. □

Note that Lemma 8.3 highlights a difference in the behaviour of $R(n, 1, k)$ and $R(n, k)$ for fixed k. In Chapter 5 it was shown that Lemma 8.3 is not true if written in terms of $R(n, k)$.

Lemma 8.4. *Let $\mathcal{C}$ be a minimal CSS on $[n]$. For some $p < n$, assume $R(p) > |\mathcal{C}| - 1$. Then $|A| < p$ for each $A \in \mathcal{C}$.*

Proof. Assume $A \in \mathcal{C}, |A| \geq p$. Then, as $R(p) > |\mathcal{C}| - 1$, the p elements of A cannot be completely separated in the remaining sets of $\mathcal{C}$. □

Let $\mathcal{C}$ be a CSS $\{A_1, \dots, A_m\}$ with $|A_i| \geq |A_j|$ if $i < j$. Recall that the cardinality sequence of $\mathcal{C}$ is the non-increasing sequence $|A_1|, \dots, |A_m|$.

Lemma 8.5. *Assume $\mathcal{C}$ is a minimal $(n+1)$CSS and $R(n+1) = R(n)$. Then*
(i) $\mathcal{C}$ contains a minimal (n)CSS in the sense that whenever a fixed element of $[n+1]$ is removed from each set in $\mathcal{C}$ in which it occurs, the remaining collection is a minimal (n)CSS.
(ii) $\mathcal{C}$ contains no singleton set.

Proof. (i) The removal of an element from each set in which it occurs in a CSS on $[n+1]$ leaves a CSS on a set of n elements. As $R(n+1) = R(n)$ any collection of n elements of $[n+1]$ require $|\mathcal{C}|$ sets to completely separate them. Hence the result.
(ii) This follows immediately from (i). □

Theorem 8.1. *Assume $\mathcal{C}$ is a minimal (n)CSS, $n \geq 5$. Then $\mathcal{C}$ contains at most one singleton set.*

Proof. If $\mathcal{C}$ is a minimal (n)CSS which contains at most one set with cardinality greater than 1 then $|\mathcal{C}| \geq n$. This contradicts Spencer's value for $R(n)$ for $n \geq 5$. Assume $\mathcal{C}$ is a minimal (n)CSS containing two sets A and B with cardinality greater than 1. Assume $\mathcal{C}$ contains two singleton sets, say $\{n-1\}$ and $\{n\}$. Note that the 2-set $\{n-1, n\}$ cannot be in $\mathcal{C}$ as its removal would give a CSS with cardinality less than $|\mathcal{C}|$. Thus both A and B contain elements other than $n-1$ or n. Recalling that $\mathcal{C}$ is a minimal (n)CSS, the proof is completed by showing that a (n)CSS $\mathcal{A}$ with $|\mathcal{A}| \leq |\mathcal{C}| - 1$ can be formed. This is achieved as follows. Remove the singleton sets $\{n-1\}$ and $\{n\}$ from $\mathcal{C}$ and also remove all other occurrences of the elements $n-1$ and n from $\mathcal{C}$. Include $n-1$ in A and n in B. Append the 2-set $\{n-1, n\}$ to form $\mathcal{A}$. $\square$

Corollary 8.1. *Assume $\mathcal{C}$ is a minimal (n)CSS, $n \geq 5$. Then $V(\mathcal{C}) \geq 2n - 1$.*

Corollary 8.2. *Minimal (n)CSSs containing a singleton set exist exactly when $n \leq 4$ or when $n = \binom{m}{\lfloor \frac{m}{2} \rfloor} + 1$ with m a positive integer. Moreover, when $n \geq 5$ these minimal (n)CSSs contain exactly one singleton set.*

Proof. The cases when $n = 1, 2, 3$ or 4 follow immediately from the fact that $R(n) = n$, so that n singleton sets form a minimal (n)CSS in each of these cases. By Lemma 4.1, for $n \geq 5$, $R(n) > R(n-1)$ only when $n = \binom{m}{\lfloor \frac{m}{2} \rfloor} + 1$. For these values of n, the addition of the singleton set $\{n\}$ to a minimal $(n-1)$CSS on $[n-1]$ forms a minimal (n)CSS on $[n]$. Theorem 8.1 asserts that there cannot be more than one singleton set in these cases. For all other values of n, Lemma 8.5 can be applied to conclude that no minimal (n)CSS can contain a singleton set. $\square$

Lemma 8.6. *If $a_1, \ldots, a_R$ is a cardinality sequence of a CSS $\mathcal{C}$ on $[n]$ then:*
(i) the sequence $n - a_R, \ldots, n - a_1$ is also a cardinality sequence of a CSS on $[n]$;
(ii) $V(\mathcal{C}) \geq 2n$ if $a_R > 1$.

Proof. (i) The complementary CSS $\mathcal{C}'$ has cardinality sequence $n - a_R, \ldots, n - a_1$.
(ii) Assume $a \in [n]$ is a 1-element. Then to ensure all elements are completely separated, a must occur in a 1-set. If $a_R > 1$ then $\mathcal{C}$ cannot contain a 1-element. Therefore $V(\mathcal{C}) \geq 2n$. $\square$

Note 8.1. This Lemma implies that $V(\mathcal{C}) \geq 2n$ for any (n, h, k)CSS with $h \geq 2$.

Consideration of the bounds on the volume of a (n, h, k)CSS yields the following lemma.

Lemma 8.7. *Let C be a minimal $(n,h,k)CSS$ and let $R = R(n,h,k)$. Then $hR \leq V(C) \leq kR$.*

8.2 $R(n)$ for $n \leq 10$

For the rest of this chapter a CSS C is said to be in **standard form** if the set of largest cardinality in C is an m-set and the first set in C is the set $[m]$.

Theorem 8.2. *1. For each $n \leq 10$, $R(n)$ has the values as shown in the row labelled R in the table below.*

2. For each $n \leq 10$, the number of non-isomorphic designs which achieve $R(n)$ is shown in the row labelled d in the table below.

3. The minimum and maximum volumes of $(n)CSS$ which achieve $R(n)$ are shown in the rows labelled V_{min} and V_{max} in the table below.

n :	1	2	3	4	5	6	7	8	9	10
R :	1	2	3	4	4	4	5	5	5	5
d :	1	1	2	6	1	1	18	7	2	2
V_{min} :	1	2	3	4	10	12	13	16	18	20
V_{max} :	1	2	6	12	10	12	22	24	27	30

The proof of this theorem is given after the following notes and corollaries. Although the second corollary is simple, it is given because of its importance in Chapter 9.

Note 8.2. 1. The values for V_{min} in the table are monotone. The values for V_{max} are not monotone.

2. For the values of n considered in the proof below, all non-isomorphic (n)CSSs are included and this has been checked by exhaustion and careful comparison of the (n)CSSs constructed. This was made possible by the limited number of choices that can be made when designing the small (n)CSSs due to the underlying structures that they must possess. (See the proof for more details of these structures).

Parts of the proof consider representations of various cardinality sequences. In particular, it has been checked that all representations of the cardinality sequences considered in the proof are isomorphic to the representations actually included. An open question which is expected to have a negative answer is the following: Let S be the cardinality sequence of a minimum (n)CSS. Is there more than one non-isomorphic representation of S? If there is a value of n and a sequence S for which the answer to the question is yes, then $n > 10$ by the proof of Theorem 8.2.

Corollary 8.3. *Let n, R and d be as defined in Theorem 8.2. Then there are exactly d non-isomorphic antichains of size n on $[R]$.*

Proof. The Corollary is a simple consequence of the following facts: the dual of any (n)CSS in R sets is an antichain of size n on $[R]$; the dual of non-isomorphic collections are non-isomorphic; and there are d non-isomorphic (n)CSSs. □

The following corollary is an immediate consequence of the theorem.

Corollary 8.4. *1. If $n \leq 4$ then $R(n) = n$.*
2. If $n = 5$ or 6 then $R(n) = 4$ and there is a unique way of achieving $R(n)$ in each case.

Proof of Theorem 8.2. The values of $R(n)$ for $n \leq 10$ are given by Lemma 4.1. The minimum volume in each case is given in Lemma 7.3. The maximum volume is derived from the minimum volume using Lemma 7.1. Thus, for the theorem, it is only necessary to derive the values of d. In this process all non-isomorphic (n)CSSs for $n \leq 10$ are constructed. All CSSs are shown in standard form. Thus a CSS labelled C' may be the standard form of a CSS isomorphic to a CSS C rather than the true complementary CSS of C.

The derivation of the non-isomorphic CSSs has been achieved by using the underlying structures that a CSS must have, based upon the values of n and $R(n)$ and the possible sizes of the largest set in the CSS. The underlying structures for fixed n include the forms of the solutions for smaller n. They also include constraints imposed by the number of 2-elements and 3-elements that may occur in the CSS based upon the volume of the CSS. Some of these structures are considered in more detail in Chapter 9. The recognition of these underlying structures allows a computationally feasible exhaustive construction of all non-equivalent CSSs with the given parameters.

The cases when $n \leq 3$ are simple and can be checked by exhaustion. The possible designs are:

n=1: $V_{min} = V_{max} = 1, R = 1.$

$$1$$

n=2: $V_{min} = V_{max} = 2, R = 2.$

$$\begin{matrix} 1 \\ 2 \end{matrix}$$

n=3: $V_{min} = 3, V_{max} = 6, R = 3.$

$$\begin{matrix} 1 \\ 2 \\ 3 \end{matrix} , \begin{matrix} 1 & 2 \\ 1 & 3 \\ 2 & 3 \end{matrix}$$

Assume $n \geq 4$. In each of the following cases $\mathcal{C}$ denotes a minimal (n)CSS in standard form. Thus it is assumed that $[m]$ is the first and largest set in $\mathcal{C}$. Note that if $|\mathcal{C}| = R$ then $V(\mathcal{C}) \leq mR$.

n=4: $V_{min} = 4, V_{max} = 12, R = 4.$

For $n = 4$, there are 3 choices for m, each of which allow $R(4) = 4$ to be achieved. In each case it is easy to check that the CSS shown is unique.

1. Assume $m = 1$. There is one possible design:

$$A: \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \end{array}$$

2. Assume $m = 2$. There are two possible designs:

$$B: \begin{array}{ll} 1 & 2 \\ 1 & 3 \\ 2 & 3 \\ 4 & \end{array}, \quad C: \begin{array}{ll} 1 & 2 \\ 1 & 3 \\ 2 & 4 \\ 3 & 4 \end{array}$$

Here $C \equiv C'$.

3. Assume $m = 3$. It can be assumed that [3] is the first set in $\mathcal{C}$. There are two possible ways of completely separating the elements of M in three sets as shown in the case when $n = 3$. This gives the following three designs when the element 4 is included:

$$D: \begin{array}{lll} 1 & 2 & 3 \\ 1 & 4 & \\ 2 & 4 & \\ 3 & & \end{array}, \quad B': \begin{array}{lll} 1 & 2 & 3 \\ 1 & 4 & \\ 2 & 4 & \\ 3 & 4 & \end{array}, \quad A': \begin{array}{lll} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 1 & 3 & 4 \\ 2 & 3 & 4 \end{array}$$

Here $D \equiv D'$.

n=5: $V_{min} = V_{max} = 10, R = 4$.

1. $m > 2$ as $R = 4$ and $V_{min} = 10$.

2. Assume $m = 3$. It can be assumed that [3] is the first set in $\mathcal{C}$. As $V(\mathcal{C}) = 10$ there are two possible cardinality sequences to check. These are S_1 : 3322 and S_2 : 3331. The sequences can be checked to see if they have a CSS representation with the help of the minimal (3)CSSs to completely separate the elements 1,2 and 3 from one another. This shows that S_2 cannot be a cardinality sequence. The

one possible design which is a representation of S_1 is:

1 2 3
1 4 5
2 4
3 5

3. Four elements cannot be completely separated in three sets since $R(n) = 4$. Hence $m \neq 4$.

n=6: $V_{min} = V_{max} = 12, R = 4$.

Following similar reasoning to that used in the $n = 5$ case, it is easy to see that the only possible design is

1 2 3
1 4 5
2 4 6
3 5 6

n=7: $V_{min} = 13, V_{max} = 22, R = 5$.

1. $m > 2$ as $V_{min} = 13$ and $R = 5$.

2. Assume $m = 3$. It can be assumed that [3] is the first set in C so that C must contain the configuration

1 2 3
1
2
3

As $m = 3$ and $13 \leq V(C) \leq 15$ there are four possible cardinality sequences to check. These are S_1 : 33333, S_2 : 33332, S_3 : 33331 and S_4 = 33322. By Lemma 8.6(ii), S_4 cannot be a cardinality sequence. By applying the uniqueness of the $n = 6$ case it can be seen that A is the unique representation of S_3. With S_2 each element must be a 2-element. By the $n = 4, m = 2$ case the elements $4, 5, 6$ and 7 must occur as shown in B, so B is the unique representation of S_2. With S_1 there must be six 2-elements and one 3-element so it can be assumed that C contains the elements 1,2 and 3 as shown in C. The elements 4,5,6 and 7 can then occur as shown in C. All other arrangements of these elements give a CSS isomorphic to C or do not give a CSS.

The three possible designs are:

$$\begin{array}{cccc} & 1 & 2 & 3 \\ A: & 1 & 4 & 5 \\ & 2 & 4 & 6 \\ & 3 & 5 & 6 \\ & 7 & & \end{array}, \quad \begin{array}{cccc} & 1 & 2 & 3 \\ B: & 1 & 4 & 5 \\ & 2 & 5 & 6 \\ & 3 & 6 & 7 \\ & 4 & 7 & \end{array}, \quad \begin{array}{cccc} & 1 & 2 & 3 \\ C: & 1 & 4 & 5 \\ & 2 & 4 & 6 \\ & 3 & 5 & 7 \\ & 1 & 6 & 7 \end{array}$$

3. Assume $m = 4$. Note that minimal (7)CSSs derived in this case are shown at the end of the $m = 4$ case. As $\mathcal{C}$ contains the 4-set [4], $\mathcal{C}$ must include a minimal (4)CSS on [4] in its last four rows to completely separate the elements 1,2,3 and 4. Also note that $13 \leq V(\mathcal{C}) \leq 20$.

There are 22 possible cardinality sequences with five terms satisfying the condition that 4 is the largest element and with the volume of the corresponding CSS being between 13 and 20 inclusive. Eight of these sequences contain a 1. Corresponding to each of these sequences containing a 1, the element of the singleton set must be chosen from the set [4] as an element of [4] must occur in each set in $\mathcal{C}$ because $R(4) = 4$. Assume that this element is 4 in each case. By the $n = 6$ case the other six elements must be completely separated as shown in E (see below). Hence the only possible cardinality sequence containing a 1 is the sequence 4,3,3,3,1 with representation E.

The other possible cardinality sequences are now listed with the volume of their corresponding CSS representation shown in brackets.

$S_1 : 44444(20)$, $S_2 : 44443(19)$, $S_3 : 44442(18)$, $S_4 : 44433(18)$, $S_5 : 44432(17)$, $S_6 : 44422(16)$, $S_7 : 44333(17)$, $S_8 : 44332(16)$, $S_9 : 44322(15)$, $S_{10} : 44222(14)$, $S_{11} : 43333(16)$, $S_{12} : 43332(15)$, $S_{13} : 43322(14)$, $S_{14} : 43222(13)$.

Assuming that $\mathcal{C}$ contains no singleton set, each of the elements of [7] must occur at least twice in a representation of any of these sequences. S_{14} cannot be a cardinality sequence by Lemma 8.6(ii). By considering the volumes of the CSSs in the $n = 4$ case, it can be concluded that if $V(\mathcal{C}) \leq 16$ then $\mathcal{C}$ must contain A from the $n = 4$ case as a subarray in its last four rows. By considering the minimal (4)CSSs and the fact that the elements 5, 6 and 7 need to be completely separated from the elements 1,2,3 and 4 it can be concluded that $\mathcal{C}$ cannot contain the subarrays A' and B' of the $n = 4$ case in its last four rows. By attempting the constructions, it can be concluded that if $\mathcal{C}$ contains C or D of the $n = 4$ case, then the only possible CSS is isomorphic to C' or H' respectively. Hence the only CSSs left to consider are those which contain A or B of the $n = 4$ case as a subarray in their last four rows.

Assume $V(\mathcal{C}) = 14$. Then each element must be a 2-element. (The cardinality sequence 43331 has already been dealt with.) It is easy to check that S_{10} and S_{13} have unique representations D and F respectively.

Assume $V(\mathcal{C}) = 15$. Then there are six 2-elements and one 3-element. It is easy to check that S_9 does not have a representation and that S_{12} has unique representation G.

Assume $V(\mathcal{C}) = 16$. Then $\mathcal{C}$ cannot contain a 4-element else the remaining elements cannot be completely separated from it. Hence $\mathcal{C}$ contains two 3-elements and five 2-elements with A as a subarray in its last four rows (as noted above). This means that two of 5, 6 and 7 are the 3-elements in $\mathcal{C}$. Then it is easy to see that S_6 and S_8 cannot be cardinality sequences and that S_{11} has unique representation H.

Assume $V(\mathcal{C}) = 17$. Then $\mathcal{C}$ must contain three 3-elements and four 2-elements and so it must contain either A or B of the $n = 4$ case as a subarray in its last four rows. It is easy to check that S_5 is not a cardinality sequence and that S_7 has unique representation I.

Assume $V(\mathcal{C}) \geq 18$. By checking each case it can be seen that S_3 cannot be a cardinality sequence and that S_1, S_2 and S_4 have unique representations equivalent to C', H' and I' respectively.

The nine possible designs are:

$$D: \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 5 & 6 & 7 \\ 2 & & 5 & \\ 3 & & 6 & \\ 4 & & 7 & \end{array}, \quad E: \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & & 5 & 6 \\ 2 & & 5 & 7 \\ 3 & & 6 & 7 \\ 4 & & & \end{array}, \quad F: \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & & 5 & 6 \\ 2 & & 5 & 7 \\ 3 & & 6 & \\ 4 & & 7 & \end{array}$$

$$G: \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & & 5 & 6 \\ 2 & & 5 & 7 \\ 3 & & 6 & 7 \\ 4 & & 7 & \end{array}, \quad H: \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & & 5 & 6 \\ 2 & & 5 & 6 \\ 3 & & 5 & 7 \\ 4 & & 6 & 7 \end{array}, \quad I: \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 5 & 6 & 7 \\ 2 & 5 & 6 & \\ 3 & 5 & 7 & \\ 4 & 6 & 7 & \end{array}$$

$$I': \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 2 & 5 & 6 \\ 1 & 3 & 5 & 7 \\ 2 & 3 & 5 & \\ 4 & & 6 & 7 \end{array}, \quad H': \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 2 & 5 & 6 \\ 1 & 3 & 5 & 7 \\ 2 & 3 & 5 & 7 \\ 4 & & 6 & 7 \end{array}, \quad C': \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 2 & 5 & 6 \\ 1 & 3 & 5 & 7 \\ 2 & 3 & 6 & 7 \\ 4 & 5 & 6 & 7 \end{array}$$

4. Assume $m = 5$. It can be assumed that $\mathcal{C}$ contains $[5]$ as its first set. The $n = 5$ case provides the unique configuration in which the elements of this set must occur in the other sets in $\mathcal{C}$. By attempting to insert the elements 6 and 7 into the array it is clear that the last two rows of $\mathcal{C}$ must contain either 3 or 4 elements. That is, there is a set $A \in \mathcal{C}$ with $|A| < 5$. Thus the CSSs in this case must be the complements of CSSs already shown above.

The four possible designs are:

$$G': \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & & 6 \\ 1 & 4 & 5 & & 6 \\ 2 & 4 & & 6 & 7 \\ 3 & 5 & & & 7 \end{array}, \quad D': \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 6 & 7 \\ 1 & 4 & 5 & 6 & 7 \\ 2 & 4 & & 6 & \\ 3 & 5 & & 7 & \end{array}$$

$$B': \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & & 6 \\ 1 & 4 & 5 & & 7 \\ 2 & 4 & & 6 & 7 \\ 3 & 5 & & 6 & 7 \end{array}, \quad F': \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 6 & 7 \\ 1 & 4 & 5 & & 6 \\ 2 & 4 & & 6 & 7 \\ 3 & 5 & & & 7 \end{array}$$

5. Assume $m = 6$. It can be assumed that C contains [6] as its first set. By the $n = 4$ case there is a unique way of completely separating the elements of [6] in four other sets. By considering the possible placement of the element 7 and complements it is clear that the two possible designs are:

$$E': \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & & 7 & \\ 1 & 4 & 5 & & 7 & \\ 2 & 4 & 6 & & 7 & \\ 3 & 5 & 6 & & & \end{array} \quad , \quad A': \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & & 7 & \\ 1 & 4 & 5 & & 7 & \\ 2 & 4 & 6 & & 7 & \\ 3 & 5 & 6 & & 7 & \end{array}$$

The methods used for the remaining values of n are very similar to those used above. Hence they are explained in less detail.

n=8: $V_{min} = 16, V_{max} = 24, R = 5.$

1. $m > 3$ as $V_{min} = 16$ and $R = 5$.

2. Assume $m = 4$. The possible cardinality sequences can be checked one at a time to see if they have a representation. The various ways that four elements can be completely separated in four sets can be applied in this process. The four possible designs are:

$$A: \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 5 & 6 & 7 \\ 2 & 5 & 8 & \\ 3 & 6 & 8 & \\ 4 & 7 & & \end{array} \quad , \quad B: \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 5 & 6 & \\ 2 & 5 & 7 & \\ 3 & 6 & 8 & \\ 4 & 7 & 8 & \end{array} \quad , \quad C: \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 5 & 6 & 7 \\ 2 & 5 & 8 & \\ 3 & 6 & 8 & \\ 4 & 7 & 8 & \end{array} \quad , \quad D: \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 2 & 5 & 8 \\ 1 & 3 & 6 & 8 \\ 2 & 3 & 7 & 8 \\ 4 & 5 & 6 & 7 \end{array}$$

Note that $D \equiv D'$.

3. Assume $m = 5$. The two possible designs are:

$$C': \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & \\ 1 & 2 & 3 & & 6 & 7 \\ 1 & 4 & 5 & & 6 & 7 \\ 2 & 4 & & & 6 & 8 \\ 3 & 5 & & & 7 & 8 \end{array} \quad , \quad B': \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & \\ 1 & 2 & 3 & & 6 & 7 \\ 1 & 4 & 5 & & 6 & 8 \\ 2 & 4 & & 6 & 7 & 8 \\ 3 & 5 & & & 7 & 8 \end{array}$$

4. Assume $m = 6$. The only possible design is

$$A': \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & & 7 & 8 \\ 1 & 4 & 5 & & 7 & 8 \\ 2 & 4 & 6 & & 7 & \\ 3 & 5 & 6 & & 8 & \end{array}$$

5. $m \not> 6$ follows from an application of the case when $n = 7$, as a row of seven elements cannot be completely separated in the four other sets in $\mathcal{C}$.

n=9: $V_{min} = 18, V_{max} = 27, R = 5$.

1. $m > 3$ as $V_{min} = 18$ and $R = 5$.

2. Assume $m = 4$. The only possible design is

$$A: \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 5 & 6 & 7 \\ 2 & 5 & 8 & 9 \\ 3 & 6 & 8 & \\ 4 & 7 & 9 & \end{array}$$

3. $m \neq 5$ can be seen by assuming $[5] \in \mathcal{C}$, applying the unique construction for completely separating 5 elements in the $n = 5$ case and then trying to add four more elements whilst maintaining complete separation.

4. $m = 6$. The only possible design is

$$A': \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 7 & 8 & 9 \\ 1 & 4 & 5 & & 7 & 8 \\ 2 & 4 & 6 & & 7 & 9 \\ 3 & 5 & 6 & & 8 & 9 \end{array}$$

5. $m \not> 6$ by applying similar reasoning to that applied to the $n = 8$ and $m > 6$ case.

n=10: $V_{min} = 20, V_{max} = 30, R = 5$.

1. $m > 3$ as $V_{min} = 20$ and $R = 5$.

2. Assume $m = 4$. The only possible design is

$$A: \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 5 & 6 & 7 \\ 2 & 5 & 8 & 9 \\ 3 & 6 & 8 & 10 \\ 4 & 7 & 9 & 10 \end{array}$$

3. $m = 5$ is not possible by similar reasons to the case when $n = 9$ and $m = 5$.

4. Assume $m = 6$. The only possible design is

$$A': \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \\ 1 & 2 & 3 & & 7 & 8 & 9 \\ 1 & 4 & 5 & & 7 & 8 & 10 \\ 2 & 4 & 6 & & 7 & 9 & 10 \\ 3 & 5 & 6 & & 8 & 9 & 10 \end{array}$$

5. $m \not> 6$ as the m elements cannot be completely separated in the four rows of $\mathcal{C}$. □

This completes the proof of Theorem 8.2. Note that a catalogue of all non-isomorphic minimal (n)CSSs for $n \leq 10$ is contained within the proof of the theorem and all of the minimum volume minimal (n)CSSs shown are fair.

8.3 $R(n, h, k)$ for $n \leq 10$

Many of the values of $R(n, h, k)$ are easily determined for $n \leq 10$ from minimum values for other types of CSSs as follows. The cases for $R(n, h, k)$ with $h = k$ can be found in Chapter 5 as the value of $R(n, k)$. They are not repeated here. In some cases $R(n, h, k) = R(n)$. As $R(n) \leq R(n, h, k)$, these cases can be determined by considering the catalogue in the proof of Theorem 8.2 and applying Lemma 8.1. Lemma 4.5 can be used to immediately state the value of $R(n, n-k, n-h)$ once the value of $R(n, h, k)$ is determined.

Putting these ideas together for a given n gives the following interpretation of the problem to be addressed. For n fixed, let A be an array which lists the values of $R(n,h,k)$ with $A(i,j)$ representing the value of $R(n,i,j)$. Here $A(i,j)$ is not defined for $i > j$. The main diagonal values $A(k,k) = R(n,k)$ are determined in Chapter 5. The other diagonal is defined by the positions $A(i,j)$ with $i+j = n+1$. The values below this diagonal can be found from the values above this diagonal by applying Lemma 4.5. Thus, given existing results, it is only necessary to derive the values above the main diagonal and on or above the other diagonal to determine all values of $R(n,h,k)$ for a fixed n. For each value of n, the values of $A(h,k)$ and hence $R(n,h,k)$ are determined in this section. Theorem 8.3 gives the values in the first row of A for each $n \leq 10$.

Theorem 8.3. *For $n \leq 10$, the following values occur for $R(n,1,k)$.*

	k								
n	1	2	3	4	5	6	7	8	9
2	2								
3	3	3							
4	4	4	4						
5	5	5	4	4					
6	6	6	4	4	4				
7	7	7	5	5	5	5			
8	8	8	6	5	5	5	5		
9	9	9	6	5	5	5	5	5	
10	10	10	7	5	5	5	5	5	5

Proof. The values when $k = 1$ are clear. The cases when $n > \frac{k^2}{2}$ follow from Lemma 4.1. Each of the remaining cases follow from an examination of the catalogue in the proof of Theorem 8.2 where it can be seen that there is an example of a $(n,1,k)$CSS as a minimum (n)CSS. Noting Lemma 8.1 completes the proof. □

The value of $R(n,h,k)$ is now considered for $h > 1$. Note that as $h > 1$ every element must occur at least twice in any CSS. Recall that Theorem 5.5 states that $R(n,2,k) = \lceil \frac{2n}{k} \rceil$ for $n \geq \binom{k+1}{2}$.

Theorem 8.4. *The following values occur for* $R(n, h, k)$ *with* $2 \leq h < k < n \leq 7$.

(i) $R(4, 2, 3) = 4$.

(ii) $R(5, 2, k) = 4$ *for* $k \leq 4$.

(iii) $R(5, 3, 4) = 5$.

(iv) $R(6, h, k) = 4$ *for* $h \leq 3$.

(v) $R(6, 4, 5) = 6$.

(vi) $R(7, h, k) = 5$ *for* $2 \leq h \leq 4$.

(vii) $R(7, 5, 6) = 7$.

Proof. Each of these cases follow from the catalogue in the proof of Theorem 8.2 and by applying Lemma 4.5, Lemma 8.1 or Theorem 8.3. □

Theorem 8.5. *The following values occur for* $R(n, h, k)$ *with* $2 \leq h < k < n = 8$.

(i) $R(8, 2, 3) = 6$.

(ii) $R(8, 2, k) = 5$ *for* $k \geq 4$.

(iii) $R(8, 3, k) = 5$ *for all* k.

(iv) $R(8, 4, k) = 5$ *for all* k.

(v) $R(8, 5, k) = 6$ *for all* k.

(vi) $R(8, 6, 7) = 8$.

Proof. (i) follows from Theorem 5.5. $R(8, 5, 6) = 6$ by (i) and Lemma 4.5. The remaining cases follow from the catalogue in the proof of Theorem 8.2 and by applying Lemma 4.5, Lemma 8.1 or Theorem 8.3. □

Theorem 8.6. *The following values occur for* $R(n, h, k)$ *with* $2 \leq h < k < n = 9$.

(i) $R(9, 2, 3) = 6$.

(ii) $R(9, 2, k) = 5$ *for* $k \geq 4$.

(iii) $R(9, 3, k) = 5$ *for all* k.

(iv) $R(9, 4, 5) = 6$.

(v) $R(9, 4, k) = 5$ *for all* $k \geq 6$.

(vi) $R(9, 5, k) = 5$ *for all* k.

(vii) $R(9, 6, k) = 6$ *for all* k.

(viii) $R(9, 7, 8) = 9$.

Proof. (i) follows from Theorem 5.5. $R(9, 6, 7) = 6$ by (i) and Lemma 4.5. The remaining cases follow from the catalogue in the proof of Theorem 8.2 and by applying Lemma 4.5, Lemma 8.1 or Theorem 8.3. The only case that needs further argument is now considered.

It can be seen that $R(9, 4, 5) > 5$ as a $(9, 4, 5)$ in 5 sets does not appear in the catalogue in the proof of Theorem 8.2. To see that $R(9, 4, 5) = 6$ consider

1	2	3	4	5
1	2	3	6	7
1	4	5	6	8
2	4	7	9	
3	5	8	9	
6	7	8	9	

□

Theorem 8.7. *The following values occur for* $R(n, h, k)$ *with*
$2 \leq h < k < n = 10$.

(i) $R(10, 2, 3) = 7$.

(ii) $R(10, 2, k) = 5$ *for* $k \geq 4$.

(iii) $R(10, h, k) = 5$ *for* $3 \leq h \leq 6$.

(iv) $R(10, 7, 8) = 7$.

(v) $R(10, 7, 9) = 7$.

(vi) $R(10, 8, 9) = 10$.

Proof. (i) follows from Theorem 5.5. $R(10,7,8) = 7$ by (i) and Lemma 4.5. The remaining cases follow from the catalogue in the proof of Theorem 8.2 and by applying Lemma 4.5, Lemma 8.1 or Theorem 8.3. □

8.4 $R(n, h, k)$ for Large n

In this section the value of $R(n,h,k)$ is determined for all $n \geq \binom{k+1}{2} - 1$ and $h \geq 2$. The main result is the following theorem.

Theorem 8.8. *$R(n,h,k) = \lceil \frac{2n}{k} \rceil$ for $n \geq \binom{k+1}{2} - 1$ and $h \geq 2$.*

Before proving this theorem it should be noted that this contrasts with the value of $R(n,k)$ as expressed in Theorem 5.2 where it is shown that $R(n,k) = \lceil \frac{2n}{k} \rceil$ for $n \geq \binom{k+1}{2}$ and $k \geq 2$, but $R(n,k) = k+2 = \lceil \frac{2n}{k} \rceil + 1$ for $n = \binom{k+1}{2} - 1$ and $k \geq 3$.

Proof (of Theorem 8.8). By adapting the argument used in Lemma 4.6 it is easy to see that $R(n,h,k) \geq \lceil \frac{2n}{k} \rceil$ if $h > 1$. The fact that this bound is achieved for $n = \binom{k+1}{2}$ follows from Theorem 5.2 and the fact that $R(n,k) \geq R(n,h,k)$ for all $h < k$. It remains to show that $R(n,h,k) = \lceil \frac{2n}{k} \rceil$ for $n = \binom{k+1}{2} - 1$ and $h \geq 2$.

Note that $\lceil \frac{2n}{k} \rceil = k + 1$ when $n = \binom{k+1}{2} - 1$. Recall Construction M and the resultant array M as defined in Subsection 5.2.2. There are exactly two blank spaces left in the array M after each of the elements of $n = \binom{k+1}{2} - 1$ are included exactly twice in M using Construction M. Further these two blank spaces must occur in the last two positions in the last column of M. This means that each row in M contains either $k - 1$ or k elements. Recall that it was proved in Subsection 5.2.2 that if M contains the element $\binom{k+1}{2}$ in these two places then M is a CSS. Thus M is a CSS if these two places are left blank and in fact M is the array representation of a (n,h,k)CSS for each h, $2 \leq h \leq k-1$. This completes the proof. □

Note 8.3. The bound on n is tight in Theorem 8.8. To see this consider the following example. When $k = 4$ and $n = 8$, $\lceil \frac{2n}{k} \rceil = 4 < R(8, 2, 4) = 5$ by Theorem 8.5. When $k = 4$ and $n = 9$, $\lceil \frac{2n}{k} \rceil = 5 = R(9, 2, 4)$ by Theorem 8.6.

8.5 Comments

There remain many questions to be addressed concerning $R(n)$ and $R(n, h, k)$. These include:

1. For $n > 10$, in how many different ways can $R(n)$ be achieved with minimum volume?
2. When, and under what circumstances, does $R(n) = R(n, k)$?
3. Are there underlying patterns in the values of d in Theorem 8.2 when extended to larger n?
4. What is the minimum size of a (n)CSS $\mathcal{C}$, if for each pair of sets A, B in $\mathcal{C}$, $A \cap B = k$ for some integer k?
5. What is the smallest volume of minimal (n, h, k)CSSs for each n?
6. When do the minimum and maximum possible volumes of minimal (n, h, k)CSSs coincide?
7. For a given n, in how many different ways can $R(n, h, k)$ be achieved, or be achieved with minimum volume?
8. When, and under what circumstances, does $R(n, h, k) = R(n, k)$?
9. For each n, is there a fair minimal (n, h, k)CSS? For each n, is there a fair minimal (n, h, k)CSS of minimum volume?
10. For each k, what is the least value of n for which $R(n, h, k) = \lceil 2n/k \rceil$?

As a final comment on (n, h, k)CSSs note the following connection with antichains. The dual of a (n, h, k)CSS $\mathcal{C}$ is an antichain on $[|\mathcal{C}|]$ of size $[n]$ in which each element of $[|\mathcal{C}|]$ occurs between h and k times inclusively. Thus the results concerning the connection between minimal (n, k)CSSs and antichains at the end of Chapter 7 can be modified to be results about the connection between minimal (n, h, k)CSSs and antichains. This modification is left to the reader.

Chapter 9

(n, k)Completely Separating Systems, Part 3

This chapter has two main aims. The first aim is to derives some underlying structures that many minimal (n, k)CSSs must possess. The second aim is to apply these structures in the cases when $k \leq 10$ to allow the full determination of $R(n, k)$ for $k \leq 10$. The achievement of these aims rely upon results in Chapters 4, 5, 7 and 8.

As $R(n, k)$ is known for all n with $k \leq 6$, throughout this chapter k is assumed to be a positive integer with $k > 6$. Thus no singleton set can occur in any (n, k)CSS considered here. This means that every element of $[n]$ must occur in at least 2 sets in any minimal (n, k)CSS and hence $V(\mathcal{C}) \geq 2n$ for each (n, k)CSS $\mathcal{C}$ in this chapter.

Particular values to be considered in this chapter include the following: $k = 7$ for $n = 17, 18$; $k = 8$ for $n = 20, 25$; $k = 9$ for $n = 27, 28, 33$; $k = 10$ for $n = 24, 25, 30, 31, 35, 36$.

Note 9.1. 1. In each case above it turns out that $R(n, k) \leq k$.

2. All other values of $R(n,k)$ for $k = 7, \ldots, 10$ have been determined in Chapters 4, 5 and 7. Thus, the determination of $R(n,k)$ for those values listed above, together with the results in Chapters 4, 5 and 7 means that each value of $R(n,k)$ is known for all n when $k \leq 10$.

3. Generally, the determination of $R(n,k)$ should include the exhibition of a sample (n,k)CSS in an appropriate number of sets. The exhibition of some of these (n,k)CSSs is included in the appendix.

Convention. For convenience of notation, $R(n,k)$ is sometimes denoted by R. It is often convenient to represent the elements of a completely separating system on an n-set by elements of $[n]$ for the 2-elements and by alphabetic characters for the 3-elements. In this case lower and upper case letters may be used when the number of 3-elements exceed 26. If 4-elements are to be used, numeric elements of $[n]$ will be used to denote them.

Recall that if $\mathcal{C}$ is a minimal (n,k)CSS and $|\mathcal{C}| = R$ then the **excess** is $E = kR - 2n$.

Note 9.2. When $kR \leq 3n$, E is the maximum number of elements of an n-set S which can occur in more than two sets in $\mathcal{C}$.

A common approach to proving the results included here is to determine the excess E for a minimal $(n,k)CSS$ $\mathcal{C}$. This is used to determine the number of 2-elements and 3-elements that may occur in $\mathcal{C}$. This information is used to impose constraints on the structure for the completely separating system, which then either improves the lower bound on $R(n,k)$ or guides in the construction of a CSS of a given size.

Definition 9.1. For R a fixed positive integer, define
$t = t(n,k) = \max\{0, \lceil 2(3n - kR)/R \rceil\}$.

Note that $t = \lceil 2(n - E)/R \rceil$ when $t > 0$. For the cases considered below, t represents the minimum number of 2-elements that must occur in at least one set in $\mathcal{C}$. This is stated in Lemma 9.1.

9.1 Structures Within Minimal (n, k)CSSs

9.1.1 Some Basic Constraints

Lemma 9.1. *Let $\mathcal{C}$ be a minimal $(n,k)CSS$ with $R \leq k$. Then there is a set in $\mathcal{C}$ which contains at least t 2-elements.*

Proof. The lemma is obvious if $3n - kr \leq 0$. Assume that $3n - kr > 0$ so that $3n > kR$. Recall that no element of $[n]$ can be a 1-element . Then, by the pigeon-hole principle, there are at least $3n - kR$ distinct 2-elements in $\mathcal{C}$ which occupy $2(3n - kR)$ places in the R sets of $\mathcal{C}$. Thus some set contains at least $\lceil \frac{2(3n-kR)}{R} \rceil = t$ 2-elements. □

The following result is a corollary of Theorem 5.2.

Corollary 9.1. *If $\mathcal{C}$ is a minimal $(n,k)CSS$ with $R(n,k) \leq k$ then $R(n,k) - t(n,k) \geq 5$.*

Proof. Immediate from Theorem 5.2 and Lemma 9.1. □

Note 9.3. 1. Corollary 9.1 asserts that if $\mathcal{C}$ is minimal then there are at most $R - 5$ 2-elements in each set of $\mathcal{C}$.

2. Corollary 9.1 is often useful when trying to construct a minimal (n, k)CSS, or to show that a given lower bound on $R(n, k)$ cannot be the actual value of $R(n, k)$. These ideas are regularly applied hereafter.

3. In the particular cases considered in Section 9.2 it is convenient to calculate t in the form mentioned after Definition 9.1.

The following corollary provides a bound on R which can be used to achieve the same result as Corollary 9.1 in terms of showing that a given lower bound on $R(n,k)$ cannot be the actual value of $R(n,k)$.

Corollary 9.2. *If* $R(n,k) \leq k$ *then*

$$R(n,k) \geq \lceil \frac{5-2k+\sqrt{(2k-5)^2+24n}}{2} \rceil. \tag{9.1}$$

Proof. For $R \leq k$, Corollary 9.1 can be applied to obtain

$$R \geq 5 + 2\frac{3n-kR}{R}.$$

This easily leads to (9.1). □

Note that 9.1 is not necessarily true if $R(n,k) > k$. For example, it is not true if $n = 27$ and $k = 6$ as $R(27,6) = 9$ by Theorem 5.1.

The following example illustrates the use of Corollary 9.1. Corollary 9.2 could be used to obtain the same result. Note that the determination of $R(n,k)$ for given n and k often takes the form of increasing the lower bound on $R(n,k)$ until a (n,k)CSS which achieves the known lower bound is constructed. This approach is illustrated here.

Example 9.1. In Chapter 5 it was shown that $11 \leq R(57,12) \leq 13$.

If $R = 11$, then $R - t = 3$ contradicting Corollary 9.1. Thus $R(57,12) \geq 12$.

The following construction is included as an example of a hand construction of a minimal (n,k)CSS. This construction shows more underlying structure than an alternate computer generated construction by McKay [18]. It is useful to develop skills in these hand constructions due to the large search space that sometimes prevents a computer search finding an example minimal (n,k)CSS in reasonable time.

It is known that $12 \leq R(57,12) \leq 13$. (See Table 5.5). To see that $R(57,12) = 12$ consider

1	2	3	4	5		a	b	c	d	e	f	g					
1	6	7	8	9		h	i	j	k	l	m	n					
2	10	11	12	13		h	i	j		o	p	q	r				
3	14	15	16	17		h	k	l		o	p	s	t				
4	18	19	20	21		i	k	m		o	q	s	u				
5	22	23	24	25		j	l	m		p	q	t	u				
a	b	c	26	6			18	22			r			w	x	y	
a	d	e	27	7	10					s	t		v	w	z		A
b	f		27	8	11	14				u			x	y	B	C	D
c	g			9	12	15	19	23					v	x	z	B	C
d	f		26		13	16	20	24	n					z	A	B	D
e	g					17	21	25	n	r			w	y	A	C	D

9.1.2 Minimal (n,k)CSSs when $t = k - 5$ or $t = k - 6$

Define $d = n - E$. For an array X, define $|\boldsymbol{X}|$ to be the number of terms in the array.

Note 9.4. 1. d is the minimum number of 2-elements that must occur in a minimal (n,k)CSS with $kR \leq 3n$.
2. A CSS C which achieves $R(n,k)$ has at least $2d/R$ 2-elements in one of its sets. Note that this provides an alternate definition of t as $t = \lceil \frac{2d}{R} \rceil$.

Consider $(n,k)CSS$s for $2n < kR \leq 3n$, with $R \leq k$ and $t = R - 5$. This case is common enough to specify the following general structure.

Theorem 9.1. *Assume that C is a minimal $(n,k)CSS$, $2n < kR \leq 3n$, with $R = k$ and with $t = R - 5$. Then, without loss of generality, C has the array* *form M shown below.*

$$
\begin{array}{cccccccccc}
1 & 2 & \dots & \dots & R-5 & a & b & c & d & e \\
1 & & & & & & & & & \\
2 & & & & W & & & X & & \\
\vdots & & & & & & & & & \\
R-5 & & & & & & & & & \\
a & b & c & & & & & & & \\
a & d & e & & Y & & & Z & & \\
b & d & & & & & & & & \\
c & e & & & & & & & &
\end{array}
$$

Here, excluding the elements $1, \dots, R-5, a, \dots, e$ *placed as shown above,*
W is the subarray of 2-elements in rows 2 to $R-4$ of M;
X is the subarray of 3-elements in rows 2 to $R-4$ of M;
Y is the subarray of 2-elements in rows $R-3$ to R of M;
Z is the subarray of 3-elements in rows $R-3$ to R of M;
and no row of M contains more than $k-5$ 2-elements.

Proof. By Lemma 9.1, Theorem 5.2 and Corollary 9.1 it can be asserted that:
(i) t is the greatest number of 2-elements appearing in any set in $\mathcal{C}$;
(ii) $\mathcal{C}$ must contain a set A (the first row above) with exactly t 2-elements;
(iii) There are t other sets each containing one of the 2-elements of A. There are $k-t=5$ elements in A other than the 2-elements, say a, b, c, d, e, and these cannot appear with the 2-elements of A elsewhere in $\mathcal{C}$. As $R=k$ they must be completely separated in the remaining $R-t-1=4$ sets. By Corollary 8.4 the elements a, b, c, d, e can be uniquely separated in four sets as shown in M. Hence the structure of M is determined as shown. □

Note 9.5. 1. In Theorem 9.1 rows R and $R-1$ can be thought of as being equivalent in terms of the interchanging of the rows not changing the underlying structure of the array. The same can be said for rows $R-2$ and $R-3$.
2. In Theorem 9.1 each row of W contains at most $R-6$ 2-elements else there

are more than t 2-elements in a row of M. Similarly each row of Y contains at most $k-5$ 2-elements.

3. Whenever the subarray

$$\begin{array}{ccc} a & b & c \\ a & d & e \\ b & d & \\ c & e & \end{array}$$

occurs in a CSS then at most one 2-element may occur in both of the rows which contain the last two rows of this subarray. Furthermore, no 2-element may occur more than once in any pair of rows containing any other rows of the subarray.

4. Whenever the subarray

$$\begin{array}{ccc} a & b & c \\ a & d & e \\ b & d & f \\ c & e & f \end{array}$$

occurs in a CSS then no 2-element can occur twice in the rows of the CSS which contains this subarray.

Corollary 9.3. *Assume C is a minimal $(n,k)CSS$ with $R=k$ and $t=R-5$. Assume M, W, Y are as defined in Theorem 9.1. Then*

(i) at most one 2-element can occur twice in Y.

(ii) $|W| \leq (R-5)(R-6)$.

(iii) $|W| \geq d-t-1 = d-R+4 = 3n-R^2-R+4$.

(iv) If no 2-element occurs twice in Y then at least $d-5R+25$ 2-elements occur in W only.

(v) If a 2-element occurs twice in Y then at least $d-5R+24$ 2-elements occur in W only.

(vi) $R \geq \frac{5+\sqrt{6n-27}}{2}$.

Proof. (i) If a 2-element occurs twice in Y it must occur in rows $R-1$ and R of M to have it completely separated from each element of $\{a,b,c,d,e\}$.

Hence there is only one 2-element that can occur twice in Y.

(ii) W contains $R-5$ rows and cannot contain more than $R-6$ 2-elements in each row by Note 9.5.

(iii) By Note 9.4 there are at least d 2-elements in C. Thus there are at least $d-(R-5)$ 2-elements other than $1,\dots,R-5$. At most one 2-element occurs only in Y so $d-(R-5)-1$ occur at least once in W. Hence the result.

(iv) Y contains at most $4(R-5)$ 2-elements. Thus the 2-elements occurring twice in W consist of at least d 2-elements less the elements $1,...,R-5$ less at most $4(R-5)$ 2-elements in Y. Hence the result.

(v) If an element occurs twice in Y then there are at most $4(R-5)$ 2-elements occurring exactly once in Y. Thus Y contains at most $4(R-5)-1$ distinct 2-elements. This is one less than in case (iv). The same type of argument as in case (iv) can be applied to obtain the result.

(vi) This follows from (ii) and (iii). □

Example 9.2. In Chapter 5 it was shown that $R(25,8) \geq 8$. Suppose $R(25,8) = 8$. Then, by part (ii) of the corollary, $|W| \leq 6$, and by part (iii) $|W| \geq 7$. This contradiction implies $R(25,8) \neq 8$.

Theorem 9.2. *Assume C is a minimal $(n,k)CSS$ with $R=k-1$ and $t=R-5$. Then C has the form of the array M shown below.*

$$\begin{array}{cccccccccccc}
1 & 2 & \dots & \dots & R-5 & a & b & c & d & e & f \\
1 & & & & & & & & & & \\
2 & & & W & & & & X & & & \\
\vdots & & & & & & & & & & \\
R-5 & & & & & & & & & & \\
a & b & c & Y & & & & Z & & & \\
a & d & e & & & & & & & & \\
b & d & f & & & & & & & & \\
c & e & f & & & & & & & &
\end{array}$$

Here, excluding the element f, W, X, Y, Z have the same meaning as in Theorem 9.1.

Proof. Corollary 8.4 asserts that there is a unique way of completely separating 6 elements in 4 sets. This unique way is as shown for the elements a, b, c, d, e, f in the last four lines of M. The rest of the proof mimics the proof of Theorem 9.1. □

Note 9.6. 1. In Theorem 9.2 the last four rows can be thought of as being equivalent in terms of the interchanging of the rows not changing the underlying structure of the array.
2. In Theorem 9.2, each row of W contains at most $R-6$ 2-elements and each row of Y contains at most $R-5$ 2-elements.

Corollary 9.4. *Assume C is a minimal $(n,k)CSS$ with $R = k-1$ and $t = R-5$. Then*
(i) No 2-element occurs twice in Y.
(ii) $|W| \leq (R-5)(R-6) = (k-6)(k-7)$.
(iii) $|W| \geq d-t = d-R+5 = 3n-kR-R+5$.
(iv) W contains at least $d-5R+25$ distinct 2-elements which occur in W only.
(v) $R \geq \frac{9+\sqrt{24n-119}}{4}$.

Proof. The proof follows the same argument as in the proof of Corollary 9.3 but applied to Theorem 9.2. □

9.2 Values of $R(n,k)$ for $k = 7, 8, 9, 10$

In Chapters 5 and 7 many of the values of $R(n,k)$ were determined for $k = 7, 8, 9$ and 10. The remaining unknown values of $R(n,k)$ for these values of k are determined in this section. This means that $R(n,k)$ is fully determined for all $k \leq 10$. A table of the values of $R(n,k)$ for $k \leq 10$ and $n \leq 41$ is included at the end of this chapter. The values of $R(n,k)$ for $n > 41$ and $k \leq 10$ have already been determined and have been included in Chapter 5. For some of the cases below, sample minimal (n,k)CSSs are included in the appendix.

In each lemma in this section $\mathcal{C}$ stands for a minimal (n,k)CSS for appropriate values of n and k. The array representation of $\mathcal{C}$ is called M.

In Chapter 5 the values of $R(n,7)$ were found for $n \leq 13$ and $n \geq 19$. In Chapter 7 the values for $R(n,7)$ were found for $14 \leq n \leq 16$.

Lemma 9.2.

(i) $R(17,7) = 7$.

(ii) $R(18,7) = 8$.

Proof. For $n = 17, 18$ examples of $(n,7)$CSSs containing the appropriate numbers of sets have been constructed. It is necessary to show that smaller $(n,7)$CSSs do not exist. In Chapter 5 it was shown that $R(17,7) \geq 6$ and $R(18,7) \geq 7$.

(i) This follows from Corollary 9.2 which implies that $R(17,7) \geq 7$.

(ii) Assume that $n = 18$ and $k = 7$. It is shown by contradiction that $R(18,7) \neq 7$. Assume that $R(18,7) = 7$. Then $R = k, d = 5$, $t = R - 5 = 2$ and the structure of Theorem 9.1 must occur as shown below with $|W| = |Y| = 2$ and with the elements 3,4 and 5 placed as shown.

$$\begin{array}{ccccccc}
1 & 2 & a & b & c & d & e \\
1 & 3 & & & & & X \\
2 & 4 & & & & & \\
a & b & c & & & & \\
a & d & e & & & & Z \\
b & d & & 5 & & & \\
c & e & & 5 & & &
\end{array}$$

Note that there must be at least three 3-elements in row 7 in Z and the sets of 3-elements in rows 6 and 7 in Z must be disjoint because of the 2-element 5 occurring in those rows. There are two possibilities depending on the second position of the element 3.

Assume that 3 occurs in row 4 or 5, say row 4. Then the five 3-elements of row 2, say f, g, h, i and j must be completely separated in rows 3, 5, 6 and 7. By Corollary 8.4 there is a unique way of completely separating 5 elements in 4 sets and by Note 9.5 this allows only one 2-element to occur twice in the same rows. Thus, as 5 is a 2-element which occurs in rows 6 and 7, the element 4 must occur in row 4 and each of rows 3 and 5 must contain three elements of $\{f, g, h, i, j\}$. Then the two remaining 3-elements in row 3, say k and l, must be completely separated in rows 5, 6 and 7. To avoid dominating 5, both k and l must occur in row 5. This is not possible because there are already 6 3-elements other than k and l which occur in row 5.

Now assume that 3 occurs in row 6 or 7, say row 7. Then the three 3-elements of Z in row 7 must be completely separated in rows 3 to 5, with two of these in each of these rows. This leaves five 3-elements in row 2 which must be completely separated in rows 3 to 6. By Corollary 8.4 there is a unique way of doing this as shown in Note 9.5. By Note 9.5 the element 4 cannot be included a second time without it being dominated by one of these 3-elements. Thus $R(18,7) \neq 7$. □

In Chapter 5 the values of $R(n,8)$ were found for $n \leq 14$ and $n \geq 26$. In Chapter 7 the values for $R(n,8)$ were found for $n = 15, \ldots, 19, 21, \ldots, 24$.

Lemma 9.3. *(i)* $R(20,8) = 8$.
(ii) $R(25,8) = 9$.

Proof. For $n = 20, 25$ examples of $(n,8)$CSSs containing the appropriate numbers of sets have been constructed. It is necessary to show that smaller $(n,8)$CSSs do not exist. In Chapter 5 it was shown that $R(20,8) \geq 7$ and $R(25,8) \geq 8$.
(i) Assume C is a minimal $(20,8)$CSS. It is shown by contradiction that $R(20,8) \neq 7$.
Assume $R(20,8) = 7$. Then $d = 4$ and $t = 2$. By Note 9.3 any set in C contains at

most two 2-elements. Then the structure of Theorem 9.2 must occur as shown.

1	2	a	b	c	d	e	f
1				W		X	
2							
a	b	c					
a	d	e					
b	d	f		Y		Z	
c	e	f					

Assume that $n = 20$. By Note 9.5.2 and without loss of generality, W must contain the 2-elements 3 and 4, with 3 in its first row and 4 in its second row and thus 3 and 4 occur once each in Y. It may also be assumed that the first row of X consists of the 3-elements g, h, i, j, k and l. By Note 9.5.4, if 3 and 4 are not in the same row of Y then g, h, i, j, k and l cannot be completely separated without one of them dominating at least one of 3 or 4.

If 3 and 4 are in the same row of Y, say row 4, then by Corollary 8.4, g, h, i, j, k and l must be completely separated in rows 3, 5, 6 and 7 with the same configuration as for a, b, c, d, e, f in rows 4 to 7. The remaining three 3-elements of row 3 must be chosen from $\{m, n, o, p, q\}$ and must be completely separated in rows 5 to 7 of M. This means that there are insufficient elements left to fill row 4 without 4 being dominated by some 3-element.

(ii) In Example 9.2 it is shown that $R(25, 8) > 8$. This completes the proof. □

In Chapter 5 the values of $R(n, 9)$ were found for $n \leq 15$ and $n \geq 34$. In Chapter 7 the values of $R(n, 9)$ were found for all n with $16 \leq n \leq 33$ except for $n = 27, 28, 33$. Here the values of $R(n, 9)$ are determined for $n = 27, 28, 33$.

Lemma 9.4. *(i)* $R(27, 9) = 9$.
(ii) $R(28, 9) = 9$.
(iii) $R(33, 9) = 10$.

Proof. For $n = 27, 28, 33$ examples of $(n, 9)$CSSs containing the appropriate numbers of sets have been constructed. It is necessary to show that smaller $(n, 9)$CSSs do not exist. In Chapter 5 it was shown that $R(n,9) \geq 8$ for $n = 27$ or 28 and $R(33,9) \geq 9$.

(i) Assume C is a minimal $(27,9)$CSS. It is shown by contradiction that $R(27,9) \neq 8$.

Assume $R(27,9) = 8$. Then $d = 9$, $t = 3$ and C has the structure of Theorem 9.2 with some positions filled as shown.

$$\begin{array}{ccccccccc}
1 & 2 & 3 & a & b & c & d & e & f \\
1 & 4 & 5 & g & h & i & j & k & l \\
2 & 6 & 7 & & & & & & \\
3 & 8 & 9 & & & & & & \\
a & b & c & 4 & & & & & \\
a & d & e & 5 & & & & & \\
b & d & f & & & & Z & & \\
c & e & f & & & & & &
\end{array}$$

The 3-elements of row 2, namely g, h, i, j, k and l must be completely separated in rows 3,4,7 and 8 in the unique way shown in Note 9.5. Consequently none of the 2-elements 6,7,8,9 can occur in rows 7 or 8. Thus row 5 must contain the elements 6 and 8 and row 6 must contain the elements 7 and 9. Then the 3-elements of row 3 or row 4 must occur only in rows 3,4,7 and 8. It is not hard to check that there must be 4 distinct 3-elements m, n, o, p which fill the remaining places in the rows 3,4,7, and 8. This leaves the 3-elements q, r which cannot be completely separated in rows 5 and 6 only. Thus $R(27,9) > 8$.

(ii) $R(28,9) > 8$ by Corollary 9.4.

(iii) In Chapter 5 it was shown that $R(33,9) \geq 9$. $R(33,9) > 9$ by Corollary 9.3.

□

In Chapter 5 the values of $R(n,10)$ were found for $n \leq 16$ and $n \geq 42$. In Chapter 7 the values of $R(n,10)$ were found for
$n = 17, \ldots, 23, 26, \ldots, 29, 32, 33, 34, 37, \ldots, 41$. The values of $R(n,10)$ to be determined here are for $n = 24, 25, 30, 31, 35, 36$.

Lemma 9.5. *(i)* $R(n,10) = 8$ *for* $n = 24, 25$.

(ii) $R(n, 10) = 9$ *for* $n = 29, 30, 31$.

(iii) $R(n, 10) = 10$ *for* $n = 35, 36$.

Proof. For $n = 24, 25, 29, 30, 31, 35, 36$ examples of (n, k)CSSs containing the appropriate numbers of sets have been constructed. It is necessary to show that smaller (n, k)CSSs do not exist.

(i) Let $n = 24$ or 25. In Chapter 5 it was shown that $R(n, 10) \geq 7$. It is shown by contradiction that $R(n, 10) \neq 7$.

Assume $R(n, 10) = 7$. Assume $\mathcal{C}$ is a $(24, 10)$CSS in 7 sets. Then $d = 2$ and $t = 1$. If a set in $\mathcal{C}$, say A, contains more than one 2-element, then each of these 2-elements must occur in different sets of $\mathcal{C}$ other than A. Then, by Corollary 8.4 the remaining elements of A cannot be completely separated in the sets of $\mathcal{C}$ which do not contain those 2-elements.

Therefore, it can be assumed that $\mathcal{C}$ has at most one 2-element in each set and thus $\mathcal{C}$ has the following partial form.

$$\begin{array}{cccccccccc} 1 & a & b & c & d & e & f & g & h & i \\ 1 & j & k & l & m & n & o & p & q & r \\ 2 & . & . & . & & & & & & \\ 2 & . & . & . & & & & & & \\ . & . & . & & & & & & & \\ . & . & . & & & & & & & \\ . & . & . & & & & & & & \end{array}$$

Assume four or more elements of $\{a, \ldots, i\}$ or four or more elements of $\{j, \ldots, r\}$ occur in row 3 (or row 4). Then these 3-elements must be completely separated in the last 3 rows of $\mathcal{C}$. This is impossible by Corollary 8.4.

Therefore each of rows 3 and 4 contain at most 6 elements of $\{a, \ldots, r\}$. Therefore, at least three elements of $\{s, t, u, v\}$ occur in each of row 3 and 4 to fill those rows. Then at least one of these elements, say s, occurs in both rows 3 and 4. This means that the element 2 is not completely separated from s. This completes the proof for $n = 24$.

Assume $n = 25$. Assume $R(n, 10) = 7$. Assume $\mathcal{C}$ is a $(25, 10)$CSS in 7 sets. Then $d = 4$ and $t = 2$. Therefore a set in $\mathcal{C}$, say A, contains at least two 2-elements which occur again in distinct sets in $\mathcal{C}$. By Corollary 8.4 it is impossible to completely separate the remaining elements of A in the number of sets which do not contain 2-elements of A. This completes the proof of this part.

(ii) Let $n = 29$. In Chapter 7 it was shown that $R(29, 10) \geq 8$. It is shown by contradiction that $R(29, 10) \neq 8$. Assume $R(29, 10) = 8$. Assume $\mathcal{C}$ is a $(29, 10)$CSS in 8 sets. Then $d = 7$ and $t = 2$. Therefore a set in $\mathcal{C}$, say A, contains at least two 2-elements which occur again in distinct sets in $\mathcal{C}$. If A contains more than two 2-elements then, by Corollary 8.4, it is impossible to completely separate the remaining elements of A in the number of sets which do not contain 2-elements of A. Thus A must contain exactly two 2-elements and $\mathcal{C}$ must have the following partial form.

1	2	a	b	c	d	e	f	g	h
1		W		X					
2									
.									
.									
.		Y		Z					
.									
.									

Here each row of W can contain at most one 2-element so Y contains at least three 2-elements which occur in Y only. Each of these three 2-elements must occur in a different pair of rows in Y and thus they occupy at least three of the ten different pairs of rows in Y. Each of the elements $a, \ldots, h$ must occur at least twice in Z to completely separate them from the elements 1 and 2 and from each other. In fact, each of the eight elements $a, \ldots, h$ must occur in a different pair of rows in Z and thus they occupy at least eight of the ten different pairs of rows in Z. This means that a 2-element which occurs in Y only must be dominated by one of $a, \ldots, h$. This is the contradiction which completes this part of the proof.

Let $n = 30$ or 31. In Chapter 5 it was shown that $R(n, 10) \geq 8$. It is shown by contradiction that $R(n, 10) \neq 8$.

Assume $\mathcal{C}$ is a $(n, 10)$ CSS in 8 sets. Then $d \geq 10$ and $t = 3$. For each value of n, each set $A \in \mathcal{C}$ can contain at most two 2-elements by an application of Corollary 8.4 to the remaining elements of A. This contradicts $t = 3$ so $R(n, 10) > 8$.

(iii) In Chapter 5 it was shown that $R(35, 10) \geq 9$. It is shown by contradiction that $R(35, 10) \neq 9$.

Assume $\mathcal{C}$ is a $(35, 10)$CSS in 9 sets. Then $d = 15$ and $t = 4$. Thus $\mathcal{C}$ must have the form of Theorem 9.2 with the meaning of W as defined in that theorem. This is expressed in the array representation of $\mathcal{C}$ shown below in the form of row 1 and the reoccurrence of the elements of row 1 in other rows. Other constraints can be imposed on $\mathcal{C}$. These are shown in the same array and are explained below in order.

1	2	3	4	a	b	c	d	e	f			
1	5	6	7		g	h	i	j	k	l		
2	8	9	10		g	h	i		m	n	o	
3	11	12	13		g	j	k		m	n		p
4	14	15		h	j	l		m	o		p	q
a	b	c		5	8	11	14					
a	d	e		6	9	12	15					
b	d	f		7	10	13						q
c	e	f		i	k	l		n	o		p	q

1. $11 \leq |W| \leq 12$ by Theorem 9.2 and Note 9.5.

2. Hence the 3-elements $g, \ldots, l$ must occur as shown in row 2. The 2-elements 5,6,7 in row 2 must reoccur in 3 other rows of the array with at least two of these in rows 6-9. Each of the 2-elements $8, \ldots, 15$ must reoccur in rows 6-9. Then the elements $g, \ldots, l$ must be completely separated in the other four rows excluding row 1. By Corollary 8.4 and Note 9.5 this can be done in a unique way in four rows with 3 elements in each row. If a 2-element of row 2, say 5, reoccurs in row

5 then the elements $g \ldots, l$ cannot be completely separated without dominating one fo the elements $8, \ldots, 15$. Therefore the occurrences of the elements $5, \ldots, 15$ and $g, \ldots, l$ must be exactly as shown in the array.

3. Without loss of generality rows 3,4 and 5 must be filled as shown with the elements $m, \ldots, q$ and these elements must reoccur as shown.

4. Row 6 needs three more 3-elements r, s, t to fill it and at least three other rows are needed to completely separate these 3-elements. However, only rows 7 and 8 are not yet filled. Thus r, s, t cannot be completely separated. Therefore $R(35, 9) > 9$.

In Chapter 5 it was shown that $R(36, 10) \geq 9$. $R(36, 10) > 9$ by Corollary 9.4(v). □

Table 9.3 at the end of this chapter provides a complete set of values on $R(n, k)$ for $k \leq 10$ and $4 \leq n \leq 41$ using the results in this chapter together with the results in Chapters 5 and 7. Recall that the values of $R(n, k)$ for $k \leq 10$ and $n > 41$ have been completely determined in Chapter 5 and are shown in Table 5.5. Exact values or improved bounds are also shown in Table 9.3 for $11 \leq k \leq 14$ and $n \leq 41$. These values are based upon results in this thesis or by the use of a search program written by McKay [18]. In the case of the search program there are two possible outcomes. Either the lower bound is achieved, which gives an exact value for $R(n, k)$, or the upper bound on $R(n, k)$ is improved by finding a sample $(n, k)CSS$ in a fewer number of sets than the previously known upper bound.

The table only shows values of $R(n, k)$ for $k \leq n/2$. The missing values can be determined by applying Lemma 4.5.

9.3 Comments

It should be noted that some results in Chapters 4, 5 and 7 allow the application of values of $R(n,k)$ derived in this chapter to be used to find values of $R(n',k')$ for appropriate n' and k' for $k > 14$. This is a matter to be considered in the future.

There are still many open problems involving separating systems. Some of these have been stated earlier in this thesis. Some immediate problems to consider include: cataloguing all minimal (n)CSSs for $11 \leq n \leq 20$; determining $R(n,k)$ for $k = 11$ or 12; determining when $R(n,k) \leq k$; determining if fair minimal completely separating systems always exist.

A particular problem that the reader may like to consider is the following: For $n \geq 11$, is $R(n,k) = k$ for $n = \lceil \frac{(k-1)^2}{2} \rceil$? There is evidence that for each value of k, this value of n is close to a borderline at which $R(n,k) \geq k$ for each larger value of n and $R(n,k)$ is smaller than k for each smaller value of n. A case in point is when $n = 12$. The question in this case is: does $R(61,12) = 12$?

An alternative statement of this problem is stated next and is based upon the connections between minimal (n,k)CSSs and antichains as expressed at the end of Chapter 7: does there exist an antichain on [12] which contains 61 sets and in which each element of [12] occurs exactly twelve times? Certainly there are many different flat antichains of size 61 on [12] (with sets of size 2 and 3). It is not clear that there exists a flat antichain or any other antichain on [12] which gives a positive answer to the question above.

Another problem to consider is the following. Define a **covering separating system** $\mathcal{K}$ to be a separating system in which every element occurs at least once. Provided that no element occurs in every set in $\mathcal{K}$, this corresponds to having a subbase for a T_0 topology in which every element of the universal set occurs in the topology. Problems which could be considered include finding the size of minimal covering separating systems with and without restrictions on the size of the sets in the system.

This chapter and the main part of this thesis is finished with the following comment. The reader may wonder why the theory in this paper was developed directly for minimal CSSs rather than developing the corresponding theory using antichains and applying the duality of antichains and CSSs. In certain cases the antichain approach can be a preferred choice. For example the generation of example minimal (n, k)CSSs by McKay [18] was carried out using the antichain approach. In other cases the ease of derivations is similar with both approaches.

There are two main reasons for the direct derivation of most of the results on minimal CSSs. Firstly, the motivation for determining $R(n, k)$ came from the consideration of union-closed collections on $[n]$ in which all of the $(n-1)$-sets are included. This provides a natural lead into considering CSSs in their own right. Secondly, there is practical evidence that it is easier to consider minimal CSSs directly in a large number of cases. In practice it is much easier to work with as few sets as possible. The construction of a minimal $(32, 9)$CSS in 9 sets is a case in point. It was much easier to insert 32 different elements into 9 sets of size 9 than it was to try to find an antichain containing 32 sets on a 9-set in which each element occurs exactly 9 times.

n	k													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
4	4	4												
5	5	5												
6	6	6	4											
7	7	7	5											
8	8	8	6	5										
9	9	9	6	6										
10	10	10	7	5	6									
11	11	11	8	6	6									
12	12	12	8	6	6	6								
13	13	13	9	7	6	7								
14	14	14	10	7	7	7	6							
15	15	15	10	8	6	7	7							
16	16	16	11	8	7	7	7	6						
17	17	17	12	9	7	7	7	7						
18	18	18	12	9	8	7	8	7	6					
19	19	19	13	10	8	7	8	7	7					
20	20	20	14	10	8	8	8	8	7	6				
21	21	21	14	11	9	7	8	8	7	7				
22	22	22	15	11	9	8	8	8	8	7	7			
23	23	23	16	12	10	8	8	8	8	7	7			
24	24	24	16	12	10	8	8	8	8	8	7	7		
25	25	25	17	13	10	9	8	9	8	8	7	7		
26	26	26	18	13	11	9	8	9	8	8	7	7-8	7	
27	27	27	18	14	11	9	9	9	9	8	8	7-8	7-8	
28	28	28	19	14	12	10	8	9	9	8	8	7	8	7
29	29	29	20	15	12	10	9	9	9	9	8	8	7-8	8
30	30	30	20	15	12	10	9	9	9	9	8	8	8	8
31	31	31	21	16	13	11	9	9	9	9	8-9	8	8	8
32	32	32	22	16	13	11	10	9	9	9	8-9	8	8	8
33	33	33	22	17	14	11	10	9	10	9	9	8	8	8
34	34	34	23	17	14	12	10	9	10	10	9	8-9	8	8
35	35	35	24	18	14	12	10	10	10	10	9	8-9	8	8
36	36	36	24	18	15	12	11	9	10	10	9	9	8-9	8
37	37	37	25	19	15	13	11	10	10	10	9-10	9	8-9	8
38	38	38	26	19	16	13	11	10	10	10	9-10	9	9	9
39	39	39	26	20	16	13	12	10	10	10	9-10	9-10	9	9
40	40	40	27	20	16	14	12	10	10	10	9-10	9-10	9	9
41	41	41	28	21	17	14	12	11	10	10	10	9-10	9	9

Table 9.1: Values of $R(n,k)$ for $4 \leq n \leq 41$ and $k \leq 14$.

Chapter 10

Appendix - Sample Minimal (n, k)CSSs

Some sample minimal CSSs are shown here. They are split into two types. The CSSs in the first collection are shown with numeric elements only and were found by a search program using results in Chapters 4 and 5. The program was written by C. Ramsay and modified by P. Lieby.

The second collection contains CSSs with alphabetic representation of elements also used. These CSSs were derived from the results in Chapter 9 and their array representations (including spaces) are arranged to highlight the underlying structures detailed in Chapter 9.

Within each collection the sample CSSs are arranged in increasing order of n and then k.

$R(16,5) = 7$, $R(13,6) = 7$

1	2	3	4	5		1	2	3	4	5	8
1	3	6	11	15		1	2	9	11	12	13
1	6	7	8	9		1	3	4	6	12	13
2	7	10	11	12	,	2	3	5	9	10	13
3	8	12	13	14		3	5	6	7	10	11
4	9	13	15	16		4	5	7	8	9	12
5	6	10	14	16		6	7	8	9	10	11

$R(22,9) = 8$

1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	10	11	22
1	7	8	9	10	18	19	20	21
2	7	10	11	12	13	14	15	16
3	8	11	12	13	17	19	20	21
4	9	14	15	17	18	20	21	22
5	12	14	16	17	18	19	21	22
6	13	15	16	17	18	19	20	22

$R(23,9) = 8$

1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	10	11	23
1	7	8	9	10	11	12	21	22
2	7	10	13	14	15	16	17	18
3	8	12	13	14	15	19	20	22
4	9	13	16	17	19	21	22	23
5	11	14	16	18	20	21	22	23
6	12	15	17	18	19	20	21	23

$R(21,10) = 7$

1	2	3	4	5	6	7	8	9	21
1	2	3	4	5	16	17	18	19	20
1	6	7	8	10	11	12	13	14	20
2	6	9	10	11	12	15	17	18	19
3	7	10	13	14	15	16	18	19	21
4	8	11	13	15	16	17	19	20	21
5	9	12	14	15	16	17	18	20	21

$R(22,10) = 7$

1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	11	12	20	21	22
1	6	7	8	13	14	15	19	21	22
2	6	9	10	13	16	17	19	20	22
3	7	9	11	14	16	18	19	20	21
4	8	11	12	13	14	15	16	17	18
5	10	12	15	17	18	19	20	21	22

$R(23,10) = 7$

1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	11	12	13	14	23
1	6	7	8	9	15	16	17	18	23
2	6	10	11	12	15	16	19	20	21
3	7	10	11	13	17	19	20	22	23
4	8	12	14	15	18	19	21	22	23
5	9	13	14	16	17	18	20	21	22

$R(25,10) = 8$

1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	11	12	13	23
1	7	8	9	10	11	14	22	23	24
2	7	11	12	15	16	17	21	23	24
3	8	12	15	18	19	21	22	24	25
4	9	13	14	15	16	17	18	19	20
5	10	13	16	18	20	21	22	23	25
6	14	17	19	20	21	22	23	24	25

$R(26,10) = 8$

1	2	3	4	5	6	7	8	9	10
1	2	6	10	11	12	13	14	15	16
1	3	7	11	16	17	18	19	20	21
1	4	8	12	17	18	22	23	24	25
1	5	9	13	19	20	22	23	24	26
2	3	4	5	14	19	21	22	25	26
6	7	8	9	11	15	17	21	23	25
10	12	13	14	15	16	18	20	24	26

$R(23, 11) = 7$

1	2	3	4	5	6	7	8	9	22	23
1	2	3	4	5	10	17	18	19	20	21
1	6	7	8	10	11	12	13	14	15	21
2	6	9	11	12	13	16	18	19	20	23
3	7	9	11	14	16	17	19	20	21	22
4	8	12	15	16	17	18	20	21	22	23
5	10	13	14	15	16	17	18	19	22	23

$R(24, 11) = 7$

1	2	3	4	5	6	7	8	9	10	11
1	2	3	4	5	12	13	21	22	23	24
1	6	7	8	14	15	16	20	22	23	24
2	6	9	10	14	17	18	20	21	23	24
3	7	9	12	13	14	15	16	17	18	19
4	8	11	12	15	17	19	20	21	22	24
5	10	11	13	16	18	19	20	21	22	23

$R(25, 11) = 7$

1	2	3	4	5	6	7	8	9	10	11
1	2	3	4	5	12	13	14	15	24	25
1	6	7	8	9	16	17	18	19	24	25
2	6	10	11	12	16	20	21	22	24	25
3	7	10	13	14	17	18	20	21	23	25
4	8	11	13	15	17	19	20	22	23	24
5	9	12	14	15	16	18	19	21	22	23

$R(26, 11) = 7$

1	17	18	19	20	21	22	23	24	25	26
1	7	8	9	10	11	12	13	14	15	16
2	3	4	7	8	10	13	17	18	20	23
2	3	5	7	9	11	14	17	19	21	24
2	4	6	8	9	12	15	18	19	22	25
3	5	6	10	11	12	16	20	21	22	26
4	5	6	13	14	15	16	23	24	25	26

$R(25,12) = 7$

1	2	3	4	5	6	7	8	22	23	24	25
1	2	3	4	5	9	10	11	18	19	20	21
1	6	7	8	9	10	11	12	13	14	15	16
2	6	12	13	16	17	19	20	21	23	24	25
3	7	10	13	15	17	18	20	21	22	24	25
4	8	11	14	16	17	18	19	21	22	23	25
5	9	12	14	15	17	18	19	20	22	23	24

$R(14,7) = 6$, $R(15,7) = 7$

a	b	c	d	e	f	g			
a	b	c		h	i	j		m	
a	d	e		h	k	l		n	
b	e	f		i	k			m	n
c	f	g		j	l			m	n
d	g			h	i	j	k	l	

1	2	3		a	b	c	d			
1	2	4				e	f	g		j
1	3	4		a		e		h		k
2	3	4		b		f		i		k
1	2			c		h	i		j	k
3			d			e	f	g	h	i
4			a	b	c	d		g		j

$R(17,7) = 7$, $R(18,7) = 8$

1	2		a	b	c	d	e	
1		f	g	h	i	j	k	
2		f	g	h		l	m	n
a	b	c		f	i	j		o
a	d	e		g	i		l	m
b	d		h	k		l	n	o
c	e		j	k		m	n	o

1	a	b	c	d	e	f		
1	g	h	i	j	k	l		
1		g	h	i		m	n	o
1		j	k	l		m	n	p
2	a	b	c		g		m	p
2	a	d	e		h	j		n
2	b	d	f		i	k		o
2	c	e	f		l		o	p

$R(16,8) = 6$, $R(17,8) = 7$

a	b	c	d	e	f	g	h		
a	b	c	h		i	j	k	l	
a	d	e		i	j		m	n	o
b	e	f		i	k		m	n	p
c	f	g		j	l		m	o	p
d	g	h		k	l		n	o	p

1	2	3	4	a		i	j	k
1	2	3	5	b	e	f		i
1	2	4	5	c	e	g		j
1	3	4	5	d	f	h		l
2	3	4	5	g	h		k	l
a	b	c	d	e	f	g	h	
a	b	c	d		i	j	k	l

$R(18,8) = 7$, $R(19,8) = 7$

1	2	a	b	c	d	e	f			1	a	b	c	d	e	f	g		
1	a	d		g	h	i	j	k		1	h	i	j	k	l	m	n		
1	b	e		g	h	l	m	n		a	b	c		k	n		o	p	q
1	c	f		i	j	l	o	p	,	a	d	e		j	m	n		o	r
2	a	e		g	k	l	m	o		b	d	f		i	l	m		p	r
2	b	f		h	j	m	n	p		c	f	g		h	k	l		q	r
2	c	d		i	k	n	o	p		e	g		h	i	j		o	p	q

$R(20,8) = 8$, $R(21,8) = 8$

1	2	a	b	c	d	e	f				1	a	b	c	d	e	f	g		
1	2	g	h	i	j	k	l				1	a	b	c		l		o	p	q
1	3	a	b	c	d		m	n			1	d	e		k	n		o	r	s
1	3	g	h	i	j		o	p			1	f	g		j	m		p	r	t
2	4	a	e		g	k		m	p	,	a	e		i	m	n		q	s	t
2	4	b	f		h	l		n	o		b	f		h	k	l		o	r	s
3	4	c	e		i	k		m	o		c	g		h	i	j		p	q	t
3	4	d	f		j	l		n	p		d		h	i	j	k	l	m	n	

$R(22,8) = 8$

1	a	b	c	d	e	f	g		
1	h	i	j	k	l	m	n		
2	f		h	i	j		o	p	q
2	g		k	l	m		r	s	t
a	b	c		h	k		o	p	q
a	d	e		i	l		o	r	s
b	d	f		j	n		p	r	t
c	e	g		m	n		q	s	t

$R(23,8) = 8$

1	2	a	b	c	d	e	f			
1	3	g	h	i	j	k	l			
2	4	g	h	i		m	n	o		
3	5	a	b	c		p	q	r		
4	5	d	e	f	j	k	l			
a	d	g	j			m	n		p	q
b	e	h	k			m	o		p	r
c	f	i	l			n	o		q	r

$R(24,8) = 8$

1	2	3	a	b	c	d	e	
1	4		f	g	h	i	j	k
2	5	6	f	g	h		o	p
3	7	8	f	i	j		o	p
a	b	c	4	5	7		l	m
a	d	e	6	8		l	m	n
b	d	g	i	k		l	n	o
c	e	h	j	k		m	n	p

$R(25,8) = 9$

1		a	b	c	d	e	f	g			
1		h	i	j	k	l	m	n			
2	a	d		h		o	p	q	r		
2	b	e		k	l				s	t	u
3	c	f		i	m		o	p		s	
3		g		j	n		q	r		t	v
a	b	c		m	n		o			u	v
d	e		h	l			p	q		s	v
f	g		k	i	j		r			t	u

$R(18,9) = 6$

a	b	c	d	e	f	g	h	i		
a	b	c	d		j	k	l	m	n	
a	e	f	g		k	m		o	p	q
b	e	h	i		l	n		o	p	r
c	f	h		j	m	n		o	p	r
d	g	i		j	k	l		p	q	r

$R(19,9) = 7$

1	2	3	4		a	b	c	d	e		
1	2	3	4		f	g	h	i	j		
1	2	4	5		f	i		k	l	m	
1	3	5	6		b	d		g		k	
2	3	4	6		c	e		k	l	m	
5	6		a	b	c		f	g	h		l
5	6		a	d	e		h	i	j		m

$R(20,9) = 7$

1	2	a	b	c	d	e	f	g			
1	2	a	b		h	i	j	k	l		
1	2	c	d		m	n	o	p	q		
1	3	e	f		j	l		m	n	o	
2	3	e	g		k		i	l		n	p
3		a	c	f	g		h	k		o	q
3		b	d		h	i	j		m	p	q

$R(21,9) = 7$

a	b	c	d	e	f	g	h	i			
a	b	c		j	k	l	m	n	o		
a	d	e		j	k	l		p	q	r	
b	f	g		j	m	n		p		s	t
c	f	h		k	n		q		s	u	
d	g	i		l	o		p	r		t	u
e	h	i		m		q	r		s	t	u

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