

New bounds for the price of anarchy under nonlinear and asymmetric costs

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Abstract

We derive new bounds for the price of anarchy under nonlinear and asymmetric costs. The bounds depend on an additional factor called the intrinsic cost of the system and therefore tend to be more accurate than the current bounds that are dependent only on the degree of asymmetry of the Jacobian and the degree of the nonlinearity of the cost function.

Key words: Price of anarchy, Nash equilibrium, System optimum.

1 Introduction

Many problems in business operations can be modeled as Nash games. It is well known that the solutions of a Nash game, i.e. the Nash equilibria may not be efficient from the system point of view. The inefficiency of Nash equilibrium has been extensively studied in recent literature [3, 4, 11, 16]. The phrase “the price of anarchy” is used by Koutsoupias and Papadimitriou [13] to characterize the degree of efficiency loss, which is the ratio of the system’s cost at a Nash equilibrium point and the overall optimal cost of the system optimization problem. Various worst case bounds of the price of anarchy have been established for the system optimization problems in which a Nash equilibrium point exists. Notable examples include the network equilibrium problem, e.g., [2, 8, 16], the competitive multi-period pricing problem, e.g., [15], and the Nash Cournot equilibria problem, e.g., [7].

Perakis [14] emphasized the importance of non-separable and asymmetric system cost in the study of the price of anarchy and established bounds for linear and nonlinear nonseparable cost functions, which depend on the degree of asymmetry and nonlinearity of the cost function. She

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also gave some examples to show that the bounds could be tight. The model under consideration is as follows. Let *the system optimization problem* be

$$\min x^\top F(x) \text{ subject to } x \in K \subset R^n, \quad (1)$$

where $F : R^n \rightarrow R^n$ is a continuously differentiable function and K is a nonempty, closed, convex and bounded subset of R^n . Suppose that the Nash equilibrium points of the system are characterized by the solution set of a variational inequality problem called *the user optimization problem*

$$\text{Find } x_u \in K \text{ such that } F(x_u)^\top (x - x_u) \geq 0, \forall x \in K. \quad (2)$$

Let x_u and x_s denote solutions of the user and system optimization problems, respectively. Let $Z_u = x_u^\top F(x_u)$ be the cost of the user-optimized problem and $Z_s = x_s^\top F(x_s) = \min_{x \in K} x^\top F(x)$ be the cost of the system optimization problem, respectively. The target is to provide bounds for Z_u/Z_s .

In this paper, we generalize the results of Perakis [14]. We show that the bounds depend on not only the degree of asymmetry and the nonlinearity of the cost function, but also an intrinsic property of the system, namely, the ratio between the minimum intrinsic cost and the maximum intrinsic cost (defined later). The new bounds are not trivial in the sense that they are generally more accurate than the old bounds and that if the above ratio reduces to zero, then the new and old bounds become identical.

The remainder of this paper is organized as follows. In Section 2, we prove our new bounds. In Section 3, we give some examples to show that the new bounds can improve the current bounds. The new bounds allow us to draw some interesting insights, which cannot be observed using the bounds in [14]. Finally, we make some concluding remarks in Section 4.

2 New bounds

2.1 The linear case

We first consider the case that the cost function F is linear. That is, $F(x) = Mx + q$, where M is an $n \times n$, positive definite and possibly asymmetric matrix, and $q^\top x \geq 0$ for all $x \in K$.

In a similar way to [14], we characterize the degree of asymmetry of a matrix M using the following constant.

Definition 2.1

$$c^2 := \|S^{-1}M\|_S^2 = \sup_{w \neq 0} \frac{\|S^{-1}Mw\|_S^2}{\|w\|_S^2} = \sup_{w \neq 0} \frac{w^\top M^\top S^{-1}Mw}{w^\top Sw}, \quad (3)$$

where

$$S = \frac{M + M^\top}{2}$$

is invertible and $\|\cdot\|_S$ denotes the S -norm of a vector, i.e. $\|x\|_S := \sqrt{x^\top Sx}$.

The constant c^2 was originally introduced by Hammond [9] and has the following property:

Lemma 2.1 *If M^2 is a positive semidefinite matrix, then $c^2 \leq 2$.*

The following theorem provides a new bound of the price of anarchy for the linear case.

Theorem 2.1 *Suppose that $F(x) = Mx + q$, M is positive definite and $q^\top x \geq 0$ for all $x \in K$. Let x_u be a solution of the user optimization problem (2) and let x_s be a solution of the system optimization problem (1). Then*

$$\frac{Z_u}{Z_s} \leq \begin{cases} c^2 & \begin{cases} \text{if } c^2 \geq 4 \\ \text{or } c^2 < 4 \text{ and } c^2 d \geq 1 \\ \text{or } c^2 < 4, c^2 d < 1, \text{ and } \frac{1-\sqrt{1-c^2 d}}{d} > 1 \end{cases} \\ \frac{d(1+\sqrt{1-c^2 d})}{(2d-1)+\sqrt{1-c^2 d}} & \text{otherwise,} \end{cases}$$

where

$$d := \frac{\inf_{x \in K} q^\top x}{\sup_{x \in K} q^\top x}.$$

Proof. Since x_u is a solution of the variational inequality problem (2) and $x_s \in K$, we have

$$(x_s - x_u)^\top F(x_u) \geq 0.$$

Thus,

$$\begin{aligned} Z_u &= x_u^\top F(x_u) \\ &\leq x_s^\top F(x_u) \\ &= x_u^\top M^\top x_s + q^\top x_s \\ &= x_u^\top M^\top S^{-1} S x_s + q^\top x_s \\ &\leq \|x_s\|_S \|x_u^\top M^\top S^{-1}\|_S + q^\top x_s \\ &\leq \|x_u\|_S \|S^{-1} M\|_S \|x_s\|_S + q^\top x_s \\ &= c \|x_u\|_S \|x_s\|_S + q^\top x_s. \end{aligned}$$

For any two vectors x and y in R^n , we have

$$\|x\|_S \|y\|_S \leq b_1 \|x\|_S^2 + b_2 \|y\|_S^2 \quad (4)$$

if $b_1, b_2 \geq 0$ and $b_1 b_2 \geq \frac{1}{4}$. It follows that

$$\begin{aligned} Z_u &\leq a_1 \|x_u\|_S^2 + a_2 \|x_s\|_S^2 + q^\top x_s \\ &= a_1 x_u^\top S x_u + a_2 x_s^\top S x_s + q^\top x_s \\ &= a_1 (x_u^\top M^\top x_u + q^\top x_u) + a_2 (x_s^\top M^\top x_s + q^\top x_s) - a_1 q^\top x_u - (a_2 - 1) q^\top x_s \\ &= a_1 Z_u + a_2 Z_s - a_1 q^\top x_u - (a_2 - 1) q^\top x_s, \end{aligned} \quad (5)$$

where $a_1, a_2 \geq 0$, and $a_1 a_2 \geq \frac{c^2}{4}$. We consider the two cases, $c^2 \geq 4$ and $c^2 < 4$, separately.

For the first case, we can further select $a_2 \geq 1$ and $a_1 < 1$ so that the last two terms in (5) are nonpositive, resulting in

$$\frac{Z_u}{Z_s} \leq \frac{a_2}{1 - a_1}.$$

Thus, we can find a good upper bound by solving

$$\begin{aligned} & \text{minimize} && \frac{a_2}{1 - a_1} \\ & \text{subject to} && a_1 a_2 \geq c^2/4 \\ & && a_2 \geq 1, 0 \leq a_1 < 1. \end{aligned} \tag{6}$$

The solution is $a_1 = 1/2$, $a_2 = c^2/2$ and $Z_u/Z_s \leq c^2$, which is the same bound as that in [14].

We now discuss the second case. Let us define *the maximum and minimum intrinsic costs* respectively by $C_{\text{sup}} := \sup_{x \in K} q^\top x$ and $C_{\text{inf}} := \inf_{x \in K} q^\top x$ and impose the condition that $0 \leq a_1 < 1$, $0 \leq a_2 \leq 1$. Then, if

$$a_1 C_{\text{inf}} + (a_2 - 1) C_{\text{sup}} \geq 0,$$

we have

$$\begin{aligned} Z_u & \leq a_1 Z_u + a_2 Z_s - a_1 C_{\text{inf}} - (a_2 - 1) C_{\text{sup}} \\ & \leq a_1 Z_u + a_2 Z_s. \end{aligned}$$

In a similar way to the first case, we can find an upper bound by solving

$$\begin{aligned} & \text{minimize} && \frac{a_2}{1 - a_1} \\ & \text{subject to} && a_1 a_2 \geq c^2/4 \\ & && a_1 C_{\text{inf}} + (a_2 - 1) C_{\text{sup}} \geq 0 \\ & && 0 \leq a_1 < 1. \end{aligned} \tag{7}$$

Let $d := C_{\text{inf}}/C_{\text{sup}}$. If $c^2 d \geq 1$, then the problem (7) becomes

$$\begin{aligned} & \text{minimize} && \frac{a_2}{1 - a_1} \\ & \text{subject to} && a_1 a_2 \geq c^2/4 \\ & && 0 \leq a_1 < 1. \end{aligned} \tag{8}$$

The solution is $a_1 = 1/2$, $a_2 = c^2/2$ and $Z_u/Z_s \leq c^2$. Otherwise, if $c^2 d < 1$ and $\frac{1 - \sqrt{1 - c^2 d}}{d} < 1$, we have solution

$$a_1 = \frac{1 - \sqrt{1 - c^2 d}}{2d}, \quad a_2 = \frac{1 + \sqrt{1 - c^2 d}}{2},$$

and

$$\frac{Z_u}{Z_s} \leq \frac{d(1 + \sqrt{1 - c^2 d})}{(2d - 1) + \sqrt{1 - c^2 d}};$$

if $c^2 d < 1$ and $\frac{1 - \sqrt{1 - c^2 d}}{d} \geq 1$, we have solution $a_1 = 1/2$, $a_2 = c^2/2$, and $Z_u/Z_s \leq c^2$. \square

Remark: In the above theorem, we have used the convention that $\frac{0}{0} = 1$; that is, when $q = 0$, we have $d = 1$. For $\sup_{x \in K} q^\top x = \infty$, we have $d = 0$ and

$$\frac{1 - \sqrt{1 - c^2 d}}{d} := \lim_{d \rightarrow 0} \frac{1 - \sqrt{1 - c^2 d}}{d} = \frac{c^2}{2},$$

$$\frac{d(1 + \sqrt{1 - c^2 d})}{(2d - 1) + \sqrt{1 - c^2 d}} := \lim_{d \rightarrow 0} \frac{d(1 + \sqrt{1 - c^2 d})}{(2d - 1) + \sqrt{1 - c^2 d}} = \frac{4}{4 - c^2}.$$

It then follows immediately that when $d = 0$, our bound reduces to

$$\frac{Z_u}{Z_s} \leq \begin{cases} c^2 & \text{if } c^2 \geq 2 \\ \frac{4}{4 - c^2} & \text{otherwise,} \end{cases}$$

which is the one obtained by Perakis [14]. \square

For the separable affine functions where the matrix M is diagonal positive definite or non-separable functions where M is symmetric positive definite, $c^2 = 1$. Roughgaden and Tardos [16] and Perakis [14] derive the bound $4/3$ for the separable and nonseparable case, respectively. They showed that the bound can be tight. We can see that our bound can also be tight, since it is a generalization of that of Perakis. Furthermore, our new bounds can give some useful information on the problem, using the quantity d . For example, when $d = 1$, that is, if the intrinsic cost is a constant, then the bound is c^2 and we have the following corollary.

Corollary 2.1 *If the intrinsic cost $q^\top x$ is a constant for all $x \in K$ and if M is a positive definite matrix, then*

$$\frac{Z_u}{Z_s} \leq c^2. \quad \square$$

Remark: The above corollary includes Corollary 2 in [14] as a special case, where the same bound was derived for the case that $q = 0$. Furthermore, if $c^2 = 1$, i.e., for the separable case or the nonseparable symmetric case, the bound is 1. Hence, we have provided a sufficient condition where there is no efficiency loss in the game. \square

2.2 The nonlinear case

In this section, we show that the results in the previous subsection can be extended to the case of nonlinear and asymmetric cost functions. Here, the Jacobian matrix is not a constant matrix M but a positive definite and asymmetric matrix $\nabla F(x)$. The positive-definiteness of the Jacobian $\nabla F(x)$ and the boundedness of K ensure that the variational inequality problem (2) has a unique solution [6].

To derive the bounds, we have to introduce two quantities for measuring the degree of asymmetry and the nonlinearity, respectively. The following definition is from Definition 2 of [14].

Definition 2.2 The quantity c^2 for the nonlinear function F , which measures the asymmetry of the Jacobian $\nabla F(x)$, is defined as

$$c^2 := \sup_{x \in K} \|S(x)^{-1} \nabla F(x)\|_{S(x)}^2, \quad (9)$$

where

$$S(x) = \frac{\nabla F(x) + \nabla F(x)^\top}{2}$$

is the symmetrized part of the Jacobian matrix $\nabla F(x)$.

The following definition was first used in [12] and [17] in the analysis of interior point methods. It finds new applications recently in the analysis of the price of anarchy [14].

Definition 2.3 The function $F : R^n \rightarrow R^n$ is said to satisfy the **Jacobian similarity condition** if there exists $\kappa \geq 1$ such that $\forall w \in R^n, \forall x, \bar{x} \in K$, there holds

$$\kappa^{-1} w^\top \nabla F(x) w \leq w^\top \nabla F(\bar{x}) w \leq \kappa w^\top \nabla F(x) w.$$

Theorem 2.2 Suppose that $F(x)$ satisfies the Jacobian similarity condition and that $F(0)^\top x \geq 0$ for all $x \in K$. Then

$$\frac{Z_u}{Z_s} \leq \begin{cases} c^2 \kappa^2 - 2(\kappa - 1) & \text{if } c^2 \geq 4 \text{ or } c^2 \kappa > 2; \\ c^2 + 2 - \frac{2}{\kappa} & \text{if } c^2 \leq 4, c^2 \kappa \leq 2 \text{ and } c^2 \kappa^2 d > 1; \\ c^2 + 2 - \frac{2}{\kappa} & \text{if } c^2 \leq 4, c^2 \kappa \leq 2, c^2 \kappa^2 d \leq 1 \text{ and } \frac{1 - \sqrt{1 - c^2 \kappa^2 d}}{\kappa^2 d} > 1/\kappa; \\ \frac{2\kappa d - (1 - \sqrt{1 - c^2 \kappa^2 d})d}{2\kappa d - (1 - \sqrt{1 - c^2 \kappa d})} & \text{otherwise,} \end{cases}$$

where $d := \frac{\inf_{x \in K} F(0)^\top x}{\sup_{x \in K} F(0)^\top x}$.

Proof. Since x_u is a solution of the variational inequality problem (2) and $x_s \in K$, using a similar argument as for the linear case, we have

$$\begin{aligned} Z_u &= x_u^\top F(x_u) \\ &\leq x_s^\top F(x_u) \\ &= x_s^\top F(x_s) + x_s^\top (F(x_u) - F(x_s)). \end{aligned}$$

Define

$$\Phi(t) := F(x_s + t(x_u - x_s))^\top x_s, \quad t \in [0, 1].$$

Then from the mean value theorem we have that there is some $\bar{t} \in [0, 1]$ such that

$$\Phi(1) - \Phi(0) = \Phi'(\bar{t})(1 - 0),$$

i.e.,

$$(F(x_u) - F(x_s))^\top x_s = (x_u - x_s)^\top \nabla F(\bar{x})^\top x_s,$$

where

$$\bar{x} = x_s + \bar{t}(x_u - x_s).$$

Thus,

$$\begin{aligned}
Z_u &\leq x_s^\top F(x_s) + x_s^\top \nabla F(\bar{x})(x_u - x_s) \\
&= x_s^\top F(x_s) - x_s^\top \nabla F(\bar{x})x_s + x_u^\top \nabla F(\bar{x})^\top S(\bar{x})^{-1} S(\bar{x})x_s \\
&\leq x_s^\top F(x_s) - x_s^\top \nabla F(\bar{x})x_s + \|x_u^\top \nabla F(\bar{x})^\top S(\bar{x})^{-1}\|_{S(\bar{x})} \|x_s\|_{S(\bar{x})} \\
&\leq x_s^\top F(x_s) - x_s^\top \nabla F(\bar{x})x_s + \|x_u\|_{S(\bar{x})} \|S(\bar{x})^{-1} \nabla F(\bar{x})\|_{S(\bar{x})} \|x_s\|_{S(\bar{x})} \\
&\leq x_s^\top F(x_s) - x_s^\top \nabla F(\bar{x})x_s + c \|x_u\|_{S(\bar{x})} \|x_s\|_{S(\bar{x})} \\
&\leq x_s^\top F(x_s) - x_s^\top \nabla F(\bar{x})x_s + a_1 \|x_u\|_{S(\bar{x})}^2 + a_2 \|x_s\|_{S(\bar{x})} \\
&= x_s^\top F(x_s) + a_1 x_u^\top \nabla F(\bar{x})x_u + (a_2 - 1)x_s^\top \nabla F(\bar{x})x_s,
\end{aligned} \tag{10}$$

where the second inequality follows from the Cauchy-Schwarz inequality, the third one from the norm inequality, the fourth one from (9), and the last one from (4) and $a_1 a_2 \geq \frac{c^2}{4}$. Applying again the mean value theorem to the functions

$$\Phi_1(t) := F(x_u + t(0 - x_u))^\top x_u, \quad t \in [0, 1],$$

and

$$\Phi_2(t) := F(x_s + t(0 - x_s))^\top x_s, \quad t \in [0, 1],$$

we claim that there are $t_1, t_2 \in [0, 1]$, such that

$$(F(x_u) - F(0))^\top (x_u - 0) = x_u^\top \nabla F(x_1)x_u, \quad \text{where } x_1 = x_u - t_1 x_u, \tag{11}$$

$$(F(x_s) - F(0))^\top (x_s - 0) = x_s^\top \nabla F(x_2)x_s, \quad \text{where } x_2 = x_s - t_2 x_s. \tag{12}$$

We consider the two cases $c^2 \geq 4$ or $c^2 \kappa > 2$ and $c^2 < 4$, $c^2 \kappa < 2$ separately.

Case 1: $c^2 \geq 4$ or $c^2 \kappa > 2$. For this case, we can further select that $a_2 \geq 1$ and $a_1 < 1$. The analysis is the same as that in [14] and we include it here just for completeness. It follows from (10)-(12) that

$$\begin{aligned}
Z_u &\leq x_s^\top F(x_s) + a_1 \kappa x_u^\top \nabla F(x_1)x_u + (a_2 - 1) \kappa x_s^\top \nabla F(x_2)x_s, \\
&= x_s^\top F(x_s) + a_1 \kappa x_u^\top (F(x_u) - F(0)) + (a_2 - 1) \kappa x_s^\top (F(x_s) - F(0)),
\end{aligned}$$

where the inequality follows from the Jacobian similarity. Using the assumption that $F(0)^\top x \geq 0$ for all $x \in K$, we obtain from the above inequality that

$$(1 - a_1 \kappa)Z_u \leq [1 + (a_2 - 1)\kappa]Z_s.$$

Thus, we can obtain an upper bound by solving

$$\begin{aligned}
&\text{minimize} && \frac{1 + (a_2 - 1)\kappa}{1 - a_1 \kappa} \\
&\text{subject to} && a_1 a_2 \geq c^2/4 \\
&&& a_2 \geq 1, 0 \leq a_1 < \kappa^{-1}.
\end{aligned} \tag{13}$$

The solution is $a_1 = \frac{1}{2\kappa}$, $a_2 = c^2\kappa/2$, and

$$\frac{Z_u}{Z_s} \leq c^2\kappa^2 - 2(\kappa - 1).$$

Case 2: $c^2 < 4$ and $c^2\kappa < 2$. For this case, we further impose the condition that $a_2 \leq 1$. It follows from (10)-(12), the fact that $a_2 \leq 1$ and the Jacobian similarity that

$$\begin{aligned} Z_u &\leq x_s^\top F(x_s) + a_1\kappa x_u^\top \nabla F(x_1)x_u + \frac{(a_2 - 1)}{\kappa} x_s^\top \nabla F(x_2)x_s, \\ &= x_s^\top F(x_s) + a_1\kappa x_u^\top (F(x_u) - F(0)) + \frac{(a_2 - 1)}{\kappa} x_s^\top (F(x_s) - F(0)). \end{aligned}$$

Now, define $C_{\text{sup}} := \sup_{x \in K} F(0)^\top x$ and $C_{\text{inf}} := \inf_{x \in K} F(0)^\top x$. Then, if

$$a_1\kappa C_{\text{inf}} + (a_2 - 1)\kappa^{-1}C_{\text{sup}} \geq 0,$$

we have

$$\begin{aligned} Z_u &\leq a_1\kappa Z_u + [1 + (a_2 - 1)\kappa^{-1}] Z_s - a_1\kappa C_{\text{inf}} - (a_2 - 1)\kappa^{-1}C_{\text{sup}} \\ &\leq a_1\kappa Z_u + [1 + (a_2 - 1)\kappa^{-1}]Z_s. \end{aligned}$$

Therefore, we can find an upper bound by solving

$$\begin{aligned} &\text{minimize} && \frac{1 + (a_2 - 1)/\kappa}{1 - a_1\kappa} \\ &\text{subject to} && a_1a_2 \geq c^2/4 \\ &&& a_1\kappa C_{\text{inf}} + (a_2 - 1)\kappa^{-1}C_{\text{sup}} \geq 0 \\ &&& 0 \leq a_1 < \kappa^{-1}. \end{aligned} \tag{14}$$

Let $d := C_{\text{inf}}/C_{\text{sup}}$. Problem (14) becomes

$$\begin{aligned} &\text{minimize} && \frac{1 + (a_2 - 1)/\kappa}{1 - a_1\kappa} \\ &\text{subject to} && a_1a_2 \geq c^2/4 \\ &&& a_1\kappa^2d + (a_2 - 1) \geq 0 \\ &&& 0 \leq a_1 < \kappa^{-1}. \end{aligned} \tag{15}$$

If $c^2\kappa^2d > 1$, then (15) has the solution $a_1 = \frac{1}{2\kappa}$, $a_2 = c^2\kappa/2$ and

$$\frac{Z_u}{Z_s} \leq c^2 + 2 - 2\kappa^{-1}.$$

Otherwise, if $c^2\kappa^2d \leq 1$ and $\frac{1 - \sqrt{1 - c^2\kappa^2d}}{\kappa^2d} < \kappa^{-1}$, we have the solution that

$$a_1 = \frac{1 - \sqrt{1 - c^2\kappa^2d}}{2\kappa^2d}, \quad a_2 = \frac{1 + \sqrt{1 - c^2\kappa^2d}}{2},$$

and

$$\frac{Z_u}{Z_s} \leq \frac{2\kappa d - (1 - \sqrt{1 - c^2 \kappa^2 d})d}{2\kappa d - (1 - \sqrt{1 - c^2 \kappa d})d};$$

if $c^2 \kappa^2 d \leq 1$ and $\frac{1 - \sqrt{1 - c^2 \kappa^2 d}}{\kappa^2 d} \geq \kappa^{-1}$, we have solution $a_1 = \frac{1}{2\kappa}$, $a_2 = c^2 \kappa / 2$, and

$$\frac{Z_u}{Z_s} \leq c^2 + 2 - 2\kappa^{-1}.$$

□

Remarks: Theorem 2.2 is a direct extension of Theorem 2.1. Note that if the cost function is linear, i.e., $F(x) = Mx + q$, then $\kappa = 1$. In this case, the bounds in Theorem 2.2 reduce exactly to those in Theorem 2.1. If $\sup_{x \in K} F(0)^\top x = \infty$ then $d = 0$ and the bound reduces to that of [14] for the nonlinear case. □

3 Examples in traffic equilibrium

From the above section, we can see that for the case that $c^2 > 4$, the bounds presented here are the same as those derived in [14]. The difference arises in the case when $c^2 \leq 4$. In this case, instead of the constraint that $a_2 \geq 1$, we use

$$a_1 C_{\text{inf}} + (a_2 - 1) C_{\text{sup}} \geq 0$$

for the linear case and

$$a_1 \kappa C_{\text{inf}} + (a_2 - 1) \kappa^{-1} C_{\text{sup}} \geq 0,$$

for the nonlinear case, in the optimization problem to derive the bound. The feasible region in these optimization problems can be larger than that in [14] and thus the bounds cannot be worse than those in [14]. Figure 1 shows the situation, where $c^2 = 1.3$, $C_{\text{sup}} = 2$ and $C_{\text{inf}} = 0.8$.

Our work adds another dimension to the problem discussed by Perakis, incorporating the effects from d (the intrinsic cost) and c (degree of asymmetry of matrix M). We present some insights using some simple examples. Figure 2 plots the new bound and the bound from Perakis [14] (the old bound for short) as functions of d with fixed $c^2 = 1.3$, and Figure 3 plots the new bound and the old bound as functions of c with fixed $d = 0.8$, respectively. In both figures we set $\kappa = 1$.

From the two figures we can see that the new bound is not worse than that of [14] and in some cases, it can be better. In Figure 2, at $d = 0$ (by convention, we have infinite max intrinsic cost), the two bounds are the same. However, as the max intrinsic cost increases, the bound gets better. An explanation to this phenomenon is that lowering the max intrinsic cost will imply that the max cost of the inefficiency of the user is not heavily penalized. Thus the user tends to make near-sighted decisions, which results in a smaller value for the ratio Z_u/Z_s . We are able to capture this characteristic from our analysis.

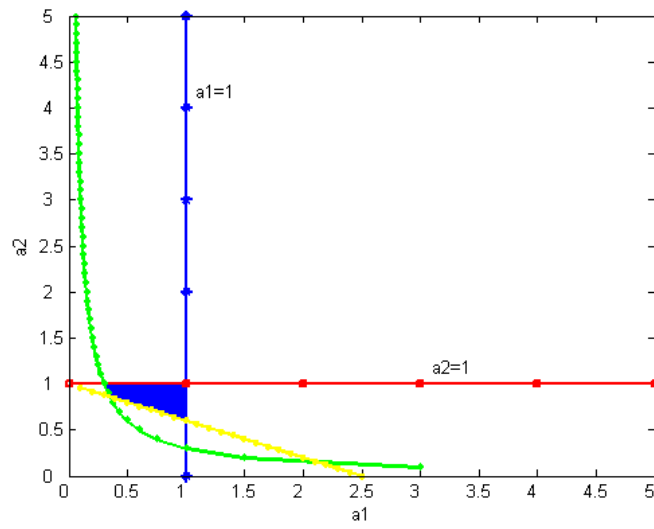


Figure 1: An illustrative figure: The feasible region of the optimization problem in [14] is the intersection of $0 \leq a_1 < 1$, $a_2 \geq 1$, and $a_1 a_2 \geq c^2/4$; while our feasible region is the intersection of $0 \leq a_1 < 1$, $0.8a_1 + a_2 \geq 1$, and $a_1 a_2 \geq c^2/4$. It is clear that our region is larger than that of [14], as shown by the shadowed part.

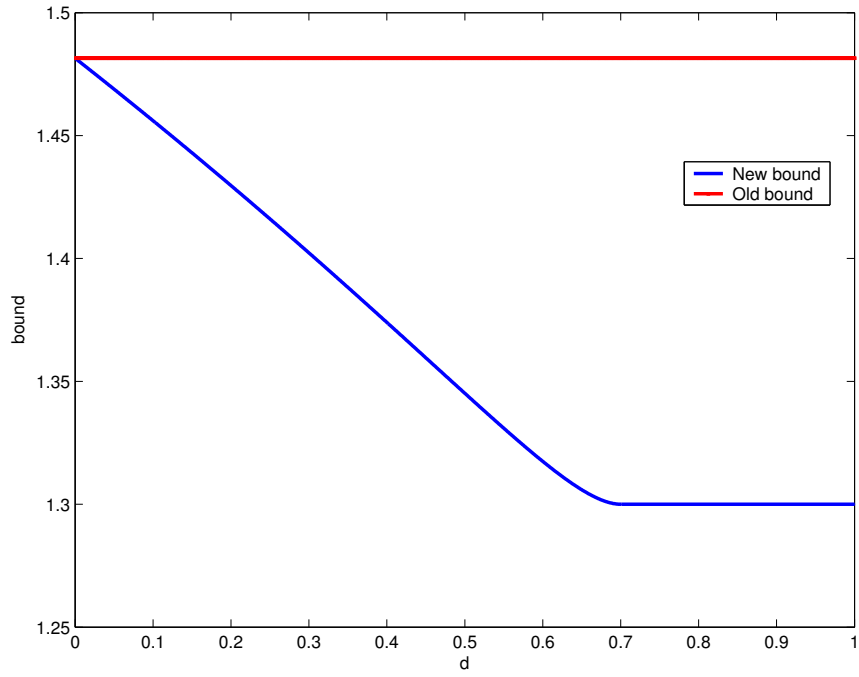


Figure 2: The bound as a function of d with fixed c .

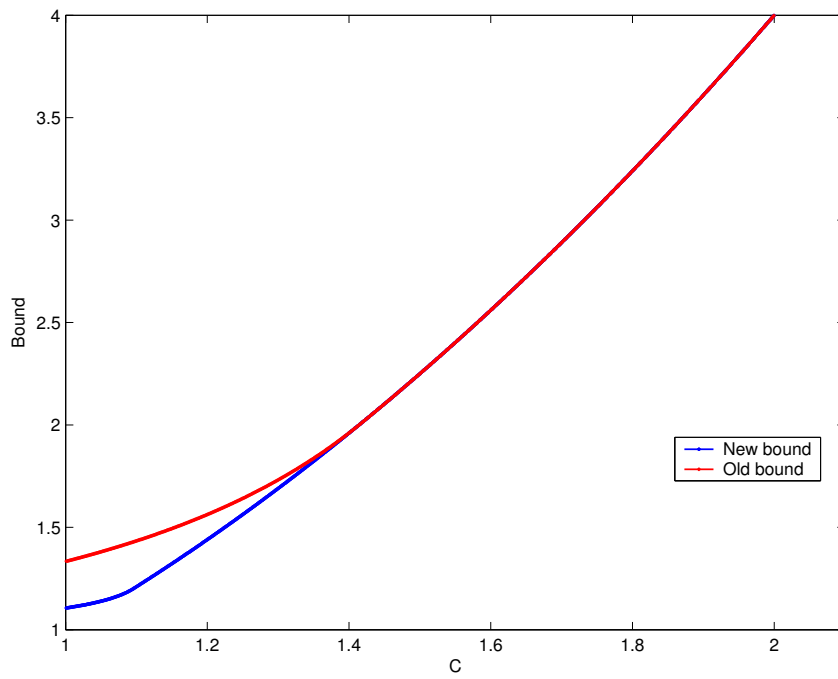


Figure 3: The bound as a function of c with fixed d .

Figure 3 suggests that having a low degree of asymmetry leads to a low ratio of Z_u/Z_s . This can be verified by both bounds. However, it is clear that our bound is more sensitive than the old bound when c is small ($c < 1.4$).

We now present some examples to illustrate our results.

Example 1. The following figure is the original example from Braess's paradox where there are 4 nodes, 5 links, an origin-destination pair from node 1 to node 4 and the demand between this O-D pair is 6. There are 3 paths from node 1 to node 4: the first one is through links a and c ; the second one is through links b and d ; and the last one is through a , e , and d . Let $x = (x_1, x_2, x_3)^\top$ denote the vector of the path flow and let the path cost be given by

$$\begin{cases} c_1 = 11x_1 + 12x_3 + 50, \\ c_2 = 11x_2 + 12x_3 + 50, \\ c_3 = 8x_1 + 8x_2 + 21x_3 + 10. \end{cases}$$

That is, the path cost function $F(x) = Mx + q$ is given by

$$M = \begin{pmatrix} 11 & 0 & 12 \\ 0 & 11 & 12 \\ 8 & 8 & 21 \end{pmatrix}, \quad q = \begin{pmatrix} 50 \\ 50 \\ 10 \end{pmatrix}.$$

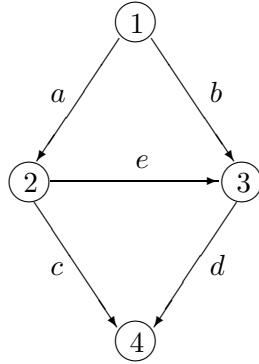


Figure 4. The Braess Network in Example 1

The user and the system optimal path flows are

$$x_u^* = \begin{pmatrix} 14/13 \\ 14/13 \\ 50/13 \end{pmatrix}, \quad x_s^* = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix},$$

and the total costs are

$$Z_u = 648, \quad Z_s = 498,$$

respectively. Thus,

$$\frac{Z_u}{Z_s} = \frac{648}{498} \approx 1.3012.$$

Using the following formula [14]

$$c^2 = \lambda_{\max} \left(S^{-\frac{1}{2}} M^\top S^{-1} M S^{-\frac{1}{2}} \right),$$

where $S = (M + M^\top)/2$, we can calculate that $c^2 = 1.2581$. The feasible set K is

$$K := \{x \in \mathbb{R}^3, x \geq 0, x_1 + x_2 + x_3 = 6\}.$$

The minimum intrinsic cost is $\inf_{x \in K} q^\top x = 60$ and the maximum intrinsic cost is $\sup_{x \in K} q^\top x = 300$. Consequently, $d = 0.2$. The bound from [14] is

$$\frac{4}{(4 - c^2)} = 1.4588,$$

and our new bound from Theorem 2.1 is

$$\frac{d(1 + \sqrt{1 - c^2 d})}{(2d - 1) + \sqrt{1 - c^2 d}} = 1.4071,$$

better than 1.4588.

Example 2. Consider the network in Example 1 but with a different cost function. The purpose of this example is to show that in practice, the bound may really rely on the constant d and thus our results make sense. The path cost functions are now

$$\begin{cases} c_1 &= 11x_1 + 12x_3 + 32, \\ c_2 &= 11x_2 + 12x_3 + 32, \\ c_3 &= 8x_1 + 8x_2 + 21x_3 + 10. \end{cases}$$

Thus,

$$M = \begin{pmatrix} 11 & 0 & 12 \\ 0 & 11 & 12 \\ 8 & 8 & 21 \end{pmatrix}, \quad q = \begin{pmatrix} 32 \\ 32 \\ 10 \end{pmatrix}.$$

In this case, $d = 10/32 = 0.3125$. The user and the system optimal path flows are

$$x_u^* = \begin{pmatrix} 32/13 \\ 32/13 \\ 14/13 \end{pmatrix}, \quad x_s^* = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}, \text{ respectively,}$$

and the total costs are

$$Z_u = 432, \quad Z_s = 390,$$

respectively.

$$\frac{Z_u}{Z_s} = \frac{432}{390} \approx 1.1077,$$

which is smaller than that in the first case where $d = 0.2$ and our bound from Theorem 2.1 is 1.3761.

4 Concluding remarks

We have derived new bounds for efficiency loss in a model where the Nash equilibrium can be characterized by a variational inequality and the system objective function is of the form $x^\top F(x)$. We focus on deriving an upper bound for the ratio Z_u/Z_s . We introduced a ratio d , which is the ratio between the minimum and the maximum intrinsic costs. We extended the results of Perakis and show that the bound for Z_u/Z_s depends on d . When d reduces to one, which corresponds to the linear and symmetric systems, there is no loss of efficiency. Our numerical results suggest that the ratio d plays an important role in determining the ratio Z_s/Z_u . We provide some insights regarding the intrinsic cost, which appears to be new to the literature. A potential application of this notion could be in the design of strategies for improving the user's solution; e.g., setting the toll price in a congested traffic networks [1, 10]. Furthermore, the numerical results also show that the new bounds are more accurate especially when the degree of asymmetry of the matrix is small.

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