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OPTIMAL INVESTMENT-CONSUMPTION PROBLEM WITH CONSTRAINT

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Abstract. In this paper, we consider an optimal investment-consumption problem subject to a closed convex constraint. In the problem, a constraint is imposed on both the investment and the consumption strategy, rather than just on the investment. The existence of solution is established by using the Martingale technique and convex duality. In addition to investment, our technique embeds also the consumption into a family of fictitious markets. However, with the addition of consumption, it leads to nonreflexive dual spaces. This difficulty is overcome by employing the so-called technique of "relaxation-projection" to establish the existence of solution to the problem. Furthermore, if the solution to the dual problem is obtained, then the solution to the primal problem can be found by using the characterization of the solution. An illustrative example is given with a dynamic risk constraint to demonstrate the method.

1. **Introduction.** The continuous-time consumption-portfolio optimization problem was pioneered by Merton [18, 19], where the dynamic programming approach is used. The solution of a nonlinear partial differential equation is constructed, and it is then verified that the solution is the value function for the original optimization problem. Cox and Huang [4, 5], Karatzas et al.[12] and Pliska [21] developed an alternative approach, which is known as the Martingale approach, to solve the

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continuous-time optimization problem. A clear advantage of the Martingale approach over the dynamic programming approach is that it gives rise to only linear partial differential equations, rather than the nonlinear partial differential equation when the dynamic programming approach is used. Another major advantage is that it is not required to assume that the wealth should be governed by Markovian dynamics. This assumption is needed when the dynamic programming approach is used.

The Martingale method was initially applied to optimal investment/consumption problems under the assumption of a complete market [1], meaning that the family of Martingale measures is a singleton. With the help of the Girsanov Theorem (see Theorem 5.2.12 in [1]), the original probability measure can be transformed into an equivalent Martingale measure under which all the stock prices discounted by the bond rate become martingales. Its proof is based on the fact that every martingale relative to a Brownian filtration can be represented by a stochastic integral with respect to the underlying Brownian motion. However, difficulties arise in the case of incomplete markets. Fortunately, the notion of equivalent Martingale introduced by Harrison and Kreps [8], Harrison and Pliska [9] and Ross [22] has opened up the possibility of solving such problems by $convex - duality$ methods. A distinctive feature of this approach is that it relates the original stochastic optimal control problem (the primal problem) to a "dual" problem such that a solution to the primal problem induces a solution to the dual problem (and vice versa). This duality dated back to Bismut [2], and it has since been exploited by many researchers, such as in [10, 11, 13], and more recently by Kramkov and Schachermayer [14]. They related the marginal utility from the terminal wealth of the optimal portfolio to the density of the Martingale measure, using powerful $convex - duality$ techniques. In particular, the minimal conditions on the agent's utility function and the financial market model are discussed by Kramkov and Schachermayer [14]. Since then, stochastic duality theory has become very successful as a method for solving portfolio selection problems. A common theme of all these papers is to take the original problem, which involves a maximization over a class of policies, and restate it in the form of the dual problem, which involves a minimization over the constructed measures. The dual problem is easier to solve than the primal problem. The convexity properties of the primal problem are critical in establishing the connection between this problem and the corresponding dual problem. Based on this connection, the solution to the primal problem can be constructed by using the solution to the dual problem.

The works mentioned above dealt with the application of the Martingale approach and convex duality to problems in which there are no portfolio constraints, that is, at every instant the investor can freely distribute the wealth among all of the assets. In reality, there are many situations where the portfolios are restricted in some way. For example, the holding of the money-market account should never be below some fixed value (see Karatzas et al. [13]), or there is a convex constraint with the strategy (see Cvitanic and Karatzas [7]). Their solution method involves a

[1]Incomplete markets in correspond to a setting which the investor has full information about many aspects of the the market, but various exogenously constraints (taxation, transaction costs, bad credit rating, legislature etc.) prevent him/her from choosing the portfolio outside a given constraint set. In fact, even without government-imposed portfolio constraints, financial markets will typically not offer tradable assets corresponding to certain sources of uncertainty (weather conditions, non-listed companies, etc.) The financial agent will still observe many of these sources, as their uncertainty evolves, but will typically not be able to trade in all of them.

completion of the incomplete market. This is called a *fictitious completion*, since the market is completed with fictitious stocks. The fictitious stocks are carefully chosen so that the optimal portfolio has no constraint on the investment of these fictitious stocks. The optimal portfolio process in the fictitious market will then provide a potential solution to the original, incomplete market. The optimal solution to the original incomplete market is then the optimal portfolio process which minimizes the expected utility of the terminal wealth.

In the literature, if the portfolio constraint is imposed on the investment strategy alone, exact solutions can be constructed [3, 17] in some special cases. For more general situations, Pirvu [20], Liu et al.[16] and Yiu [23, 24] considered the optimal portfolio problem with risk constraint, which is imposed on the whole investment-consumption portfolio rather than just on the investment strategy. Under the assumption that there is a smooth solution to the associated HJB equation, numerical methods are developed. However, no result on the existence of solution was reported. Motivated by this, we aim to use the Martingale approach and convex duality to investigate the existence of the optimal investment and consumption strategy with constraint imposed on the whole investment-consumption portfolio. This is different from [7] who considered mainly constraints on investment. In [7], the investment is embedded into a family of fictitious markets without constraint, where the family of fictitious markets are characterized by the elements in a convex subspace of a Hilbert space. Then the problem is transformed into solving the minimization problem in this subspace. In this paper, our method embed not only investment but also consumption into the fictitious market. In particular, we construct a mapping for the consumption from the original market to the fictitious market. However, if we simply consider the dual objective function with the subspace of the Hilbert Space, the dual objective function does not satisfy the coercive condition due to the embedding of the consumption, yet the coercive condition is needed in establishing the existence of the solutions to the dual problem. In view of this, we consider the parameter set characterizing the fictitious markets in L^1. Although the L^1 space leads to a non reflexive dual space to work with, we can make use of the so-called technique of "relaxation-projection" [15] to tackle it.

The rest of the paper is organized as follows. In Section 2, we present the model and formulation of the optimal portfolio problem. The original constrained problem is embedded into a family of markets without constraint, which is the primal problem. In Section 3, we investigate the properties of the fictitious market so that the optimal strategy will coincide with that in the original market. If such a market exists, then we deal with this problem by using the Martingale approach. The dual optimal problem, which aims to find such a market, is investigated in Section 4. Then, the existence of the optimal investment-consumption strategy is established via solving this dual problem. Finally, some discussions and an example with an investment-consumption constraint are given in Section 5. For logarithmic utility function, we derive the optimal solution from both primal and dual problems.

2. Model and problem.

Suppose that an agent is allowed to invest its surplus in a financial market consisting of a risk-free asset (bond) and d risky assets (stocks). Specifically, the price process of the risk-free asset is given by

$$dP^0(t) = rP^0(t)dt, \qquad r > 0,$$

and the price processes of the risky assets evolve according to the system of the stochastic differential equations given below,

$$dP_i(t) = P_i(t)(\mu_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_j(t)), \qquad \mu_i > r, \qquad i = 1, \cdots, d,$$

where $W(t) = (W_1(t), ..., W_d(t))^\top$ is a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathrm{P})$, $\mu(t) = (\mu_1(t), \cdots, \mu_d(t))^\top$ are the appreciation rates for the risky assets, and the volatility matrix $\sigma(t) = \{\sigma_{i,j}(t)\}_{1\le i,j\le d}$ is invertible. Throughout the paper, the supperscript "$\top$" denotes the transpose of a vector or a matrix. Let $\{\mathcal{F}_t\}_{t\ge 0}$ be the P-augmentation of the filtration $\sigma(W(s), 0 \le s \le t)$.

Let $\pi(t) = (\pi_1(t), \pi_2(t), ..., \pi_d(t))^\top$ denote the proportional risky investments and let $c(t)$ denote the consumption with a proportional rate. Besides the trading strategy, we regard consumption as proportion of wealth, rather than the dollar amounts, so as to avoid the situation of bankruptcy. A strategy $(\pi(t), c(t))$ is called *admissible* if $(\pi(t), c(t))$ is $\mathcal{F}_t$ progressively measurable.

Let $X^{\pi,c}(t)$ denote the wealth process corresponding to $(\pi(t), c(t))$. It evolves according to

$$\begin{cases} \frac{dX^{\pi,c}(t)}{X^{\pi,c}(t)} &= (r + \pi^\top(t)(\mu(t) - r(t)\mathbf{1}) - c(t))dt + \pi^\top(t)\sigma(t)dW(t) \\ &= (r - c(t))dt + \pi^\top(t)\sigma(t)dW_0(t), \\ X(0) &= x, \end{cases} \tag{2.1}$$

where $\mathbf{1} = (\underbrace{1, 1, ..., 1}_{d})^\top, W_0(t) = W(t) + \int_0^t \sigma^{-1}(s)(\mu(s) - r(s)\mathbf{1})ds.$

In practice, there often exist constraints on the strategy, such as the constraint on no short selling, the constraint on no borrowing, the risk constraint on $(\pi(\cdot), c(\cdot))$ (see [23, 16]). Suppose that the strategy $(\pi(\cdot), c(\cdot))$ is confined to a convex set B at time t, denoted by

$$\mathcal{A}_x := \{\text{admissible } (\pi, c),\ (\pi(t), c(t)) \in B\}.$$

Let $U_1(\cdot) : (0, \infty) \to \mathbb{R}$ and $U_2(\cdot) : (0, \infty) \to \mathbb{R}$ be both strictly increasing, concave functions satisfying

$$\begin{aligned} &U_1'(0_+) = \infty, \qquad U_2'(0_+) = \infty, \\ &U_1'(\infty) = 0,\ U_2'(\infty) = 0, \end{aligned} \tag{2.2}$$

where " $'$ " denotes the derivative of a function. Furthermore, it is assumed that $U_1'(x)$ and $U_2'(x)$ are non-decreasing on $\mathbb{R}_+ = (0, \infty)$, and that for any $\alpha \in (0, 1)$, there exists a $\beta \in (1, \infty)$ such that

$$\alpha U_i'(x) \ge U_i'(\beta x),\ i = 1, 2, \qquad \forall\ x\ \in (0, \infty), \tag{2.3}$$

which are, respectively, equivalent to

$$I_i(\alpha y) \le \beta I_i(y),\ i = 1, 2, \qquad \forall\ y\ \in (0, \infty), \tag{2.4}$$

where for each "i=1,2", the function $I_i : \mathbb{R}_+ \to \mathbb{R}_+$, denotes the inverse of $U_i'(\cdot)$. This assumption is for later use. Moreover, we suppose that $U_1(\cdot)$ and $U_2(\cdot)$ satisfy (2.4) with the same constants.

Define the utility function

$$J(x, \pi, c) \quad = \quad E\int_0^T U_1(c(t)X^{\pi,c}(t))dt + U_2(X^{\pi,c}(T)), \qquad (\pi, c) \in \mathcal{A}_x. \tag{2.5}$$

The objective is to maximize the utility function, yielding

$$V(x) = \sup_{(\pi,c)\in\mathcal{A}_x} J(x,\pi,c). \tag{2.6}$$

This problem is the *initial* problem we consider in this work. Without constraint, the Martingale approach is a widely used approach to this optimal control problem. The key idea is to construct the optimal strategy with Martingale representation. Here, when the strategy is constrained within certain convex set, the technique we use is to embed the strategy into a set of fictitious markets and then construct a new utility function without constraint in this market. We will show that the optimal strategy in a certain class of fictitious markets is optimal to the original one.

2.1. **The embedding (primal) problem.** Let $\gamma := \gamma(t) = (\gamma_1(t), ..., \gamma_d(t))^\top \in \mathbb{R}^d$ and $\widetilde{\gamma} := \widetilde{\gamma}(t) = (\gamma_1(t), ..., \gamma_d(t), \gamma_{d+1}(t))^\top \in \mathbb{R}^{d+1}$ be $\mathcal{F}_t$-progressively measurable process. The 1-norm and 2-norm of $\gamma, \tilde{\gamma}$ and $\gamma(t), \tilde{\gamma}(t)$ are defined as follows:

$$\|\widetilde{\gamma}\|_1 = E\int_0^T \sum_{i=1}^{d+1} |\gamma_i(t)|dt, \ \|\widetilde{\gamma}(t)\|_1 = \sum_{i=1}^{d+1} |\gamma_i(t)|, \tag{2.7}$$

$$\|\gamma\|_1 = E\int_0^T \sum_{i=1}^{d} |\gamma_i(t)|dt, \ \|\gamma(t)\|_1 = \sum_{i=1}^{d} |\gamma_i(t)|, \tag{2.8}$$

$$\|\gamma\|_2 = (E\int_0^T \sum_{i=1}^{d} |\gamma_i(t)|^2 dt)^{\frac{1}{2}}, \ \|\gamma(t)\|_2 = (\sum_{i=1}^{d} |\gamma_i(t)|^2)^{\frac{1}{2}}. \tag{2.9}$$

Denote

$$\begin{aligned} &\delta(\widetilde{\gamma}) \equiv \delta(\widetilde{\gamma} \mid B) := \sup_{b\in B}(-b\cdot\widetilde{\gamma}) := \sup_{(\pi,c)\in B} -(\pi^\top\gamma + c\gamma_{d+1}), \\ &\tilde{B} = \{\widetilde{\gamma} : \ \delta(\widetilde{\gamma}) < \infty\}, \qquad \widetilde{\gamma} \in \mathbb{R}^{d+1}. \end{aligned} \tag{2.10}$$

It is assumed that $\delta(\cdot|B)$ is continuous on $\widetilde{B}$ and bounded below on $\mathcal{R}^{d+1}$ by δ_0. Define

$$\begin{aligned} \mathcal{H} &= \{\widetilde{\gamma} : \ \|\widetilde{\gamma}\|_1 < \infty, \ \widetilde{\gamma}(t) \text{ is progressively measurable with respect to } \mathcal{F}_t\}, \\ \mathcal{C} &= \{\widetilde{\gamma} \in \mathcal{H} : \gamma_{d+1}(t) < 1, \widetilde{\gamma}(t,\omega) \in \tilde{B}, \ \textit{a.e. on } [0,T]\times\Omega\}, \\ \mathcal{D} &= \{\widetilde{\gamma} \in \mathcal{C} : \ E\int_0^T \delta(\widetilde{\gamma}(t))dt < \infty\}. \end{aligned} \tag{2.11}$$

We introduce a set of fictitious markets $\mathcal{M}^{\widetilde{\gamma}}$, $\widetilde{\gamma} \in \mathcal{H}$, below. In the market $\mathcal{M}^{\widetilde{\gamma}}$, the dynamics of risk free and risky assets evolve as follows:

$$\begin{cases} dP_0^{\widetilde{\gamma}}(t) = (r(t) + \delta(\widetilde{\gamma}(t)))P_0^{\widetilde{\gamma}}(t)dt, \\ dP_i^{\widetilde{\gamma}}(t) = P_i^{\widetilde{\gamma}}(t)(\mu_i(t) + \delta(\widetilde{\gamma}(t)) + \gamma_i(t))dt + \sum_{j=1}^d \sigma_{ij}(t)dW_j(t)). \end{cases}$$

To continue, we need the following notations.

For each $\widetilde{\gamma} \in \mathcal{H}$ and $\mathcal{F}_t$ progressively measurable (π, c), let $X_{\widetilde{\gamma}}^{\pi,c}(t)$ denote the wealth in $\mathcal{M}^{\widetilde{\gamma}}$, corresponding to (π, c). Now we construct a project from $X_{\widetilde{\gamma}}^{\pi,c}$ in $\mathcal{M}^{\widetilde{\gamma}}$ to $X^{\pi,\frac{c}{1-\gamma_{d+1}}}$ in the original market. That is, the strategy (π, c) in $\mathcal{M}^{\widetilde{\gamma}}$ is corresponding to $(\pi, \frac{c}{1-\gamma_{d+1}})$ in the original market, or equivalently, the strategy (π, c) in the original market is corresponding to $(\pi, (1-\gamma_{d+1})c)$ in $\mathcal{M}^{\widetilde{\gamma}}$.

In the market $\mathcal{M}^{\widetilde{\gamma}}$, corresponding to (π, c) in the original market, the dynamics of $X^{\pi,c}_{\widetilde{\gamma}}(t)$ evolves according to

$$\frac{dX^{\pi,c}_{\widetilde{\gamma}}(t)}{X^{\pi,c}_{\widetilde{\gamma}}(t)} = (r(t) - c(t))dt + (\delta(\widetilde{\gamma}(t)) + \pi^\top(t)\gamma(t) + \gamma_{d+1}c(t))dt + \pi^\top(t)\sigma(t)dW_0(t). \tag{2.12}$$

Let

$$R_{\widetilde{\gamma}}(t) = \exp\{-\int_0^t [r(s) + \delta(\widetilde{\gamma}(s))]ds\},$$
$$M_{\widetilde{\gamma}}(t) = \exp\{-\int_0^t \theta_{\widetilde{\gamma}}(s)dW(s) - \frac{1}{2}\int_0^t \|\theta_{\widetilde{\gamma}}(s)\|_2^2 ds\} := \exp\{-\zeta_{\widetilde{\gamma}}(t)\}, \tag{2.13}$$
$$H_{\widetilde{\gamma}}(t) = R_{\widetilde{\gamma}}(t)M_{\widetilde{\gamma}}(t), \tag{2.14}$$

where

$$\theta_{\widetilde{\gamma}}(t) = \sigma^{-1}(t)(\mu(t) - r(t)\mathbf{1}) + \sigma^{-1}(t)\gamma(t). \tag{2.15}$$

Then, applying Ito Lemma leads to

$$H_{\widetilde{\gamma}}(t)X^{\pi,c}_{\widetilde{\gamma}}(t) + \int_0^t H_{\widetilde{\gamma}}(s)c(s)X^{\pi,c}_{\widetilde{\gamma}}(s)ds = x + \int_0^t H_{\widetilde{\gamma}}(s)(\pi^\top(s)\sigma(s) - \theta^\top_{\widetilde{\gamma}}(s))dW(s). \tag{2.16}$$

Now we define a new utility function in $\mathcal{M}^{\widetilde{\gamma}}$,

$$J_{\widetilde{\gamma}}(x, \pi, c) = E\int_0^T U_1(c(t)X^{\pi,(1-\gamma_{d+1})c}_{\widetilde{\gamma}}(t))dt + U_2(X^{\pi,(1-\gamma_{d+1})c}_{\widetilde{\gamma}}(T)), \tag{2.17}$$

and let

$$V_{\widetilde{\gamma}}(x) = \sup_{(\pi,c)} J_{\widetilde{\gamma}}(x, \pi, c). \tag{2.18}$$

Remark 1. (a) If $(\pi(t), c(t)) \in \mathcal{A}_x$, by comparing (2.1) with (2.12), we have

$$X^{\pi,c}(t) \le X^{\pi,(1-\gamma_{d+1})c}_{\widetilde{\gamma}}(t). \tag{2.19}$$

Thus, it follows from (2.18) that

$$V(x) \le V_{\widetilde{\gamma}}(x). \tag{2.20}$$

(b) Suppose that the following conditions are satisfied.
 (i) $(\pi(t), c(t)) \in \mathcal{A}_x$, and
 (ii) $\delta(\widetilde{\gamma}(t)) + \gamma(t)\pi(t) + \gamma_{d+1}(t)c(t) = 0$.
Then, it follows from (2.12) that $X^{\pi,(1-\gamma_{d+1})c}_{\widetilde{\gamma}}$ is equal to $X^{\pi,c}$ in the initial market. Thus,

$$J_{\widetilde{\gamma}}(x, \pi(t), c(t)) = J(x, \pi(t), c(t)) \le V(x). \tag{2.21}$$

(c) In particular, if $V_{\widetilde{\gamma}}(x) = J_{\widetilde{\gamma}}(x, \pi(t), c(t))$ and $(\pi(t), c(t))$ are such that the conditions stated in (b) are satisfied, then

$$V(x) \ge V_{\widetilde{\gamma}}(x), \tag{2.22}$$

which, together with (2.20), leads to

$$V(x) = V_{\widetilde{\gamma}}(x). \tag{2.23}$$

By this remark, the problem with constraint can be reduced to the optimal control problem (2.18) if there exists a $\widetilde{\gamma} \in \mathcal{H}$ such that the conditions specified in (b) of Remark 1 are satisfied.

In the following section, we will investigate the conditions specified in (b) of Remark 1.

3. **The optimal portfolio in a fictitious market.** In Lemma 3.1, we shall show that the conditions specified in (b) of Remark 1 are satisfied. Lemma 3.3 verifies the validity of the conditions specified in (c) (equivalently, Problem (2.18)). The main results are given in Theorem 3.4.

Lemma 3.1. *For any $(C(t), \xi)$, where $C(t) \geq 0$ and $\xi > 0$ a.e., if there exists a $\widetilde{\lambda} \in \mathcal{D}$ such that for any $\tilde{\gamma} \in \mathcal{H}$,*

$$\begin{aligned} & E\left[H_{\widetilde{\gamma}}(T)\xi + \int_0^T H_{\widetilde{\gamma}}(t)(1-\gamma_{d+1}(t))C(t)dt\right] \leq \\ & E\left[H_{\widetilde{\lambda}}(T)\xi + \int_0^T H_{\widetilde{\lambda}}(t)(1-\lambda_{d+1}(t))C(t)dt\right] = x, \end{aligned} \tag{3.1}$$

then there exists $(\pi, c) \in \mathcal{A}_x$ such that

$$X^{\pi,c}(T) = \xi, \;\; c(t)X^{\pi,c}(t) = C(t), \tag{3.2}$$

and

$$\delta(\widetilde{\lambda}(t)) + \lambda(t)\pi(t) + \lambda_{d+1}(t)c(t) = 0. \tag{3.3}$$

The proof is given in Appendix.

The next lemma is needed in the proof of Lemma 3.3.

Lemma 3.2. *For any $(C(t), \xi)(C(t) \geq 0, \xi > 0$ a.e.$)$ and the fictitious market $\mathcal{M}^{\tilde{\gamma}}$, there exists a $\mathcal{F}_t$ progressively measurable $(\pi_{\tilde{\gamma}}, c_{\tilde{\gamma}})$ and $X^{\pi_{\tilde{\gamma}}, c_{\tilde{\gamma}}}(t)$ such that*

$$(X^{\pi_{\tilde{\gamma}}, c_{\tilde{\gamma}}}(T), c(t)X^{\pi_{\tilde{\gamma}}, c_{\tilde{\gamma}}}(t)) = (\xi, C(t)).$$

Proof. Denote

$$X_{\widetilde{\gamma}}(t) \;\; := \;\; \frac{1}{H_{\widetilde{\gamma}}(t)} E[\int_t^T H_{\widetilde{\gamma}}(s)C(s)ds + H_{\widetilde{\gamma}}(T)\xi|\mathcal{F}_t] \tag{3.4}$$

and

$$\begin{aligned} M_0(t) \;\; &:= \;\; E[\int_0^T H_{\widetilde{\gamma}}(s)C(s)ds + H_{\widetilde{\gamma}}(T)\xi|\mathcal{F}_t] \\ &= \;\; H_{\widetilde{\gamma}}(t)X_{\widetilde{\gamma}}(t) + \int_0^t H_{\widetilde{\gamma}}(s)C(s)ds. \end{aligned} \tag{3.5}$$

Obviously, $M_0(t)$ is a $\mathcal{F}_t$-Martingale. From the Martingale representation theory, we have

$$M_0(t) \;\; = \;\; x + E\int_0^t \psi^*(s)dW(s), \tag{3.6}$$

where $\psi^*(s)$ is $\mathcal{F}_t$ progressively measurable and satisfying $E\int_0^T \|\psi^*(s)\|_2^2 ds < \infty$. As $X_{\tilde{\gamma}}(0) = x$ and $X_{\tilde{\gamma}}(T) = \xi$, it follows from (3.5), (3.6) and (2.16) that

$$\begin{aligned} X_{\tilde{\gamma}}(T) &= \xi, \\ \psi^*(s) &= H_{\tilde{\gamma}}(s)(\sigma(s)\pi(s) - \theta_{\tilde{\gamma}}(s))^\top. \end{aligned} \tag{3.7}$$

Let π that satisfies (3.7) be referred as $\pi_{\tilde{\gamma}}$.
Define

$$c_{\tilde{\gamma}}(t) = \frac{C(t)}{X_{\tilde{\gamma}}(t)}, \tag{3.8}$$

which makes sense as, from (3.4), $X_{\tilde{\gamma}}(t) > 0$, $t \in [0, T]$, holds a.e in the market $\mathcal{M}^{\tilde{\gamma}}$. The construction above shows that $X_{\tilde{\gamma}}(t)$ is corresponding to (π_γ, c_γ), namely, $X_{\tilde{\gamma}}(t) = X^{\pi_{\tilde{\gamma}}, c_{\tilde{\gamma}}}(t)$. □

Sometimes we also write $J_{\tilde{\gamma}}(x, \pi, c)$ as $J_{\tilde{\gamma}}(x, C(t), \xi)$ if (π, c) is the strategy corresponding to $(C(t), \xi)$ in the market $\mathcal{M}_{\tilde{\gamma}}$.

Lemma 3.3. *Denote*

$$\begin{aligned} C^*_{\tilde{\gamma}}(t) &:= I_1((1-\gamma_{d+1}(t))y_{\tilde{\gamma}}H_{\tilde{\gamma}}(t)), \\ \xi^*_{\tilde{\gamma}}(T) &:= I_2(y_{\tilde{\gamma}}H_{\tilde{\gamma}}(T)), \end{aligned} \tag{3.9}$$

where $y_{\tilde{\gamma}} > 0$ satisfies

$$E\left[\int_0^T I_1((1-\gamma_{d+1}(t))H_{\tilde{\gamma}}(t)y_{\tilde{\gamma}})(1-\gamma_{d+1}(t))H_{\tilde{\gamma}}(t)dt + I_2(H_{\tilde{\gamma}}(T)y_{\tilde{\gamma}})H_{\tilde{\gamma}}(T)\right] = x.$$

Then,

$$V_{\tilde{\gamma}}(x) = E\int_0^T U_1(\frac{C^*_{\tilde{\gamma}}(t)}{1-\gamma_{d+1}(t)})dt + U_2(\xi^*_{\tilde{\gamma}}(T)). \tag{3.10}$$

Proof. From Lemma 3.2, there exists a $(\pi^*_{\tilde{\gamma}}(t), c^*_{\tilde{\gamma}}(t)) \in \mathcal{A}_x$ such that

$$c^*_{\tilde{\gamma}}(t)X^{\pi^*_{\tilde{\gamma}}, c^*_{\tilde{\gamma}}}_{\tilde{\gamma}} = C^*_{\tilde{\gamma}}(t) \text{ and } X^{\pi^*_{\tilde{\gamma}}, c^*_{\tilde{\gamma}}}_{\tilde{\gamma}}(T) = \xi^*_{\tilde{\gamma}}(T).$$

Thus, for any $\mathcal{F}_t$-progressively measurable $(\pi(t), c^{\tilde{\gamma}}(t))$,

$$\begin{aligned} & E[\int_0^T U_1(\frac{c^*_{\tilde{\gamma}}}{1-\gamma_{d+1}(t)}X^{\pi^*_{\tilde{\gamma}}, c^*_{\tilde{\gamma}}}_{\tilde{\gamma}})dt + U_2(\xi^*_{\tilde{\gamma}}(T)) \\ & -E\int_0^T U_1(\frac{c(t)}{1-\gamma_{d+1}(t)}X^{\pi,c}_{\tilde{\gamma}}(t))dt + U_2(\xi^{\pi,c}(T))] \\ & \geq E[\int_0^T (1-\gamma_{d+1}(t))y_{\tilde{\gamma}}H_{\tilde{\gamma}}(t)(\frac{C^*_{\tilde{\gamma}}(t)}{1-\gamma_{d+1}(t)} - \frac{c(t)}{1-\gamma_{d+1}(t)}X^{\pi,c}_{\tilde{\gamma}}(t))dt \\ & +y_{\tilde{\gamma}}H_{\tilde{\gamma}}(t)(\xi^*_{\tilde{\gamma}}(T) - X^{\pi,c}_{\tilde{\gamma}}(T))] \\ & \geq y_{\tilde{\lambda}}(x - E\int_0^T (1-\gamma_{d+1}(t))H_{\tilde{\gamma}}(t)\frac{c(t)X^{\pi,c}_{\tilde{\gamma}}(t)}{1-\gamma_{d+1}(t)}dt + H_{\tilde{\gamma}}(T)X^{\pi,c}_{\tilde{\gamma}}(T)) \\ & \geq 0. \end{aligned} \tag{3.11}$$

Therefore,

$$V_{\tilde{\gamma}}(x) \leq E\int_0^T U_1(\frac{C^*_{\tilde{\gamma}}(t)}{1-\gamma_{d+1}(t)})dt + U_2(\xi^*_{\tilde{\gamma}}(T)). \tag{3.12}$$

By the construction of $(\pi^*_{\widetilde{\gamma}}(t), c^*_{\widetilde{\gamma}}(t))$, we have

$$V_{\widetilde{\gamma}}(x) = J_{\widetilde{\gamma}}(x, \pi^*_{\widetilde{\gamma}}, \frac{c^*_{\widetilde{\gamma}}}{1-\gamma_{d+1}}). \tag{3.13}$$

□

We now present the following theorem.

Theorem 3.4. *Suppose there exists a* $\widetilde{\lambda} \in \mathcal{D}$ *such that*

$$\begin{aligned} & E\Big[H_{\widetilde{\gamma}}(T)\xi^*_{\widetilde{\lambda}}(T) + \int_0^T H_{\widetilde{\gamma}}(t)(1-\gamma_{d+1}(t))C^*_{\widetilde{\lambda}}(t)dt\Big] \\ \leq\ & E\Big[H_{\widetilde{\lambda}}(T)\xi^*_{\widetilde{\lambda}}(T) + \int_0^T H_{\widetilde{\lambda}}(t)(1-\lambda_{d+1}(t))C^*_{\widetilde{\lambda}}(t)dt\Big] \\ =\ & x, \qquad \forall\ \widetilde{\gamma} \in \mathcal{D}. \end{aligned} \tag{3.14}$$

Then, $V(x) = V_{\widetilde{\lambda}}(x)$*. Furthermore,* $C^*_{\widetilde{\lambda}}(t)$ *and* $\xi^*_{\widetilde{\lambda}}(t)$ *are the optimal consumption and the final wealth, respectively.*

Proof. From Lemma 3.1, $(\pi^*_{\widetilde{\lambda}}(t), c^*_{\widetilde{\lambda}}(t)) \in \mathcal{A}_x$ satisfies (3.3), thus $X^{(\pi^*_{\widetilde{\lambda}}, c^*_{\widetilde{\lambda}})}_{\widetilde{\lambda}}(t)$ is in the initial market. It follows from Lemma 3.3 that

$$\begin{aligned} V_{\widetilde{\lambda}}(x) &= J_{\widetilde{\lambda}}(x, \pi^*_{\widetilde{\lambda}}(t), c^*_{\widetilde{\lambda}}(t)) \\ &\underline{\underline{(2.13)}}\ J(x, \pi^*_{\widetilde{\lambda}}(t), \frac{c^*_{\widetilde{\lambda}}(t)}{1-\lambda_{d+1}}) \\ &\leq V(x). \end{aligned} \tag{3.15}$$

Together with (2.20), we have $V_{\widetilde{\lambda}}(x) = V(x)$. Thus, it can be seen from (3.15) that $(\pi^*_{\widetilde{\lambda}}(t), \frac{c^*_{\widetilde{\lambda}}(t)}{1-\lambda_{d+1}(t)})$ is the optimal strategy. □

The main idea of Theorem 3.4 can be further explained as follows. Find a class of $\tilde{\lambda}$ and construct the solution in the market $\mathcal{M}^{\widetilde{\lambda}}$. Then, we verify that it is also optimal in the initial market. However, it is difficult to show the existence of such an $\widetilde{\lambda}$ and $\mathcal{M}^{\widetilde{\lambda}}$. The difficulty is simplified through the duality method to be presented in the following section.

4. The dual problem.

In this section, we introduce the dual problem with the market $\mathcal{M}^{\widetilde{\gamma}}$. We first show the existence of the optimal strategy. Then, the connection of the value function and the optimal solution between the primal and its dual problem will be established.

For $i = 1, 2$, the conjugate function $\widetilde{U}_i(y)(y > 0)$ of $U_i(x)$ is defined by

$$\widetilde{U}_i(y) := \max_x U_i(x) - xy = U_i(I(y)) - yI(y).$$

Clearly,

$$U_i(x) := \min_y \widetilde{U}_i(y) + xy = \widetilde{U}_i(U_i'(x)) + xU_i'(x).$$

Observe that $\widetilde{U}_i(y), i = 1, 2$, are strictly convex and decreasing in their domain. Denote

$$\begin{aligned}\bar{\mathcal{D}} &= \{\widetilde{\gamma} \in \mathcal{D}:\ E\int_0^T I_1((1-\gamma_{d+1}(t))H_{\tilde{\gamma}}(t)y)(1-\gamma_{d+1}(t))H_{\tilde{\gamma}}(t)dt \\ &\quad +I_2(t, H_{\tilde{\gamma}}(t)y)H_{\tilde{\gamma}}(t) < \infty, \forall\ y \in (0, +\infty)\}. \end{aligned} \tag{4.1}$$

Suppose that $U_1(\cdot)$ and $U_2(\cdot)$ satisfy (2.2) with the same constants α and β, it follows from [13] that

$$\begin{aligned}\{\widetilde{\gamma} \in \bar{\mathcal{D}}\} \quad \Leftrightarrow \quad &\{\widetilde{\gamma} \in \mathcal{D};\ E\int_0^T I_1((1-\gamma_{d+1}(t))H_{\tilde{\gamma}}(t)y)(1-\gamma_{d+1}(t))H_{\tilde{\gamma}}(t)dt \\ &+I_2(t, H_{\tilde{\gamma}}(t)y)H_{\tilde{\gamma}}(t) < \infty,\ \exists\ y \in (0, +\infty)\}.\end{aligned}$$

Let the function $\widetilde{J}(y, \widetilde{\gamma}) : \mathcal{H} \to \mathbb{R} \cup \infty$ be defined by

$$\widetilde{J}(y,\widetilde{\gamma}) := \begin{cases} E\Big[\int_0^T \widetilde{U}_1(y(1-\gamma_{d+1}(t))\exp\{-\int_0^t (r(s)+\delta(\widetilde{\gamma}(s)))ds - \zeta_\gamma(t)\})dt \\ +\widetilde{U}_2(y\exp\{-\int_0^T (r(t)+\delta(\widetilde{\gamma}(t)))dt - \zeta_\gamma(T)\})\Big],\ \widetilde{\gamma} \in \mathcal{C}, \\ \infty, \qquad\qquad \text{otherwise.} \end{cases}$$

The dual problem is defined by

$$\widetilde{V}(y) := \inf_{\tilde{\gamma}\in\mathcal{H}} \widetilde{J}(y, \tilde{\gamma}). \tag{4.2}$$

By duality, the optimal control problem is reduced to solving the optimal control problem with the parameter $\tilde{\gamma}(t, \omega)$ defined on the subset of $L^1([0, T] \times \mathcal{F})$. The main results of this work are presented in the following as a theorem.

Theorem 4.1. *(1) There exists a* $\widetilde{\lambda}_y \in \bar{\mathcal{D}}$*, such that*

$$\widetilde{V}(y) := \inf_{\gamma\in\bar{D}} \widetilde{J}(y, \tilde{\gamma}) = \widetilde{J}(y, \tilde{\lambda}_y). \tag{4.3}$$

(2) The value function $V(x)$ *and the dual value function* $\widetilde{V}(y)$ *form a conjugate pair; namely,*

$$\begin{aligned} U(x) &= \inf_y \widetilde{V}(y) + xy, \\ \widetilde{V}(y) &= \sup_x U(x) - xy. \end{aligned} \tag{4.4}$$

(3) For any given x, suppose that there exist y_x *and* $\widetilde{\lambda}_{y_x}$ *satisfying, respectively,*

$$\begin{aligned} \widetilde{V}(y_x) &= U(x) - xy_x, \\ \widetilde{V}(y_x) &= \widetilde{J}(y_x, \widetilde{\lambda}_{y_x}), \end{aligned}$$

then,

$$\begin{aligned} &E\int_0^T (1-\lambda_{y_x,d+1}(t))H_{\tilde{\lambda}_{y_x}}(t)I_1((1-\lambda_{y_x,d+1}(t))H_{\tilde{\lambda}_{y_x}}(t)y_x)dt \\ &+H_{\tilde{\lambda}_{y_x}}(T)I_2((1-\lambda_{y_x,d+1}(t))H_{\tilde{\lambda}_{y_x}}(T)y_x) = x, \end{aligned} \tag{4.5}$$

where $\lambda_{y_x,d+1}(t)$ *is the* $d+1$ *element of* $\tilde{\lambda}_{y_x}(t)$*.*

(4) $(C^*(t), X^*(T)) := (I_1(y_x H_{\tilde{\lambda}_{y_x}}(t)(1-\lambda_{y_x,d+1}(t))), I_2(y_x H_{\tilde{\lambda}_{y_x}}(T)(1-\lambda_{y_x,d+1}(T))))$

are, respectively, the optimal consumption and the final wealth process in the primal problem.

The proof will be given in following subsection.

Remark 2. Define $\tilde{\mathbf{0}} = (\underbrace{0, \cdots, 0}_{d+1})^\top$. From Cvitanic and Karatzas [7], it follows that $\widetilde{J}(y, \widetilde{\mathbf{0}}) < \infty$ for any $y > 0$, and hence (4.3) is well defined.

Remark 3. From Theorem 4.1, we see that the optimal control problem is to seek a $\widetilde{\lambda} \in \bar{\mathcal{D}}$ and $y_x \in \mathbb{R}_+$ such that

$$\widetilde{V}(y_x) = \widetilde{J}(y_x, \widetilde{\lambda}),$$

where y_x satisfies

$$E\Big[\int_0^T (1 - \lambda_{y_x,d+1}(t))H_{\tilde{\lambda}}(t)I_1((1 - \lambda_{y_x,d+1}(t))y_x H_{\tilde{\lambda}}(t))dt$$
$$+H_{\tilde{\lambda}}(T)I_2((1 - \lambda_{y_x,d+1}(t))y_x H_{\tilde{\lambda}}(T))\Big] = x.$$

4.1. **The proof of Theorem 4.1.**

4.1.1. *The proof of part* (1) *of Theorem 4.1.* As the dual space is not reflexive, the problem is often solved by using the so-called technique of "relaxation-projection" [15]. We adopt some technical notations and a lemma given in [6] as follows:
$\mathcal{F}$: $\sigma(\bigcup_{0\le t\le T} \mathcal{F}_t)$.
$\mathcal{P}_L$: The Lebesgue measure on [0,T].
$\mathcal{P}_L \times \mathcal{P}$: The unique measure on the measurable space $(T \times \Omega, \mathcal{B}[0,T] \times \mathcal{F})$ satisfying the property $(\mathcal{P}_L \times \mathcal{P})(A \times B) = \mathcal{P}_L(A) \times \mathcal{P}(B)$ for all $A \in \mathcal{B}[0,T], B \in \mathcal{F}$.
$\mathcal{L}^*$: The class of $\mathcal{P}_L \times \mathcal{P}$-null sets in $\mathcal{B}([0,T]) \times \mathcal{F}_T$.
$\mathcal{M}$: The σ-field generated by the $\mathcal{F}_t$- progressively measurable processes.
$\mathcal{M}^* = \sigma(\mathcal{M} \bigcup \mathcal{L}^*)$: The smallest σ-field containing $\mathcal{M}$ and $\mathcal{L}^*$.
$L^1(\mathcal{P}_L \times \mathcal{P}) = L^1([0,T] \times \Omega, \mathcal{M}^*, \mathcal{P}_L \times \mathcal{P})$: The set of $\mathcal{M}^*$-measurable integrable processes.

The following lemma was excerpted from [6].

Lemma 4.2. *(i)* $\mathcal{M}^* = \{A \in \mathcal{B}[0,T] \times \mathcal{F} : \exists\ B \in \mathcal{M}$ *such that* $A \vartriangle B \in \mathcal{L}^*\}$, *where the symbol* $A \vartriangle B$ *denotes the symmetric difference of* A *and* B, *i.e.,* $A \vartriangle B = (A \setminus B) \cup (B \setminus A)$.
(ii) Suppose that X*:* $[0,T] \times \Omega \to \mathbb{R}$ *is* $(\mathcal{B}[0,T] \times \mathcal{F})$*-measurable. Then,* X *is* $\mathcal{M}^*$*-measurable if and only if there exists a progressive process* Y *such that* $X = Y$, $\mathcal{P}_L \times \mathcal{P}$ *a.e. on* $[0,T] \times \Omega$.

Denote

$$\mathcal{C}_e = \{\widetilde{\gamma} : \widetilde{\gamma} \textit{ is } \mathcal{M}^* \textit{ measurable},\ \widetilde{\gamma}(t) \in \tilde{B}$$
$$\textit{and } \gamma_{d+1}(t) < 1\ (\mathcal{P}_L \times \mathcal{P}) - a.e. \textit{ on } [0,T] \times \Omega\};$$

$$
\begin{aligned}
\overline{\mathcal{C}}_e &= \{\widetilde{\gamma} : \widetilde{\gamma} \textit{ is } \mathcal{M}^* \textit{ measurable}, \ \widetilde{\gamma}(t) \in \tilde{B} \\
&\quad \textit{and } \gamma_{d+1}(t) \leq 1 \ (\mathcal{P}_L \times \mathcal{P}) - \textit{a.e. on } [0,T] \times \Omega\}; \\
\mathcal{D}_e &= \{\widetilde{\gamma} \in \overline{\mathcal{C}}_e : \ E\int_0^T \delta(\widetilde{\gamma}(t))dt < \infty\}; \\
\bar{\mathcal{D}}_e &= \{\widetilde{\gamma} \in \mathcal{D}_e : \ E\int_0^T I_1((1-\gamma_{d+1}(t))H_{\tilde{\gamma}}(t)y)(1-\gamma_{d+1}(t))H_{\tilde{\gamma}}(t)dt \\
&\quad + I_2(H_{\tilde{\gamma}}(T)y)H_{\tilde{\gamma}}(T) < \infty. \ \forall \ y \in (0,+\infty)\}.
\end{aligned} \tag{4.6}
$$

Remark 4. From the notations and Lemma 4.2, for any process $\tilde{\gamma}_1(t) \in \bar{\mathcal{D}}_e$, there exists a progressive measurable process $\tilde{\gamma}_2(t) \in \bar{\mathcal{D}}$ such that $\tilde{\gamma}_1 = \tilde{\gamma}_2$ $\mathcal{P}_L \times \mathcal{P}$ a.e. on $[0,T] \times \Omega$. Thus, we will look for $\widetilde{\gamma}$ in $\bar{\mathcal{D}}_e$ instead of $\bar{\mathcal{D}}$.

Now we consider the function $\widetilde{J}(y,\tilde{\gamma})$ defined by (4.2) with a larger domain $\mathcal{M}^*$,

$$
\widetilde{J}(y,\tilde{\gamma}) = \begin{cases} E[\int_0^T \widetilde{U}_1(y(1-\gamma_{d+1}(t))\exp\{\int_0^t(-r(s)-\delta(\tilde{\gamma}_e(s)))ds - \zeta_{\tilde{\gamma}}(t)\})dt \\ +\widetilde{U}_2(y\exp\{\int_0^T(-r(t)-\delta(\widetilde{\gamma}(t)))dt - \zeta_{\tilde{\gamma}}(T)\})], \ \tilde{\gamma} \in \mathcal{C}_e, \\ \infty, \qquad \text{otherwise.} \end{cases}
$$

and the dual problem of (2.18) is

$$
\widetilde{V}_e(y) := \inf_{\tilde{\gamma} \in \mathcal{M}^*} \widetilde{J}(y,\tilde{\gamma}). \tag{4.7}
$$

The next two lemmas will be needed in the proof of Theorem 4.1.

Lemma 4.3. *(Excerpted from Theorem 1 in [15]) Let $F : L^1(S,\Sigma,\mu) \to \mathbb{R} \cup +\infty$ be a convex function, where (S,Σ,μ) is a measure space with μ-finite and nonnegative and Σ complete. If F is lower semicontinuous in the topology of convergence in measure, then it attains a minimum on any convex set $K \in L^1(\mu)$ that is closed and norm-bounded.*

Lemma 4.4. *Suppose there exists $(0,...,0_d,k_{d+1}) \in B$, where $(k_{d+1} \in \mathbb{R}_+)$, and $U_i(\infty) = \infty$, i=1,2. Then,*

$$
\lim_{\|\tilde{\gamma}\|_1 \to \infty} \widetilde{J}(y,\tilde{\gamma}) = \infty, \ \forall \ y \in (0,\infty). \tag{4.8}
$$

Proof. By the convexity of $\widetilde{U}_i(\cdot), \ i=1,2$, it follows from the application of Jensen's inequality that

$$\begin{aligned}
\widetilde{J}(y,\tilde{\gamma}) &\geq E\Big[\int_0^T \widetilde{U}_1(y(1-\gamma_{d+1}(t))\exp\{\int_0^t -\delta_{\tilde{\gamma}(s)}ds-\zeta_\gamma(t)\})dt \\
&\quad +\tilde{U}_2(y\exp\{\int_0^T -\delta(\tilde{\gamma}(t))dt-\zeta_{\tilde{\gamma}}(T)\})\Big] \\
&\geq \int_0^T \widetilde{U}_1(y\exp(E\{\ln(1-\gamma_{d+1}(t))-\int_0^t \delta_{\tilde{\gamma}}(s)ds+\zeta_\gamma(t)\}))dt \\
&\quad +\widetilde{U}_2(y\exp\{E\int_0^T -\delta_{\tilde{\gamma}}(t)dt-\zeta_{\tilde{\gamma}}(T)\}) \\
&= \int_0^T \widetilde{U}_1(y\exp(E\{\ln(1-\gamma_{d+1}(t))-E\int_0^t (\delta_{\tilde{\gamma}}(s) \\
&\quad +\frac{1}{2}\|\theta(s)+\sigma^{-1}(s)\gamma(s)\|_2^2)ds\}))dt \\
&+ \widetilde{U}_2(y\exp(-E\{\int_0^T (\delta_{\tilde{\gamma}}(t)dt+\frac{1}{2}\int_0^T \|\theta(s)+\sigma^{-1}(s)\gamma(s)\|_2^2)dt\})). \quad (4.9)
\end{aligned}$$

Before showing $\lim_{\|\tilde{\gamma}\|_1\to\infty}\widetilde{J}(y,\tilde{\gamma})=\infty$, let us first show the validity of the following relation

$$\lim_{\|\tilde{\gamma}\|_1\to\infty}\int_0^T \delta_{\tilde{\gamma}}(t)dt+\frac{1}{2}E\int_0^T \|\theta(t)+\sigma^{-1}(t)\gamma(t)\|_2^2dt=\infty. \quad (4.10)$$

Indeed, by the assumptions of the theorem, we have

$$\delta_{\tilde{\gamma}}(t)\geq k_{d+1}|\gamma_{d+1}(t)|-k_{d+1}. \quad (4.11)$$

Since

$$\begin{aligned}
&\int_0^T \delta_{\tilde{\gamma}}(t)dt+\frac{1}{2}E\int_0^T \|\theta(t)+\sigma^{-1}(t)\gamma(t)\|_2^2dt \\
\geq &\int_0^T (k_{d+1}|\gamma_{d+1}(t)|-k_{d+1})dt+\frac{1}{2}E\int_0^T \|\theta(t)+\sigma^{-1}(s)\gamma(t)\|_2^2dt, \quad (4.12)
\end{aligned}$$

it suffices to prove

$$\lim_{\|\tilde{\gamma}\|_1\to\infty}\int_0^T k_{d+1}|\gamma_{d+1}(t)|dt+\frac{1}{2}E\int_0^T \|\theta(t)+\sigma^{-1}(t)\gamma(t)\|_2^2dt=\infty. \quad (4.13)$$

By

$$\|\tilde{\gamma}\|_1=\|\gamma\|_1+E\int_0^T |\gamma_{d+1}(t)|dt, \quad (4.14)$$

we have

$$\|\tilde{\gamma}\|_1\to\infty, \quad (4.15)$$

which implies that at least one of the following two statements is valid:

$$(i) \qquad \|\gamma\|_1\to\infty; \quad (4.16)$$

or

$$(ii) \qquad E\int_0^T |\gamma_{d+1}(t)|dt\to\infty. \quad (4.17)$$

If $E\int_0^T |\gamma_{d+1}(t)|dt \to \infty$, it follows from (4.11) that (4.10) holds. On the other hand, if $\|\gamma\|_1 \to \infty$, then $\|\gamma\|_2 \to \infty$, and hence

$$\|\theta + \sigma^{-1}\gamma\|_2 = \infty. \tag{4.18}$$

Therefore, (4.10) holds. $\widetilde{U}_2(0_+) = \infty$ together with (4.9) gives (4.8). ☐

Before we begin with the proof of Theorem 4.1, we need the following lemma.

Lemma 4.5. *Denote*

$$K_y := \{\tilde{\gamma}, \widetilde{J}(y,\tilde{\gamma}) \le \widetilde{J}(y,0), \tilde{\gamma} \in \overline{\mathcal{C}}_e\}.$$

Then, there exists a $\widetilde{\lambda}_y \in K_y$ *such that*

$$\widetilde{V}(y) = \widetilde{J}(y,\widetilde{\lambda}_y) = \inf_{\tilde{\gamma}\in\bar{\mathcal{C}}_e} \widetilde{J}(y,\tilde{\gamma}). \tag{4.19}$$

Proof. The proof is proceeded in two steps:

1. We first show that $\widetilde{J}(y,\tilde{\gamma}) : \bar{\mathcal{C}}_e \to \mathbb{R}\cup\infty$ is convex and lower-semicontinuous. The lower-semicontinuity is, for any $\tilde{\gamma}$ and $\tilde{\gamma}^n \in \mathcal{M}^*$, with $\lim_{n\to\infty} \|\tilde{\gamma}^n - \tilde{\gamma}\|_1 = 0$,

$$\widetilde{J}(\tilde{\gamma}) \le \liminf_{n\to\infty} \widetilde{J}(\tilde{\gamma}^n). \tag{4.20}$$

   By the arguments similar to Cvitanic and Karatzas [7], the lower-semicontinuouty can be obtained. Thus, we only need to verify the convexity of $\widetilde{J}(y,\tilde{\gamma})$. Rewrite

$$(1-\gamma_{d+1}(t))\exp\{-\int_0^t (r(s)+\delta(\tilde{\gamma}(s)))ds - \zeta_{\tilde{\gamma}}(t)\}$$

   as

$$\exp\{-\int_0^t (r(s)+\delta(\tilde{\gamma}(s)))ds - \zeta_{\tilde{\gamma}}(t) + \ln(1-\gamma_{d+1}(t))\}.$$

   Since $\widetilde{U}_1(\exp(\cdot))$ and $\widetilde{U}_2(\exp(\cdot))$ are convex and decreasing, the conclusion follows readily from the convexity of $\delta(\tilde{\gamma}(t))$, $\zeta_{\tilde{\gamma}}(t)$ and $-\ln(1-\gamma_{d+1}(t))$.
2. Now we show that K_y is norm bounded, convex and closed in the topology of convergence in measure. In fact, the convexity of K_y is clear by virtue of the convexity of $J(y,\tilde{\gamma})$. Closure follows from Fatou's lemma and the fact that any sequence converging in measure has a subsequence converging a.e. By Lemma 4.4, it follows from (4.8) that there exists a constant M such that if $\|\tilde{\gamma}\|_1 > M$, then $\widetilde{J}(y,\tilde{\gamma}) \ge \widetilde{J}(y,\tilde{\mathbf{0}}) + 1$, where $\tilde{\mathbf{0}}$ is defined by Remark 2. Now, following an argument similar to that given for showing the closure, we can show that the norm is bounded by $D+1$.

Now, by Lemma 4.4 and Lemma 4.3, we conclude that there exists a $\widetilde{\lambda}_y \in K_y$ such that $\widetilde{V}_e(y) = \widetilde{J}(y,\widetilde{\lambda}_y)$. ☐

Proof of Part (1) of Theorem 4.1.

Proof. By Lemma 4.5, it suffices to show that $\widetilde{\lambda}_y \in \bar{\mathcal{D}}_e$. In deed, let $\lambda_{y,d+1}$ be the $d+1$ element of $\tilde{\lambda}_y$, then

$$
\begin{aligned}
&\widetilde{J}(y,\widetilde{\lambda}_y)\\
\geq\ & \int_0^T \widetilde{U}_1(y\exp\{E(\ln(1-\lambda_{y,d+1}(t)) - \int_0^t [r(s)+\delta(\tilde{\lambda}(s))\\
&+\frac{1}{2}\|\theta(s)+\sigma^{-1}(s)\lambda_y(s)\|_2^2]ds)\})dt\\
&+\widetilde{U}_2(y\exp\{-E\int_0^T [(r(t)+\delta(\tilde{\lambda}_y(t)))+\frac{1}{2}\|\theta(t)+\sigma^{-1}(t)\lambda_y(t)\|_2^2]dt\})\\
\geq\ & \int_0^T \widetilde{U}_1(y\exp E\ln(1-\lambda_{y,d+1}(t)))dt+\\
&\widetilde{U}_2(y\exp(E\int_0^T -(r(t)+\delta(\lambda_y(t)))dt)). \qquad (4.21)
\end{aligned}
$$

Since $\widetilde{U}_1(0_+)=\infty$, we have

$$\lambda_{y,d+1}(t,\omega)<1\ \mathcal{P}_L\times\mathcal{P}\text{ a.e, on }[0,T]\times\Omega. \qquad (4.22)$$

Thus, $\widetilde{\lambda}_y\in\mathcal{C}_e$ and $\widetilde{U}_2(0_+)=\infty$, meaning that

$$\widetilde{J}(y,\widetilde{\lambda}_y)=\infty, \qquad \text{if } \widetilde{\lambda}_y\ \in\mathcal{M}^*\setminus\ \mathcal{D}_e. \qquad (4.23)$$

Therefore, $\widetilde{\lambda}_y\in\mathcal{D}_e$. Now, by Theorem 12.3 in [12], we have $\widetilde{\lambda}_y\in\bar{\mathcal{D}}_e$. Thus, the validity of Part (1) of the theorem follows readily from Remark 4. □

4.1.2. *The proof of part (2) of Theorem 4.1.* To prove Part (2), we need the following proposition.

Proposition 1. *Assume that there exists $y>0$ satisfying*

$$E\Big[\int_0^T (1-\lambda_{y,d+1}(t))H_{\tilde{\lambda}_y}(t)I_1((1-\lambda_{y,d+1}(t))yH_{\tilde{\lambda}_y}(t))dt+H_{\tilde{\lambda}_y}(T)I_2(yH_{\tilde{\lambda}_y}(T))\Big]=x,$$

and

$$\widetilde{V}(y)=\widetilde{J}(y,\widetilde{\lambda}), \qquad (4.24)$$

or equivalently, for any $\tilde{\gamma}\in\mathcal{H}$,

$$
\begin{aligned}
&E\Big[\int_0^T \widetilde{U}_1((1-\lambda_{y,d+1}(t))yH_{\widetilde{\lambda}_y}(t))dt+\widetilde{U}_2(yH_{\widetilde{\lambda}_y}(T))\Big]\\
\leq\ & E\Big[\int_0^T \widetilde{U}_1((1-\gamma_{d+1}(t))yH_{\widetilde{\gamma}}(t))dt+\widetilde{U}_2(yH_{\widetilde{\gamma}}(T))\Big]. \qquad (4.25)
\end{aligned}
$$

If there exist $(C^y(t),\xi^y(T))$ such that

$$(C^y(t),\xi^y(T))=(I_1(y(1-\lambda_{y,d+1}(t))H_{\tilde{\lambda}_y}(t)),I_2(yH_{\tilde{\lambda}_y}(T))),$$

then there exist

$$(\pi^y(t),c^y(t))\in\mathcal{A}_x$$

and $X^{\pi^y,c^y}_{\tilde{\lambda}_y}(t)$ such that

$$(c^y(t)X^{\pi^y,c^y}_{\tilde{\lambda}_y}(t),X^{\pi^y,c^y}_{\tilde{\lambda}_y}(T))=(C^y(t),\xi^y(T)),$$

and

$$\delta_{\widetilde{\lambda}_y}(t) + \lambda_y \pi^y(t) + \lambda_{y,d+1}(t)c^y(t) = 0.$$

The proof is given in Appendix.

Now let us return to the proof of Part (2) of the theorem.

Proof. Note that

$$\begin{aligned} & J(x,c,\pi) = E\Big\{\int_0^T [U_1(c(t)X^{\pi,c})]dt + U_2(X^{\pi,c}(T))\Big\} \\ & \leq \widetilde{J}(y,\widetilde{\gamma}) + yE\Big\{\int_0^T (1-\gamma_{d+1}(t))H_{\tilde{\gamma}}(t)yc(t)X^{\pi,c}(t)dt + H_{\tilde{\gamma}}(T)yX^{\pi,c}(T)\Big\} \\ \leq \;& \widetilde{J}(y,\widetilde{\gamma}) + xy - yE\int_0^T X^{\pi,c}(t)[\delta(\widetilde{\gamma}(t)) + \pi(t)\gamma(t) + c(t)\gamma_{d+1}(t)]dt \\ \leq \;& \widetilde{J}(y,\widetilde{\gamma}) + xy. \end{aligned} \tag{4.26}$$

Thus,

$$V(x) \leq \widetilde{V}(y) + xy. \tag{4.27}$$

By part (1) of the theorem, there exists a $\widetilde{\lambda}_y \in \bar{\mathcal{D}}$ such that

$$\widetilde{V}(y) = \widetilde{J}(y,\widetilde{\lambda}_y).$$

Let

$$(C^y(t),\xi^y(T)) = (I_1((1-\lambda_{y,d+1}(t))H_{\widetilde{\lambda}_y}(t)y), I_2((1-\lambda_{y,d+1}(T))H_{\widetilde{\lambda}_y}(T)y)). \tag{4.28}$$

Then, by Proposition 1, there exists a $(\pi^y(t),c^y(t)) \in \mathcal{A}_x$ with

$$(c^y(t)X^{\pi^y,c^y}(t), X^{\pi^y,c^y}(T)) = (C^y(t),\xi^y(T))$$

such that

$$\begin{aligned} &\delta_{\widetilde{\lambda}_y}(t) + \lambda_y\pi^y(t) + \lambda_{y,d+1}c^y(t) = 0, \\ &X_{\widetilde{\lambda}_y}^{\pi^y,(1-\lambda_{y,d+1})c^y}(t) = X^{\pi^y,c^y}(t). \end{aligned} \tag{4.29}$$

Then, $(\pi^y(t),c^y(t))$ ensures the two sides of (4.26) are equal. Thus,

$$\begin{aligned} \widetilde{V}(y) &= \widetilde{J}(y,\widetilde{\lambda}_y) = J(x,\pi^y,c^y) - xy \\ &\leq \sup_x [V(x) - xy]. \end{aligned} \tag{4.30}$$

Together with (4.27), the proof of Part (2) of the theorem is finished. As a by-product, the proof of Part (4) of the theorem has already been included in the proof for Part (2). To be more specific, from (4.30), we have

$$J(x,\pi^y,c^y) = \widetilde{V}(y) + xy.$$

By this, together with the fact that $J(x,\pi^y,c^y) \leq V(x) \leq \widetilde{V}(y) + xy$, we obtain

$$V(x) = J(x,\pi^y,c^y).$$

□

4.1.3. *The proof of part (3) of Theorem 4.1.*

Proof. Denote

$$f_x(y) = \widetilde{V}(y) + xy, \tag{4.31}$$

and $M = T\delta_0$. Then, $EH_{\tilde{\gamma}}(t) \leq e^M$. Applying Jensen's inequality, we obtain

$$\begin{aligned}\widetilde{J}(y,\gamma) &\geq \int_0^T \widetilde{U}_1(E((1-\lambda_{d+1}(t))H_{\tilde{\lambda}}(t)y))dt + \widetilde{U}_2(E(H_{\tilde{\lambda}}(T)y)) \\ &\geq \int_0^T \widetilde{U}_1(ye^M E\ln(1-\lambda_{d+1}(t)))dt + \widetilde{U}_2(ye^M). \end{aligned}\tag{4.32}$$

As $U_2(\infty) = \infty$ implies $\widetilde{U}_2(0) = \infty$, we have

$$f_x(0^+) = \lim_{y\to 0} \widetilde{J}(y,\tilde{\gamma}) = \infty. \tag{4.33}$$

From (4.33), the fact that

$$f_x(\infty) = \infty, \tag{4.34}$$

and Remark 2, it follows that $f_x(y)$ attains its infimum in $(0,\infty)$.
Denote

$$y_x = \mathrm{argmin}_y \widetilde{V}(y) + xy, \tag{4.35}$$

and

$$g(z) = y_x xz + \widetilde{J}(y_x z, \widetilde{\lambda}_{y_x}). \tag{4.36}$$

Then,

$$\begin{aligned}\inf_z g(z) &= \inf_z \{y_x xz + \widetilde{J}(y_x z, \widetilde{\lambda}_{y_x})\} \\ &= \inf_y \{yx + \widetilde{J}(y, \widetilde{\lambda}_{y_x})\} \\ &\geq y_x x + \widetilde{V}(y_x) \\ &= y_x x + \widetilde{J}(y_x, \widetilde{\lambda}_{y_x}), \end{aligned}\tag{4.37}$$

the last equality follows from Theorem 4.1(1). Thus,

$$\inf_z g(z) = g(1). \tag{4.38}$$

Follow an argument similar to that given for the proof of Lemma 12.3 in Cvitanic and Karatzas [7], and use (2.4), (4.8) and the assumption that $U_1(\cdot)$ and $U_2(\cdot)$ satisfy (2.4) with the same constants, we can show that $g(y)$ is well defined and differentiable everywhere in its domain. Furthermore, $g'(1) = 0$, that is

$$\begin{aligned}& xy_x \\ = \;& y_x\Big(E\int_0^T I_1((1-\lambda_{y_x,d+1}(t))H_{\widetilde{\lambda}_{y_x}}(t)y_x)(1-\lambda_{y_x,d+1}(t))H_{\widetilde{\lambda}_{y_x}}(t)dt \\ + \;& I_2(H_{\widetilde{\lambda}_{y_x}}(T)y_x)H_{\widetilde{\lambda}_{y_x}}(T)\Big).\end{aligned}$$

Hence,

$$\begin{aligned} & x \\ = \quad & E\int_0^T I_1((1-\lambda_{y_x,d+1}(t))H_{\widetilde{\lambda}_{y_x}}(t)y_x)(1-\lambda_{y_x,d+1}(t))H_{\widetilde{\lambda}_{y_x}}(t)dt \\ + \quad & I_2(H_{\widetilde{\lambda}_{y_x}}(T)y_x)H_{\widetilde{\lambda}_{y_x}}(T). \end{aligned}$$

□

5. **Example and discussions.**

5.1. **Relative value at risk (RVaR) constraint example.** We consider a case when $(c(t), \pi(t))$ are constrained by a convex set K. Here, the dynamic risk constraint is imposed as a portfolio constraint. Here, r(t), $\mu(t)$ and $\sigma(t)$ are assumed to be constants. They are written, respectively, as r, μ and σ. Assume that all the parameters are constants.

Given an arbitrary but fixed time t, $(\pi(s), c(s))$ are approximated as constants from t to $t+\Delta t$. Then, conditioned on time t,

$$\begin{aligned} X_{t+\Delta t} &= \int_t^{t+\Delta t} X(t)\exp(((r+\pi^\top(t)(\mu-r\mathbf{1})-c(t))dt+\pi^\top(t)\sigma dW_0(t))) \\ &= X(t)\exp(((r+\pi^\top(t)(\mu-r\mathbf{1})-c(t))\Delta t \\ &\quad +\pi^\top(t)\sigma(W_0(t+\Delta t)-W_0(t))). \end{aligned} \tag{5.1}$$

For a given confidence level k, the dynamic relative value at risk (RVaR) is defined by

$$P(\frac{X(t)-X(t+\Delta t)}{X(t)} \le RVaR) = k. \tag{5.2}$$

Thus,

$$RVaR = 1-\exp((r+\pi(t)(\mu-r\mathbf{1})-c(t))\Delta t+\Phi^{-1}(1-k)\sqrt{\Delta t\pi^\top(t)\Sigma\pi(t)}). \tag{5.3}$$

Suppose that the maximal risk is constrained to be less or equal to a level R, that is,

$$1-\exp((r+(\mu-r\mathbf{1})\pi(t)-c(t))\Delta t+\Phi^{-1}(1-k)\sqrt{\Delta t\pi^\top(t)\Sigma\pi(t)}) \le R, \tag{5.4}$$

where R is a given constant.

We consider the logarithmic utility function with primal embedding problem and the dual problem. Here, we assume that

$$U_1(x) = U_2(x) = \log x. \tag{5.5}$$

Let k be larger than 0.5. Then, this constraint set B is a convex closed set with respect to (π, c). $\delta(\cdot, \overline{B})$ is bounded below by 0. Thus the condition of Lemma 4.4 is satisfied.

5.1.1. *The dual problem.* We have

$$
\begin{aligned}
I_1(y) &= \frac{1}{y}, \\
\widetilde{U}(y) &= \inf_x(\log x - xy) = -\log y - 1.
\end{aligned} \tag{5.6}
$$

From Theorem 4.1 (3), the problem is to look for a $(\widetilde{\lambda}, y_{\widetilde{\lambda}})$ such that

$$
\begin{aligned}
& E\Big[\int_0^T \widetilde{U}((1-\lambda_{d+1}(t))H_{\widetilde{\lambda}}(t)y_{\widetilde{\lambda}})dt + \widetilde{U}(H_{\widetilde{\lambda}}(T)y_{\widetilde{\lambda}})\Big] \\
= \; & \inf_{\widetilde{\gamma}} E\Big[\int_0^T \widetilde{U}((1-\gamma_{d+1}(t))H_{\widetilde{\gamma}}(t)y_{\widetilde{\lambda}})dt + \widetilde{U}(H_{\widetilde{\gamma}}(T)y_{\widetilde{\lambda}})\Big] \\
= \; & \inf_{\widetilde{\gamma}} E\Big[\int_0^T \log\frac{1}{(1-\gamma_{d+1}(t))H_{\widetilde{\gamma}}(t)y_{\widetilde{\lambda}}}dt + \log\frac{1}{H_{\gamma}(T)y_{\widetilde{\lambda}}} - (1+T)\Big]
\end{aligned} \tag{5.7}
$$

and

$$
\begin{aligned}
& E\Big[\int_0^T (1-\lambda_{d+1}(t))H_{\tilde{\lambda}}(t)I_1((1-\lambda_{d+1}(t))y_{\widetilde{\lambda}}H_{\tilde{\lambda}}(t))dt \\
& +H_{\tilde{\lambda}}(T)I_2((1-\lambda_{d+1}(T))y_{\widetilde{\lambda}}H_{\tilde{\lambda}}(T))\Big] = x.
\end{aligned} \tag{5.8}
$$

It follows from (5.8) that

$$
y_{\widetilde{\lambda}} = \frac{T+1}{x}. \tag{5.9}
$$

Substituting (5.9) into (5.7) gives

$$
\widetilde{\lambda} = \mathrm{argmin}_{\widetilde{\gamma}\in\bar{D}} E\Big[\int_0^T (\log\frac{1}{1-\gamma_{d+1}(t)} + \log\frac{1}{H_{\widetilde{\gamma}}(t)})dt + \log\frac{1}{H_{\widetilde{\gamma}}(T)}\Big]. \tag{5.10}
$$

It follows from the expression of $H_{\widetilde{\gamma}}(t)$ that

$$
\begin{aligned}
\widetilde{\lambda} \;&=\; \mathrm{argmin}_{\widetilde{\gamma}\in\bar{D}} E\Big[\int_0^T (-\log(1-\gamma_{d+1}(t)) + \int_0^t (\delta(\widetilde{\gamma}(s)) + \frac{1}{2}\|\theta(s) \\
&\quad +\frac{1}{\sigma}\gamma(s)\|_2^2)ds)dt + \int_0^T (\delta(\widetilde{\gamma}(t)) + \frac{1}{2}\|\theta(t) + \frac{1}{\sigma}\gamma(t)\|_2^2)dt\Big] \\
&=\; \mathrm{argmin}_{\widetilde{\gamma}\in\bar{D}} E\int_0^T \Big(-\log(1-\gamma_{d+1}(t)) + (\delta(\widetilde{\gamma}(t)) + \frac{1}{2}\|\theta(t) \\
&\quad +\frac{1}{\sigma}\gamma(t)\|_2^2)(T-t+1)\Big)dt.
\end{aligned}
$$

Then, it is sufficient to consider the static optimization given below:

$$
\begin{aligned}
\widetilde{\lambda}(t) \;&=\; \mathrm{argmin}_{\widetilde{\gamma}(t)\in\tilde{B},\gamma_{d+1}(t)<1}\{-\log(1-\gamma_{d+1}(t)) + (\delta(\widetilde{\gamma}(t)) + \frac{1}{2}\|\theta(t) \\
&\quad +\frac{1}{\sigma}\gamma(t)\|_2^2)(T-t+1)\}.
\end{aligned}
$$

From the expression of (3.4), we have

$$
H_{\widetilde{\lambda}}(t)X_{\widetilde{\lambda}}(t) = \frac{x(T+1-t)}{T+1}
$$

and

$$H_{\widetilde{\lambda}}(T)X^{\pi,c}_{\widetilde{\lambda}}(T) + \int_0^T H_{\tilde{\lambda}}(t)c(t)X_{\tilde{\lambda}}(t)dt = x.$$

Thus, the optimal strategy π can be obtained from

$$\sigma\pi - \theta_{\widetilde{\lambda}} = \mathbf{0},$$

where $\mathbf{0} = (\underbrace{0, \cdots, 0}_{d})^\top$. The optimal consumption is

$$\begin{aligned} C(t) &= I_1(y_{\tilde{\lambda}}(1-\lambda_{d+1}(t))H_{\tilde{\lambda}}(t)) \\ &= \frac{1}{y_{\tilde{\lambda}}(1-\lambda_{d+1}(t))H_{\tilde{\lambda}}(t)} \\ &= \frac{x}{(T+1)(1-\lambda_{d+1}(t))H_{\tilde{\lambda}(t)}} \\ &= \frac{X_{\widetilde{\lambda}}(t)}{(1-\lambda_{d+1}(t))(1+T-t)}. \end{aligned}$$

In other words,

$$c(t) = \frac{1}{(1-\lambda_{d+1}(t))(1+T-t)}.$$

5.1.2. *The primal problem.* For the logarithmic utility function, the optimal strategy can be solved from the primal problem with Theorem 3.4.
Consider the primal problem (3.14). Then, by Theorem 3.4, we have

$$y_{\widetilde{\lambda}} = \frac{T+1}{x},$$

and hence the primal problem is reduced to

$$\widetilde{\lambda} = \arg \sup_{\widetilde{\gamma}\in\tilde{B}} E\int_0^T [\log(1-\gamma_{d+1}(t)) + \log H_{\widetilde{\gamma}}(t)]dt + \log H_{\widetilde{\gamma}}(T), \tag{5.11}$$

which is equivalent to (5.10).

5.2. **Discussions on the consumption function.** In [7], when the consumption $C(t), t \in [0,T)$ is included, an additional no-bankruptcy constraint is required. Here, we relax the bankruptcy constraint by using $c(t)$ being proportional to the wealth. As a result, the wealth dynamics (2.1) shows that $X(t), t \in [0,T]$, is strictly positive whenever the initial wealth is strictly positive. Thus, the no-bankruptcy constraint become redundant in our setting. Indeed, any proportional strategy $c(t)$ gives rise to a monetary amount equal to $C(t) = c(t)X(t)$. On the other hand, from Section 10 in [7] and Theorem 9.1, we learn that the no-bankruptcy constraint implies the optimal wealth $X^*(t) > 0, t \in [0,T]$. Therefore, for any admissible strategy with a strictly positive wealth process $X(t)$, we can recast the consumption function into a proportional consumption strategy defined by $c(t) = \frac{C(t)}{X(t)}$. Consequently, the optimal consumption function in [7] can be represented fully by a proportional consumption strategy proposed here.

Moreover, when there is no constraint on $c(t)$, we have, from the notation of $\delta(\widetilde{\gamma})$,

$$\begin{aligned} \tilde{B} &= \{\widetilde{\gamma} \in B : \ \delta(\widetilde{\gamma}) < \infty\} \\ &= \{\widetilde{\gamma} \in B : \ \delta(\gamma) < \infty, \gamma_{d+1} = 0\}. \end{aligned} \tag{5.12}$$

Thus, $\delta(\tilde{\gamma}) = \delta(\gamma)$ and our model reduces to the case in [7]. Therefore, this work serves as an extension.

6. **Concluding remarks.** In this paper, we have studied the optimal investment-consumption problem with a closed convex constraint on both investment and consumption. The initial problem is first embedded into a family of fictitious markets parameterized by $\tilde{\gamma}$ and the unconstrained optimal investment-consumption strategies are sought. A specific market has been identified such that the optimal strategy under this market coincides with the optimal strategy to the original constrained problem. We have proven the existence of such a market using the theory of duality. Furthermore, we have demonstrated by using the logarithmic utility function how to construct the optimal strategy from the solution to the dual problem. In addition, we have relaxed the no-bankruptcy constraint by using a consumption function being proportional to the wealth. As a future extension, it is of interest to study other utility functions and include further stochastic processes in the model.

7. **Appendix. Proof of Lemma 3.1.**

Proof. For an arbitrary but fixed $\widetilde{\gamma} \in \mathcal{H}$, let

$$\breve{\delta}^{\widetilde{\gamma}}(\widetilde{\lambda}(s)) = \begin{cases} -\delta(\widetilde{\lambda}(s)), & \tilde{\gamma} = \tilde{\mathbf{0}}, \\ \delta(\widetilde{\gamma}(s) - \widetilde{\lambda}(s)), & \text{otherwise.} \end{cases}$$

where $\tilde{\mathbf{0}} = (\underbrace{0, \cdots, 0}_{d+1})^\top$. Denote

$$\begin{aligned} \widetilde{\lambda}^{\tilde{\gamma}}_{\varepsilon,n}(t) &= \widetilde{\lambda}^{\tilde{\gamma}}(t) + \varepsilon(\widetilde{\gamma}(t) - \widetilde{\lambda}(t))1_{t\le\tau_n}, \qquad 0 < \varepsilon \le \varepsilon_n, \\ L^{\tilde{\gamma}}(t) &= \int_0^t \breve{\delta}^{\gamma}(\widetilde{\lambda}(s))ds, \\ N^{\tilde{\gamma}}(t) &= \int_0^t \sigma^{-1}(\gamma(s) - \lambda(s))ds, \end{aligned} \tag{A.1}$$

where

$$1_{t\le\tau_n} = \begin{cases} 1, & \text{if } t \le \tau_n(\omega), \\ 0, & \text{otherwise,} \end{cases}$$

for any $\omega \in \Omega$.

Define a sequence of stopping times

$$\begin{aligned} \tau_n \quad &:= T \wedge \inf\{t \in [0,T]; \int_0^t \|\theta(s) + \sigma^{-1}\lambda(s)\|_2^2 ds \ge n; \\ &\text{or } \int_0^t \|\sigma^{-1}(\gamma(s) - \lambda(s))\|_2^2 ds \ge n; \\ &\text{or } \int_0^t X^2(s)R^2_{\widetilde{\lambda}}(s)\|\sigma^{-1}(\gamma(s) - \lambda(s)) + (L^{\widetilde{\gamma}}(s) + N^{\widetilde{\gamma}}(s)^\top\sigma^*(s)\pi(s)\|^2 ds \ge n; \\ &\text{or } |N^{\widetilde{\gamma}}(t)| \ge n; \text{ or } |L^{\widetilde{\gamma}}(t)| \ge n; \text{ or } |\gamma_{d+1} - \lambda_{d+1}| \ge n; \\ &\text{or } \lambda_{d+1} \ge 1 - \frac{1}{n}; \text{ or } \lambda_{d+1} \le -n\}, \end{aligned} \tag{A.2}$$

for $n \in N$.

Denote $\varepsilon_n = \frac{1}{2n}$. Thus,

$$
\begin{aligned}
& \frac{H_{\widetilde{\lambda}^{\gamma}_{\varepsilon,n}(t)}}{H_{\widetilde{\lambda}}(t)} \\
= \;& \exp\{-\int_0^{t \bigwedge \tau_n} (\delta(\widetilde{\gamma}(s) + \varepsilon(\widetilde{\gamma}(t) - \widetilde{\lambda}(t))) - \delta(\widetilde{\gamma}(s))ds - \varepsilon N^{\tilde{\gamma}}(t \wedge \tau_n) \\
& -\frac{\varepsilon^2}{2}\int_0^{t \bigwedge \tau_n} \|\sigma^{-1}(s)(\gamma(s) - \lambda(s))\|_2^2 ds)\} \\
\geq \;& \exp\{-\varepsilon(L^{\tilde{\gamma}}(t \wedge \tau_n) + N^{\tilde{\gamma}}(t \wedge \tau_n) - \frac{\varepsilon^2}{2}\int_0^{t \bigwedge \tau_n} \|\sigma^{-1}(\gamma(s) - \lambda(s))\|_2^2 ds\} \\
\geq \;& e^{-3\varepsilon n}.
\end{aligned} \tag{A.3}
$$

Similarly,

$$
\begin{aligned}
\frac{H_{\widetilde{\lambda}^{\gamma}_{\varepsilon,n}(t)}}{H_{\widetilde{\lambda}}(t)} &\leq e^{\int_0^{t \bigwedge \tau_n} \delta(\widetilde{\lambda}(s))ds + \varepsilon n} \\
&= e^{L^{\tilde{\lambda}}(t \wedge \tau_n) + \varepsilon n}.
\end{aligned} \tag{A.4}
$$

Let $\lambda_{\varepsilon,n,d+1}(t)$ be the $d+1$ element of $\lambda_{\varepsilon,n}(t)$. Then,

$$
\begin{aligned}
& H_{\widetilde{\lambda}}(T)\frac{\xi}{\varepsilon}(1 - \frac{H_{\widetilde{\lambda}^{\gamma}_{\varepsilon,n}}(T)}{H_{\widetilde{\lambda}}(T)}) \\
& + \int_0^T H_{\widetilde{\lambda}}(t)\frac{C(t)}{\varepsilon}((1 - \lambda_{d+1}(t)) - \frac{(1 - \lambda_{\varepsilon,n,d+1}(t))}{H_{\widetilde{\lambda}}(t)} H_{\widetilde{\lambda}_{\varepsilon,n}}(t))dt \\
\leq \;& \sup_{\varepsilon} \frac{1 - e^{-3\varepsilon n}}{\varepsilon}[H_{\widetilde{\lambda}}(T)\xi + \int_0^T H_{\widetilde{\lambda}}(t)C(t)((1 - \lambda_{d+1}(t)))dt] \\
& + \sup_{\varepsilon} e^{2\varepsilon n + \int_0^t \delta(\widetilde{\gamma}(s))ds} \int_0^T H_{\widetilde{\lambda}}(t)C(t)(\lambda_{d+1}(t) - \gamma_{d+1}(t))dt \\
\leq \;& K_n[H_{\widetilde{\lambda}}(T)\xi + \int_0^T H_{\widetilde{\lambda}}(t)C(t)(1 - \lambda_{d+1}(t))dt \\
& + \int_0^T H_{\widetilde{\lambda}}(t)C(t)(\lambda_{d+1}(t) - \gamma_{d+1}(t))dt],
\end{aligned} \tag{A.5}
$$

where

$$
K_n = \max\{\sup_{\varepsilon} \frac{1 - e^{-3\varepsilon n}}{\varepsilon}, \sup_{\varepsilon} e^{L^{\tilde{\lambda}}_{\tau_n}(t) + 2\varepsilon n}\}. \tag{A.6}
$$

By Fatou's lemma, we have

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\frac{x(\tilde{\lambda})-x(\tilde{\lambda}_{\varepsilon,n})}{\varepsilon}\\
= \;& E\limsup_{\varepsilon\to 0}[H_{\tilde{\lambda}}(T)\frac{\xi}{\varepsilon}(1-\frac{H_{\tilde{\lambda}^{\gamma}_{\varepsilon,n}}(T)}{H_{\tilde{\lambda}}(T)})\\
&\qquad\qquad + \int_0^T H_{\tilde{\lambda}}(t)\frac{C(t)}{\varepsilon}(1-\lambda(t))(1-\frac{H_{\tilde{\lambda}}(t)}{H_{\tilde{\lambda}_{\varepsilon,n}}(t)})dt]\\
&+\int_0^T H_{\tilde{\lambda}}(t)C(t)\frac{\lambda_{\varepsilon,n,d+1}(t)-\lambda_{d+1}(t)}{\varepsilon}\frac{H_{\tilde{\lambda}}(t)}{H_{\tilde{\lambda}_{\varepsilon,n}}(t)}dt\\
\leq \;& E[H_{\tilde{\lambda}}(T)\xi(L^{\tilde{\gamma}}(\tau_n)+N^{\tilde{\gamma}}(\tau_n))]+\int_0^T H_{\tilde{\lambda}}(t)C(t)(1-\lambda_{d+1}(t))(L^{\tilde{\gamma}}(t\wedge\tau_n)\\
&+N^{\tilde{\gamma}}(t\wedge\tau_n))dt+\int_0^T H_{\tilde{\lambda}}(t)C(t)((\gamma_{d+1}(t)-\lambda_{d+1}(t))1_{t\wedge\tau_n}dt\\
= \;& E\int_0^{\tau_n} H_{\tilde{\lambda}}(t)X_{\tilde{\lambda}}(t)[\pi^*(t)(\gamma(t)-\lambda(t))+\frac{C(t)}{X_{\tilde{\lambda}}(t)}(\gamma_{d+1}(t)-\lambda_{d+1}(t))]dt\\
&+dL^{\tilde{\gamma}}(t\wedge\tau_n).
\end{aligned} \tag{A.7}$$

The last equality of the above equation follows from

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\int_0^T H_{\tilde{\lambda}}(t)\frac{C(t)}{\varepsilon}(\frac{\widetilde{\lambda}_{\varepsilon,n}(t)-\widetilde{\lambda}(t)}{H_{\tilde{\lambda}}(t)}H_{\tilde{\lambda}_{\varepsilon,n}}(t))dt\\
= \;& \int_0^T H_{\tilde{\lambda}}(t)\limsup_{\varepsilon\to 0}C(t)\frac{\widetilde{\lambda}_{\varepsilon,n}(t)-\widetilde{\lambda}(t)}{\varepsilon}\frac{H_{\tilde{\lambda}_{\varepsilon,n}}(t)}{H_{\tilde{\lambda}}(t)}dt\\
= \;& \int_0^T H_{\tilde{\lambda}}(t)\limsup_{\varepsilon\to 0}C(t)(\gamma_{d+1}(t)-\lambda_{d+1}(t))[\frac{H_{\tilde{\lambda}_{\varepsilon,n}}(t)}{H_{\tilde{\lambda}}(t)}-1+1_{(t\leq\tau_n)}]dt\\
= \;& \int_0^T H_{\tilde{\lambda}}(t)C(t)(\gamma_{d+1}(t)-\lambda_{d+1}(t))\lim_{\varepsilon\to 0}[\frac{H_{\tilde{\lambda}_{\varepsilon,n}}(t)}{H_{\tilde{\lambda}}(t)}-1]dt\\
&+\int_0^T H_{\tilde{\lambda}}(t)C(t)(\gamma_{d+1}(t)-\lambda_{d+1}(t))1_{t\leq\tau_n}dt\\
= \;& \int_0^T H_{\tilde{\lambda}}(t)C(t)(\gamma_{d+1}(t)-\lambda_{d+1}(t))1_{t\leq\tau_n}dt,
\end{aligned} \tag{A.8}$$

and

$$\begin{aligned}
&E[H_{\tilde{\lambda}}(T)\xi(L^{\tilde{\gamma}}(\tau_n)+N^{\tilde{\gamma}}(\tau_n))]\\
&+\int_0^T H_{\tilde{\lambda}}(t)c(t)X_{\tilde{\lambda}}(t)(1-\lambda_{d+1}(t))(L^{\tilde{\gamma}}(t\wedge\tau_n)+N^{\tilde{\gamma}}(t\wedge\tau_n))dt\\
= \;& E\int_0^{\tau_n} H_{\tilde{\lambda}}(t)X_{\tilde{\lambda}}(t)[\pi(t)(\gamma(t)-\lambda(t))dt+dL^{\tilde{\gamma}}(t)],
\end{aligned} \tag{A.9}$$

with $X^{\pi,c}(T)=\xi$, the proof of (A.9) is similar to that gives for Step 3 of Theorem 9.1 in Karatzas and et al. [13].

Then,

$$
\begin{aligned}
& \limsup_{\varepsilon\to 0} \frac{x(\lambda)-x(\lambda_{\varepsilon,n})}{\varepsilon} \\
\leq \; & E[H_{\widetilde{\lambda}}(T)\xi(L^{\tilde{\gamma}}(t\wedge\tau_n)+N^{\tilde{\gamma}}(t\wedge\tau_n)) \\
& +\int_0^T H_{\widetilde{\lambda}}(t)c(t)X_{\widetilde{\lambda}}(t)((1-\lambda_{d+1}(t))(L^{\tilde{\gamma}}(t\wedge\tau_n)+N^{\tilde{\gamma}}(t\wedge\tau_n))dt \\
& +\int_0^T H_{\widetilde{\lambda}}(t)c(t)X_{\widetilde{\lambda}}(t)(\gamma_{d+1}(t)-\lambda_{d+1}(t))1_{t\wedge\tau_n}dt \\
= \; & E\int_0^{\tau_n}\Big[H_{\widetilde{\lambda}}(t)X_{\widetilde{\lambda}}(t)[\pi(t)(\gamma(t)-\lambda(t))+c(t)(\gamma_{d+1}(t)-\lambda_{d+1}(t))]dt+dL^{\tilde{\gamma}}(t)\Big],
\end{aligned}
$$

where $c(t)=\frac{C(t)}{X_{\widetilde{\lambda}}(t)}$. Now, by using the same procedure as that gives for Step 3 in the proof of Theorem 9.1 given in Cvitanic and Karatzas [7], it follows that for any $\widetilde{\rho}\in\mathcal{H}$, we have

$$\delta(\widetilde{\rho}(t))+\pi(t)\rho(t)+\rho_{d+1}(t)c(t)\geq 0, \tag{A.10}$$

and $(\pi(t),c(t))\in\mathcal{A}_x$. Meanwhile, $\widetilde{\gamma}=0$ leads to

$$\delta_{\widetilde{\lambda}(t)}+\pi(t)\lambda(t)+\lambda_{d+1}(t)c(t)\leq 0. \tag{A.11}$$

From (A.10) and (A.11), $X_{\widetilde{\lambda}}^{\pi,(1-\lambda_{d+1})c}(t)$ is equal to $X^{\pi,c}(t)$ in the initial market. □

Proof of Proposition 1.

Proof. Let $\tilde{\gamma}$ be replaced by $\tilde{\lambda}_{\varepsilon,n}$ in the statement and assume that $\varepsilon_n=\frac{1}{2n}$ and $0<\varepsilon\leq\varepsilon_n$. Then,

$$
\begin{aligned}
& \limsup_{\varepsilon\to 0}\frac{1}{\varepsilon}E\{\int_0^T[\tilde{U}_1((1-\lambda_{\varepsilon,n,d+1}(t))H_{\tilde{\lambda}_{\varepsilon,n}}(t)y) \\
& -\tilde{U}_1((1-\lambda_{y,d+1}(t))H_{\tilde{\lambda}_y}(t)y)]dt+\tilde{U}_2(U_2'^{-1}(H_{\tilde{\lambda}_{\varepsilon,n}}(T)y)) \\
& -\tilde{U}_2(U_2'^{-1}(H_{\tilde{\lambda}_y}(T)y))\} \\
\leq \; & \limsup_{\varepsilon\to 0}\frac{y}{\varepsilon}E[\int_0^T I_1(y(1-\lambda_{\varepsilon,n,d+1}(t))H_{\tilde{\lambda}_{\varepsilon,n}})((1-\lambda_{y,d+1}(t))H_{\tilde{\lambda}_y} \\
& -(1-\tilde{\lambda}^{\varepsilon,n}(t))H_{\tilde{\lambda}_{\varepsilon,n}}(t))dt+I_2(T,y^{\varepsilon,n}H_{\tilde{\lambda}_{\varepsilon,n}}(t))((1-\lambda_{y,d+1}(t))H_{\tilde{\lambda}_y}(t) \\
& -(1-\lambda_{\varepsilon,n,d+1}(t))H_{\tilde{\lambda}_{\varepsilon,n}}(t))] \\
\leq \; & \limsup_{\varepsilon\to 0} yE[\int_0^T I_1(y(1-\lambda_{\varepsilon,n,d+1}(t))e^{-3\varepsilon n}H_{\tilde{\lambda}})Q_{\varepsilon,n}(t)dt \\
& +I_2(T,ye^{-3n}H_{\tilde{\lambda}}(T)Q_{\varepsilon,n}(T))]
\end{aligned}
$$

$$
\begin{aligned}
&= \limsup_{\varepsilon\to 0} yE[\int_0^T I_1(y(1-\lambda_{y,d+1}(t))H_{\tilde{\lambda}_y}(t))Q_{\varepsilon,n}(t)dt + I_2(T, yH_{\tilde{\lambda}_y}(T)Q_{\varepsilon,n}(T))] \\
&= y_\lambda E[\int_0^T H_{\tilde{\lambda}_y}(t)C^{\widetilde{\lambda},0}(t)(1-\lambda_{y,d+1}(t))(L^{\tilde{\lambda}_{\varepsilon,n}}(t\wedge\tau_n) + N^{\tilde{\lambda}_{\varepsilon,n}}(t\wedge\tau_n))dt \\
&\quad + \int_0^T H_{\tilde{\lambda}_y}(t)C^{\widetilde{\lambda},0}(t)(\lambda_{\varepsilon,n,d+1}(t) - \lambda_{y,d+1}(t))1_{t\wedge\tau_n}dt \\
&\quad + H_{\tilde{\lambda}_y}(T)\xi^{\widetilde{\lambda}_y}(T)(L^{\tilde{\lambda}_{\varepsilon,n}}(\tau_n) + N^{\tilde{\lambda}_{\varepsilon,n}}(\tau_n))].
\end{aligned}
\tag{A.12}
$$

Here,

$$\xi^{\widetilde{\lambda}}(T) = I_2(yH_{\tilde{\lambda}_y}(T)),$$

$$C^{\widetilde{\lambda},0}(t) = I_1(y(1-\lambda_{y,d+1})H_{\tilde{\lambda}_y}),$$

$$Q_{\varepsilon,n}(t) = \frac{(1-\lambda_{\varepsilon,n,d+1}(t))H_{\tilde{\lambda}_{\varepsilon,n}}(t) - (1-\lambda_{y,d+1}(t))H_{\lambda_y}(t)}{\varepsilon}.$$

The second inequality follows from Fatou's lemma, as it is bounded above by

$$
\begin{aligned}
Q^n &= K^n yE\Big[\int_0^T I_1(ye^{-3n}(1-\lambda_{y,d+1}(t)-\varepsilon_0)H_{\tilde{\lambda}_y})(1-\lambda_{y,d+1}(t))H_{\tilde{\lambda}_y}(t)dt \\
&\quad + I_2(T, ye^{-3n}H_{\tilde{\lambda}_y}(T))\Big].
\end{aligned}
\tag{A.13}
$$

Here,

$$K^n = \max\{\limsup_{0<\varepsilon<\varepsilon_n} \frac{e^{3\varepsilon n}-1}{\varepsilon}, e^{2\varepsilon n + \int_0^T \delta(\tilde{\lambda}_{\varepsilon,n}(s))ds}\}. \tag{A.14}$$

The remaining part follows in a similar way as that given for the proof of Lemma 3.1, showing that there exists a $(\pi^y(t), c^y(t)) \in \mathcal{A}_x$ satisfying

$$\delta_{\widetilde{\lambda}_y}(t) + \lambda_y(t)\pi^y(t) + \lambda_{y,d+1}(t)c^y(t) = 0.$$

□

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