Numerical Solution of First-Order Linear Fredholm Integro-Differential Equations using Conjugate Gradient Method

Elayaraja Aruchunan
Department of Engineering and Science,
School of Foundation and Continuous Studies
Curtin University of Technology, Sarawak Malaysia.
Email: elayarajah@curtin.edu.my

Jumat Sulaiman
[2]School of Scince and Technology,
Universiti Malaysia Sabah,Malaysia.
Email: jumat@ums.edu.my

***Abstract*— This paper presents the numerical solution of the linear Fredholm integro-differential equation of first order discretized by using finite difference and trapezoidal methods. The application of Conjugate Gradient (CG) method will be examined to solve the generated linear system. The formulation and implementation of Gauss-Seidel (GS) method are also presented. The numerical results were included to demonstrate the effectiveness of CG method compared to GS method. Analysis is accompanied by an example that demonstrates the comparison of iterative methods.**

Keywords: *Fredholm integro-differential, Quadrature-Difference, Conjugate-Gradient.*

I. INTRODUCTION

In the recent years mathematical modeling of real-life usually results in functional equations, e.g. partial equation, integral and integro-differential equations (IDE), these equations arises in fluid dynamics, biological models and chemical kinetics [1]. IDE is an equation that the unknown function appears under the sign of integration and it also contains the derivatives of the unknown function. It can be classified into Fredholm equations and Volterra equations. The upper bound of the region for integral part of Volterra type is variable, while it is a fixed number for that of Fredholm type. However, it this paper, we focus on Fredholm integro-differential. Generally, first-order linear Fredholm integro-differential equations can be defined as follows

$$y'(x)=q(x)y(x)+\int_a^b K(x,t)y(t)dt+f(x),$$
$$x,t\in\Gamma=[a,b]$$
(1)
$$y(a)=y_a$$

where the functions $f(x)$, $q(x)$ and the kernel $K(x,t)$ are known and $y(x)$ is the solution to be determined.

In many application areas, it is necessary to use the numerical approach to obtain an approximation solution for the problem (1). To solve a linear integro-differential equation numerically, discretization of integral equation to the solution of system of linear algebraic equations is the basic concept used by researchers to solve integro-differential problems. By considering numerical techniques, there are many methods can be used to discretize problem (1) such as compact finite difference [2], Wavelet-Galerkin[3], rationalized Haar functions [4], Lagrange interpolation [5], Tau [6] and quadrature-difference [7].

Consequently, iterative methods under the category of Krylov subspaces have been proposed widely to be one of the feasible and successful classes of numerical algorithms for solving linear systems. There are many Krylov subspaces iterative methods can be considered such as Conjugate Gradient (CG) method [8], Generalized Minimal Residual (GMRES) method [9], Conjugate Gradient Squared method [10], Bi-Conjugate Gradient Stabilized (Bi-CGSTAB) method [11] and Orthogonal Minimization (ORTHOMIN) method [12].

The purpose of this paper is to examine CG iterative methods for solving linear algebraic equations produced by the discretization of the first-order linear Fredholm integro-differential equations by using finite difference and quadrature methods. The differential term is approximated by finite difference method, and the integral term is discretized by quadrature method. For the numerical tests, we compare the performance of the Gauss-Seidel(GS) and CG iterative methods.

The remainder of this paper is organized in following way. In Section 2, the formulation of the finite difference and quadrature methods are elaborated. The latter section of this paper discussed the formulations of the CG iterative methods in solving dense linear systems generated from discretization of the Eq. (1). Meanwhile, some numerical results are shown in fourth section to assert the effectiveness of the proposed method and concluding remarks are given in Section 5.

II. METHODOLOGY

A. *Approximation Equation*

As afore-mentioned, a discretization method based on quadrature and finite difference methods was used to construct approximation equations for problem (1).

1) Quadrature method

The formulas of quadrature method, in general have the form

$$\int_a^b y(t)dt = \sum_{j=0}^{n} A_j y(t_j) + \varepsilon_n(y)$$

(2)

where, $t_j \quad (j = 0,1,\ldots,n)$ are the abscissas of the partition points of the integration interval $[a,b]$ or quadrature (interpolation) nodes, $A_j \quad (j = 0,1,\ldots,n)$ are numerical coefficients that do not depend on the function $y(t)$ and $\varepsilon_n(y)$ is the truncation error of Eq. (2). To facilitate in formulating the approximation equations for problem (1), further discussion will restrict onto repeated trapezoidal (RT) method, which is based on linear interpolation formulas with equally spaced data. Based on RT method, numerical coefficients A_j are satisfied following relation

$$A_j = \begin{cases} \frac{1}{2}h, & j = 0,n \\ h, & j = 1,2,\cdots,n-1 \end{cases}$$

(3)

where the constant step size, h is defined as

$$h = \frac{b-a}{n}.$$

(4)

and n is the number of subintervals in the interval $[a,b]$. Based on Eq. (4), we have the discrete set of points be given as $x_i = a + ih$.

2) Finite difference method

In this paper, two approximations formula were used where by forward difference approximation as follow

$$y'(x_i) = \frac{y_{i+1} - y_i}{h}$$

(5)

and backward difference approximation is as follow,

$$y'(x_i) = \frac{y_i - y_{i-1}}{h}$$

(6)

where $y'(x_i)$ is approximated by the gradient of the line passing $(x_i, y'(x_i))$. By applying Eqs. (2), (5) and (6) into Eq. (1), a system of linear algebraic equations obtains for approximation values $y(x)$ at the nodes $x_0, x_1, \ldots, x_n$. The generated linear systems can be easily shown as

$$M \underset{\sim}{y} = \underset{\sim}{f}$$

(7)

3) Conjugate Gradient Method (CG)

As mentioned in the previous section, the CG method will be used to solve a system of linear equations. CG method is proposed by Hestenes and Stiefel [13] and was originally developed as a direct method designed to solve $n \times n$ positive definite linear system [13]. The implementation of CG method may be described by the following algorithm.

Algorithm: CG method

Computer $r_0 := f - My_0$, $p_0 := r_0$.
For $j = 0,1,\ldots,$ *until convergence Do:*

$$\alpha_j := (r_j, r_j)/(Mp_j, p_j)$$
$$y_{j+1} := y_j + \alpha_j p_j$$
$$r_{j+1} := r_j - \alpha_j M p_j$$
$$\beta_j := (r_{j+1}, r_{j+1})/(r_j, r_j)$$
$$p_{j+1} := r_{j+1} + \beta_j p_j$$

End do

The vectors p_j are multiples of the p_j 's of algorithm.

III. DISCUSSION

In order to compare the performances of the methods described in the previous section, several experiments were carried out on the following problems [14].

$$y'(x) = xe^x + e^x - x + \int_0^1 xy(t)\,dt, \qquad y(0) = 0$$

with exact solution $y(x) = xe^x$.

There are parameters considered in numerical comparison such as number of iterations, execution time and maximum absolute error. As comparisons, the Gauss-Seidel (GS) method acts as the control of comparison of numerical results. Throughout the simulations, the convergence test considered the tolerance error of the $\varepsilon = 10^{-16}$. Figures 1 and 2 show number of iterations and execution time versus mesh size respectively.

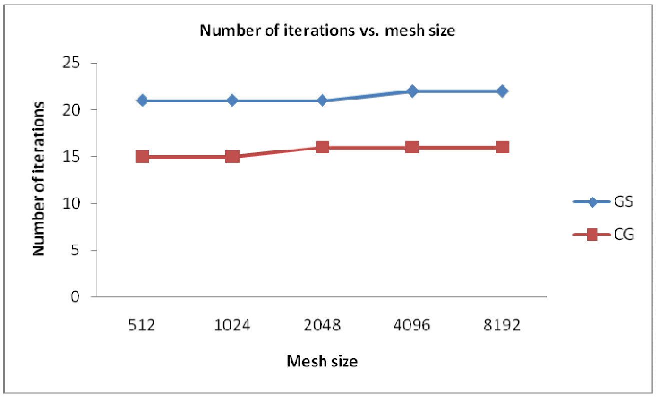

Figure 1. Number of iterations versus mesh size of the GS and CG methods

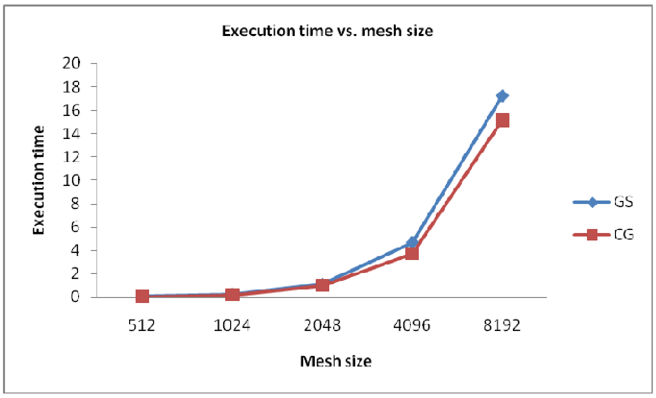

Figure 2. The execution time (seconds) versus mesh size of the GS and CG methods

Results of numerical simulations, which were obtained from implementations of the GS and CG iterative methods, have been recorded in Tables 1.

TABLE 1: COMPARISON OF A NUMBER OF ITERATIONS, EXECUTION TIME AND MAXIMUM ABSOLUTE ERROR FOR THE ITERATIVE METHODS

Methods	**Number of iteration**				
	Mesh Size				
	512	1024	2048	4096	8192
GS	21	21	21	22	22
CG	15	15	16	16	16
Methods	**Execution time (seconds)**				
	Mesh Size				
	512	1024	2048	4096	8192
GS	0.10	0.27	1.12	4.70	17.23
CG	0.04	0.16	0.97	3.71	15.14
Methods	**Maximum absolute error**				
	Mesh Size				
	512	1024	2048	4096	8192
GS	5.34E-03	2.67E-03	1.33E-03	6.67E-04	3.34E-04
CG	5.01E-03	2.47E-03	1.23E-03	6.51E-04	3.20E-04

IV. CONCLUSION

Through the results in Table 1, number of iterations of the GS and CG methods have decreased approximately 23.81%-28.57% compared to GS method, see Figure 1. For the execution time, CG method is much faster about 12.13%-75.00% compared to GS method, see Figure 2. The accuracy of CG method is in good agreement. Overall, the numerical results have shown that the CG method is more superior interm of number of iterations and the execution time than GS method.

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