On the solution of the Riccati differential equation arising from the LQ optimal control problem[*]

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Abstract

In this paper we consider the matrix Riccati differential equation (RDE) that arises from linear-quadratic (LQ) optimal control problems. In particular, we establish explicit closed formulae for the solution of the RDE with a terminal condition using particular solutions of the associated algebraic Riccati equation. We discuss how these formulae change as assumptions are progressively weakened. An application to LQ optimal control is briefly analysed.

Keywords: Riccati differential equation, LQ optimal control, Hamiltonian differential equation, extreme solutions of the algebraic Riccati equation, sign-controllability.

1 Introduction

The matrix differential Riccati equation is uniquitous in systems and optimal control/filtering theory and, for this reason, it has always received a great deal of attention, see e.g. [2] and the bibliography therein. For example, the solution of the classical finite-horizon linear-quadratic (LQ) optimal control problem is traditionally expressed in terms of the solution of a matrix Riccati differential equation with a terminal condition (see for example the classic textbooks [1, 23]).

From the Seventies, a significant stream of literature emerged regarding the problem of devising numerically reliable and efficient algorithms for the integration of the Riccati differential equation, see e.g. [10, 18]. Some of these methods, that make use of the so-called *Bernoulli substitution* technique, consist in exploiting the solution of a larger linear differential equation (the so-called *Hamiltonian differential equation*), whose size is twice the size of the Riccati equation to be solved. Another important class of algorithms is based on the so-called *Chandrasekhar decomposition* [21, 18], whereby the solution of the Riccati differential equation is obtained by solving a pair of coupled matrix linear differential equations. Other important approaches aimed at solving the so-called periodic differential Riccati equation include the multi-shot and iterative algorithms, see [31, 17, 30] and [11], respectively. See also [2].

Another area of intense research activity, which originated in the early Eighties and flourished in the Nineties, was centered on the attempt to characterise the solutions of the differential Riccati equation – particularly in the linear time-invariant (LTI) case – by using non-recursive formulae. The first and most general formula introduced in the literature exploits the solution of the Hamiltonian differential equation via a Bernoulli substitution as described above. However, this formula was soon found to be unsuitable for studying important properties of the solutions of the Riccati differential equation, such as finite escape times, convergence and mechanisms of attraction. Another drawback associated with Bernoulli substitutions is the difficulty that arises in determining an expression for the variations of the solution of the Riccati differential equation as an explicit function of variations of some parameters of the problem. Therefore, over the last twenty years, many researchers have focused their attention to finding alternative closed-form representations of the solution of the Riccati differential equation, [6, 5, 25]. Most of the representations proposed so far in the literature exploit the extreme (i.e., stabilising and anti-stabilising) solutions of the associated *algebraic* Riccati equation. These representations can be used to study the dependence of the solution of the Riccati differential equation on certain parameters of the problem in the cases when the functional dependence of the solution of the associated *algebraic* Riccati equation on such parameters is computable (as it happens, for example, in [7, 8]). The problem associated with these alternative representations is that the

extreme solutions of the algebraic Riccati equation both exist only if the underlying system (in the case of the Riccati equation arising from the finite-horizon LQ problem) is controllable.

The aim of this paper is to obtain more general representations for the solution of the Riccati differential equation, by progressively removing the assumptions on the problem data. We begin by considering a formula that is valid under the very mild assumption of sign-controllability, [29, 28]: this is the weakest form of controllability introduced so far in the literature, as it encompasses controllability, stabilisability and anti-stabilisability. Moreover, sign-controllability is also the weakest known assumption for which the associated algebraic Riccati equation is guaranteed to admit a symmetric solution. Under this assumption, a formula parameterising in finite terms all the trajectories originating from the Hamiltonian differential equation, introduced by the same authors in [14], is exploited to derive a non-recursive formula for the solution of the Riccati differential equation. This parameterisation of the trajectories of the Hamiltonian differential equation generalises those proposed in [15] and [27] for controllable and stabilisable systems, respectively; see also [13, 12] for the discrete-time counterpart. In the particular case of stabilisability, it is also shown that using the Kalman controllability decomposition it is possible to reduce the size of the algebraic Riccati equation to be solved. Furthermore, particular cases are also identified where a solution of the algebraic Riccati equation exists even when the sign-controllability assumption does not hold. This task is accomplished by introducing a first new sign-controllability form, that gives a sign-controllable part whose dimension is the largest that can be obtained. In the second part of the paper, the sign-controllability assumption is relaxed further, and a second sign-controllability form is exploited to the end of employing a solution of an algebraic Riccati equation to parameterise the trajectories of the sole sign-controllable part of the Hamiltonian differential equation, while the nonsign-controllable part is handled using the aforementioned Bernoulli substitution method. In this way, it is proven that if the nonsign-controllable part of the system has no common modes with the sign-controllable part of the Hamiltonian matrix, and such sign-controllable part has no modes on the imaginary axis, the solution of the Riccati differential equation can still be represented in a non-recursive fashion. In the last part of the paper, the applications of these results to optimal control problems are discussed. In particular, we show that using an approach that is similar in spirit to that proposed in [15, 24, 27, 14] very general linear-quadratic optimal control problems with affine constraints on the state vector at the end-points and with the most general type of quadratic performance index can be solved in closed form.

2 Problem formulation

In this paper we consider the problem of finding closed-form representations for the solutions $P(t)$, $t \in [0, T]$ of the matrix Riccati differential equation

$$\dot{P}(t) + A^\top P(t) + P(t) A - (P(t) B + S) R^{-1} (P(t) B + S)^\top + Q = 0 \qquad (1)$$

with terminal condition

$$P(T) = P_T, \qquad (2)$$

where $T > 0$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, with $m \leq n$; the matrices $Q \in \mathbb{R}^{n \times n}$, $S \in \mathbb{R}^{n \times m}$ and $R \in \mathbb{R}^{m \times m}$ are such that the so-called *Popov matrix*

$$\Pi := \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix}$$

is symmetric and positive semidefinite, and R is positive definite; finally, the terminal condition $P_T \in \mathbb{R}^{n \times n}$ is symmetric and positive semidefinite.

We recall that the *Popov triple* is defined as the triple $\Sigma := (A, B, \Pi)$, see e.g. [16].

The matrix Riccati differential equation (1) is concisely denoted by RDE(Σ), while the differential problem (1-2) is referred to as the Riccati differential problem and denoted by RDP(Σ, P_T). The solution of RDE(Σ) and RDP(Σ, P_T) are related to the solution $X, \Lambda : (-\infty, T] \longrightarrow \mathbb{R}^{n \times n}$ of the Hamiltonian differential equation HDE(Σ)

$$\begin{bmatrix} \dot{X}(t) \\ \dot{\Lambda}(t) \end{bmatrix} = H \begin{bmatrix} X(t) \\ \Lambda(t) \end{bmatrix}, \quad \text{where} \quad H = \begin{bmatrix} A - B R^{-1} S^\top & -B R^{-1} B^\top \\ -Q + S R^{-1} S^\top & -A^\top + S R^{-1} B^\top \end{bmatrix}, \quad (3)$$

and to the solution of the Hamiltonian differential problem HDP(Σ, P_T), i.e., a solution $X(t), \Lambda(t)$ of HDE(Σ) that also satisfies the boundary condition

$$\begin{bmatrix} X(T) \\ \Lambda(T) \end{bmatrix} = \begin{bmatrix} I_n \\ P_T \end{bmatrix}. \qquad (4)$$

The relation between the solutions of RDE(Σ) and RDP(Σ, P_T) with those of HDE(Σ) and HDP(Σ, P_T) is stated in precise terms as follows.

Theorem 1 ([9], pp. 274-275) *Let $X, \Lambda : (-\infty, T] \longrightarrow \mathbb{R}^{n \times n}$ be the solutions of the Hamiltonian differential problem HDP(Σ, P_T). Then*

1. *$X(t)$ is non-singular for all $t \in (-\infty, T]$;*

2. *the solution $P(t)$ of the RDP(Σ, P_T) is*

$$P(t) = \Lambda(t) X^{-1}(t), \qquad t \in (-\infty, T]. \qquad (5)$$

Moreover, if $P(t)$ is a solution of the RDE(Σ) on $(-\infty, T]$ and $X(t) : (-\infty, T] \longrightarrow \mathbb{R}^{n \times n}$ is a solution of the matrix differential equation

$$\dot{X}(t) = \left(A - B\,R^{-1}\,(B^\top\,P(t) + S^\top) \right) X(t),$$

then $X(t)$ and $\Lambda(t) = P(t)\,X(t)$ are a solution of HDE(Σ).

Theorem 1 provides a first general way to characterise the solution of both RDE(Σ) and RDP(Σ, P_T). Indeed, linearity of HDE(Σ) yields

$$\begin{bmatrix} X(t) \\ \Lambda(t) \end{bmatrix} = e^{H\,t}\,C, \tag{6}$$

where $C \in \mathbb{R}^{2n \times n}$ is an integration constant that can be found by imposing the boundary condition (4) on (6). As such, the solution of HDP(Σ, P_T) is given by

$$\begin{bmatrix} X(t) \\ \Lambda(t) \end{bmatrix} = e^{H\,(t-T)} \begin{bmatrix} I \\ P_T \end{bmatrix}.$$

If we partition the exponential matrix $e^{H\,(t-T)}$ as $\begin{bmatrix} E_1(t) & E_2(t) \\ E_3(t) & E_4(t) \end{bmatrix}$, where each $E_i(t)$ is an $n \times n$ time varying matrix, we can express $X(t)$ and $\Lambda(t)$ as $X(t) = E_1(t) + E_2(t)\,P_T$ and $\Lambda(t) = E_3(t) + E_4(t)\,P_T$, which enable the solution of RDP(Σ, P_T) to be written as

$$P(t) = \Lambda(t)\,X^{-1}(t) = (E_3(t) + E_4(t)\,P_T)\,(E_1(t) + E_1(t)\,P_T)^{-1}. \tag{7}$$

This approach for solving the Riccati differential equation is known in the literature as *Bernoulli substitution*, [18, 25], and constitutes the basis of the so-called *Davison-Maki numerical method*, [10]. Expression (7) has the advantage of being very general. On the other hand, since there is no explicit way to express the submatrices $E_i(t)$ as a function of the problem data Σ, this formula is not suitable for studying properties like convergence, limiting behaviour of the solution and mechanisms of attraction, [6, 5]. Moreover, (7) does not allow to analyse how the solution varies in terms of perturbation of Σ or P_T, which is a fundamental question in many practical and theoretical problems. For this reason, remarkable efforts have been devoted to finding more explicit representations for the solutions of the Riccati differential equation, see e.g. [5] and the references therein. In this paper, we present new ways to represent the solution of the Riccati differential problem under mild assumptions that are progressively weakened, and generalise the ones previously introduced in the literature in several directions.

3 Closed-form solutions under Sign-Controllability assumption

3.1 Background material

Consider the continuous-time algebraic Riccati equation CARE(Σ)

$$P A + A^\top P - (S + P B) R^{-1} (S^\top + B^\top P) + Q = 0. \tag{8}$$

To any solution $P = P^\top \in \mathbb{R}^{n \times n}$ of CARE(Σ) corresponds the closed-loop matrix

$$A_P := A - B R^{-1} (S^\top + B^\top P). \tag{9}$$

The set of eigenvalues $\sigma(A_P)$ of A_P is a subset of the spectrum of the Hamiltonian matrix H, [26, Theorem 6]. The following esistence properties of solutions to algebraic Riccati equations will be used in the sequel, [26, Lemma 7 and Theorem 6] and [32, Lemma 8 p. 629].

• If the pair (A, B) is stabilizable and H has no purely imaginary eigenvalues, CARE(Σ) has a unique solution $P^+ = (P^+)^\top \geq 0$ which is stabilising, i.e., all the eigenvalues of A_P have strictly negative real part, and P^+ is maximal among the solutions of CARE(Σ).[1]

• If (A, B) is anti-stabilizable and H has no eigenvalues on the imaginary axis, CARE(Σ) has a unique solution $P^- = (P^-)^\top \leq 0$ which is anti-stabilising, i.e., all the eigenvalues of A_P have strictly positive real part, and P^- is minimal among the solutions of the CARE(Σ).

• If (A, B) is controllable and H has no eigenvalues on the imaginary axis, the extreme solutions of CARE(Σ), P^+ and P^-, both exist. The *gap matrix* $\Delta := P^+ - P^- > 0$ satisfies the identity $A_{P^+} = -\Delta^{-1} A_{P^-}^\top \Delta$.

• If the pair (A, B) is sign-controllable, i.e., if the set of uncontrollable eigenvalues of A does not contain pairs of elements in the form $(\lambda, -\bar{\lambda})$, [29, 19], CARE($\Sigma$) admits an unmixed solution P, i.e., a solution such that if $\lambda, -\bar{\lambda} \in \sigma(A_P)$ implies $\mathfrak{Re}(\lambda) = 0$, [29, 28]. Moreover, given a solution $P = P^\top$ of CARE(Σ) and the corresponding closed-loop matrix A_P defined in (9), the Lyapunov equation

$$A_P Y + Y A_P^\top + B R^{-1} B^\top = 0 \tag{10}$$

has a unique solution $Y = Y^\top \in \mathbb{R}^{n \times n}$ if and only if P is strongly unmixed, which means that A_P does not contain mirrored pairs, i.e., $\lambda \in \sigma(A_P)$ implies $-\bar{\lambda} \notin \sigma(A_P)$, [14].[2]

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[1] Maximal and minimal are here referred to the standard ordering of symmetric matrices, i.e., $M_1 \geq M_2$ if and only if $M_1 - M_2 \geq 0$.

[2] Since A_P is real, this is equivalent to the fact that $\sigma(A_P)$ does not contain opposite pairs $(\lambda, -\lambda)$. Clearly, if P is unmixed and none of the eigenvalues of A_P lay on the imaginary axis, then P is also strongly unmixed. [22, Theorem 5.2.2].

Remark 1 Sign-controllability is the weakest form of controllability: It is weaker than the assumption of reachability, and even than that of stabilisability or anti-stabilisability of the pair (A, B). Indeed, it generically holds even in the extreme case of $B = 0$.

3.2 Solution of sign-controllable RDE

The following theorem establishes an explicit expression for the solution of the Riccati differential equation with terminal condition under sign-controllability assumption.

Theorem 2 *Assume that (A, B) is sign-controllable and H has no eigenvalues on the imaginary axis. Let P be a strongly unmixed solution of CARE(Σ), let A_P be given by (9) and Y be the corresponding solution of (10). The matrix $P(t) = \Lambda(t) X^{-1}(t)$, with*

$$X(t) = e^{-A_P(T-t)}\Big(I_n - Y(P - P_T)\Big) + Y e^{A_P^\top(T-t)}(P - P_T), \tag{11}$$

$$\Lambda(t) = P e^{-A_P(T-t)}\Big(I_n - Y(P - P_T)\Big) + (PY - I_n) e^{A_P^\top(T-t)}(P - P_T), \tag{12}$$

is the solution of RDP(Σ, P_T).

Proof: First, recall that sign-controllability of (A, B) guarantees that an unmixed solution P of CARE(Σ) exists. Moreover, in view of the absence of purely imaginary eigenvalues of H, P is also strongly unmixed, since $\sigma(A_P) \subset \sigma(H)$. Let P be a strongly unmixed solution of CARE(Σ), let A_P be the corresponding closed-loop system matrix and Y be the corresponding solution of (10). By adapting the proof of Theorem 1 in [14], it is easy to see that the set of trajectories of HDE(Σ) is parameterised in $\Phi, \Psi \in \mathbb{R}^{n \times n}$ as

$$\begin{bmatrix} X(t) \\ \Lambda(t) \end{bmatrix} = \begin{bmatrix} I_n \\ P \end{bmatrix} e^{A_P t}\, \Phi + \begin{bmatrix} Y \\ PY - I_n \end{bmatrix} e^{A_P^\top(T-t)}\, \Psi. \tag{13}$$

By imposing the boundary conditions $X(T) = I_n$ and $\Lambda(T) = P_T$ on (13), we find $\Phi = e^{-A_P T}(Y P_T - Y P + I_n)$ and $\Psi = P - P_T$. Thus, the corresponding $X(t)$ and $\Lambda(t)$ are given by (11) and (12). Hence, in view of Theorem 1, $X(t)$ is invertible for all $t \in (-\infty, T]$, and $P(t) = \Lambda(t) X^{-1}(t)$ is the solution of RDP(Σ, P_T). ∎

Remark 2 Using the technique described in the proof of Theorem 2, it is also possible to solve the Riccati differential equation with an initial condition

$$P(0) = P_0 = P_0^\top \geq 0 \tag{14}$$

instead of the terminal condition (2). In this case it suffices to impose the boundary conditions $X(0) = I_n$ and $\Lambda(0) = P_0$ on (13). This yields $\Phi = I_n - Y(P - P_0)$ and $\Psi = e^{-A_P^\top T}(P - P_0)$.

The matrix $P(t) = \Lambda(t)\,X^{-1}(t)$, with

$$
\begin{aligned}
X(t) &= e^{A_P t}\Big(I_n - Y\,(P - P_0)\Big) + Y e^{-A_P^\top t}(P - P_0), \\
\Lambda(t) &= P\,e^{A_P t}\Big(I_n - Y\,(P - P_0)\Big) + (P\,Y - I_n)e^{-A_P^\top t}(P - P_0)
\end{aligned}
$$

is the solution of the Riccati differential equation (1) with the initial condition (14). All the other results that are presented in this paper on the solution of the Riccati differential equation with terminal condition can be adapted in a similar way to the case of an initial condition.

3.3 A sign-controllability form

When (A, B) is not sign-controllable, two important sign-controllability forms can be defined. The first has the largest possible sign-controllable part. Let (A, B) be in the Kalman controllability form

$$
A = \begin{bmatrix} A_u & 0 \\ A_{uc} & A_c \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 0 \\ B_c \end{bmatrix}, \tag{15}
$$

where (A_c, B_c) is controllable. Consider a non-singular matrix T such that $T^{-1}A_u T$ is in Jordan form, partitioned as $T^{-1}A_u T = \mathrm{diag}(A'_u, A''_u)$. The Jordan blocks corresponding to the eigenvalues of A_u are divided between A'_u and A''_u according to the following rule:

 i) the Jordan blocks corresponding to eigenvalues on the imaginary axis are placed in A'_u;

 ii) every Jordan block B_λ corresponding to an eigenvalue λ of A_u such that $-\lambda$ is not an eigenvalue of A_u is included in A''_u;

 iii) for each pair of eigenvalues $(\lambda, -\lambda)$ of A_u, we denote the corresponding Jordan blocks by B_λ and $B_{-\lambda}$. If $\dim(B_\lambda) \leq \dim(B_{-\lambda})$, we include B_λ in A'_u and $B_{-\lambda}$ in A''_u; Otherwise, we include $B_{-\lambda}$ in A'_u and B_λ in A''_u.

In this way, A''_u has no pairs of eigenvalues in the form $(\lambda, -\lambda)$, nor eigenvalues with zero real part. Now, changing coordinates with $U := \mathrm{diag}(T, I)$ gives

$$
U^{-1}AU = \begin{bmatrix} T^{-1}A_u T & 0 \\ A_{uc}T + A_c & A_c \end{bmatrix} = \left[\begin{array}{cc|c} A'_u & 0 & 0 \\ 0 & A''_u & 0 \\ \hline A'_{uc} & A''_{uc} & A_c \end{array}\right], \qquad U^{-1}B = \left[\begin{array}{c} 0 \\ 0 \\ \hline B_c \end{array}\right], \tag{16}
$$

where $\begin{bmatrix} A'_{uc} & A''_{uc} \end{bmatrix} := A_{uc}T + A_c$. The pair $\left(\begin{bmatrix} A''_u & 0 \\ A''_{uc} & A_c \end{bmatrix}, \begin{bmatrix} 0 \\ B_c \end{bmatrix} \right)$ is sign-controllable, since (A_c, B_c) is controllable and A''_u has no mirrored pairs of eigenvalues with respect to the imaginary axis, nor eigenvalues on the imaginary axis. Notice that this definition of sign-controllability form provides a sign-controllable part with the largest possible size.

Remark 3 Consider the matrix differential Riccati equation (1). Consider the change of coordinate matrix T_s that transforms the pair (A, B) into the sign-controllability form (or in the controllability form). Define the new Popov triple Σ_s characterised by the matrices $T_s^{-1} A T_s$, $T_s^{-1} B$, $T_s^\top Q T$, $T^\top S$ and R. It is easy to establish that, given a solution $P_s(t)$ of RDE(Σ_s), the matrix $T_s^{-\top} P_s(t) T_s^{-1}$ is a solution of RDE(Σ), and *vice-versa*. Moreover, in order to obtain the solution $P(t)$ of RDP(Σ) with boundary condition $P(T) = P_T$, one can solve RDP(Σ_s) in $P_s(t)$, with boundary condition $P_s(T) = T_s^\top P_T T_s$, and then set $P(t) = T_s^{-\top} P_s(t) T_s^{-1}$.

When (A, B) is not sign-controllable, but the nonsign-controllable part is unobservable from the Popov matrix, a parameterisation of the solutions of HDE(Σ) based on the solutions of standard algebraic equations can still be established. Let (A, B) be in the sign-controllability form as constructed above, i.e.,

$$A = \begin{bmatrix} A_r & 0 \\ A_{rs} & A_s \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 0 \\ B_s \end{bmatrix}, \tag{17}$$

where (A_s, B_s) is sign-controllable; let n_s be the order of A_s, and let $n_r := n - n_s$ be the dimension of the residual (i.e., nonsign-controllable) part. Let the nonsign-controllable subsystem be unobservable from the Popov matrix, i.e., Q and S are partitioned accordingly to this basis as $Q = \text{diag}(0_{n_r \times n_r}, Q_s)$ and $S = [\, 0_{n_r \times m} \quad S_s^\top \,]^\top$ and $A_{rs} = 0$. If the Hamiltonian matrix referred to the sole sign-controllable part

$$H_s := \begin{bmatrix} A_s - B_s R^{-1} S_s^\top & -B_s R^{-1} B_s^\top \\ -Q_s + S_s R^{-1} S_s^\top & -A_s^\top + S_s R^{-1} B_s^\top \end{bmatrix}$$

has no eigenvalues on the imaginary axis, a matrix P_s and a matrix Y_s exist satisfying the algebraic Riccati equation referred to the sole sign-controllable part

$$P_s A_s + A_s^\top P_s - (S_s + P_s B_s) R^{-1} (S_s^\top + B_s^\top P_s) + Q_s = 0$$

and the corresponding closed-loop Lyapunov equation

$$A_{P_s} Y_s + Y_s A_{P_s}^\top + B_s R^{-1} B_s^\top = 0, \tag{18}$$

respectively, where $A_{P_s} := A_s - B_s R^{-1} (S_s^\top + B_s^\top P_s)$. As such, $P = \text{diag}(0_{n_r \times n_r}, P_s)$ and $Y = \text{diag}(0_{n_r \times n_r}, Y_s)$ are the solutions of CARE(Σ) and of the corresponding closed-loop Lyapunov equation referred to the complete system, respectively, as can be proved by means of a simple substitution. By using these matrices in (13), we obtain a parameterisation of the set of trajectories of HDE(Σ), and therefore the result established in Theorem 2 can still be applied in this case using P and Y obtained above. The fact that H_s has no eigenvalues on the imaginary axis is necessary to ensure that a solution Y_s of (18) exists. Indeed, suppose by contradiction

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that H_s has an eigenvalue at $i\,\omega$, with $\omega \in \mathbb{R}$. It follows that $i\,\omega$ is an eigenvalue of A_{P_s}, and, as a consequence, it is an eigenvalue of $A_{P_s}^\top$, as well. Let $v \in \mathbb{C}^n$ denote the eigenvector of $A_{P_s}^\top$ corresponding to such eigenvalue, i.e., $A_{P_s}\,v = i\,\omega\,v$. Let Y_s be a solution of (18). By premultiplying and postmultiplying (18) by v^* and v, respectively, we get

$$-i\,\omega\,v^*\,Y_s\,v + v^*\,Y_s\,i\,\omega\,v + v^*\,B_s\,R^{-1}\,B_s^\top\,v = v^*\,B_s\,R^{-1}\,B_s^\top\,v = 0,$$

which implies that v lies in the null-space of B_s. Therefore, v lies in the null-space of $A_s^\top$, as well, and this means that $i\,\omega$ is an uncontrollable eigenvalue of the pair (A_s, B_s). This is a contradiction, since the pair (A_s, B_s) is sign-controllable.

3.4 Stabilisability

The case where the pair (A, B) is stabilisable is clearly included, as a particular case, under the sign-controllability assumption. In this case, one can choose as strongly unmixed solution of CARE(Σ) the stabilising solution P^+, and the result in Theorem 2 still applies with $P = P^+$ and $A_P = A_{P^+}$. In the case where (A, B) is stabilisable, an alternative expression for the solution of the RDE(Σ) and RDP(Σ, P_T) can be found by resorting to two elements of the lattice of all symmetric solutions of the algebraic Riccati equation restricted to the sole controllable part. This method has the remarkable computational advantage of reducing the order of the involved algebraic Riccati equation, which, being the sole nonlinear matrix equation to be solved in this procedure, is the sole critical element in terms of numerical robustness, see e.g. [3].

Let $\mathcal{R}$ be the reachable subspace from the origin, whose dimension is here denoted by n_c; let $n_u := n - n_c$. Consider a change of coordinate basis matrix T_s such that its first n_c columns span $\mathcal{R}$. In the new basis, matrices A, B are partitioned as in (15), while Q and S are partitioned as

$$Q = \begin{bmatrix} Q_u & Q_{uc} \\ Q_{uc}^\top & Q_c \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} S_u \\ S_c \end{bmatrix}. \tag{19}$$

The strongly unmixed solution P^+ of CARE(Σ) can be partitioned correspondingly as

$$P^+ = \begin{bmatrix} P_u^+ & P_{uc}^+ \\ (P_{uc}^+)^\top & P_c^+ \end{bmatrix},$$

and it is easily seen that P_c^+ is a solution of the algebraic Riccati equation of the sole controllable part

$$P_c^+\,A_c + A_c^\top\,P_c^+ - (S_c + P_c^+\,B_c)\,R^{-1}\,(S_c + P_c^+\,B_c)^\top + Q_c = 0. \tag{20}$$

On the other hand, since the pair (A_c, B_c) appearing in (20) is controllable, the ARE (20) admits an anti-stabilising solution P_c^-, such that $\sigma(A_{P_c^+}) = -\sigma(A_{P_c^-})$. Moreover, the gap matrix $\Delta_c := P_c^+ - P_c^-$ is positive definite, and therefore invertible.

Define also the two $n \times n$ matrices $A^\star$ and $P^\star$ as

$$A^\star := \begin{bmatrix} -A_u^\top & -Q_{uc}^\top + S_u R^{-1} S_c^\top - A_{uc}^\top P_c^- + S_u R^{-1} B_c^\top P_c^- \\ 0 & A_c - B_c R^{-1} (B_c^\top P_c^- + S_c^\top) \end{bmatrix} \tag{21}$$

and $P^\star := \mathrm{diag}(I_{n_u}, P_c^-)$. Finally, let $J := \mathrm{diag}(0_{n_u \times n_u}, I_{n_c})$.

Theorem 3 *Assume that (A, B) is stabilisable and H has no eigenvalues on the imaginary axis. The matrix $P(t) = T_s^{-\top} \Lambda(t) X^{-1}(t) T_s^{-1}$, with*

$$X(t) = e^{A_{P+} t} \hat{\Phi} + J e^{A^\star (t-T)} \hat{\Psi} \tag{22}$$

$$\Lambda(t) = P^+ e^{A_{P+} t} \hat{\Phi} + P^\star e^{A^\star (t-T)} \hat{\Psi} \tag{23}$$

where

$$\hat{\Phi} := e^{-A_{P+} T} \left(I_n + \begin{bmatrix} 0 & 0 \\ 0 & \Delta_c^{-1} \end{bmatrix} (T_s^\top P_T T_s - P^+) \right) \tag{24}$$

and

$$\hat{\Psi} := \begin{bmatrix} I_{n_u} & -P_{uc}^+ \Delta_c^{-1} \\ 0 & -\Delta_c^{-1} \end{bmatrix} (T_s^\top P_T T_s - P^+) \tag{25}$$

is the solution of RDP(Σ, P_T).

Proof: The functions $X, \Lambda : (-\infty, T] \longrightarrow \mathbb{R}^{n \times n}$ defined by

$$\begin{bmatrix} X(t) \\ \Lambda(t) \end{bmatrix} = \begin{bmatrix} I_n \\ P^+ \end{bmatrix} e^{A_{P+} t} \Phi + \begin{bmatrix} J \\ P^\star \end{bmatrix} e^{A^\star (t-T)} \Psi \tag{26}$$

represent a parameterisation of all the solutions of the Hamiltonian differential equation (3) in Φ and Ψ. This fact can be proved by direct substitution of (26) into HDE(Σ) written with all submatrices of H partitioned as in (15) and (19). Moreover, all the solutions of the Hamiltonian differential equation (3) can be expressed through (26) for a suitable choice of Φ and Ψ. In fact, the images of $S_1 := [\, I_n \quad P^+ \,]^\top$ and $S_2 := [\, J \quad P^\star \,]^\top$ are H-invariant subspaces, since it can be proved by direct substitution that $H S_1 = S_1 A_{P+}$ and $H S_2 = S_2 A^\star$. The eigenvalues of H restricted to $im\, S_1$ coincide with the eigenvalues of A_{P+}, while the eigenvalues of H restricted to $im\, S_2$ are the eigenvalues of $A^\star$. However, from the discussion above it follows that the eigenvalues of A_{P+} and those of $A^\star$ are opposite, i.e., $\sigma(A_{P+}) = -\sigma(A^\star)$. As a result of this, $im\, S_1 \cap im\, S_2 = \{0\}$, which implies that (26) represents a set of $2 n^2$ linearly independent

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trajectories of HDE(Σ). To find the solution of RDP(Σ, P_T), we need to find the matrices Φ and Ψ to be used in (26) so that (26) represents the solution of HDP(Σ, P_T). By imposing the boundary conditions in (26) as indicated in Remark 3, we get $e^{A_{P^+}\,T}\,\Phi + J\,\Psi = I_n$ and $P^+ e^{A_{P^+}\,T}\,\Phi + P^\star\Psi = T_s^\top\,P_T\,T_s$. Matrix Φ can be computed from the first and substituted into the second, to get

$$(P^\star - P\,J)\Psi = T_s^\top\,P_T\,T_s - P^+. \tag{27}$$

Matrix $P^\star - P^+\,J$ can be written as

$$P^\star - P^+\,J = \begin{bmatrix} I_{n_u} & -P_{uc}^+ \\ 0 & P_c^- - P_c^+ \end{bmatrix},$$

and is easily seen to be invertible, and yields

$$\Psi = \begin{bmatrix} I_{n_u} & -P_{uc}^+\Delta_c^{-1} \\ 0 & -\Delta_c^{-1} \end{bmatrix} (T_s^\top\,P_T\,T_s - P^+).$$

By substition of this value of Ψ into $\Phi = e^{-A_{P^+}\,T}(I_n - J\,\Psi)$, we get (24). ∎

4   Dropping the sign-controllablity assumption

As already observed, sign-controllability is a very weak and general system-theoretic assumption. Nevertheless, this assumption could still result to be restrictive in some situations. Therefore, in this section we show that some results can be extended even when the sign-controllability assumption does not hold. In other terms, we investigate a further generalisation of the results obtained for sign-controllable pairs can be achieved under milder assumptions, that do not guarantee the existence of a solution of CARE(Σ) and of the corresponding Lyapunov equation. To the best of the authors' knowledge, sign-controllability is the weakest assumption on the pair (A, B) for which CARE(Σ) is guaranteed to admit a symmetric solution. Therefore, without the sign-controllability assumptions, CARE(Σ) might not admit a symmetric solution. The parameterisation of the trajectories of HDE(Σ) involves the (strongly unmixed) solution of the algebraic Riccati equation restricted to the sole sign-controllable part of the system. However, the sign-controllability form needed here must be constructed in a different way than that presented in Section 3.3, as will become clear in Theorem 4.

Let (A, B) be in the Kalman controllability canonical form (15). Define $\Lambda := \sigma(A_u) \cap \sigma(-A_u)$. There exists $T \in \mathbb{R}^{n \times n}$ such that $T^{-1}A_u T = \mathrm{diag}(A_u', A_u'')$ with $\sigma(A_u') = \Lambda$. Changing coordinates with $U := \mathrm{diag}(T, I)$ gives (16) where, again, the pair $\left(\begin{bmatrix} A_u'' & 0 \\ A_{uc}'' & A_c \end{bmatrix}, \begin{bmatrix} 0 \\ B_c \end{bmatrix} \right)$ is sign-controllable. Notice that, differently from the sign-controllability form introduced in Section

3.3, in this case for each pair of uncontrollable eigenvalues the corresponding Jordan blocks B_λ and $B_{-\lambda}$ are both placed in the nonsign-controllable part. Hence, the size of the resulting sign-controllable part is smaller than that obtained using the sign-controllability form as defined in Section 3.3. Now, let the pair (A, B) be in this second sign-controllability form,

$$A = \begin{bmatrix} A_r & 0 \\ A_{rs} & A_s \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ B_s \end{bmatrix},$$

where the pair (A_s, B_s) is sign-controllable, and let Q and S be partitioned accordingly, i.e.,

$$Q = \begin{bmatrix} Q_r & Q_{rs} \\ Q_{rs}^\top & Q_s \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} S_r \\ S_s \end{bmatrix}. \tag{28}$$

Moreover, define $F_{sr} := A_{rs} - B_s R^{-1} S_r^\top$, $F_{ss} := A_s - B_s R^{-1} S_s^\top$, $G_r := -Q_r + S_r R^{-1} S_r^\top$, $G_s := -Q_s + S_r R^{-1} S_s^\top$, $G_{rs} = -Q_{rs} + S_s R^{-1} S_s^\top$ and $V := -B_s R^{-1} B_s^\top$. With respect to this basis, the Hamiltonian differential equation can be written as

$$\begin{bmatrix} \dot{X}_r(t) \\ \dot{X}_s(t) \\ \dot{\Lambda}_r(t) \\ \dot{\Lambda}_s(t) \end{bmatrix} = \bar{H} \begin{bmatrix} X_r(t) \\ X_s(t) \\ \Lambda_r(t) \\ \Lambda_s(t) \end{bmatrix}, \quad \text{where} \quad \bar{H} = \begin{bmatrix} A_r & 0 & 0 & 0 \\ F_{sr} & F_{ss} & 0 & V \\ G_r & G_s & -A_r^\top & -F_{sr}^\top \\ G_s^\top & G_{rs} & 0 & -F_{ss}^\top \end{bmatrix}.$$

In the former, the time varying matrices $X(t)$ and $\Lambda(t)$ have been partitioned according to the sign-controllability form. The Hamiltonian matrix H_s referred to the sole sign-controllable part is

$$H_s = \begin{bmatrix} F_{ss} & V \\ G_{rs} & -F_{ss}^\top \end{bmatrix}.$$

In the next theorem, it is shown that if A_r and H_s have no common eigenvalues, and if H_s has no eigenvalues on the imaginary axis, a characterisation of the solutions of HDE(Σ) is still possible. Notice that if instead of the sign-controllability form defined in this section we used that presented in Section 3.3, for each pair $(\lambda, -\lambda)$ of uncontrollable eigenvalues, the corresponding Jordan blocks B_λ and $B_{-\lambda}$ would be divided between the sign-controllable and the nonsign-controllable part. As a result, the sign-controllable part would have uncontrollable eigenvalues that are opposite in sign to some nonsign-controllable eigenvalues. However, since uncontrollable eigenvalues and their opposites are also eigenvalues of the Hamiltonian matrix, it follows that the spectrum of the Hamiltonian matrix of the sign-controllable part H_s would include the pair $(\lambda, -\lambda)$, and therefore it would end up having eigenvalues in common with the nonsign-controllable part A_r. As a result, the Sylvester equations used in the next theorem to construct the parameterisation of the trajectories of HDE(Σ) would not be guaranteed to admit solutions.

Theorem 4 *Let the Popov triple be in the second sign-controllability form, as defined in this section. Assume $\sigma(A_r) \cap \sigma(H_s) = \emptyset$ and that H_s has no eigenvalues on the imaginary axis. Let $U_1, U_2 \in \mathbb{R}^{n_r \times n_s}$ be given by the solution of the Sylvester equation*

$$A_r^\top \begin{bmatrix} U_1 & U_2 \end{bmatrix} + \begin{bmatrix} U_1 & U_2 \end{bmatrix} H_s = \begin{bmatrix} G_s & -F_{sr}^\top \end{bmatrix}, \tag{29}$$

and let $U_3, U_4 \in \mathbb{R}^{n_r \times n_s}$ be given by the solution of the Sylvester equation

$$-A_r^\top \begin{bmatrix} U_3 & U_4 \end{bmatrix} + \begin{bmatrix} U_3 & U_4 \end{bmatrix} H_s^\top = -\begin{bmatrix} F_{sr}^\top & G_s \end{bmatrix}. \tag{30}$$

The set of trajectories originating from HDE(Σ) is parameterised in $\Gamma \in \mathbb{R}^{2\,n_r \times n}$, $\Phi \in \mathbb{R}^{n_s \times n}$ and $\Psi \in \mathbb{R}^{n_s \times n}$ as

$$\begin{bmatrix} X_r(t) \\ X_s(t) \\ \Lambda_r(t) \\ \Lambda_s(t) \end{bmatrix} = \begin{bmatrix} 0 & I_{n_r} \\ 0 & U_3^\top \\ I_{n_r} & U \\ 0 & U_4^\top \end{bmatrix} e^{\begin{bmatrix} -A_r^\top & \Xi \\ 0 & A_r \end{bmatrix} t} \Gamma + \begin{bmatrix} 0 \\ I_{n_s} \\ U_1 + U_2\,P \\ P \end{bmatrix} e^{A_P\,t} \Phi + \begin{bmatrix} 0 \\ Y \\ U_1 Y + U_2(PY - I_{n_s}) \\ PY - I_{n_s} \end{bmatrix} e^{A_P^\top (T-t)} \Psi \tag{31}$$

where $\Xi := -U_1\,F_{sr} + G_r - U_2\,G_s^\top$ and $U := U_1\,U_3^\top + U_2\,U_4^\top$.

Proof: We construct three changes of coordinates for the Hamiltonian differential equation, in order to obtain a block-diagonal structure for H. Since $\sigma(A_r) \cap \sigma(H_s) = \emptyset$, the Sylvester equation (29) admits a unique solution $\begin{bmatrix} U_1 & U_2 \end{bmatrix}$. Consider the non-singular matrix

$$T_1 = \begin{bmatrix} I_{n_r} & 0 & 0 & 0 \\ 0 & 0 & I_{n_s} & 0 \\ 0 & I_{n_r} & U_1 & U_2 \\ 0 & 0 & 0 & I_{n_s} \end{bmatrix}, \quad \text{so that} \quad T_1^{-1} = \begin{bmatrix} I_{n_r} & 0 & 0 & 0 \\ 0 & -U_1 & I_{n_s} & -U_2 \\ 0 & I_{n_r} & 0 & 0 \\ 0 & 0 & 0 & I_{n_s} \end{bmatrix}.$$

By taking (29) into account, we get

$$H_1 := T_1^{-1} \bar{H} T_1 = \begin{bmatrix} A_r & 0 & 0 & 0 \\ \Xi & -A_r^\top & 0 & 0 \\ F_{sr} & 0 & F_{ss} & V \\ G_s^\top & 0 & G_{rs} & -F_{ss}^\top \end{bmatrix},$$

where $\Xi := -U_1\,F_{sr} + G_r - U_2\,G_s^\top$. Since $\sigma(A_r) \cap \sigma(H_s) = \emptyset$, the Sylvester equation (30) admits a unique solution $\begin{bmatrix} U_3 & U_4 \end{bmatrix}$. Consider the non-singular matrix

$$T_2 = \begin{bmatrix} 0 & I_{n_r} & 0 & 0 \\ I_{n_r} & 0 & 0 & 0 \\ 0 & U_3^\top & I_{n_s} & 0 \\ 0 & U_4^\top & 0 & I_{n_s} \end{bmatrix}, \quad \text{so that} \quad T_2^{-1} = \begin{bmatrix} 0 & I_{n_r} & 0 & 0 \\ I_{n_r} & 0 & 0 & 0 \\ -U_3^\top & 0 & I_{n_s} & 0 \\ -U_4^\top & 0 & 0 & I_{n_s} \end{bmatrix}.$$

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The new change of coordinates over H_1 yields

$$H_2 = T_2^{-1} H_1 T_2 = \begin{bmatrix} -A_r^\top & \Xi & 0 & 0 \\ 0 & A_r & 0 & 0 \\ 0 & 0 & F_{ss} & V \\ 0 & 0 & G_{rs} & -F_{ss}^\top \end{bmatrix},$$

which is block-diagonal. A further change of coordinates is performed using

$$T_3 = \begin{bmatrix} 0 & I_{n_r} & 0 & 0 \\ I_{n_r} & 0 & 0 & 0 \\ 0 & 0 & I_{n_s} & 0 \\ 0 & 0 & 0 & I_{n_s} \end{bmatrix}, \quad \text{so that} \quad H_3 = T_3^{-1} H_2 T_3 = \left[\begin{array}{cc|cc} A_r & 0 & 0 & 0 \\ \Xi & -A_r^\top & 0 & 0 \\ \hline 0 & 0 & F_{ss} & V \\ 0 & 0 & G_{rs} & -F_{ss}^\top \end{array}\right],$$

The submatrix in the right-bottom of H_3 is exactly H_s. Now, consider the Hamiltonian differential equation in this new basis:

$$\begin{bmatrix} \dot{X}_1(t) \\ \dot{\Lambda}_1(t) \\ \dot{X}_2(t) \\ \dot{\Lambda}_2(t) \end{bmatrix} = H_3 \begin{bmatrix} X_1(t) \\ \Lambda_1(t) \\ X_2(t) \\ \Lambda_2(t) \end{bmatrix},$$

where

$$\begin{bmatrix} X_r(t) \\ X_s(t) \\ \Lambda_r(t) \\ \Lambda_s(t) \end{bmatrix} = T_1 T_2 T_3 \begin{bmatrix} X_1(t) \\ \Lambda_1(t) \\ X_2(t) \\ \Lambda_2(t) \end{bmatrix} = \begin{bmatrix} X_1(t) \\ U_3^\top X_1(t) + X_2(t) \\ U X_1(t) + \Lambda_1(t) + U_1 X_2(t) + U_2 \Lambda_2(t) \\ U_4^\top X_1(t) + \Lambda_2(t) \end{bmatrix}.$$

Since the pair (A_s, B_s) is sign-controllable and H_s has no eigenvalues on the imaginary axis, we can compute the strongly unmixed solution of the sole sign-controllable part P, the closed-loop matrix and the solution of the corresponding closed-loop Lyapunov equation A_P and Y, respectively. Hence, due to the block-diagonal structure of H_3, the parameterisation of the solutions of the Hamiltonian differential equation in the new basis is decoupled as

$$\begin{bmatrix} X_1(t) \\ \Lambda_1(t) \\ X_2(t) \\ \Lambda_2(t) \end{bmatrix} = \begin{bmatrix} I_{n_r} & 0 \\ 0 & I_{n_r} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} e^{\left[\begin{smallmatrix} -A_r^\top & \Xi \\ 0 & A_r \end{smallmatrix}\right] t} \Gamma + \begin{bmatrix} 0 \\ 0 \\ I_{n_s} \\ P \end{bmatrix} e^{A_P t} \Phi + \begin{bmatrix} 0 \\ 0 \\ Y \\ PY - I_{n_s} \end{bmatrix} e^{A_P^\top (T-t)} \Psi.$$

Writing this equation with respect to the original basis gives (31). ∎

Our aim is now to show how the result established in Theorem 4 can be exploited to find a representation of the solution of the Riccati differential equation with terminal condition RDP(Σ, P_T). The solution to this problem is given in the following theorem.

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Theorem 5 *Let T_s be a change of basis matrix that transforms the pair (A, B) into the second sign-controllability form as defined in Section 4, and let matrices Q and S be partitioned as in (28). Assume $\sigma(A_r) \cap \sigma(H_s) = \emptyset$ and that H_s has no eigenvalues on the imaginary axis. Let $U_1, U_2, U_3, U_4 \in \mathbb{R}^{n_r \times n_s}$, be the solutions of (29) and (30), respectively. Let*

$$
\begin{bmatrix} X_r(T) \\ X_s(T) \\ \Lambda_r(T) \\ \Lambda_s(T) \end{bmatrix} = \begin{bmatrix} M_r \\ M_s \\ N_r \\ N_s \end{bmatrix}, \quad \text{where} \quad \begin{bmatrix} M_r \\ M_s \end{bmatrix} = I_n \ \text{and} \ \begin{bmatrix} N_r \\ N_s \end{bmatrix} = T_s^\top P_s T_s. \tag{32}
$$

The solution $P(t)$ of $RDP(\Sigma)$ is given by

$$
P(t) = T_s^{-\top} \begin{bmatrix} \Lambda_r(t) \\ \Lambda_s(t) \end{bmatrix} \begin{bmatrix} X_r(t) \\ X_s(t) \end{bmatrix}^{-1} T_s^{-1},
$$

where $X_s(t)$, $X_r(t)$, $\Lambda_s(t)$ and $\Lambda_r(t)$ are given by (31) with

$$
\begin{bmatrix} \Phi \\ \Psi \\ \Gamma_1 \\ \Gamma_2 \end{bmatrix} = \begin{bmatrix} e^{-A_P T}(I - YP) & e^{-A_P T} Y & 0 & 0 \\ P & -I & 0 & 0 \\ U_1 & (U_1 + U_2 P)(I-Y) + U_2 & -\Omega - e^{A_r T} U & e^{A_r T} \\ 0 & 0 & e^{-A_r T} & 0 \end{bmatrix} \begin{bmatrix} M_s - U_3^\top M_r \\ N_s - U_4^\top M_r \\ M_r \\ N_r \end{bmatrix}.
$$

Proof: Consider $HDP(\Sigma, P_T)$, and consider the change of coordinate matrix T_s that takes the original Popov triple into the second sign-controllability form. By imposing (32) on X_r and X_s in (31) we obtain

$$
\begin{bmatrix} X_r(T) \\ X_s(T) \end{bmatrix} = \begin{bmatrix} 0 & I \\ 0 & U_3^\top \end{bmatrix} e^{\begin{bmatrix} -A_r^\top & \Xi \\ 0 & A_r \end{bmatrix} T} \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} + \begin{bmatrix} 0 \\ I_{n_s} \end{bmatrix} e^{A_P T} \Phi + \begin{bmatrix} 0 \\ Y \end{bmatrix} \Psi = \begin{bmatrix} M_r \\ M_s \end{bmatrix}.
$$

By partitioning $e^{\begin{bmatrix} -A_r^\top & \Xi \\ 0 & A_r \end{bmatrix} T}$ as $\begin{bmatrix} e^{-A_r^\top T} & \Omega \\ 0 & e^{A_r T} \end{bmatrix}$ we obtain the two equations

$$
\begin{bmatrix} e^{A_r T} \Gamma_2 \\ U_3^\top e^{A_r T} \Gamma_2 + e^{A_P T} \Phi + Y \Psi \end{bmatrix} = \begin{bmatrix} M_r \\ M_s \end{bmatrix},
$$

which lead to $\Gamma_2 = e^{-A_r T} M_r$ and

$$
\begin{bmatrix} e^{A_P T} & Y \end{bmatrix} \begin{bmatrix} \Phi \\ \Psi \end{bmatrix} = M_s - U_3^\top M_r. \tag{33}
$$

By imposing the boundary condition on Λ_r and Λ_s in (31) we obtain

$$
\begin{bmatrix} \Lambda_r(T) \\ \Lambda_s(T) \end{bmatrix} = \begin{bmatrix} I & U \\ 0 & U_4^\top \end{bmatrix} \begin{bmatrix} e^{-A_r^\top T} & \Omega \\ 0 & e^{A_r T} \end{bmatrix} \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} + \begin{bmatrix} U_1 + U_2 P \\ P \end{bmatrix} e^{A_P T} \Phi
$$

$$
+ \begin{bmatrix} U_1 Y + U_2(PY - I) \\ PY - I \end{bmatrix} \Psi = \begin{bmatrix} N_r \\ N_s \end{bmatrix}
$$

which leads to

$$
\left[
\begin{array}{c}
e^{-A_r\,T}\Gamma_1 + (\Omega + U\,e^{A_r\,T})\Gamma_2 + (U_1 + U_2\,P)e^{A_P\,T}\Phi + [U_1\,Y + U_2(P\,Y - I)]\Psi \\
U_4^\top e^{A_r\,T}\Gamma_2 + P\,e^{A_P\,T}\Phi + (P\,Y - I)\Psi
\end{array}
\right]
=
\left[
\begin{array}{c}
N_r \\
N_s
\end{array}
\right]
$$

The first block-equation enables Γ_1 to be computed given the values of Φ, Ψ and Γ_2. The second gives

$$
P\,e^{A_P\,T}\,\Phi + (P\,Y - I)\,\Psi = N_s - U_4^\top\,M_r. \tag{34}
$$

From (33) and (34) we obtain

$$
\left[
\begin{array}{cc}
e^{A_P\,T} & Y \\
P\,e^{A_P\,T} & P\,Y - I
\end{array}
\right]
\left[
\begin{array}{c}
\Phi \\
\Psi
\end{array}
\right]
=
\left[
\begin{array}{c}
M_s - U_3^\top\,M_r \\
N_s - U_4^\top\,M_r
\end{array}
\right]. \tag{35}
$$

The Schur complement of the first matrix on the left hand side $(P\,Y - I) - P\,e^{A_P\,T}\,e^{-A_P\,T}\,Y = -I$ is invertible. As such, (35) can be inverted:

$$
\begin{aligned}
\left[
\begin{array}{c}
\Phi \\
\Psi
\end{array}
\right]
&=
\left[
\begin{array}{cc}
e^{A_P\,T} & Y \\
P\,e^{A_P\,T} & P\,Y - I
\end{array}
\right]^{-1}
\left[
\begin{array}{c}
M_s - U_3^\top\,M_r \\
N_s - U_4^\top\,M_r
\end{array}
\right] \\
&=
\left[
\begin{array}{cc}
e^{-A_P\,T}(I - Y\,P) & e^{-A_P\,T}\,Y \\
P & -I
\end{array}
\right]
\left[
\begin{array}{c}
M_s - U_3^\top\,M_r \\
N_s - U_4^\top\,M_r
\end{array}
\right]
\end{aligned}
$$

and

$$
\begin{aligned}
\Gamma_1 &= e^{A_r\,T}\left(N_r - (\Omega\,e^{-A_r\,T} + U)M_r + (U_1 + U_2\,P)\,e^{A_P\,T}\Phi + [U_1\,Y + U_2(P\,Y - I)]\,\Psi\right) \\
&= e^{A_r\,T}\left(N_r - (\Omega\,e^{-A_r\,T} + U)M_r + U_1\,(M_s - U_3^\top M_r)\right. \\
&\quad \left. + (U_1 + U_2\,P - U_1\,Y - U_2\,P\,Y + U_2)(N_s - U_4^\top M_r)\right).
\end{aligned}
$$

∎

5 Applications to LQ optimal control

In this section, we investigate how the material developed in this paper, and in particular in Section 4, can be utilised to solve very general LQ optimal control problems. In particular, in [14] it was shown how to use parameterised expressions of the trajectories generated by the Hamiltonian differential equation to solve the most general form of LQ optimal control problem with affine constraints on the state vector at the end points and with a quadratic performance index; in [14] this very general problem is tackled under the assumption of sign-controllability. Here we show how to use the results of Section 4 to cases where the pair (A, B) might not

be sign-controllable. For the sake of argument, however, here we restrict our attention to the so-called fixed end-point LQ problem. Consider the LTI state differential equation

$$\dot{x}(t) = A\, x(t) + B\, u(t), \tag{36}$$

where, for all $t \geq 0$, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input. Moreover, it is required that

$$x(0) = \bar{x}_0 \in \mathbb{R}^n. \tag{37}$$

$$x(T) = \bar{x}_T \in \mathbb{R}^n. \tag{38}$$

The problem here is to find a measurable $u(t)$, $t \in [0, T)$, and an absolutely continuous $x(t)$, $t \in [0, T]$, minimising the quadratic performance index

$$J(x, u) := \int_0^T \left[x^\top(t) \ \ u^\top(t) \right] \Pi \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt, \tag{39}$$

under the constraints (36) and (37-38). We show how the result in Theorem 4 can be utilised to the end of constructing the optimal control law. For the sake of simplicity, we consider the system to be already in the sign-controllable form, so that no preliminary coordinate changes are needed. If $u(t)$ and $x(t)$ are optimal for the fixed end-point LQ problem, then a costate function $\lambda(t) \in \mathbb{R}^n$ exists such that $x(t)$, $\lambda(t)$ and $u(t)$ satisfy for all $t \in [0, T)$ the following set of equations:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} A - B\,R^{-1}\,S^\top & -B\,R^{-1}\,B^\top \\ -Q + S\,R^{-1}\,S^\top & -A^\top + S\,R^{-1}\,B^\top \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix}, \tag{40}$$

$$u(t) = -R^{-1}\left(S^\top\,x(t) + B^\top\,\lambda(t) \right), \tag{41}$$

$$x(0) = \bar{x}_0, \tag{42}$$

$$x(T) = \bar{x}_T, \tag{43}$$

These equations follow as a particular case of those in [14, Lemma 1]. Equation (40) differs from the Hamiltonian differential equation (3) only for the dimension of the unknown vector. As such, the set of all the trajectories satisfying (40) is given by (31), where now Φ, Ψ and Γ are vectors. By partitioning $\bar{x}_0 = \begin{bmatrix} \bar{x}_{r0}^\top & \bar{x}_{s0}^\top \end{bmatrix}^\top$ and $\bar{x}_T = \begin{bmatrix} \bar{x}_{rT}^\top & \bar{x}_{sT}^\top \end{bmatrix}^\top$ conformably with the sign-controllability form, by imposing (42-43) on (40) we get

$$\begin{bmatrix} \Gamma_2 \\ U_3^\top \Gamma_2 + \Phi + Y\, e^{A_P^\top T}\Psi \end{bmatrix} = \begin{bmatrix} \bar{x}_{r0} \\ \bar{x}_{s0} \end{bmatrix}$$

and

$$\begin{bmatrix} e^{A_r\,T}\Gamma_2 \\ U_3^\top e^{A_r\,T}\Gamma_2 + e^{A_P\,T}\Phi + Y\,\Psi \end{bmatrix} = \begin{bmatrix} \bar{x}_{rT} \\ \bar{x}_{sT} \end{bmatrix}$$

By eliminating Γ_2 from these two equations we obtain the condition $\bar{x}_{rT} = e^{A_r\,T}\,\bar{x}_{r0}$, which is not surprising. In fact, the fixed end-point problem can be solved only if the nonsign-controllable part of the state, which is obviously also uncontrollable, reaches the target $\bar{x}_{rT}$ starting from the initial value $\bar{x}_{r0}$ by itself. The remaining equations lead to the linear relation

$$\begin{bmatrix} I & Y\,e^{A_P^\top T} \\ e^{A_P\,T} & Y \end{bmatrix} \begin{bmatrix} \Phi \\ \Psi \end{bmatrix} = \begin{bmatrix} \bar{x}_{s0} - U_3^\top \bar{x}_{r0} \\ \bar{x}_{sT} - U_3^\top \bar{x}_{r0} \end{bmatrix} \tag{44}$$

which cannot be inverted in general. In fact, within the sign-controllable subsystem there are in general uncontrollable modes. Therefore, the fact that the condition $\bar{x}_{rT} = e^{A_r\,T}\,\bar{x}_{r0}$ is satisfied does not guarantee that the fixed end-point LQ problem admits solutions, because there are other uncontrollable modes in the sign-controllable part. Therefore, the $2n \times 2n$ matrix in the left-hand side of (44) is not invertible in general. Hence, the other condition that needs to be satisfied for the LQ problem to be solvable is that

$$\begin{bmatrix} \bar{x}_{s0} - U_3^\top \bar{x}_{r0} \\ \bar{x}_{sT} - U_3^\top \bar{x}_{r0} \end{bmatrix} \in \mathrm{im} \begin{bmatrix} I & Y\,e^{A_P^\top T} \\ e^{A_P\,T} & Y \end{bmatrix}.$$

If this condition holds true, the remaining parameters Φ and Ψ can be determined by pseudo-inversion of (44). These values of Φ, Ψ, Γ_2 (and with arbitrary Γ_1) can be used in (31), which at that point provides the trajectory of (40) satisfying (42) and (43). Then, the state and costate can be replaced in (41) to provide a control law which is optimal for the LQ control problem at hand, since $x(t)$, $\lambda(t)$ and $u(t)$ satisfy all the necessary and sufficient conditions required to guarantee optimality.

Concluding remarks

The formulae established in this paper for the solution of the matrix Riccati differential equation, which generalise those proposed so far in the literature, can be used to extend the analysis of finite escape time of the solution of the Riccati differential equation carried out in [4, p.1240]. Moreover, important properties of the dynamic behaviour of the solution of RDE(Σ), such as convergence and mechanisms of attraction, which have been studied in the literature under the assumptions of controllability and stabilisability, can be generalised using the results presented here.

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