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Discrete time market with serial correlations and optimal myopic strategies*

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Abstract

The paper studies discrete time market models with serial correlations. We found a market structure that ensures that the optimal strategy is myopic for the case of both power or log utility function. In addition, discrete time approximation of optimal continuous time strategies for diffusion market is analyzed. It is found that the performance of optimal myopic diffusion strategies cannot be approximated by optimal strategies with discrete time transactions that are optimal for the related discrete time market model.

Key words: control, economics, stochastic processes, optimal portfolio.

1 Introduction

The paper investigates discrete time stochastic market models. We consider an optimal investment problem that includes as a special case a problem where $\mathbf{E}U(X_T)$ is to be maximized, where X_T represents the total wealth at final time T and where $U(\cdot)$ is a utility function. We consider two types of utility function: $U(x) = \ln x$ and $U(x) = \delta^{-1}x^{\delta}$, where $\delta < 1$ and $\delta \neq 0$. For continuous time market models, these utilities have a special significance, in particular, because the optimal strategies for them can be *myopic*. In that case they do not require future distributions of parameters and do not depend on terminal time. In fact, the optimal strategies for power utilities

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for continuous time are myopic under some additional assumptions when the risk free rate, the appreciation rate, and the volatility matrix are random processes that are supposed to be currently observable (may be, with unknown prior distributions and evolution law). Besides, these parameters must be independent of the driving Brownian motion (i.e., it is the case of "totally unhedgeable" coefficients, according to Karatzas and Shreve (1998), Chapter 6). The solution that leads to myopic strategies goes back to Merton (1969); the case of random coefficients was discussed in Karatzas and Shreve (1998) and Dokuchaev and Haussmann (2001).

The real stock prices are presented as time series, so the discrete time (multi-period) models are more natural than continuous time models. On the other hand, continuous-time models give a good description of distributions and often allows explicit solutions of optimal investment problems.

For the real market, a formula for an optimal strategy derived for a continuous-time model can often be effectively used after the natural discretization. However, this strategy will not be optimal for time series observed in the real market. Therefore, it is important to extend the class of discrete time models that allow myopic and explicit optimal portfolio strategies. The problem of discrete-time portfolio selection has been studied in the literature, such as in Smith (1967), Chen et al. (1971), Leland (1968), Mossin (1968), Merton (1969), Samuelson (1969), Fama (1970), Hakansson (1971), Elton and Gruber (1974,1975), Winkler and Barry (1975), Francis (1976), Dumas and Luciano (1991), Östermark (1991), Grauer and Hakansson (1993), Pliska (1997), and Li and Ng (2000). If a discrete time market model is complete, then the martingale method can be used (see, e.g., Pliska (1997)). Unfortunately, a discrete time market model can be complete only under very restrictive assumptions. For incomplete discrete time markets, the main tool is dynamic programming that requires solution of Bellman equation starting at terminal time. For the general case, this procedure requires to calculate the conditional densities at any step (see, e.g., Pliska (1997) or Gikhman and Skorohod (1979)). This is why the optimal investment problems for discrete time can be more difficult than for continuous time setting that often allows explicit solutions.

There are several special cases when investment problem allows explicit solution for discrete time, and, for some cases, optimal strategies are myopic (see Leland (1968), Mossin (1968), Hakansson (1971)). However, the optimal strategy is not myopic and it cannot be presented explicitly for power utilities in general case. Hakansson (1971) showed that the optimal strategy is not myopic for $U(x) = \sqrt{x}$ if returns have serial correlation and evolve as a Markov process.

In the present paper, we study the optimal investment problem for a incomplete discrete time market under some general assumptions. We found a wide class of models with serial correlation such that the optimal strategies are myopic for both power and log utilities. In fact, the basic restrictions for this class of models are similar to the ones that ensure optimality of myopic strategy in continuous setting. We present an algorithm for calculation of optimal strategies. These strategies are analogs of Merton's optimal strategies for diffusion market model. Note that these strategies are different from the strategies constructed via the natural discretization of the Merton's strategies.

In addition, we found the following interesting consequence: the difference between the optimal expected utilities for discrete time and continuous time models does not disappear if the number of periods (or frequency of adjustments) grows. In particular, we found that the optimal expected utility calculated for continuous time market cannot be approximated by piecewise constant strategies with possible jumps at given times $\{t_k\}_{k=1}^T$, even if $T \to +\infty$ and $t_k - t_{k+1} \to 0$ (see Corollary 5.1 below).

Our model includes the case when the risk-free rate may have correlation with the risky asset. For simplicity, we considered single stock market only, but the generalization for the multi-stock case is straightforward.

2 The market model

We consider a model of a market consisting of the risk-free bond or bank account with price B_k and the risky stock with price S_k, $k = 0, 1, 2, \dots, T$, where $T \geq 1$ is a given integer. The initial prices $S_0 > 0$ and $B_0 > 0$ are given non-random variables.

We assume that

$$\begin{aligned} S_k &= \rho_k S_{k-1}(1+\xi_k), \\ B_k &= \rho_k B_{k-1}, \qquad\qquad k = 1, 2, \dots, T. \end{aligned} \tag{1}$$

Here ξ_k and ρ_k are random variables. We assume that $\xi_k > -1$ and $\rho_k \geq 1$ for all k.

We are given a standard probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where Ω is the set of all elementary events, $\mathcal{F}$ is a complete σ-algebra of events, and $\mathbf{P}$ is the probability measure.

Let us describe our main assumptions about the distributions of ξ_k and ρ_k.

Let $\mathcal{Z}$ be a metric space.

We assume that the following condition is satisfied.

Condition 2.1 *There exists a sequence $\{\theta_k\}_{k=1}^{T-1}$ of random variables that take values in $\mathcal{Z}$ such that*

(i) The pairs $\{(\xi_k, \rho_k)\}_{k=1}^{T}$ are mutually conditionally independent given $\theta_1, \dots, \theta_{T-1}$;

(ii) For any $k = 2, .., T-1$, the pair (ξ_k, ρ_k) does not depend on $\theta_k, \dots, \theta_{T-1}$ conditionally given $\theta_1, \dots, \theta_{k-1}$. In addition, the pair (ξ_1, ρ_1) does not depend on $\theta_1, \dots, \theta_{T-1}$ unconditionally.

(iii) θ_k, S_k, and B_k, are currently observable, i.e., they are known at time k. (Therefore, ξ_k and ρ_k are currently observable);

(iv) For $k = 0, \dots, T-1$, the conditional distributions of (ξ_{k+1}, ρ_{k+1}) given $(\theta_1, \dots, \theta_k)$ are known. (If $k = 0$, then the unconditional distribution is known).

Note that Condition 2.1(iv) can be relaxed (see Remark 4.1 below).

In this model, $\{\theta_k\}_{k=1}^{T-1}$ defines mutual dependence of the pairs $\{(\xi_k, \rho_k)\}$ (see Remark 2.1 below). In fact, the vector $(\theta_1, \dots, \theta_k)$ together with conditional distribution of (ξ_{k+1}, ρ_{k+1}) given $(\theta_1, \dots, \theta_k)$ describes information about ξ_{k+1} available at time k.

Remark 2.1 *Condition 2.1(i)-(ii) is satisfied, for instance, if there exists a metric space $\mathcal{W}$ and measurable functions $F_k(\cdot) : \mathcal{Z}^{k-1} \times \mathcal{W} \to \mathbf{R}^2$ such that $(\xi_k, \rho_k) = F_k(\theta_1, \dots, \theta_{k-1}, w_k)$, $k = 1, \dots, T$, where w_k are mutually independent random variables that take values in $\mathcal{W}$ and do not depend on $\{\theta_k\}_{k=1}^{T-1}$.*

We assume that the following more technical condition is satisfied.

Condition 2.2 *(i) $\mathbf{E}\{\rho_t^T \xi_k^T \,|\, \theta_1, \dots, \theta_{k-1}\} < +\infty$ a.s. for $k = 1, \dots, T$, $t \le k$ (if $k = 1$, then we mean the unconditional expectation); and* *(ii) For any $k = 1, \ldots, T$, there exist random variables $M'_k > 0$ and $M''_k > 0$ such that they are measurable with respect to the σ-algebra generated by $(\theta_1, ..., \theta_{k-1})$, and such*

$$\mathbf{P}(\xi_k \geq M'_k \,|\, \theta_1, \ldots, \theta_{k-1}) > 0, \quad \mathbf{P}(\xi_k \leq -M''_k \,|\, \theta_1, \ldots, \theta_{k-1}) > 0 \quad a.s.$$

(Again, if $k = 1$, then we mean the unconditional probability).

Let $X_0 = 1$ be the initial wealth of an investor at time $k = 0$, and let X_k be the wealth at time $k \geq 0$. We set that

$$X_k = \beta_k B_k + \gamma_k S_k, \tag{2}$$

where β_k is the quantity of the bond portfolio and where γ_k is the quantity of the stock portfolio. The pair (β_k, γ_k) describes the state of the bond-stocks securities portfolio at time k. We call the sequences of these pairs portfolio strategies.

We consider the problem of trading or choosing a portfolio strategy. Some constraints will be imposed on current operations in the market.

Definition 2.1 *A portfolio strategy $\{(\beta_k, \gamma_k)\}$ is said to be admissible if*

$$\mathbf{E}\left\{|\beta_k B_k| + |\gamma_k S_k| \,\Big|\, \theta_1, \ldots, \theta_k\right\} < +\infty \quad a.s, \quad k = 0, 1, \ldots, T-1,$$

and there exist measurable functions $\Phi_k : \mathbf{R}^{2k} \times \mathcal{Z}^k \to \mathbf{R}^2$ such that

$$(\beta_k, \gamma_k) = \Phi_k(S_1, \ldots, S_k, B_1, \ldots, B_k, \theta_1, \ldots, \theta_k).$$

(In other words, β_k and γ_k are defined by $\{\xi_m, \rho_m, \theta_m\}_{m=1}^k$, and they do not depend on the "future", or on S_{k+l}, B_{k+l}, θ_{k+l} for all $l > 0$.)

The main constraint in choosing a portfolio strategy is the so-called condition of self-financing.

Definition 2.2 *A portfolio strategy $\{(\beta_k, \gamma_k)\}$ is said to be self-financing, if*

$$X_{k+1} - X_k = \beta_k \left(B_{k+1} - B_k\right) + \gamma_k \left(S_{k+1} - S_k\right), \quad k = 0, 1, \ldots, T-1. \tag{3}$$

Remark 2.2 We do not impose additional conditions on strategies such as restrictions on short selling; furthermore, we assume that shares are divisible arbitrarily, and that the current prices are available at the time of transactions without delay. A more realistic model would allow (β_k, γ_k) depend on the history up to $k-1$ instead of k. In any case, we shall ignore these difficulties here.

For the trivial, risk-free, "keep-only-bonds" portfolio strategy, the portfolio contains only the bonds, $\gamma_k \equiv 0$, and the corresponding wealth is $X_k \equiv \beta_0 B_k / B_0 \equiv \prod_{m=1}^{k} \rho_m$.

Let $R_k \triangleq \prod_{m=1}^{k} \rho_m$.

Definition 2.3 *The process* $\widetilde{X}_k \triangleq R_k^{-1} X_k$ *is called the discounted wealth.*

Set $\widetilde{S}_k \triangleq R_k^{-1} S_k$, $k > 1$, $\widetilde{S}_0 \triangleq S_0$. Clearly, $\widetilde{S}_k = \widetilde{S}_{k-1}(1 + \xi_k)$.

Proposition 2.1 *(Dokuchaev and Savkin (2002), or Dokuchaev (2002), p.17). Let* $\{(X_k, \gamma_k)\}_{k=1}^{T}$ *be a sequence, and let the sequence* $\{(\beta_k, \gamma_k)\}_{k=1}^{T}$ *be an admissible portfolio strategy, where* $\beta_k = (X_k - \gamma_k X_k) B_k^{-1}$. *Then the strategy* $\{(\beta_k, \gamma_k)\}_{k=1}^{T}$ *is self-financing if and only if the process* $\widetilde{X}_k = R_k^{-1} \widetilde{X}_k$ *evolves as*

$$\widetilde{X}_{k+1} - \widetilde{X}_k = \gamma_k (\widetilde{S}_{k+1} - \widetilde{S}_k), \quad k = 0, 1, \ldots, T-1.$$

It follows from this proposition that the sequence $\{\gamma_k\} = \{\gamma_k\}_{k=0}^{T-1}$ alone suffices to specify self-financing portfolio strategy $\{(\beta_k, \gamma_k)\}$.

Definition 2.4 *Let* $\bar{\Sigma}$ *be the set of all sequences* $\{\gamma_k\} = \{\gamma_k\}_{k=0}^{T-1}$ *such that the corresponding self-financing portfolio strategy* $\{(\beta_k, \gamma_k)\}$ *is admissible.*

We shall call a sequence $\{\gamma_k\} \in \bar{\Sigma}$ self-financing strategy.

The following lemma will be useful.

Lemma 2.1 *Let* $\{\gamma_k'\} \in \bar{\Sigma}$ *be an arbitrary self-financing strategy, and let* $\widetilde{X}_k'$ *be the corresponding discounted wealth. Then the following holds for* $k = 0, \ldots, T-1$*:*

(i) If $\mathbf{P}(\widetilde{X}_k' < 0) > 0$, *then* $\mathbf{P}(\widetilde{X}_{k+1}' < 0) > 0$*;*

(ii) If $\mathbf{P}(\widetilde{X}_k' \leq 0, \widetilde{\gamma}_k' \neq 0) > 0$, *then* $\mathbf{P}(\widetilde{X}_{k+1}' < 0) > 0$.

3 Problem statement

Let $T > 0$ be a given integer, and let the initial wealth $X_0 = 1$ be given. Let $U(\cdot)$ be a utility function such that either $U(x) = \ln x$ or $U(x) = \delta^{-1}x^{\delta}$, where $\delta < 1$ and $\delta \neq 0$. More precisely, we assume that

$$U(x) = \begin{cases} \ln x, & x > 0 \\ -\infty, & x \le 0 \end{cases} \quad \text{or} \quad U(x) = \begin{cases} \delta^{-1}x^{\delta}, & x \ge 0 \ \text{or} \ x = 0,\, \delta > 0, \\ -\infty, & x < 0 \ \text{or} \ x = 0,\, \delta < 0. \end{cases} \tag{4}$$

Let $\lambda \in [0, 1]$ be given.

We assume that Conditions 2.1-2.2 are satisfied.

We may state our general problem as follows: Find a self-financing strategy $\{\gamma_k\} \in \bar{\Sigma}$ which solves the following optimization problem:

$$\text{Maximize} \quad \mathbf{E}U(R_T^{\lambda}\widetilde{X}_T) \quad \text{over} \quad \{\gamma_k\} \in \bar{\Sigma} \tag{5}$$

$$\text{subject to} \begin{cases} X_0 = 1, \\ X_k \ge 0 \quad \text{a.s.}, \quad k = 1, 2, \ldots, T. \end{cases} \tag{6}$$

Note that Condition 2.2(i) ensures that the expectation $\mathbf{E}U(R_T^{\lambda}\widetilde{X}_T)$ is well defined for any admissible strategy, and it can take values in $[-\infty, +\infty)$. Let us show this. By the definition of admissible strategy, $\mathbf{E}|R^{\lambda}\widetilde{X}_T| \le \mathbf{E}|X_T| < +\infty$. Hence $\mathbf{E}U^{+}(R_T^{\lambda}\widetilde{X}_T) < +\infty$, where $U^{+}(x) = \max(0, U(x))$. Therefore, the expectation in (5) is well defined.

Remark 3.1 *Clearly,*

$$R_T^{\lambda}\widetilde{X}_T = \begin{cases} X_T, & \lambda = 1 \\ \widetilde{X}_T, & \lambda = 0. \end{cases}$$

Therefore, our setting covers optimization of the total terminal wealth as well as optimization of the discounted terminal wealth.

Additional definitions

For $(z_1, \ldots, z_k) \in \mathcal{Z}^k$, $k \ge 1$, let

$$\Delta_k(z_1, \ldots, z_k) \triangleq \left\{ v \in \mathbf{R} : \quad \mathbf{P}\left(v\xi_{k+1} \ge -1 \,\middle|\, (\theta_1, \ldots, \theta_k) = (z_1, \ldots, z_k) \right) = 1 \right\},$$

and let $\Delta_0 \triangleq \{ v \in \mathbf{R} : \quad \mathbf{P}\left(v\xi_{k+1} \ge -1 \right) = 1 \}$.

Let the mapping $H_k : \mathbf{R} \times \mathcal{Z}^k \to \mathbf{R} \cup \{-\infty\}$, $k \geq 1$, be defined as

$$H_k(v, z_1, \ldots, z_k) \triangleq \mathbf{E}\left\{U\left(\rho_{k+1}^{\lambda}[1 + v\xi_{k+1}]\right) \,\middle|\, (\theta_1, \ldots, \theta_k) = (z_1, \ldots, z_k)\right\},$$

and let the mapping $H_0 : \mathbf{R} \to \mathbf{R} \cup \{-\infty\}$ be defined as $H_0(v) \triangleq \mathbf{E}U\left(\rho_1^{\lambda}[1 + v\xi_1]\right)$.

The expectation in the definition for H_k, $k \geq 0$, is well defined, because Condition 2.2(i) implies that

$$\mathbf{E}\left\{U^+\left(\rho_{k+1}^{\lambda}[1 + v\xi_{k+1}]\right) \,\middle|\, \theta_1, \ldots, \theta_k\right\} < +\infty \quad \text{a.s.}. \tag{7}$$

Remark 3.2 *For conciseness, we allow using functions $f(\theta_1, \ldots, \theta_k)$ or $f(z_1, \ldots, z_k)$ in statements for $k \geq 0$ sometimes without making special comments for $k = 0$, meaning that f does not depend on $\{\theta_i\}$ or $\{z_i\}$ in that case. In particular, the conditional expectation $\mathbf{E}\{\,\cdot\,|\,\theta_1, \ldots, \theta_k\}$ mentioned for $k \geq 0$ means the unconditional expectation $\mathbf{E}\{\,\cdot\,\}$ if $k = 0$.*

Lemma 3.1 *For any $k = 1, \ldots, T-1$, there exists a unique function $u_k : \mathcal{Z}^k \to \mathbf{R}$ such that $u_k(z_1, \ldots, z_k) \in \Delta_k(z_1, \ldots, z_k)$ for all $(z_1, ..., z_k)$, and*

$$H_k(\mu_k, \theta_1, \ldots, \theta_k) \geq H_k(v, \theta_1, \ldots, \theta_k) \qquad \forall v \in \Delta_k(\theta_1, \ldots, \theta_k), \tag{8}$$

where

$$\mu_k = u_k(\theta_1, \ldots, \theta_k), \qquad k = 1, \ldots, T-1, \tag{9}$$

Furthermore, there exists the unique $u_0 = \mu_0 \in \Delta_0$ such that $H_0(\mu_0) \geq H_k(v)$ ($\forall v \in \Delta_0$).

Time invariant presentation for μ_k

Let $D \triangleq [1, +\infty) \times (-1, +\infty)$. Let $\mathcal{M}$ be the set of all probability measures on D.

Let $\mathcal{P}_{\xi_{k+1}, \rho_{k+1}}(\theta_1, \ldots, \theta_k, \cdot) \in \mathcal{M}$ be the probability measure on D generated by (ξ_{k+1}, ρ_{k+1}) in the conditional probability space given $(\theta_1, \ldots, \theta_k)$.

It can be useful to represent $\mu_k = u_k(\theta_1, \ldots \theta_k)$ as

$$u_k(\theta_1, \ldots, \theta_k) = \mathcal{U}\left(\mathcal{P}_{\xi_{k+1}, \rho_{k+1}}(\theta_1, \ldots, \theta_k, \cdot)\right), \tag{10}$$

where $\mathcal{U} : \mathcal{M} \to \mathbf{R}$ is some mapping.

Clearly,

$$H_k(v, z_1, \dots, z_k) = \int_D \mathcal{P}_{\xi_{k+1},\rho_{k+1}}(z_1, \dots, z_k, dx \times dy) U(x^\lambda[1 + vy]). \tag{11}$$

Therefore, u_k can be represented as (10), where the mapping $\mathcal{U} : \mathcal{M} \to \mathbf{R}$ is defined such that

$$\int_D \mathcal{P}(dx \times dy) U(x^\lambda[1 + \mathcal{U}(\mathcal{P})y]) \ge \int_D \mathcal{P}(dx \times dy) U(x^\lambda[1 + vy]) \qquad \forall v \in \mathbf{R}.$$

Some properties of μ_k

We shall need the following lemma.

Lemma 3.2 *(i) If* $\mathbf{E}\{\xi_{k+1} \,|\, \theta_1, \dots, \theta_k\} = 0$*, then* $\mu_k = 0$*;*

(ii) If $\mathbf{E}\{\xi_{k+1} \,|\, \theta_1, \dots, \theta_k\} > 0$*, then* $\mu_k > 0$*;*

(iii) If $\mathbf{E}\{\xi_{k+1} \,|\, \theta_1, \dots, \theta_k\} < 0$*, then* $\mu_k = \inf\{v \in \Delta_k(\theta_1, \dots, \theta_k)\}$*.*

For the case of $k = 0$*, we mean that the expectations here are unconditional.*

4 The main result

Theorem 4.1 *Let the portfolio strategy* $\{(\beta_k, \gamma_k)\}$ *be defined as a self-financing closed-loop strategy such that*

$$\gamma_k = \mu_k X_k S_k^{-1}, \qquad \beta_k = \frac{X_k - \gamma_k S_k}{B_k}, \quad k = 0, 1, \dots, T-1, \tag{12}$$

where X_k *is the corresponding wealth, and where* μ_k *are defined in Lemma 3.1. Then this portfolio strategy is admissible,* $X_k \ge 0$ *for* $k = 1, \dots, T$ *a.s.,* $\{\gamma_k\} \in \bar{\Sigma}$*, and it is the unique optimal strategy for problem (5)-(6).*

Let $\widetilde{X}_k = R_k^{-1} X_k$ be the corresponding discounted wealth for strategy (9), (12). By (12),

$$\gamma_k = \mu_k \widetilde{X}_k \widetilde{S}_k^{-1}, \tag{13}$$

and Proposition 2.1 implies that

$$\widetilde{X}_{k+1} - \widetilde{X}_k = \gamma_k[\widetilde{S}_{k+1} - \widetilde{S}_k] = \gamma_k \widetilde{S}_k \xi_{k+1} = \mu_k \widetilde{X}_k \xi_{k+1}. \tag{14}$$

By Theorem 4.1, the strategy with $\mu_k \equiv 0$ cannot outperform strategy (9), (12). The function U is monotonic, and $\rho_t^\lambda \geq 1$. Hence $\mathbf{E}U(R_T^\lambda \widetilde{X}_T) \geq U(1) = U(X_0)$.

Note that strategy (9), (12) is myopic: it does not depend at time k on distributions of $\xi_{k+2}, \xi_{k+3}, \ldots$, and it does not depend on T. In particular, it follows that $\mathbf{E}U(\rho^\lambda \widetilde{X}_m) \geq U(1)$ for all $m = 1, \ldots T$, since Theorem 4.1 defines the same optimal strategies for the case when T is replaced for m.

In addition, this strategy allows time invariant presentation via (10) as $\mu_k = \mathcal{U}\left(\mathcal{P}_{\xi_{k+1},\rho_{k+1}}(\theta_1, \ldots, \theta_k, \cdot)\right)$, where $\mathcal{U} : \mathcal{M} \to \mathbf{R}$ is a time independent functional on the set of probability measures.

Strategy (9), (12) is an analog Merton's myopic strategy for continuous time diffusion market (see equation (18) below).

Remark 4.1 *In fact, we do not need to know the entire conditional distribution of* (ξ_{k+1}, ρ_{k+1}) *given* $\theta_1, \ldots, \theta_k$ *for calculation of the optimal strategy at time* k*: we need only the single parameter* $u_k(\theta_1, \ldots, \theta_k)$ *of this distribution. This parameter* u_k *can be considered as an analog of the so-called "market price of risk" process defined for the continuous time market (in particular, the market price of risk is* $\sigma(t)^{-1}[a(t) - r(t)]$ *for market (15) described below).*

Remark 4.2 *If* $U(x) = \ln x$*, then Theorem 4.1 holds without Condition 2.1(i)-(ii) (see, e.g., Hakannson (1971)).*

Remark 4.3 *The model described by Conditions 2.1-2.2 covers many important cases. However, Condition 2.1(i) is not satisfied for a case when* $\{(\xi_k, \rho_k)\}$ *is a general Markov process. That is why the result does not contradict the conclusion of Hakannson (1971) that serial correlation may prevent the optimal strategy to be myopic for* $U(x) = \sqrt{x}$*. Our class of models is different.*

Remark 4.4 *Theorem 4.1 can be extended for a market with* n *stocks. In that case, minimization in (8) will be with respect to a* n*-dimensional vector.*

5 Discretization of continuous time model and Merton's strategies

Let us consider continuous time market with two assets, bond with price $P_0(t)$ and stock with price $P(t)$, $t \in [0, \tau]$, such that

$$\begin{aligned} &\tfrac{dP_0}{dt}(t) = r(t)P_0(t), \\ &dP(t) = P(t)[a(t)dt + \sigma(t)d\mathrm{w}(t)]. \end{aligned} \tag{15}$$

Here $\tau > 0$ is given terminal time, $\mathrm{w}(t)$ is a Wiener process, $r(t)$, $a(t)$, and $\sigma(t)$, are some scalar random processes, $P_0(0) = P(0) = 1$. We also assume that $|\sigma(t)| \geq \text{const } > 0$, $r(t) \geq 0$, and that the process (a, r, σ) is bounded and does not depend on $\mathrm{w}(\cdot)$.

Let $\bar{\Sigma}^c$ denotes the class of all random processes $\gamma^c(t)$, $t \in [0, \tau]$, that are square integrable and adapted to the filtration generated by $(r(t), a(t), \sigma(t), P_0(t), P(t))$. We assume that any process $\gamma^c(\cdot) \in \bar{\Sigma}^c$ represents the quantity of stock in portfolio and uniquely defines a self-financing portfolio such that

$$\widetilde{X}^c(t) = X^c(0) + \int_0^t \gamma^c(s)d\widetilde{P}(s), \tag{16}$$

where $\widetilde{P}(t) \triangleq P_0(t)^{-1}P(t)$ is the discounted stock price, $\widetilde{X}^c(t) \triangleq P_0(t)^{-1}X^c(t)$ is the discounted wealth, $X^c(t)$ represents the wealth at time t. (These definitions for continuous time market are given in more details in Katatzas and Shreve (1998) or in the author's book (2002)). The definition of $\bar{\Sigma}^c$ implies that the market parameters are observed for this model.

We assume that we are given initial wealth $X^c(0) = 1$ and we are given utility function $U(x) = \ln x$ or $U(x) = \delta^{-1}x^\delta$, where $\delta < 1$, $\delta \neq 0$.

Let us consider the optimal investment problem

$$\text{Maximize} \quad \mathbf{E}U(\widetilde{X}^c(\tau)) \quad \text{over} \quad \gamma^c(\cdot) \in \bar{\Sigma}^c. \tag{17}$$

The solution of a similar problem was first obtained by Merton (1969). Merton's optimal strategy can be presented as

$$\gamma^M(t)^\top \triangleq \delta_* \frac{a(t) - r(t)}{P(t)\sigma(t)^2} X^M(t), \tag{18}$$

where $\delta_* = (1-\delta)^{-1}$, and where $\gamma^M(t)$ is the quantity of the stock portfolio, $X^M(t)$ is the corresponding wealth (for the case of random (r, a, σ), see, e.g., Karatzas and Shreve (1998) or Proposition 5.1 from Dokuchaev and Haussmann (2001)). If $U(x) = \ln x$ then we assume that $\delta = 0$ and $\delta_* = 1$.

If $\mathbf{E}\int_0^\tau [a(t) - r(t)]^2 \sigma(t)^{-2} dt > 0$, then

$$\mathbf{E}U(\widetilde{X}^M(\tau)) = \max_{\gamma^c(\cdot)\in\bar{\Sigma}^c} \mathbf{E}U(\widetilde{X}^c(\tau)) > \mathbf{E}U(X^c(0)). \tag{19}$$

In particular, the optimal expected utility for $U(x) = \ln x$ is

$$\mathbf{E}\ln \widetilde{X}^M(\tau) = \max_{\gamma^c(\cdot)\in\bar{\Sigma}^c} \mathbf{E}\ln \widetilde{X}^c(\tau) = \ln X^c(0) + \frac{1}{2}\mathbf{E}\int_0^\tau \frac{[a(t) - r(t)]^2}{\sigma(t)^2} dt. \tag{20}$$

Strategy (18) is said to be myopic because it does not require future distributions of parameters and does not depend on τ. It gives explicit solution of problem (17) for the case of observed parameters (r, a, σ). Discrete time strategy (9), (12) is an analog of strategy (18) since it is also based on the currently observed market parameters θ_k.

Let $T > 0$ be a given integer. Let $\bar{\Sigma}^c_{PC}(T) \subset \bar{\Sigma}^c$ be the set of all processes $\gamma^c(\cdot)$ that are piecewise constant and right continuous such that jumps may occur only at times t_k, where

$$t_0 = 0, \quad t_k = t_{k-1} + h, \quad k = 1, \ldots, T,$$

where $h = \tau/T$. For this class of strategies, buying or selling stocks can happens at time t_k only.

It is easy to see that the problem

$$\text{Maximize} \quad \mathbf{E}U(\widetilde{X}^c(\tau)) \quad \text{over} \quad \gamma^c(\cdot) \in \bar{\Sigma}^c_{PC}(T) \tag{21}$$

is equivalent to discrete time problem (5)-(6) with $X_0 = X^c(0) = 1$, $\lambda = 0$, and

$$B_k \equiv P_0(t_k), \quad S_k = P(t_k), \qquad k = 0, 1, \ldots, T. \tag{22}$$

In particular, any strategy $\{\gamma_k\} \in \bar{\Sigma}$ for problem (5)-(6), (22) defines the unique strategy $\gamma^c(\cdot) \in \bar{\Sigma}^c_{PC}(T)$ such that

$$\gamma^c(t) = \gamma_k, \quad t \in [t_k, t_{k+1}), \quad k = 0, 1, \ldots, T-1, \tag{23}$$

and (16) implies that

$$\widetilde{X}^c(t_{k+1}) - \widetilde{X}^c(t_k) = \gamma_k[\widetilde{P}(t_{k+1}) - \widetilde{P}(t_k)].$$

The natural discretization for strategy (18) gives values $\gamma_k^M \triangleq \gamma^M(t_k)$ of piecewise approximation for $\gamma^M(\cdot)$ in $\bar{\Sigma}_{PC}^c(T)$ as

$$\gamma_k^M = \delta_* \frac{a(t_k) - r(t_k)}{P(t_k)\sigma(t_k)^2} X^M(t_k). \tag{24}$$

However, the corresponding wealth $X_{PC}^M(t)$ for this piecewise strategies may be very different from the optimal wealth $X^M(t)$. Let us show this. By (16), $\widetilde{X}_{PC}^M(t_{k+1}) = \widetilde{X}_{PC}^M(t_k) + \gamma_k^M \widetilde{S}_k \xi_{k+1}$, where $\widetilde{X}_{PC}^M(t) = P_0(t)^{-1} X_{PC}^M(t)$ is the corresponding discounted wealth, $\widetilde{S}_k \triangleq \widetilde{P}(t_k)$,

$$\xi_{k+1} = \frac{S_{k+1}}{\rho_{k+1} S_k} - 1, \quad \rho_{k+1} = \frac{B_{k+1}}{B_k}.$$

For instance, if $a(t_k) < r(t_k)$, then $\gamma_k^M < 0$ and $\mathbf{P}\left(\widetilde{X}_{PC}^M(t_{k+1}) < 0\right) > 0$, since $\mathbf{P}(\xi_{k+1} > K) > 0$ for all $K > 0$. By Lemma 2.1, it follows that, in contrast with (19), $\mathbf{E}U(X_{PC}^M(\tau)) = -\infty$ in this case.

It is no surprise that the strategy defined by (24), (23) is not optimal for problem (21). Let us describe the optimal strategies for (21) and for the corresponding discrete time problem (5)-(6), (22).

Case of piecewise constant coefficients

Let us consider discrete time problem (5)-(6) with $\lambda \in [0, 1]$ that corresponds to model (22), where $T > 0$ is given, and where $P_0(t)$ and $P(t)$ are processes defined by (15). In this subsection, we assume that $(r(t), a(t), \sigma(t))$ is a random process such that its paths are right continuous and piecewise constant processes that can have jumps at times t_k only. In particular, $(r(t+0), a(t+0), \sigma(t+0)) \equiv (r(t), a(t), \sigma(t))$, and the values of $(r(t), a(t), \sigma(t)) = (r(t_k), a(t_k), \sigma(t_k))$ are known at time t_k for all $t \in [t_k, t_{k+1})$. We also assume that the process $(r(\cdot), a(\cdot), \sigma(\cdot))$ does not depend on w$(\cdot)$.

Let $\mathcal{Z} = \mathbf{R}^3$,

$$\theta_k = \begin{pmatrix} \theta_k^{(r)} \\ \theta_k^{(a)} \\ \theta_k^{(\sigma)} \end{pmatrix} \triangleq \begin{pmatrix} r(t_k)h \\ \left[a(t_k) - \frac{\sigma(t_k)^2}{2}\right] h \\ \sigma(t_k)\sqrt{h} \end{pmatrix}.$$

We assume that the random vector θ_k is currently observed, i.e., its value is known at time t_k.

Clearly, Condition 2.1 is satisfied with

$$\rho_{k+1} = e^{\theta_k^{(r)}}, \quad \xi_{k+1} = \exp\left(-\theta_k^{(r)} + \theta_k^{(a)} + \theta_k^{(\sigma)} w_{k+1}\right) - 1,$$

where $w_{k+1} \triangleq h^{-1}[\mathrm{w}(t_{k+1}) - \mathrm{w}(t_k)]$.

Let

$$H_k(v, \theta_k) = H_k(v, \theta_1, \ldots, \theta_k) \triangleq \mathbf{E}\{U\left(\rho_{k+1}^{\lambda}[1 + v\xi_{k+1}]\right) \mid \theta_1, \ldots, \theta_k\}. \tag{25}$$

We have that

$$H_k(v, \theta_k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-s^2/2} U\left(e^{\lambda\theta_k^{(r)}}\left[1 + v\left(e^{-\theta_k^{(r)} + \theta_k^{(a)} + \theta_k^{(\sigma)} s} - 1\right)\right]\right) ds. \tag{26}$$

To obtain μ_k from (9), we need to find v that ensures the maximum of $H_k(v, \theta_k)$ given θ_k. Since ξ_{k+1} has conditionally log-normal distribution with support $(0, +\infty)$, we have that $\mathbf{P}(\xi_{k+1} < -\varepsilon \,|\, \theta_1, \ldots, \theta_k) > 0$ and $\mathbf{P}(\xi_{k+1} > \varepsilon^{-1} \,|\, \theta_1, \ldots, \theta_k) > 0$ for any $\varepsilon \in (0, 1)$. By Lemma 3.2(i), it follows that $\Delta_k(\theta_1, \ldots, \theta_k) \equiv [0, 1]$ for all k. Therefore, it suffices to consider maximization only over $v \in [0, 1]$, and $\mu_k \in [0, 1]$ a.s.. If one take $\mu_k \notin [0, 1]$, then $\gamma_k \notin [0, P(t_k)^{-1}]$, $\mathbf{P}(\widetilde{X}_{k+1} < 0) > 0$, and, by Lemma 2.1(i), $\mathbf{E}U(R_T^{\lambda}\widetilde{X}_T) = -\infty$.

Further, we have that

$$\mathbf{E}\{\xi_{k+1} \,|\, \theta_1, \ldots, \theta_k\} = e^{a(t_k) - r(t_k)} - 1.$$

By Lemma 3.2 (ii),(iv), we have that if $a(t_k) - r(t_k) \leq 0$, then $\mu_k = 0$.

Corollary 5.1 *If $\sup_t[a(t) - r(t)] < 0$ then optimal strategy (9), (12) is $\mu_k \equiv 0$ ($\forall k$). In that case, optimal expected utility (20) cannot be approximated by piecewise constant strategies from $\bar{\Sigma}_{PC}^{c}(T)$ even if $T \to +\infty$ and $h = \tau/T \to 0$, because*

$$\sup_{\gamma^c(\cdot) \in \bar{\Sigma}_{PC}^c(T)} \mathbf{E}U(\widetilde{X}^c(\tau)) = \sup_{\{\gamma_k\} \in \bar{\Sigma}} \mathbf{E}U(\widetilde{X}_T) = \mathbf{E}U(X(0)) < \sup_{\gamma^c(\cdot) \in \bar{\Sigma}^c} \mathbf{E}U(\widetilde{X}^c(\tau))$$

for all $T > 0$.

We leave for further research the question of whether or not approximation of optimal continuous time expected utilities via discrete time strategies is possible for other classes of parameters.

The restriction $\mu_k \in [0, 1]$ can be avoided only for a model with additional restrictions on the range of ξ_{k+1}.

Numerical example

The scalar optimization problem can be solved numerically. For instance, let

$$h = 1/6, \quad a(t_k) = 0.14, \quad r(t_k) = 0.07, \quad \sigma(t_k) = 0.9. \tag{27}$$

We have that $\theta_k = (0.0023, 0.0117, 0.367)$. If $U(x) = \ln x$, then the maximum of $H_k(v, \theta_k)$ over v is achieved at $\mu_k = 0.5864$ for all $\lambda \in [0, 1]$. The value (24) of Merton's strategy after discretization is $\gamma_k^M = 0.086 \cdot X^M(t_k)/P(t_k)$, with these parameters and with $\lambda = 0$.

If U is replaced for $U(x) = -2x^{-1/2}$ then the maximum of $H(v, \theta_k)$ is achieved at $\mu_k = 0.3864$ given (27) for all $\lambda \in [0, 1]$. The value (24) of Merton's strategy is $\gamma_k^M = 0.057 \cdot X^M(t_k)/P(t_k)$, with these parameters and $\lambda = 0$.

Case of infinite dimensional range for θ_k

For some models, the space $\mathcal{Z}$ can be infinity dimensional. For example, let us consider model (22) under assumptions that

$$(r(t), a(t), \sigma(t)) = \alpha(t) + \int_0^{\max(0, t-h)} \alpha_0(t, s) d\mathrm{w}_0(s),$$

where $\mathrm{w}_0(\cdot)$ is a N-dimensional Wiener process that does not depend on $\mathrm{w}(\cdot)$, $h = \tau/T$, and where $\alpha(\cdot) : [0, \tau] \to \mathbf{R}^3$ and $\alpha_0(\cdot) : [0, \tau] \times [0, \tau - h] \to \mathbf{R}^{3\times N}$ are known deterministic measurable bounded functions. Then Condition 2.1 is satisfied with $\mathcal{Z} = C([0, h]; \mathbf{R}^N)$, $\theta_k = \mathrm{w}_0(\cdot + t_k)|_{[0,h]}$, $t_k = kh$, $h = \tau/T$. In that case, $(\xi_k, \rho_k) = F_k(\theta_1, \dots, \theta_{k-1}, w_k)$, where $w_k = \mathrm{w}(\cdot + t_k)|_{[0,h]}$, and where $F_k : \mathcal{Z}^{k-1} \times C([0, h]; \mathbf{R}) \to \mathbf{R}^2$ are some deterministic measurable mappings. Calculations of μ_k under this conditions is straightforward and depends on the choice of (α, α_0).

6 Proofs

We need the following auxiliary lemma.

Lemma 6.1 *Let $k \in \{0, \dots, T-1\}$ be given, and let $M'_{k+1} > 0$ and $M''_{k+1} > 0$ be such that Condition 2.2 (ii) is satisfied for this k. Then the following holds:*

(i) For any $v \notin [-1/M'_{k+1}, 1/M''_{k+1}]$*,* $H_k(v, \theta_1, \ldots, \theta_k) = -\infty$ *a.s.;*

(ii) $\Delta_k(\theta_1, \ldots, \theta_k) \subseteq [-1/M'_{k+1}, 1/M''_{k+1}]$ *a.s.; and*

(iii) $\mu_k \in [-1/M'_{k+1}, 1/M''_{k+1}]$ *and* $\mu_k \xi_{k+1} \geq -1$ *a.s..*

Proof. It suffices to prove (i) only, since (ii) and (iii) follow immediately. We have that if $v \notin [-1/M'_{k+1}, 1/M''_{k+1}]$, then

$$\mathbf{P}(v\xi_{k+1} < -1 \,|\, \theta_1, \ldots, \theta_k) > 0 \quad \text{a.s..}$$

From this inequality and (7), it follows for this v that $H_k(v, \theta_1, \ldots, \theta_k) = -\infty$ a.s. $\square$

Proof of Lemma 2.1. Clearly,

$$\mathbf{P}(\widetilde{X}'_{k+1} < 0) \;=\; \int_{\mathcal{Z}^T} \mathcal{P}_\Theta(dz) \mathbf{P}\left(\widetilde{X}'_{k+1} < 0 \,\middle|\, \Theta = z\right),$$

and

$$\begin{aligned}
\mathbf{P}\left(\widetilde{X}'_{k+1} < 0 \,\middle|\, \Theta = z\right) &= \mathbf{P}\left(\widetilde{X}'_k + \gamma'_k \xi_{k+1} < 0 \,\middle|\, \Theta = z\right) \\
&\geq \mathbf{P}\left(\widetilde{X}'_k < 0, \gamma'_k \xi_{k+1} \leq 0 \,\middle|\, \Theta = z\right) \\
&= \mathbf{P}\left(\widetilde{X}'_k < 0 \,\middle|\, \Theta = z\right) \mathbf{P}\left(\gamma'_k \xi_{k+1} \leq 0 \,\middle|\, \widetilde{X}'_k < 0\,, \Theta = z\right) \\
&\geq \mathbf{P}\left(\widetilde{X}'_k < 0 \,\middle|\, \Theta = z\right) p(z).
\end{aligned}$$

Here

$$\begin{aligned}
p(z) &\triangleq \mathbf{P}\left(\gamma'_k \leq 0,\, \xi_{k+1} \geq M'_{k+1} \,\middle|\, \widetilde{X}'_k < 0\,, \Theta = z\right) \\
&+ \mathbf{P}\left(\gamma'_k > 0,\, \xi_{k+1} \leq -M''_{k+1} \,\middle|\, \widetilde{X}'_k < 0\,, \Theta = z\right).
\end{aligned}$$

Let $p_1(z) \triangleq \mathbf{P}\left(\gamma'_k \leq 0 \,\middle|\, \widetilde{X}'_k < 0\,, \Theta = z\right)$ and $p_2(z) \triangleq \mathbf{P}\left(\gamma'_k > 0 \,\middle|\, \widetilde{X}'_k < 0\,, \Theta = z\right)$. By Condition 2.1(ii), we have that

$$\begin{aligned}
p(z) &= \mathbf{P}\left(\xi_{k+1} \geq M'_{k+1} \,\middle|\, \widetilde{X}'_k < 0\,, \Theta = z\right) p_1(z) \\
&+ \mathbf{P}\left(\xi_{k+1} \leq -M''_{k+1} \,\middle|\, \widetilde{X}'_k < 0\,, \Theta = z\right) p_2(z) \\
&= \mathbf{P}\left(\xi_{k+1} \geq M'_{k+1} \,\middle|\, \Theta = z\right) p_1(z) + \mathbf{P}\left(\xi_{k+1} \leq -M''_{k+1} \,\middle|\, \Theta = z\right) p_2(z).
\end{aligned}$$

Note that $p_1(z)+p_2(z)=1$. By Condition 2.2(ii), we have that $\mathcal{P}_\Theta(z:\ p(z)>0)=1$. Then statement (i) follows. The proof of statement (ii) is similar, if one uses that

$$\begin{aligned}
&\mathbf{P}\left(\widetilde{X}'_{k+1}<0\,\Big|\,\Theta=z\right)\geq\mathbf{P}\left(\widetilde{X}'_k\leq 0,\gamma'_k\xi_{k+1}<0\,\Big|\,\Theta=z\right)\\
&=\mathbf{P}\left(\widetilde{X}'_k<0\,,\gamma'_k\neq 0\,\Big|\,\Theta=z\right)\mathbf{P}\left(\gamma'_k\xi_{k+1}\leq 0\,\Big|\,\widetilde{X}'_k<0\,,\gamma'_k\neq 0\,,\Theta=z\right)\\
&\geq\mathbf{P}\left(\widetilde{X}'_k\leq 0\,,\gamma'_k\neq 0\,\Big|\,\Theta=z\right)\Big[\mathbf{P}\left(\gamma'_k<0,\ \xi_{k+1}\geq M'_{k+1}\,\Big|\,\widetilde{X}'_k\leq 0\,,\gamma'_k\neq 0\,,\Theta=z\right)\\
&\qquad\qquad+\mathbf{P}\left(\gamma'_k>0,\ \xi_{k+1}\leq -M''_{k+1}\,\Big|\,\widetilde{X}'_k\leq 0\,,\gamma'_k\neq 0\,,\Theta=z\right)\Big].
\end{aligned}$$

□

Proof of Lemma 3.1. Let $\bar{M}'_k\triangleq\sup M'_k$ and $\bar{M}''_k=\inf M''_k$, where M'_k and M''_k are such that Condition 2.2(ii) is satisfied. (The cases when $\bar{M}'_k=+\infty$ and $\bar{M}''_k=1$ are not excluded). By Lemma 6.1, it follows that

$$\Delta_k(\theta_1,\ldots,\theta_k)\subseteq\cap_{\{(M'_k,M''_k)\}}[-1/M'_k,1/\bar{M}''_k]=[-1/\bar{M}'_k,1/\bar{M}''_k]\quad\text{a.s..}$$

Let $v\in[-1/\bar{M}'_k,1/\bar{M}''_k]$. Then

$$\mathbf{P}(v\xi_{k+1}<-1\,|\,\theta_1,\ldots,\theta_k)=0\quad\text{a.s.,}$$

and $v\in\Delta_k(\theta_1,\ldots,\theta_k)$. Hence $\Delta_k(\theta_1,\ldots,\theta_k)=[-1/\bar{M}'_k,1/\bar{M}''_k]$ a.s., and it is compact.

The function $U(x)$ is strictly concave, hence $H_k(v,\theta_1,\ldots,\theta_k)$ is also strictly concave in v given $(\theta_1,\ldots,\theta_k)$. Then the proof follows. □

Proof of Lemma 3.2. Since $H_k(\cdot,\theta_1,\ldots,\theta_k)$ is strictly concave, we have that if $\frac{\partial H_{k+1}}{\partial v}(v,\theta_1,\ldots,\theta_k)=0$ then $u_k(\theta_1,\ldots,\theta_k)=v$.

Assume that if $U(x)=\ln x$ then $\delta=0$. Clearly,

$$\frac{\partial U}{\partial v}(x^\lambda[1+vy])=\frac{y}{(1+vy)^{\delta+1}}.$$

Hence

$$\frac{\partial H_k}{\partial v}(v,\theta_1,\ldots,\theta_k)|_{v=0}=\int_D\mathcal{P}_{\xi_{k+1},\rho_{k+1}}(\theta_1,\ldots,\theta_k,dx\times dy)y=\mathbf{E}\{\xi_{k+1}\,|\,\theta_1,\ldots,\theta_k\}.$$

This implies (i)-(iii). □

Proof of Theorem 4.1. We shall consider first the case when $U(x)=\delta^{-1}x^\delta$, where $\delta\neq 0$, $\delta<1$.

By (14),

$$\widetilde{X}_k = \widetilde{X}_{k-1} + \mu_{k-1}\widetilde{X}_{k-1}\xi_k, \quad k = 1,\dots,T,$$

and it follows that

$$\widetilde{X}_m = \prod_{k=1}^{m}(1+\mu_{k-1}\xi_k), \quad m = 1,\dots,T. \tag{28}$$

Let $\mathcal{P}_\Theta(\cdot)$ be the probability distribution on $\mathcal{Z}^{T-1}$ generated by $\Theta \triangleq (\theta_1,\dots,\theta_{T-1})$.

Let us show that $\{\gamma_k\} \in \bar{\Sigma}$. By Lemma 6.1, it follows that $|\mu_k| \le c$ given $(\theta_1,\dots,\theta_k)$, where $c = c(\theta_1,\dots,\theta_k) \in [0,+\infty)$. By Condition 2.2 (i), it follows that $\mathbf{E}\{|\widetilde{X}_k|\,|\,\theta_1,\dots,\theta_k\} < +\infty$, $\mathbf{E}\{|\gamma_k S_k|\,|\,\theta_1,\dots,\theta_k\} < +\infty$ a.s., and, therefore, $\mathbf{E}\{|\beta_k B_k|\,|\,\theta_1,\dots,\theta_k\} < +\infty$ a.s. for $k = 0,1,2,\dots,T-1$. Hence strategy (9), (12) is admissible and $\{\gamma_k\} \in \bar{\Sigma}$.

By Lemma 6.1 again, it follows that μ_k is such that $\gamma_{k-1}\xi_k \ge -1$, hence $\widetilde{X}_k \ge 0$ a.s. for all $k = 1,\dots,T$.

Let us show that $\mathbf{E}U(R_m^\lambda\widetilde{X}_m) \ge U(1)$ for all $m \in \{1,\dots,T\}$. By (8), it follows that

$$\mathbf{E}\left\{U(\rho_k^\lambda[1+\mu_{k-1}\xi_k])|\theta_1,\dots,\theta_{k-1}\right\} \ge \mathbf{E}\left\{U(\rho_k^\lambda[1+0\cdot\xi_k])|\theta_1,\dots,\theta_k\right\} \ge U(1) \quad \text{a.s.} \tag{29}$$

Let $\bar{U}(x) \triangleq \delta U(x) = x^\delta$.

We have

$$\begin{aligned}
\mathbf{E}\bar{U}(R_m^\lambda\widetilde{X}_m) &= \int_{\mathcal{Z}^{T-1}}\mathcal{P}_\Theta(dz)\mathbf{E}\{\bar{U}(R_m^\lambda\widetilde{X}_m)|\Theta = z\} \\
&= \int_{\mathcal{Z}^{T-1}}\mathcal{P}_\Theta(dz)\mathbf{E}\left\{\prod_{k=1}^{m}\bar{U}(\rho_k^\lambda[1+\mu_{k-1}\xi_k])|\Theta = z\right\} \\
&= \int_{\mathcal{Z}^{T-1}}\mathcal{P}_\Theta(dz)\mathbf{E}\left\{\prod_{k=1}^{m}\bar{U}(\rho_k^\lambda[1+u_{k-1}(z_1,\dots,z_{k-1}])\xi_k)|\Theta = z\right\} \\
&= \int_{\mathcal{Z}^{T-1}}\mathcal{P}_\Theta(dz)\prod_{k=1}^{m}\mathbf{E}\left\{\bar{U}(\rho_k^\lambda[1+u_{k-1}(z_1,\dots,z_{k-1})\xi_k])|\Theta = z\right\},
\end{aligned}$$

where $z = (z_1,\dots,z_{T-1})$. By (29), we have that if $\delta > 0$ then $\mathbf{E}\bar{U}(R_m^\lambda\widetilde{X}_m) \ge \bar{U}(1)^m = \bar{U}(1)$, and if $\delta < 0$ then $\mathbf{E}\bar{U}(R_m^\lambda\widetilde{X}_m) \le \bar{U}(1)^m = \bar{U}(1)$. In both cases,

$$\mathbf{E}U(R_m^\lambda\widetilde{X}_m) \ge U(1) > -\infty, \quad m = 1,\dots T. \tag{30}$$

Let us show that the strategy $\{\gamma_k\}$ is optimal in the class $\bar\Sigma$. Consider an arbitrary self-financing strategy $\{\gamma'_k\}\in\bar\Sigma$. Let $\widetilde X'_k\ge 0$ be the corresponding normalized wealth, $\widetilde X'_0=1$.

If $\mathbf{P}(\widetilde X'_T<0)>0$ then $\mathbf{E}U(R^\lambda_T\widetilde X'_T)=-\infty$, and, by (30), it follows that $\mathbf{E}U(R^\lambda_T\widetilde X_T)>\mathbf{E}U(R^\lambda_T\widetilde X'_T)$. Therefore, it suffices to consider the case when $\mathbf{P}(\widetilde X'_T<0)=0$. Set

$$\mu'_k\triangleq\begin{cases}\gamma'_k\widetilde S_k\widetilde{X'_k}^{-1}, & \widetilde X'_k\neq 0\\ 0, & \widetilde X'_k=0,\end{cases}\qquad k=0,1,\ldots,T-1.$$

If $\mathbf{P}(\widetilde X'_k=0,\gamma'_k\neq 0)>0$ for $k\in\{0,1,\ldots,T-1\}$, then Lemma 2.1(ii) implies that $\mathbf{P}(\widetilde X'_{k+1}<0)>0$, and Lemma 2.1(i) implies that $\mathbf{P}(\widetilde X'_T<0)>0$. Hence $\mathbf{P}(\widetilde X'_k=0,\gamma'_k\neq 0)=0$ for all $k=0,1,\ldots,T-1$. Therefore, $\gamma'_k=\mu'_k\widetilde X'_k\widetilde S_k^{-1}$ a.s. for all $k=0,1,\ldots,T-1$, and

$$\widetilde X'_k=\widetilde X'_{k-1}+\mu'_{k-1}\widetilde X'_{k-1}\xi_k,\quad k=1,\ldots,T,$$
$$\widetilde X'_T=\prod_{k=1}^T(1+\mu'_{k-1}\xi_k).$$

By the definition of admissible strategies, it follows that μ'_0 is non-random, and there exist mappings $u'_k:\mathcal{Z}^k\times\mathbf{R}^{2k}\to\mathbf{R}$, $k=0,1,\ldots,T-1$, such that $\mu'_k=u'_k(\theta_1,\ldots,\theta_k,W_k)$, where

$$W_k\triangleq(\rho_1,\xi_1,\ldots,\rho_k,\xi_k).$$

(For $k=0$, we mean that $\mu'_0=u'_0$ is a constant, according to the convention declared in Remark 3.2). We have that

$$\begin{aligned}\mathbf{E}\bar U(R^\lambda_T\widetilde X'_T)&=\int_{\mathcal{Z}^{T-1}}\mathcal{P}_\Theta(dz)\mathbf{E}\{\bar U(R^\lambda_m\widetilde X'_T)\,|\,\Theta=z\}\\&=\int_{\mathcal{Z}^{T-1}}\mathcal{P}_\Theta(dz)\mathbf{E}\left\{\prod_{k=1}^T\bar U(\rho_k^\lambda[1+\mu'_{k-1}\xi_k])\,|\,\Theta=z\right\}\\&=\int_{\mathcal{Z}^{T-1}}\mathcal{P}_\Theta(dz)\mathbf{E}\left\{\prod_{k=1}^T\bar U(\rho_k^\lambda[1+u'_{k-1}(z_1,\ldots,z_{k-1},W_{k-1})\xi_k])\,|\,\Theta=z\right\},\end{aligned}$$

where $z=(z_1,\ldots,z_{T-1})$ and $\Theta\triangleq(\theta_1,\ldots,\theta_{T-1})$.

For $z=(z_1,\ldots,z_{T-1})\in\mathcal{Z}^{T-1}$, let

$$\begin{aligned}V_k(z)&\triangleq\bar U\left(\bar\rho_k(z)^\lambda[1+u_{k-1}(z_1,\ldots,z_{k-1})\bar\xi_k(z)]\right),\\V'_k(z)&\triangleq\bar U\left(\bar\rho_k(z)^\lambda[1+u'_{k-1}(z_1,\ldots,z_{k-1},\bar W_{k-1}(z))\bar\xi_k(z)]\right),\end{aligned}\qquad(31)$$

where $\left(\bar{\xi}_k(z), \bar{\rho}_k(z), \bar{W}_k(z)\right)$ is the random vector (ξ_k, ρ_k, W_k) being considered in the conditional probability space under the condition $\Theta = z$. Then

$$\mathbf{E}\bar{U}(R_T^\lambda \widetilde{X}_T) = \int_{\mathcal{Z}^T} \mathcal{P}_\Theta(dz)\mathbf{E}\Big\{\prod_{k=1}^T V_k(z) \,\Big|\, \Theta = z\Big\}, \tag{32}$$

and

$$\mathbf{E}\bar{U}(R_T^\lambda \widetilde{X}_T') = \int_{\mathcal{Z}^T} \mathcal{P}_\Theta(dz)\mathbf{E}\Big\{\prod_{k=1}^T V_k'(z) \,\Big|\, \Theta = z\Big\}. \tag{33}$$

Let $(\Omega, \mathcal{F}_z, \mathbf{P}_z)$ be the conditional probability space under the condition $\Theta = z$. Let $\mathbf{E}_z$ be the corresponding expectation, i.e., $\mathbf{E}_z\{\,\cdot\,\} \triangleq \mathbf{E}\{\cdot \,|\, \Theta = z\}$.

Let $\mathbf{E}_{W_k}$ denotes the conditional expectation $\mathbf{E}_z\{\,\cdot \,|\, \bar{W}_k(z)\}$ in the space $(\Omega, \mathcal{F}_z, \mathbf{P}_z)$ for $k > 0$, and let $\mathbf{E}_{W_0} = \mathbf{E}_z$. For $z \in \mathcal{Z}^{T-1}$, we introduce the following sequences of random variables defined on the probability space $(\Omega, \mathcal{F}_z, \mathbf{P}_z)$:

$$\begin{aligned}
&\psi_T(z) \triangleq \mathbf{E}_{W_{T-1}} V_T(z),\\
&\psi_k(z) \triangleq \mathbf{E}_{W_{k-1}}[V_k(z)\psi_{k+1}(z)], \qquad k = T-1, \ldots, 1,
\end{aligned}$$

and

$$\begin{aligned}
&\psi_T'(z) \triangleq \mathbf{E}_{W_{T-1}} V_T'(z),\\
&\psi_k'(z) \triangleq \mathbf{E}_{W_{k-1}}[V_k'(z)\psi_{k+1}(z)], \qquad k = T-1, \ldots, 1.
\end{aligned}$$

Clearly,

$$\mathbf{E}\Big\{\prod_{k=1}^T V_k(z) \,\Big|\, \Theta = z\Big\} = \mathbf{E}\Big\{V_1(z)\mathbf{E}_{W_1}\Big[V_2(z)\mathbf{E}_{W_2}\Big[\cdots \mathbf{E}_{W_{T-1}} V_T(z)\Big]\Big] \,\Big|\, \Theta = z\Big\},$$

and

$$\mathbf{E}\Big\{\prod_{k=1}^T V_k'(z) \,\Big|\, \Theta = z\Big\} = \mathbf{E}\Big\{V_1'(z)\mathbf{E}_{W_1}\Big[V_2'(z)\mathbf{E}_{W_2}\Big[\cdots \mathbf{E}_{W_{T-1}} V_T'(z)\Big]\Big] \,\Big|\, \Theta = z\Big\}.$$

Hence

$$\mathbf{E}\Big\{\prod_{k=1}^T V_k(z) \,\Big|\, \Theta = z\} = \mathbf{E}\Big\{\psi_1(z) \,\Big|\, \Theta = z\Big\}, \tag{34}$$

$$\mathbf{E}\Big\{\prod_{k=1}^T V_k'(z) \,\Big|\, \Theta = z\Big\} = \mathbf{E}\Big\{\psi_1'(z) \,\Big|\, \Theta = z\Big\}. \tag{35}$$

We have assumed that (ξ_k, ρ_k), $k = 1, \dots, T$, are independent in $(\Omega, \mathcal{F}_z, \mathbf{P}_z)$ (i.e., they are conditionally independent under the condition $\Theta = z$). By (31), $\psi_k(z)$ are non-random for any k and $z \in \mathcal{Z}^{T-1}$.

We have proved that $\widetilde{X}_k \geq 0$ a.s. for $k = 1, \dots, T$. By restrictions imposed in (6), we have that $\widetilde{X}'_k \geq 0$ a.s. It follows that $\psi_k(z) \geq 0$ and $\psi'_k(z) \geq 0$ $\mathcal{P}_\Theta$– a.e., i.e., almost everywhere with respect to the measure $\mathcal{P}_\Theta$. (In other words, $\mathcal{P}_\Theta\left(z \in \mathcal{Z}^{T-1} :\ \psi_k(z) < 0\right) = 0$ and $\mathcal{P}_\Theta\left(z \in \mathcal{Z}^{T-1} :\ \psi'_k(z) < 0\right) = 0$.)

(i) Let us assume for certainty that $\delta < 0$. By (8) and by Condition 2.1(ii), it follows that, in the space $(\Omega, \mathcal{F}_z, \mathbf{P}_z)$,

$$\begin{aligned}&\mathbf{E}_{W_{T-1}} \bar{U}(\bar{\rho}_T(z)^\lambda [1 + u_{T-1}(z_1, \dots, z_{T-1}) \bar{\xi}_T(z))\\ &\leq \mathbf{E}_{W_{T-1}} \bar{U}(\bar{\rho}_T(z)^\lambda [1 + u'_{T-1}(z_1, \dots, z_{T-1}, \bar{W}_{T-1}(z)) \bar{\xi}_T(z)]) \quad \text{a.s.} \quad \mathcal{P}_\Theta - \text{a.e..}\end{aligned}$$

Hence

$$\psi_T(z) \leq \psi'_T(z) \quad \text{a.s. in } (\Omega, \mathcal{F}_z, \mathbf{P}_z) \quad \mathrm{P}_\Theta - \text{a.e..}$$

By (8) and by Condition 2.1(ii), again, it follows that, in the space $(\Omega, \mathcal{F}_z, \mathbf{P}_z)$,

$$\begin{aligned}&\mathbf{E}_{W_{k-1}} \bar{U}(\bar{\rho}_k(z)^\lambda [1 + u_{k-1}(z_1, \dots, z_{k-1}) \bar{\xi}_k(z)])\\ &\qquad \leq \mathbf{E}_{W_{k-1}} \bar{U}(\rho_k(z)^\lambda [1 + u'_{k-1}(z_1, \dots, z_{k-1}, \bar{W}_{k-1}(z)) \bar{\xi}_k(z)]) \quad \text{a.s.}\end{aligned}$$

$\mathcal{P}_\Theta$ – a.e.. Hence

$$\mathbf{E}_{W_{k-1}} V_k(z) \leq \mathbf{E}_{W_{k-1}} V'_k(z) \quad \text{a.s. in } (\Omega, \mathcal{F}_z, \mathbf{P}_z) \quad \mathcal{P}_\Theta - \text{a.e..} \tag{36}$$

Let $k \in \{2, \dots, T\}$. Let us show that if

$$\psi_k(z) \leq \psi'_k(z) \quad \text{a.s. in } (\Omega, \mathcal{F}_z, \mathbf{P}_z) \quad \mathrm{P}_\Theta - \text{a.e.,} \tag{37}$$

then

$$\psi_{k-1}(z) \leq \psi'_{k-1}(z) \quad \text{a.s. in } (\Omega, \mathcal{F}_z, \mathbf{P}_z) \quad \mathcal{P}_\Theta - \text{a.e..} \tag{38}$$

Let (37) be satisfied. Remind that $\psi_k(z)$ is non-random for any $z \in \mathcal{Z}^{T-1}$. By (36), it follows that

$$\begin{aligned}\psi_{k-1}(z) &= \mathbf{E}_{W_{k-1}}[V_k(z)\psi_k(z)]\\ &\leq \mathbf{E}_{W_{k-1}}[V'_k(z)\psi_k(z)]\\ &\leq \mathbf{E}_{W_{k-1}}[V'_k(z)\psi'_k(z)] = \psi'_{k-1}(z) \quad \text{a.s. in } (\Omega, \mathcal{F}_z, \mathbf{P}_z) \quad \mathcal{P}_\Theta - \text{a.e..}\end{aligned}$$

Hence (38) follows from (37). Thus, $\psi_1(z) \le \psi_1'(z)$ $\mathcal{P}_\Theta$-a.e.. By (32)-(35), it follows that $\mathbf{E}\bar{U}(R_T^\lambda \widetilde{X}_T) \le \mathbf{E}\bar{U}(R_T^\lambda \widetilde{X}_T')$. Hence $\mathbf{E}U(R_T^\lambda \widetilde{X}_T) \ge \mathbf{E}U(R_T^\lambda \widetilde{X}_T')$.

(ii) Let us assume that $\delta > 0$. The proof given above for $\delta < 0$ can be reused with a small modification to show that $\mathbf{E}\bar{U}(R_T^\lambda \widetilde{X}_T) \ge \mathbf{E}\bar{U}(R_T^\lambda \widetilde{X}_T')$ and $\mathbf{E}U(R_T^\lambda \widetilde{X}_T) \ge \mathbf{E}U(R_T^\lambda \widetilde{X}_T')$. Therefore, $\{\gamma_k\}$ is the optimal strategy in $\bar{\Sigma}$ for $\delta \in (-\infty, 1)\backslash\{0\}$.

The uniqueness of the optimal strategy follows from the fact that $U(x)$ is strictly concave and that the set of all admissible processes $\{(\gamma_t, \widetilde{X}_t)\}$ is convex.

Let us consider the case of $U(x) = \ln x$ that is simpler. Let $\{\gamma_k'\} \in \bar{\Sigma}$ be a strategy that generates the discounted wealth $\widetilde{X}_k'$ given $X_0 = 1$. Again, it suffices to consider only strategies such that if $\gamma_k' \neq 0$ then $X_k' > 0$ a.s., otherwise, by Lemma 2.1 (ii), $P(\widetilde{X}_T' \le 0) > 0$ and $\mathbf{E}U(R_T^\lambda \widetilde{X}_T') = -\infty$. Set $\mu_k' \triangleq \gamma_k'/(\widetilde{X}_k' S_k)$ if $\gamma_k' \neq 0$ and $\mu_k' \triangleq 0$ if $\gamma_k' = 0$. Similarly to (28), we have that

$$\widetilde{X}_T' = \prod_{k=1}^{T}(1 + \mu_{k-1}'\xi_k).$$

Therefore, the unique maximum of the expected utility

$$\begin{aligned}\mathbf{E}U(R_T^\lambda \widetilde{X}_T') = \mathbf{E}\ln(R_T^\lambda \widetilde{X}_T') &= \lambda \mathbf{E}\ln R_T + \mathbf{E}\sum_{k=1}^{T}\ln(1 + \mu_{k-1}'\xi_k) \\ &= \lambda \mathbf{E}U(R_T) + \mathbf{E}\sum_{k=1}^{T}U(1 + \mu_{k-1}'\xi_k)\end{aligned}$$

is achieved for $\{\mu_k'\} = \{\mu_k\}$. Then the proof of Theorem 4.1 and Remark 4.2 follows for the case of $U(x) = \ln x$.

This completes the proof of Theorem 4.1. $\square$

Proof of Corollary 5.1. The optimal piecewise constant strategy in $\bar{\Sigma}_{PC}^c(T)$ corresponds the optimal strategy $\mu_k \equiv 0$ for the discrete time market for any $T > 0$ and $h = \tau/T$. Therefore, the optimal strategy in $\bar{\Sigma}_{PC}^c(T)$ gives expected utility $\mathbf{E}U(\widetilde{X}^c(\tau)) = \mathbf{E}U(X^c(0)) = \mathbf{E}U(X_0)$ for all T and $h = \tau/T$. $\square$

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Author's corrections for the PROOFS:

1. P.4, L.140: insert: Dokuchaev (2002), p.17). It must be: (Dokuchaev and Savkin (2002), or Dokuchaev (2002), p.17).

(It is also addressing the query from Author Query Form.)

2. P.5, L.171: It must be $\mathbf{E}|R^\lambda \widetilde{X}_T| \le \mathbf{E}|X_T| < +\infty$. ($\le$ instead of $\ge$).
3. P.5, L.205: it must be $\mu_k = u_k(\theta_1, \ldots, \theta_k)$. (instead of $\mu_k = \mu_k(\theta_1, \ldots, \theta_k)$)
4. P.9, L.392: it must be $\gamma_k^M = 0.086 \cdot X^M(t_k)/P(t_k)$ (instead of $\gamma_k^M = 0.086$).
5. P.9, L.395: it must be $\gamma_k^M = 0.057 \cdot X^M(t_k)/P(t_k)$. (instead of $\gamma_k^M = 0.057$).
6. P.10, L.403: it must be $\theta_k = \mathrm{w}_0(\cdot + t_k)|_{[0,h]}$, (instead of $\theta_k = \mathrm{w}_0(\cdot - t_k)|_{[0,h]}$,).
7. P.10, L.404: it must be $w_k = \mathrm{w}(\cdot + t_k)|_{[0,h]}$, (instead of $w_k = \mathrm{w}(\cdot - t_k)|_{[0,h]}$,).
8. P.11, L.456: it must be $\delta < 1$ (instead of $\delta > -1$).