A New Exact Penalty Function Method for Continuous Inequality Constrained Optimization Problems*

Changjun Yu[†‡], Kok Lay Teo[†], Liansheng Zhang[‡] and Yanqin Bai[‡]

[†]Department of Mathematics and Statistics
Curtin University of Technology
Kent Street, Bentley 6102, WA, Australia

[‡]Department of Mathematics
Shanghai University
99, Shangda Road, 200444, Shanghai, China

Abstract

In this paper, a computational approach based on a new exact penalty function method is devised for solving a class of continuous inequality constrained optimization problems. The continuous inequality constraints are first approximated by smooth function in integral form. Then, we construct a new exact penalty function, where the summation of all these approximate smooth functions in integral form, called the constraint violation, is appended to the objective function. In this way, we obtain a sequence of approximate unconstrained optimization problems. It is shown that if the value of the penalty parameter is sufficiently large, then any local minimizer of the corresponding unconstrained optimization problem is a local minimizer of the original problem. For illustration, three examples are solved using the proposed method. From the solutions obtained, we observe that the values of their objective functions are amongst the smallest when compared with those obtained by other existing methods available in the literature. More importantly, our method finds solution which satisfies the continuous inequality constraints.

1 Introduction

Many practical problems in engineering, such as circuit design and control system design (see, for example, [16], [17] and [5]), can be formulated as continuous inequality constrained optimization problems in the form given below:

$$\min \; f(x) \tag{1.1a}$$

$$\text{subject to} \qquad \phi_j(x,\omega) \leq 0, \; \forall\, \omega \in \Omega, \; j = 1, \; \ldots \;, \; m, \tag{1.1b}$$

where $x \in \mathbb{R}^n$ is the decision parameter vector, Ω is a compact interval in $\mathbb{R}$, $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable in x, and for each $j = 1, \; \ldots \;, \; m$, $\phi_j : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function in x and ω. Let this problem be referred to as Problem (P). This problem is also known as a semi-infinity optimization problem.

Since there are infinite many inequality constraints in (1.1b), conventional constrained optimization methods are not applicable to solving this problem directly. In [8], a constrained transcription method is introduced, where the continuous inequality constraints are first transformed into equivalent equality constraints in integral form. However, the integrands are nonsmooth. Thus, a local smoothing technique is used to approximate these nonsmooth integrands by smooth functions. In this way, Problem (P) is approximated by a sequence of optimization problems involving inequality constraints in integral form, where each of which can be solved by using conventional constrained optimization methods. There are two parameters, ϵ and τ, involved in these approximate constrained optimization problems, where $\epsilon > 0$ controls the accuracy of the approximation and $\tau > 0$

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controls the feasibility. It is shown that, for any $\epsilon > 0$, if $\tau > 0$ is sufficiently small, then the solution obtained satisfies the continuous inequality constraints. Furthermore, the global optimal solution of the approximate constrained optimization problem converges to the global optimal solution of the original problem as $\epsilon \to 0$. However, it is not known if a local optimal solution of the approximate constrained optimization problem will converge to a local optimal solution of the original problem. In [20], the smooth approximate functions in integral form are appended to the objective function by using the concept of the penalty function. This leads to a sequence of unconstrained optimization problems in integral form, where each of which is solvable by conventional unconstrained optimization techniques. Convergence results and the shortcomings similar to those reported in [8] are also valid. In [21] and [24], discretization methods and nonlinear Lagrangian functions are developed respectively. For all these algorithms, the feasibility condition is often missed in actual numerical calculation.

In [9], [15], [23] and [22], numerical algorithms based on Newton method was developed to solve semi-infinite programming problems, where the KKT system is formulated as a system of nonsmooth equations. However, the number of Lagrange multipliers in KKT system is not known a priori. For this, a different formulation of KKT system is introduced in [3], where the equivalent nonsmooth function of the continuous inequality constraints are approximated by smooth functions. Then, a projected Newton-Type algorithm is used to solve the new KKT system.

For a semi-infinite optimization problem, where the objective function is quadratic and the continuous inequality constraints are linear, it is found that dual parametrization methods are effective, (see, for example, [7], [12], [11] and [13]), where the dual problem of the linear-quadratic semi-infinite optimization problem, called the primal problem, is transformed into an equivalent finite dimensional nonlinear programming problem. The global solution of the primal problem can be obtained from that of the parameterized dual problem.

For optimization problems with conventional smooth inequality constraints, the penalty function method is, in general, recognized as an efficient method. However, to ensure that the solution obtained is feasible, the penalty parameter σ is required to go to $+\infty$. This is clearly unsatisfactory. Thus, exact penalty functions are introduced for these inequality constrained optimization problems,(see, for example, [2] and [19]). A main advantage of the exact penalty function method is that a minimizer of the objective function could be obtained without requiring the penalty parameter σ to go to $+\infty$. In [6], by adding a new variable ϵ, a new exact penalty function is introduced to deal with equality constrained minimization problem. Under some mild assumptions, it is shown in [6] that, if the value of the penalty parameter σ is sufficient large, then every local minimizer of the penalty problem with finite objective value (*i.e.* $f_\sigma(x^*, \epsilon^*)$ is finite) is of the form $(x^*, 0)$, where x^* is a local minimizer of the original problem.

In this paper, a new exact penalty function approach is proposed for solving semi-infinite optimization problems, where a objective function is to be minimized subject to continuous inequality constraints. This approach is motivated by the idea reported in [20]. However, the summation of the integrals of the exact penalty functions, rather than the summation of the integrals of the smooth approximate functions, is appended to the objective function forming a new objective function. This gives rise to a sequence of unconstrained optimization problems. In this way, the error caused by taking the smooth approximation of the continuous inequality constraints, as it is done in [20], can be avoided. Furthermore, any local minimizer of the unconstrained optimization problem when the penalty parameter is sufficiently large is a local minimizer of the original problem. This property is not shared by the approaches reported in [20], [21], [8] or [24]. Clearly, this is a major advancement in the study of the solution methods for semi-infinite optimization problems.

The rest of the paper is organized as follows. In Section 2, we give a new exact penalty function and analyze its convergent properties. In Section 3, we devise an algorithm for solving Problem (P) via solving a finite sequence of unconstrained optimization problems. Several examples are solved by using the algorithm proposed. Section 4 concludes the paper.

2 New exact penalty function method

Consider Problem (P). For each $x \in \mathbb{R}^n$, $\max\{\phi_j(x,\omega),0\}$ is a continuous function of ω, since ϕ_j is continuously differentiable. Define

$$S_\epsilon = \{(x,\epsilon) \in \mathbb{R}^n \times \mathbb{R}_+ : \phi_j(x,\omega) \leq \epsilon^\gamma W_j,\ \forall\ \omega \in \Omega,\ j = 1,\ \dots\ ,\ m\} \tag{2.1}$$

where $\mathbb{R}_+ = \{\alpha \in \mathbb{R}\ :\ \alpha \geq 0\}$, $W_j \in (0,1)$, $j = 1,\ \dots\ ,\ m$, are fixed constants and γ is a positive real number. Clearly, Problem (P) is equivalent to the following problem, which is denoted as Problem $(\hat{P})$.

$$\min\ f(x) \tag{2.2a}$$

subject to

$$(x,\epsilon) \in S_0 \tag{2.2b}$$

where $S_0 = S_\epsilon$ with $\epsilon = 0$.

We assume that the following conditions are satisfied:

$(A1.)$ There exists a global minimizer of Problem (P), implying that $f(x)$ is bounded from below on S_0.

$(A2.)$ The number of distinct local minimum values of the objective function of Problem (P) is finite.

$(A3.)$ Let $L(P)$ denote the set of all local minimizers of Problem (P). If $x^* \in L(P)$, then $L_{x^*} = \{x \in L(P) : f(x) = f(x^*)\}$ is a compact set.

Motivated by the exact penalty function introduced in [6] and the constraint transcription method for converting continuous inequality constraints into a sequence of inequality constraints in integral form (see [8] and [25]), we introduce a new exact penalty function $f_\sigma(x,\epsilon)$ defined below.

$$f_\sigma(x,\epsilon) = \begin{cases} f(x) & \text{if } \epsilon = 0, \phi_j(x,\omega) \leq 0\ (\omega \in \Omega) \\ f(x) + \epsilon^{-\alpha}\Delta(x,\epsilon) + \sigma\epsilon^\beta & \text{if } \epsilon > 0 \\ +\infty & \text{otherwise} \end{cases} \tag{2.3}$$

where $\Delta(x,\epsilon)$, which is referred to as the constraint violation, is defined by

$$\Delta(x,\epsilon) = \sum_{j=1}^{m} \int_\Omega \left[\max\left\{0, \phi_j(x,\omega) - \epsilon^\gamma W_j\right\}\right]^2 d\omega \tag{2.4}$$

α and γ are positive real numbers, $\beta > 2$, and $\sigma > 0$ is a penalty parameter. We now introduce a surrogate optimization problem, which is referred to as Problem (P_σ), as follows.

$$\min\ f_\sigma(x,\epsilon) \tag{2.5a}$$

subject to

$$(x,\epsilon) \in \mathbb{R}^n \times [0,+\infty) \tag{2.5b}$$

Intuitively, during the process of minimizing $f_\sigma(x,\epsilon)$, if σ is increased, ϵ^β should be reduced, meaning that ϵ should be reduced as β is fixed. Thus $\epsilon^{-\alpha}$ will be increased, and hence the constraint violation will also be reduced. This means that the value of

$\Big[\max\big\{0, \phi_j(x,\omega) - \epsilon^\gamma W_j\big\}\Big]^2$ must go down, leading to the satisfaction of the continuous inequality constraints, i.e.,

$$\phi_j(x,\omega) - \epsilon^\gamma W_j \le 0, \ \forall\, \omega \in \Omega, \ j = 1, \ \dots\ , m$$

In the next section, we will prove that, under some mild assumptions, if the parameter σ_k is sufficient large ($\sigma_k \to +\infty$ as $k \to +\infty$) and $(x^{(k),*}, \epsilon^{(k),*})$ is a local minimizer of Problem (P_{σ_k}), then $\epsilon^{(k),*} \to \epsilon^* = 0$, and $x^{(k),*} \to x^*$ with x^* being a local minimizer of Problem (P). The importance of this result is quite obvious.

2.1 Convergence analysis

Taking the gradients of $f_\sigma(x,\epsilon)$ with respect to x and ϵ gives

$$\frac{\partial f_\sigma(x,\epsilon)}{\partial x} = \frac{\partial f(x)}{\partial x} + 2\epsilon^{-\alpha} \sum_{j=1}^{m} \int_\Omega \max\big\{0, \phi_j(x,\omega) - \epsilon^\gamma W_j\big\} \frac{\partial \phi_j(x,\omega)}{\partial x} d\omega \tag{2.6}$$

$$\begin{aligned}
\frac{\partial f_\sigma(x,\epsilon)}{\partial \epsilon} &= -\alpha\epsilon^{-\alpha-1} \sum_{j=1}^{m} \int_\Omega \Big[\max\big\{0, \phi_j(x,\omega) - \epsilon^\gamma W_j\big\}\Big]^2 d\omega \\
&\quad -2\gamma\epsilon^{\gamma-\alpha-1} \sum_{j=1}^{m} \int_\Omega \max\big\{0, \phi_j(x,\omega) - \epsilon^\gamma W_j\big\} W_j d\omega + \sigma\beta\epsilon^{\beta-1} \\
&= \epsilon^{-\alpha-1}\Big\{ -\alpha \sum_{j=1}^{m} \int_\Omega \Big[\max\big\{0, \phi_j(x,\omega) - \epsilon^\gamma W_j\big\}\Big]^2 d\omega \\
&\quad +2\gamma \sum_{j=1}^{m} \int_\Omega \max\big\{0, \phi_j(x,\omega) - \epsilon^\gamma W_j\big\}(-\epsilon^\gamma W_j) d\omega \Big\} + \sigma\beta\epsilon^{\beta-1}
\end{aligned} \tag{2.7}$$

For every positive integer k, let $(x^{(k),*}, \epsilon^{(k),*})$ be a local minimizer of Problem (P_{σ_k}). To obtain our main result, we need

Lemma 2.1. *Let $(x^{(k),*}, \epsilon^{(k),*})$ be a local minimizer of Problem (P_{σ_k}). Suppose that $f_{\sigma_k}(x^{(k),*}, \epsilon^{(k),*})$ is finite and that $\epsilon^{(k),*} > 0$. Then*

$$(x^{(k),*}, \epsilon^{(k),*}) \notin S_\epsilon$$

where S_ϵ is defined by (2.1).

Proof. Since $(x^{(k),*}, \epsilon^{(k),*})$ is a local minimizer of Problem (P_{σ_k}) and $\epsilon^{(k),*} > 0$, we have

$$\frac{\partial f_{\sigma_k}(x^{(k),*}, \epsilon^{(k),*})}{\partial \epsilon} = 0 \tag{2.8}$$

On a contrary, we assume that the conclusion of the lemma is false. Then, we have

$$\phi_j(x^{(k),*}, \epsilon^{(k),*}) \le (\epsilon^{(k),*})^\gamma W_j, \forall\, \omega \in \Omega, \ j = 1, \ \dots\ , m.$$

Thus, by (2.7) and (2.8), we obtain

$$0 = \frac{\partial f_{\sigma_k}(x^{(k),*}, \epsilon^{(k),*})}{\partial \epsilon} = \beta\sigma_k\epsilon^{\beta-1} > 0$$

This is a contradiction, and hence completing the proof. □

To continue, we introduce

Definition 2.2. *Let $\bar{x}$ be such that $\frac{\partial \phi_j(\bar{x},\omega)}{\partial x}$, $j = 1, \ldots, m$, are linearly independent for each $\omega \in \Omega$. Then, it is said that the constraint qualification is satisfied for the continuous inequality constraints (1.1b) at $x = \bar{x}$.*

Let the conditions of Lemma 2.1 be satisfied. Then, we have

Theorem 2.3. *Suppose that $(x^{(k),*}, \epsilon^{(k),*})$ is a local minimizer of Problem (P_{σ_k}) such that $f_{\sigma_k}(x^{(k),*}, \epsilon^{(k),*})$ is finite and $\epsilon^{(k),*} > 0$. If $(x^{(k),*}, \epsilon^{(k),*}) \to (x^*, \epsilon^*)$ as $k \to +\infty$, and the constraint qualification is satisfied for the continuous inequality constraints (1.1b) at $x = x^*$, then $\epsilon^* = 0$ and $x^* \in S_0$.*

Proof. From Lemma 2.1, it follows that $(x^{(k),*}, \epsilon^{(k),*}) \notin S_{\epsilon^{(k),*}}$. Furthermore,

$$\begin{aligned} & \frac{\partial f_{\sigma_k}(x^{(k),*}, \epsilon^{(k),*})}{\partial x} \\ = \; & \frac{\partial f(x^{(k),*})}{\partial x} \\ & +2(\epsilon^{(k),*})^{-\alpha} \sum_{j=1}^{m} \int_{\Omega} \max\left\{0, \phi_j(x^{(k),*}, \omega) - (\epsilon^{(k),*})^{\gamma} W_j\right\} \frac{\partial \phi_j(x^{(k),*}, \omega)}{\partial x} d\omega \\ = \; & 0 \end{aligned} \tag{2.9}$$

$$\begin{aligned} & \frac{\partial f_{\sigma_k}(x^{(k),*}, \epsilon^{(k),*})}{\partial \epsilon} \\ = \; & -\alpha(\epsilon^{(k),*})^{-\alpha-1} \sum_{j=1}^{m} \int_{\Omega} \left[\max\{0, \phi_j(x^{(k),*}, \omega) - (\epsilon^{(k),*})^{\gamma} W_j\}\right]^2 d\omega \\ & -2\gamma(\epsilon^{(k),*})^{\gamma-\alpha-1} \sum_{j=1}^{m} \int_{\Omega} \max\{0, \phi_j(x^{(k),*}, \omega) - (\epsilon^{(k),*})^{\gamma} W_j\} W_j d\omega \\ & +\sigma_k \beta (\epsilon^{(k),*})^{\beta-1} \\ = \; & (\epsilon^{(k),*})^{-\alpha-1} \Big\{ -\alpha \sum_{j=1}^{m} \int_{\Omega} \left[\max\{0, \phi_j(x^{(k),*}, \omega) - (\epsilon^{(k),*})^{\gamma} W_j\}\right]^2 d\omega \\ & +2\gamma \sum_{j=1}^{m} \int_{\Omega} \max\{0, \phi_j(x^{(k),*}, \omega) - (\epsilon^{(k),*})^{\gamma} W_j\}(-(\epsilon^{(k),*})^{\gamma} W_j) d\omega \Big\} \\ & +\sigma_k \beta (\epsilon^{(k),*})^{\beta-1} \\ = \; & 0 \end{aligned} \tag{2.10}$$

Suppose that $\epsilon^{(k),*} \to \epsilon^* \neq 0$. Then, by (2.10), we observe that its first term tends to a finite value, while the last term tends to infinity as $\sigma_k \to +\infty$, when $k \to +\infty$. This is impossible for the validity of (2.10). Thus, $\epsilon^* = 0$.
Now, by (2.9), we obtain

$$\begin{aligned} & (\epsilon^{(k),*})^{\alpha} \frac{\partial f(x^{(k),*})}{\partial x} + 2 \sum_{j=1}^{m} \int_{\Omega} \max\left\{0, \phi_j(x^{(k),*}, \omega) - (\epsilon^{(k),*})^{\gamma} W_j\right\} \frac{\partial \phi_j(x^{(k),*}, \omega)}{\partial x} d\omega \\ = & 0 \end{aligned} \tag{2.11}$$

Thus,

$$\begin{aligned}
&\lim_{k\to+\infty}\Big\{(\epsilon^{(k),*})^{\alpha}\frac{\partial f(x^{(k),*})}{\partial x}\\
&\qquad +2\sum_{j=1}^{m}\int_{\Omega}\max\big\{0,\phi_j(x^{(k),*},\omega)-(\epsilon^{(k),*})^{\gamma}W_j\big\}\frac{\partial\phi_j(x^{(k),*},\omega)}{\partial x}d\omega\Big\}\\
=\;\; & 2\sum_{j=1}^{m}\int_{\Omega}\max\big\{0,\phi_j(x^{*},\omega)\big\}\frac{\partial\phi_j(x^{*},\omega)}{\partial x}d\omega=0
\end{aligned}\tag{2.12}$$

Since the constraint qualification is satisfied for the continuous inequality constraints (1.1b) at $x=x^*$, it follows that, for each $j=1,\ \ldots,\ m$,

$$\max\big\{0,\phi_j(x^*,\omega)\big\}=0$$

for each $\omega\in\Omega$. This, in turn, implies that, for each $j=1,\ \ldots,\ m$, $\phi_j(x^*,\omega)\le 0$, $\forall\ \omega\in\Omega$. The proof is complete. □

Corollary 1. *If $x^{(k),*}\to x^*\in S_0$ and $\epsilon^{(k),*}\to\epsilon^*=0$, then $\Delta(x^{(k),*},\epsilon^{(k),*})\to\Delta(x^*,\epsilon^*)=0$*

Proof. The conclusion follows readily from the definition of $\Delta(x,\epsilon)$ and the continuity of $\phi_j(x,\omega)$. □

In [6], the construction of the form of the exact penalty function $f_\sigma(x,\omega)$ is such that it is continuously differentiable in S_ϵ when $\epsilon>0$. Its limit is continuous on the part of the boundary when its values are finite. In particular, $f_\sigma(x,0)$ is finite when x is such that $\phi_j(x,\omega)\le 0$, $\forall\ \omega\in\Omega$, $j=1,\ \ldots\ ,\ m$. In what follow, we shall turn our attention to the exact penalty function constructed in (2.3). We shall see that, under some mild conditions, $f_\sigma(x,\omega)$ is continuously differentiable with continuous limits.

Theorem 2.4. *Assume that $\phi_j(x^{(k),*},\omega)=o((\epsilon^{(k),*})^{\delta})$, $\delta>0$, $j=1,\ \ldots,\ m$. Suppose that $\gamma>\alpha$, $\delta>\alpha$, $-\alpha-1+2\delta>0$, $2\gamma-\alpha-1>0$. Then*

$$f_{\sigma_k}(x^{(k),*},\epsilon^{(k),*})\xrightarrow[x^{(k),*}\to x^*\in S_0]{\epsilon^{(k),*}\to\epsilon^*=0}f_{\sigma_k}(x^*,0)=f(x^*)\tag{2.13}$$

$$\nabla_{(x,\epsilon)}f_{\sigma_k}(x^{(k),*},\epsilon^{(k),*})\xrightarrow[x^{(k),*}\to x^*\in S_0]{\epsilon^{(k),*}\to\epsilon^*=0}\nabla_{(x,\epsilon)}f_{\sigma_k}(x^*,0)=(\nabla f(x^*),0)\tag{2.14}$$

Proof. By virtue of the conditions of the theorem, it follows that, for $\epsilon\ne 0$,

$$\begin{aligned}
&\lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0\\ x^{(k),*}\to x^*\in S_0}} f_{\sigma_k}(x^{(k),*},\epsilon^{(k),*})\\
=\;\; & \lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0\\ x^{(k),*}\to x^*\in S_0}}\Big\{f(x^{(k),*})\\
&+(\epsilon^{(k),*})^{-\alpha}\sum_{j=1}^{m}\int_{\Omega}\Big[\max\big\{0,\phi_j(x^{(k),*},\omega)-(\epsilon^{(k),*})^{\gamma}W_j\big\}\Big]^2d\omega+\sigma_k(\epsilon^{(k),*})^{\beta}\Big\}\\
=\;\; & f(x^*)+\lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0\\ x^{(k),*}\to x^*\in S_0}}\frac{\sum_{j=1}^{m}\int_{\Omega}\Big[\max\big\{0,\phi_j(x^{(k),*},\omega)-(\epsilon^{(k),*})^{\gamma}W_j\big\}\Big]^2d\omega}{(\epsilon^{(k),*})^{\alpha}}
\end{aligned}\tag{2.15}$$

For the second term of (2.15), it is clear from Lemma 2.1 that

$$\begin{aligned}
&\lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0\\ x^{(k),*}\to x^*\in S_0}} \frac{\sum_{j=1}^{m}\int_{\Omega}\Big[\max\big\{0,\phi_j(x^{(k),*},\omega)-(\epsilon^{(k),*})^{\gamma}W_j\big\}\Big]^2 d\omega}{(\epsilon^{(k),*})^{\alpha}} \\
= &\lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0\\ x^{(k),*}\to x^*\in S_0}} \sum_{j=1}^{m}\int_{\Omega}\big[(\epsilon^{(k),*})^{-\frac{\alpha}{2}}\phi_j(x^{(k),*},\omega)-(\epsilon^{(k),*})^{\gamma-\frac{\alpha}{2}}W_j\big]^2 d\omega
\end{aligned} \tag{2.16}$$

Since $\gamma>\alpha$ and $\delta>\alpha$, we have

$$\lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0\\ x^{(k),*}\to x^*\in S_0}} \sum_{j=1}^{m}\int_{\Omega}\big[(\epsilon^{(k),*})^{-\frac{\alpha}{2}}\phi_j(x^{(k),*},\omega)-(\epsilon^{(k),*})^{\gamma-\frac{\alpha}{2}}W_j\big]^2 d\omega=0 \tag{2.17}$$

Combining (2.15) and (2.17) gives

$$\lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0\\ x^{(k),*}\to x^*\in S_0}} f_{\sigma_k}(x^{(k),*},\epsilon^{(k),*})=f_{\sigma_k}(x^*,0)=f(x^*) \tag{2.18}$$

Similarly, we have

$$\begin{aligned}
&\lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0\\ x^{(k),*}\to x^*\in S_0}} \nabla_{(x,\epsilon)}f_{\sigma_k}(x^{(k),*},\epsilon^{(k),*}) \\
= &\lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0\\ x^{(k),*}\to x^*\in S_0}} \Big[\nabla_x f_{\sigma_k}(x^{(k),*},\epsilon^{(k),*})\ \nabla_\epsilon f_{\sigma_k}(x^{(k),*},\epsilon^{(k),*})\Big]^T
\end{aligned} \tag{2.19}$$

where

$$\begin{aligned}
&\lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0\\ x^{(k),*}\to x^*\in S_0}} \nabla_x f_{\sigma_k}(x^{(k),*},\epsilon^{(k),*}) \\
= &\lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0\\ x^{(k),*}\to x^*\in S_0}} \Big\{\frac{\partial f(x^{(k),*})}{\partial x} \\
&+2(\epsilon^{(k),*})^{-\alpha}\sum_{j=1}^{m}\int_{\Omega}\max\big\{0,\phi_j(x^{(k),*},\omega)-(\epsilon^{(k),*})^{\gamma}W_j\big\}\frac{\partial\phi_j(x^{(k),*},\omega)}{\partial x}d\omega\Big\} \\
= &\nabla_x f(x^*)+\lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0\\ x^{(k),*}\to x^*\in S_0}} 2\sum_{j=1}^{m}\int_{\Omega}\big[(\epsilon^{(k),*})^{-\alpha}\phi_j(x^{(k),*},\omega) \\
&\qquad -(\epsilon^{(k),*})^{\gamma-\alpha}W_j\big]\frac{\partial\phi_j(x^{(k),*},\omega)}{\partial x}d\omega \\
= &\nabla_x f(x^*)
\end{aligned} \tag{2.20}$$

while

$$
\begin{aligned}
& \lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0\\ x^{(k),*}\to x^*\in S_0}} \nabla_\epsilon f_{\sigma_k}(x^{(k),*},\epsilon^{(k),*}) \\
= & \lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0\\ x^{(k),*}\to x^*\in S_0}} \Big\{(\epsilon^{(k),*})^{-\alpha-1}\Big\{-\alpha\sum_{j=1}^m\int_\Omega\Big[\max\{0,\phi_j(x^{(k),*},\omega) \\
& \qquad\qquad -(\epsilon^{(k),*})^\gamma W_j\}\Big]^2 d\omega \\
& +2\gamma\sum_{j=1}^m\int_\Omega \max\left\{0,\phi_j(x^{(k),*},\omega)-(\epsilon^{(k),*})^\gamma W_j\right\}((-\epsilon^{(k),*})^\gamma W_j)d\omega\Big\} \\
& +\sigma_k\beta(\epsilon^{(k),*})^{\beta-1}\Big\} \\
= & \lim_{\substack{\epsilon^{(k),*}\to\epsilon^*=0\\ x^{(k),*}\to x^*\in S_0}} \Big\{-\alpha\sum_{j=1}^m\int_\Omega\left[\phi_j(x^{(k),*},\omega)(\epsilon^{(k),*})^{-\frac{\alpha+1}{2}}-(\epsilon^{(k),*})^{\gamma-\frac{\alpha+1}{2}}W_j\right]^2 d\omega \\
& +2\gamma\sum_{j=1}^m\int_\Omega\left[\phi_j(x^{(k),*},\omega)-(\epsilon^{(k),*})^\gamma W_j\right]\left((-\epsilon^{(k),*})^\gamma W_j\right)(\epsilon^{(k),*})^{-\alpha-1}d\omega\Big\} \\
= & \; 0
\end{aligned} \tag{2.21}
$$

Thus, the proof is complete. □

Results presented in Theorem 2.3, Corollary 1 and Theorem 2.4 form the foundation for constructing a computational method to be presented in section 3.

Theorem 2.5. *Let $\epsilon^{(k),*}\to\epsilon^*=0$, $x^{(k),*}\to x^*\in S_0$ be such that $f_{\sigma_k}(x^*,\epsilon^*)$ is finite. Then, x^* is a local minimizer of Problem (P).*

Proof. On a contrary, assume that x^* is not a local minimizer of Problem (P). Then, there must exists a feasible point y^* of Problem (P), satisfying $y^*\in\mathcal{N}_\delta(x^*)$, such that

$$f(y^*)<f(x^*) \tag{2.22}$$

where $\mathcal{N}_\delta(x^*)$ is a $\delta-$ neighborhood of x^* in S_0 for some $\delta>0$. Since $(x^{(k),*},\epsilon^{(k),*})$ is a local minimizer of Problem (P_{σ_k}), there exists a sequence $\{\xi^k\}$, such that

$$f_{\sigma_k}(x,\epsilon^{(k),*})\geq f_{\sigma_k}(x^{(k),*},\epsilon^{(k),*})$$

for any $x\in\mathcal{N}_{\xi^k}(x^{(k),*})$. Now, we construct a sequence $\{y^{(k),*}\}$ satisfying

$$\|y^{(k),*}-x^{(k),*}\|\leq\frac{\xi^k}{k}$$

Clearly,

$$f_{\sigma_k}((y^{(k),*},\epsilon^{(k),*})\geq f_{\sigma_k}((x^{(k),*},\epsilon^{(k),*}) \tag{2.23}$$

Letting $k\to+\infty$, we have

$$
\begin{aligned}
\lim_{k\to+\infty}\|y^{(k),*}-y^*\| &\leq \lim_{k\to+\infty}\|y^{(k),*}-x^{(k),*}\|+\lim_{k\to+\infty}\|x^{(k),*}-x^*\|+\|x^*-y^*\| \\
&\leq 0+0+\delta
\end{aligned} \tag{2.24}
$$

However, $\delta>0$ is arbitrary. Thus,

$$\lim_{k\to+\infty}y^{(k),*}=y^* \tag{2.25}$$

Letting $k \to +\infty$ in (2.23), it follows from the continuity of f and (2.25) that

$$\begin{aligned}&\lim_{k\to+\infty} f_{\sigma_k}(y^{(k),*}, \epsilon^{(k),*})\\ =&f_{\sigma_k}(y^*, 0) = f(y^*) \geq \lim_{k\to+\infty} f_{\sigma_k}(x^{(k),*}, \epsilon^{(k),*}) = f_{\sigma_k}(x^*, 0) = f(x^*)\end{aligned} \tag{2.26}$$

This is a contradiction to (2.22), hence completing the proof. □

The exactness of the penalty function is given in the following theorem.

Theorem 2.6. *If the conditions of Theorem 2.3 and Theorem 2.4 hold, then there exists a* $k_0 > 0$, *such that* $\epsilon^{(k),*} = 0,\ x^{(k),*} \in L(P)$, *for* $k \geq k_0$.

Proof. On a contrary, we assume that the conclusion is false. Then, there exists a subsequence of $\{(x^{(k),*}, \epsilon^{(k),*})\}$, which is denoted by the original sequence such that for any $k_0 > 0$, there exists a $k' > k_0$ satisfying $\epsilon^{(k'),*} \neq 0$. By Theorem 2.3, we have

$$\epsilon^{(k),*} \to \epsilon^* = 0,\ x^{(k),*} \to x^* \in S_0,\ \text{as } k \to +\infty$$

Since $\epsilon^{(k),*} \neq 0$ for all k, it follows from dividing (2.10) by $(\epsilon^{(k),*})^{\beta-1}$ that

$$\begin{aligned}&(\epsilon^{(k),*})^{-\alpha-\beta}\Big\{-\alpha\sum_{j=1}^{m}\int_\Omega\Big[\max\{0, \phi_j(x^{(k),*}, \omega) - (\epsilon^{(k),*})^\gamma W_j\}\Big]^2 d\omega\\ &+2\gamma\sum_{j=1}^{m}\int_\Omega \max\{0, \phi_j(x^{(k),*}, \omega) - (\epsilon^{(k),*})^\gamma W_j\}\big((-\epsilon^{(k),*})^\gamma W_j\big)d\omega\Big\} + \sigma_k\beta = 0\end{aligned} \tag{2.27}$$

This is equivalent to

$$\begin{aligned}&(\epsilon^{(k),*})^{-\alpha-\beta}\Big\{-\alpha\sum_{j=1}^{m}\int_\Omega\Big[\max\{0, \phi_j(x^{(k),*}, \omega) - (\epsilon^{(k),*})^\gamma W_j\}\Big]^2 d\omega\\ &+2\gamma\sum_{j=1}^{m}\int_\Omega\Big[\max\{0, \phi_j(x^{(k),*}, \omega) - (\epsilon^{(k),*})^\gamma W_j\}\big((-\epsilon^{(k),*})^\gamma W_j\big)\\ &+\max\big\{0, \phi_j(x^{(k),*}, \omega) - (\epsilon^{(k),*})^\gamma W_j\big\}\phi_j(x^{(k),*}, \omega)\\ &-\max\big\{0, \phi_j(x^{(k),*}, \omega) - (\epsilon^{(k),*})^\gamma W_j\big\}\phi_j(x^{(k),*}, \omega)\Big]d\omega\Big\} + \sigma_k\beta = 0\end{aligned} \tag{2.28}$$

Rearranging (2.28) yields

$$\begin{aligned}&(\epsilon^{(k),*})^{-\alpha-\beta}(2\gamma-\alpha)\Big\{\sum_{j=1}^{m}\int_\Omega\Big[\max\big\{0, \phi_j(x^{(k),*},\omega)\\ &\qquad -(\epsilon^{(k),*})^\gamma W_j\big\}\Big]^2 d\omega\Big\} + \sigma_k\beta\\ =\ & 2\gamma(\epsilon^{(k),*})^{-\alpha-\beta}\sum_{j=1}^{m}\int_\Omega \max\big\{0, \phi_j(x^{(k),*}, \omega) - (\epsilon^{(k),*})^\gamma W_j\big\}\phi_j(x^{(k),*}, \omega)d\omega\end{aligned} \tag{2.29}$$

Letting $k \to +\infty$ in (2.29) gives

$$2\gamma(\epsilon^{(k),*})^{-\alpha-\beta}\sum_{j=1}^{m}\int_\Omega \max\big\{0, \phi_j(x^{(k),*}, \omega) - (\epsilon^{(k),*})^\gamma W_j\big\}\phi_j(x^{(k),*}, \omega)d\omega \to +\infty \tag{2.30}$$

Define

$$y^k = (\epsilon^{(k),*})^{-\alpha-\beta}\sum_{j=1}^{m}\max\big\{0, \phi_j(x^{(k),*}, \omega) - (\epsilon^{(k),*})^\gamma W_j\big\} \tag{2.31}$$

From (2.30) and (2.31), we have

$$y^k \to +\infty \ , \ \text{as } k \to +\infty \tag{2.32}$$

Define

$$z^k = y^k / \|y^k\| \tag{2.33}$$

Clearly

$$\lim_{k\to+\infty} \|z^k\| = \|z^*\| = 1 \tag{2.34}$$

Dividing (2.11) by $\|y^k\|$ yields

$$\begin{aligned} &\frac{\frac{\partial f(x^{(k),*})}{\partial x}}{\|y^k\|} + \frac{2(\epsilon^{(k),*})^{-\alpha}}{\|y^k\|} \sum_{j=1}^{m} \int_\Omega \max\left\{0, \phi_j(x^{(k),*}, \omega)\right. \\ &\qquad\qquad\qquad\qquad \left. -(\epsilon^{(k),*})^\gamma W_j\right\} \frac{\partial \phi_j(x^{(k),*}, \omega)}{\partial x} d\omega = 0 \end{aligned} \tag{2.35}$$

For each $j = 1, \ \ldots, \ m$, define

$$\phi_j^{(k),\min} = \min\{\frac{\partial \phi_j(x^{(k),*}, \omega)}{\partial x} \ : \ \omega \in \Omega\}$$

$$\phi_j^{(k),\max} = \max\{\frac{\partial \phi_j(x^{(k),*}, \omega)}{\partial x} \ : \ \omega \in \Omega\}$$

Note that $x^{(k),*} \to x^*$ as $k \to +\infty$ and that $\dfrac{\partial f(x)}{\partial x}$ and, for each $j = 1, \ \ldots, \ m$, ϕ_j and $\dfrac{\partial \phi_j(\cdot\ , \omega)}{\partial x}$ are continuous in $\mathbb{R}^n$ for each $\omega \in \Omega$, where Ω is a compact set.
Then, it can be shown that there are constants $\hat{K}$, $\underline{K}$ and $\overline{K}$, independent of k, such that

$$\|\frac{\partial f(x^{(k),*})}{\partial x}\| \le \hat{K} \tag{2.36}$$

$$\phi_j^{(k),\min}, \ \phi_j^{(k),\max} \ \in \ [\underline{K}, \overline{K}] \tag{2.37}$$

For all $k = 1, 2, \ldots$.
By substituting (2.31) and (2.33) into (2.35), we obtain

$$\frac{\frac{\partial f(x^{(k),*})}{\partial x}}{\|y^k\|(\epsilon^{(k),*})^\beta} + 2\int_\Omega z^k \phi_j^{(k),*} d\omega = 0 \ , \ \phi_j^{(k),*} \in [\underline{K}, \overline{K}] \tag{2.38}$$

Since

$$\begin{aligned} \frac{1}{\|y^k\|(\epsilon^{(k),*})^\beta} &= \frac{1}{\|(\epsilon^{(k),*})^{-\alpha-\beta} \sum_{j=1}^{m} \max\left\{0, \phi_j(x^{(k),*}, \omega) - (\epsilon^{(k),*})^\gamma W_j\right\}\|(\epsilon^{(k),*})^\beta} \\ &= \frac{1}{\|\sum_{j=1}^{m} \max\left\{0, \phi_j(x^{(k),*}, \omega) - (\epsilon^{(k),*})^\gamma W_j\right\}\|(\epsilon^{(k),*})^{-\alpha}} \end{aligned} \tag{2.39}$$

From Theorem 2.4, we have $\phi_j(x^{(k),*}, \omega) = o((\epsilon^{(k),*})^\delta)$ and $\gamma > \alpha, \ \delta > \alpha$. Thus

$$\begin{aligned} &\lim_{k\to+\infty} \quad \|\sum_{j=1}^{m} \max\left\{0, \phi_j(x^{(k),*}, \omega) - (\epsilon^{(k),*})^\gamma W_j\right\}\|(\epsilon^{(k),*})^{-\alpha} \\ &= \quad \|\sum_{j=1}^{m} \max\left\{0, (\epsilon^*)^{\delta-\alpha} - (\epsilon^*)^{\gamma-\alpha} W_j\right\}\| \\ &= \quad 0 \end{aligned} \tag{2.40}$$

$$\frac{1}{\|y^k\|(\epsilon^{(k),*})^\beta} \to +\infty, \quad k \to +\infty \tag{2.41}$$

From (2.36) and (2.41), it is clear that

$$\frac{\frac{\partial f(x^{(k),*})}{\partial x}}{\|y^k\|(\epsilon^{(k),*})^\beta} \to +\infty, \quad k \to +\infty \tag{2.42}$$

However, for $\int_\Omega z^k \phi_j^{(k),*} d\omega$, we have $\|z^k\| = 1$. Thus, it follows from (2.37) that $\int_\Omega z^k \phi_j^{(k),*} d\omega$ is bounded uniformly with respect to k. This is a contradiction to (2.38). Thus, the proof is complete. $\square$

We may now conclude that, under some mild assumptions and the constraint qualification condition, when the parameter σ is sufficiently large, a local minimizer of Problem (P_σ) is a local minimizer of Problem (P).

3 Algorithm and numerical results

Here we use the optimization tool box *fmincon* within MATLAB environment to solve the optimization Problem (P_σ), where the integral appeared in $f_\sigma(x, \epsilon)$ is calculated by using the *Simpson's Rule*. For *Simpson's Rule*, the global error is of order h^4, where h is the discretization step size. Thus, the required accuracy of the integrations can be easily achieved if the discretization step size is sufficient small.
In the following, we give definitions to the terms used.
σ − The penalty parameter which is to be increased in every iteration.
$\bar{\omega}$ −The point at which $\max\limits_{1\le j\le m} \phi_j(x^{(k),*}, \bar{\omega}) = \max\limits_{1\le j\le m} \max\limits_{\omega\in\Omega} \phi_j(x^{(k),*}, \omega)$.
g − The value of $\max\limits_{1\le j\le m} \max\limits_{\omega\in\Omega} \phi_j(x^{(k),*}, \omega)$.
f − The objective function value.
ϵ − A new variable which is introduced in the construction of the exact penalty function.
ϵ^* − A lower bound of $\epsilon^{(k),*}$, which is introduced for avoiding $\epsilon^{(k),*} \to 0$.
With the new exact penalty function, we can construct an efficient algorithm, which is given below

Algorithm 1
Step 1 set $\sigma^{(1)} = 10$, $\epsilon^{(1)} = 0.1$, $\epsilon^* = 10^{-9}$, $\beta > 2$, choose an initial point (x_0, ϵ_0), the iteration index $k = 0$. The values of γ and α are chosen depending on the specific structure of Problem (P) concerned.
Step 2 Solve Problem (P_{σ_k}), and let $(x^{(k),*}, \epsilon^{(k),*})$ be the minimizer obtained.
Step 3 If $\epsilon^{(k),*} > \epsilon^*$, $\sigma^{(k)} < 10^8$,
set $\sigma^{(k+1)} = 10 \times \sigma^{(k)}$, $k = k+1$. Go to **Step 2** with $(x^{(k),*}, \epsilon^{(k),*})$ as the new initial point in the new optimization process
Else set $\epsilon^{(k),*} = \epsilon^*$, then go to **Step 4**
Step 4 Check the feasibility of $x^{(k),*}$ (i.e., whether or not $\max\limits_{1\le j\le m} \max\limits_{\omega\in\Omega} \phi_j(x^{(k),*}, \omega) \le 0$).
If $x^{(k),*}$ is feasible, then it is a local minimizer of Problem (P). Exit.
Else go to **Step 5**
Step 5: Adjust the parameters α, β and γ such that conditions of Lemma 2.1 are satisfied. Set $\sigma^{(k+1)} = 10\sigma^{(k)}$, $\epsilon^{(k+1)} = 0.1\epsilon^{(k)}$, $k := k+1$. Go to **Step 2**.

Remark 1. *In **Step 3**, if $\epsilon^{(k),*} > \epsilon^*$, we obtain from Theorem 2.3 and Theorem 2.6 that $x^{(k),*}$ cannot be a feasible point, meaning that the penalty parameter σ may not be large enough. Thus we need to increase σ. If $\sigma_k > 10^8$, but still $\epsilon^{(k),*} > \epsilon^*$, then we should adjust* *the value of α, β and γ, such that conditions assumed in the Theorem 2.4 are satisfied. Go to* ***Step 2****.*

Remark 2. *Clearly, we cannot check the feasibility of $\phi_j(x,\omega) \leq 0$, $j = 1, \ldots, m$, for every $\omega \in \Omega$. In practice, we choose a set $\hat{\Omega}$, which contains a dense enough of points in Ω. Check the feasibility of $\phi_j(x,\omega) \leq 0$ over $\hat{\Omega}$ for each $j = 1, \ldots, m$.*

Remark 3. *Although we have proved that a local minimizer of the exact penalty function optimization problem (P_{σ_k}) will converge to a local minimizer of the original problem (P), we need, in actual computation, set a lower bound $\epsilon^* = 10^{-9}$ for $\epsilon^{(k),*}$ so as to avoid the situation of being divided by $\epsilon^{(k),*} = 0$, leading to infinity.*

Example 1 The following example is taken from *Gonzaga* (1980), and it was also used for testing the numerical algorithms in [20] [21] and [24]. In this problem, the objective function:

$$f(x) = \frac{x_2(122 + 17x_1 + 6x_3 - 5x_2 + x_1x_3) + 180x_3 - 36x_1 + 1224}{x_2(408 + 56x_1 - 50x_2 + 60x_3 + 10x_1x_3 - 2x_1^2)} \tag{3.1}$$

is to be minimized subject to

$$\phi(x,\omega) \leq 0 \ , \ \forall \ \omega \in \Omega \tag{3.2}$$

$$0 \leq x_1, x_3 \leq 100, \ 0.1 \leq x_2 \leq 100 \tag{3.3}$$

where $\Omega = [10^{-6}, 30]$ and $(i = \sqrt{-1})$, while

$$\phi(x,\omega) = \Im T(x,\omega) - 3.33[\Re T(x,\omega)]^2 + 1.0$$

$$T(x,\omega) = 1 + H(x,i\omega)G(i\omega)$$

$$H(x,s) = x_1 + x_2/s + x_3 s$$

$$G(s) = \frac{1}{(s+3)(s^2+2s+2)}$$

Here, $\Im T(x,\omega)$ and $\Re T(x,\omega)$ are, respectively, the imaginary and real parts of $T(x,\omega)$. The initial point is (50 50 50). Actually, we can start from any point within the boundedness constraints (3.3).

For the continuous inequality constraint (3.2), the corresponding exact penalty function $f_\sigma(x,\epsilon)$ is defined by (2.3) with the constraint violation $\Delta(x,\epsilon)$ given by

$$\Delta(x,\epsilon) = \int_\Omega \left[\max\left\{0, \Im T(x,\omega) - 3.33[\Re T(x,\omega)]^2 + 1.0 - \epsilon^\gamma W_j\right\}\right]^2 d\omega$$

Simpson's Rule with $\Omega = [10^{-6}, 30]$ being divided into 3000 equal subintervals is used to evaluate the integral. The value obtained is highly accurate. Also, these discretized points define a dense subset $\hat{\Omega}$ of Ω. We check the feasibility of the continuous inequality constraint by evaluating the values of the function ϕ over $\hat{\Omega}$. Results obtained are given in Table 1 and Table 2.

σ	$\bar{\omega}$	g	f
10	5.35	1.7599e-005	0.178251096
10^2	5.64	8.2356e-006	0.174782133
10^3	5.63	-2.0612e-005	0.174778004

Table 1: Result for Example 1

σ	x_1	x_2	x_3	ϵ
10	21.796685	49.5750243	31.7018582	0.000264
10^2	17.3494249	48.9435269	34.5556544	0.0001
10^3	17.3937883	48.7713471	34.5227014	0.00001

Table 2: Result for Example 1

As we can see, as the penalty parameter, σ, is increased, the minimizer approaches to the boundary of the feasible region. When σ is sufficiently large, we obtain a feasible point. It has the same objective function value as that obtained in [21]. However, for the minimizer obtained in [21], there are some minor violations of the continuous inequality constraints (3.2).

Example 2 Consider the problem:

$$\begin{array}{ll} \min & x_1^2 + (x_2 - 3)^2 \\ \text{subject to} & x_2 - 2 + x_1 \sin(\frac{t}{x_2 - \omega}) \le 0, \;\; \forall\, t \in [0, \pi] \\ & -1 \le x_1 \le 1, \;\; 0 \le x_2 \le 2 \end{array}$$

where ω is a parameter which controls the frequency of the constraint. As in [21], ω is chosen as 2.032.

In this case, the corresponding exact penalty function $f_\sigma(x, \epsilon)$ is defined by (2.3) with the constraint violation given by

$$\Delta(x, \epsilon) = \int_0^\pi \left[\max \left\{ 0, x_2 - 2 + x_1 \sin(\frac{t}{x_2 - \omega}) - \epsilon^\gamma W_j \right\} \right]^2 dt$$

Simpson's Rule with interval $[0, \pi]$ being divided into 1000 equal subintervals is used to evaluate the integral. These discretized points also form a dense subset $\hat{\Omega}$ of the interval $[0, \pi]$. The feasibility check is carried over $\hat{\Omega}$. By using Algorithm 1 with the initial point taken as (x_1^0, x_2^0), the solution obtained is $(x_1^*, x_2^*) = (0, 2)$ with the objective function value $f^* = 1$. The results are presented in Table 3 and Table 4.

σ	$\bar{\omega}$	g	f
10	1.41	3.735773915e-008	1.000000669
10^2	1.41	3.735773916e-008	1.000006691
10^3	1.41	3.735773916e-008	1.00006691
10^4	1.41	3.735773916e-008	1.000669101
10^5	1.049	2.45667159e-007	1.000011501

Table 3: Result for Example 2

σ	x_1	x_2	ϵ
10	3.735773981e-008	2.0000	5.481e-004
10^2	3.735773981e-008	2.0000	5.481e-004
10^3	3.735773981e-008	2.0000	5.481e-004
10^4	3.735773981e-008	2.0000	5.481e-004
10^5	-5.504846644e-006	1.9999	10^{-7}

Table 4: Result for Example 2

It is observed that for sufficiently large σ, the minimizer obtained is such that the continuous inequality constraints are satisfied for all $t \in [0, \pi]$.

Example 3 Consider the problem:

$$\begin{array}{ll} \min & (x_1 + x_2 - 2)^2 + (x_1 - x_2)^2 + 30[\min\{0, x_1 - x_2\}]^2 \\ \text{subject to} & x_1 \cos t + x_2 \sin t - 1 \leq 0, \ \forall\, t \in [0, \pi] \end{array}$$

Again, *Simpson's Rule* with the interval $[0, \pi]$ being partitioned into 1000 equal subintervals is used to evaluate the corresponding constraint violation in the exact penalty function. These discretized points also define a dense subset $\hat{\Omega}$ of the interval $[0, \pi]$, which is to be used for checking the feasibility of the continuous inequality constraint. Now, by using Algorithm 1 with the initial point taken as $[0.5, 0.5]$, the result obtained are reported in Table 5 and Table 6.

σ	$\bar{\omega}$	g	f
10	0.786	0.02497208416	0.3292584852
10^2	0.786	0.00400356933	0.3409679661
10^3	0.78	-0.00029665527	0.3437506884
10^4	0.78	-0.00000024678	0.3432592109

Table 5: Result for Example 3

σ	x_1	x_2	ϵ
10	0.7247764975	0.7247530305	0.04447211922
10^2	0.7100525572	0.7098229283	0.006961707112
10^3	0.7113565666	0.7024091525	0.000000009999
10^4	0.7115629913	0.7026219620	0.00000000100

Table 6: Result for Example 3

By comparing our results with those obtained in [4, 20, 21, 8], it is observed that the objective values are almost the same. However, for our minimizer, it is a feasible point while those obtained in [4, 20, 21, 8] are not.

4 Conclusions

In this paper, a new exact penalty method is proposed for solving an optimization problem with continuous inequality constraints. Compared with the existing schemes, our algorithm can be classified as an outer approximation method as the optimal solution is approached from outside to the feasible region. Thus, there is no need to find an interior point to start with. Furthermore, our method is based on exact penalty function, so the penalty parameter σ doesn't need to go to ∞. Another very important properties of this method is that all the minimizers obtained are feasible. Numerical testing shows that the proposed exact penalty method is effective when compared with other existing methods.

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E-mail address: **yuchangjun@126.com**
E-mail address: **k.l.teo@curtin.edu.au**
E-mail address: **zhangls@staff.shu.edu.cn**
E-mail address: **yqbai@shu.edu.cn**