Proof of Theorem 1: Recall that, by construction, v_i can be viewed as the virtual reference that, if injected into the feedback system (P/C_i), would reproduce $z = (u, y)$. Then, provided that the disturbances n_u and n_y belong to $\mathcal{N}(\mathbb{Z}_+)$, for any index i corresponding to an $\mathcal{N}$-stable feedback system (P/C_i), there exist positive reals $\widetilde{\alpha}_1, \widetilde{\alpha}_2, \widetilde{\delta}_1, \widetilde{\delta}_2$ such that

$$\|u^t\| \leq \widetilde{\alpha}_1 \left\|v_i^t\right\| + \widetilde{\delta}_1, \quad \|y^t\| \leq \widetilde{\alpha}_2 \left\|v_i^t\right\| + \widetilde{\delta}_2, \quad \forall t \in \mathbb{Z}_+.$$

In other words, for indexes corresponding to $\mathcal{N}$-stabilizing controller, $\mathcal{N}$-stability of the system (P/C_i) is always unfalsified by the I/O pair (v_i, z), regardless of the switching sequence $\sigma(t)$, $t \in \mathbb{Z}_+$. As a consequence, by virtue of the second property in Def. 3, one concludes that $V_i(\cdot)$ remains bounded. Therefore, under problem feasibility, the HSL Lemma holds. Further, the test functional V_f, related to the final switched-on controller C_f, is bounded. This, along with the assumption B2, implies that $\mathcal{N}$-stability of (P/C_f) is unfalsified by (v_f, z). Then, see also (6), there exist finite nonnegative constants α_1, α_2, δ_1 and δ_2 such that

$$\|u^t\| \leq \alpha_1 \left\|v_f^t\right\| + \delta_1, \quad \|y^t\| \leq \alpha_2 \left\|v_f^t\right\| + \delta_2, \quad \forall t \in \mathbb{Z}_+. \quad (23)$$

As the virtual reference v_f converges exponentially to the true reference r, there exists a finite nonnegative constant δ such that

$$\left\|v_f^t\right\| \leq \|r^t\| + \left\|v_f^t - r^t\right\| \leq \|r^t\| + \delta.$$

Consequently, one can conclude that

$$\|u^t\| \leq \alpha_1 \|r^t\| + \beta_1, \quad \|y^t\| \leq \alpha_2 \|r^t\| + \beta_2. \quad (24)$$

where $\beta_1 := \alpha_1 \delta + \delta_1$ and $\beta_2 := \alpha_2 \delta + \delta_2$, viz. $\mathcal{N}$-stability unfalsification of (P/C_f) by (r, z). Thus, as (24) holds for the data from every possible input, the ASC $(P/C_{\sigma(\cdot)})$ turns out to be input-output stable with respect to the norm $\mathcal{N}$.

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Exponential Stability With L_2-Gain Condition of Nonlinear Impulsive Switched Systems

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***Abstract*—In this technical note, we consider exponential stability and stabilization problems of a general class of nonlinear impulsive switched systems with time-varying disturbances. By using the switched Lyapunov function method, sufficient conditions expressed as algebraic inequality constraints and linear matrix inequalities are obtained. They ensure that the nonlinear impulsive switched systems are not only exponentially stable but also satisfy the L_2-gain condition. Based on the stability results obtained, an effective computational method is devised for the construction of switched linear stabilizing feedback controllers. A numerical example is presented to illustrate the effectiveness of the results obtained.**

***Index Terms*—Exponentially stable, linear matrix inequality (LMI), nonlinear impulsive switched systems.**

I. INTRODUCTION

Stability issues are fundamentally important for any dynamical systems, and there is no exception for switched systems. As a consequence, theory and methods for stability of linear switched systems have been extensively studied by many researchers, leading to the publication of many research papers (see, for example, [1], [3]–[6], and references therein). For switched controllers of dynamical systems, many results are also available in the literature. In particular, a design method for such control systems is reported in [2]. However, a switched system often consists of nonlinear subsystems and there may also exist some impulses and disturbances when the switched system is switching amongst its subsystems. The existence of impulses, switching events and disturbances will cause oscillations and instability, leading to poor performance. Thus, it is important to consider stability problems of

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nonlinear switched systems with impulsive effects, often called nonlinear impulsive switched systems.

In the past two decades, H_∞ control for various kinds of dynamical systems has received considerable attention, and many papers on H_∞ control theory (see, for example, [9], [10]) and H_∞ controller design (see, for example, [11], [12]) have been published. Furthermore, there are many applications of H_∞ control in practical systems, such as target tracking systems [13] and networked control systems [14].

In this technical note, we consider a general class of nonlinear impulsive switched systems with nonlinear impulsive increments. This class of systems cover those considered in [7], [8] and [10] as special cases, where only linear impulsive switched systems with linear impulsive increments are involved. Clearly, the results obtained as well as the techniques used in the papers mentioned above cannot be applied to the nonlinear impulsive switched systems considered in this technical note. The main contributions of the current technical note include: (i) new stability results for a much general class of nonlinear impulsive switched systems with nonlinear impulsive increments, and (ii) an effective design method for the construction of switched stabilizing feedback controllers.

II. Nonlinear Impulsive Switched System

Consider the following class of nonlinear impulsive switched systems:

$$\begin{cases} \dot{x}(t) = A_{i_k}x(t) + B_{i_k}w(t) + C_{i_k}u(t) \\ \quad +\phi_{i_k}(t, x(t)), & t \neq \tau_k, \\ \Delta x(t) = D_k x(t) + \psi_k(t, x(t)), & t = \tau_k, \\ z(t) = E_{i_k}x(t), \\ x(\tau_0^+) = x_0 \end{cases} \tag{1}$$

where $x(t) \in R^n$ is the state, $w(t) \in R^p$ is the disturbance input, $u(t) \in R^r$ is the control input, and $z(t) \in R^q$ is the controlled output. $\phi_{i_k}(t, x(t)) : [t_0, \infty) \times R^n \to R^n$, which is globally Lipschitz continuous, and $\psi_k(t, x(t)) : [t_0, \infty) \times R^n \to R^n$ are nonlinear functions, and $\phi_{i_k}(t, 0) \equiv \psi_k(t, 0) \equiv 0$ for all $t \in [\tau_0, \infty)$. $A_{i_k} \in R^{n\times n}, B_{i_k} \in R^{n\times p}, C_{i_k} \in R^{n\times r}, D_k \in R^{n\times n}, E_{i_k} \in R^{q\times n}$ are known matrices. $\Delta x(\tau_k) = x(\tau_k^+) - x(\tau_k^-) = x(\tau_k^+) - x(\tau_k)$, with $x(\tau_k^+) = \lim_{t\downarrow\tau_k} x(t)$ and $x(\tau_k) = x(\tau_k^-) = \lim_{t\uparrow\tau_k} x(t)$, meaning that the solution of the nonlinear impulsive switched system (1) is left continuous. $i_k \in \{1, 2, \ldots, m\}$ is a discrete state parameter, where k is a nonnegative integer and m is a positive integer. τ_k, $k = 1, 2, \ldots$, are impulsive switching time points satisfying $0 = \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_k < \ldots < \tau_\infty = \infty$. Under the control of a switching signal, coupling with the impulsive effects, system (1) enters from the i_{k-1} subsystem to the i_k subsystem at the time point $t = \tau_k$.

We now introduce the following assumptions and definitions.

Assumption 1: $\|\psi_k(t, x(t))\| \leq \rho_k\|x(t)\|$ for all $t \in [\tau_0, \infty)$, where $\rho_k = \rho(I + D_k)$, $\|\cdot\|$ denotes the Euclidean norm and $\rho(\cdot)$ is the spectral radius.

Assumption 2: There exist nonnegative scalars $g_{i_k} \geq 0$ for $t \in [\tau_0, \infty)$ such that $\phi_{i_k}(t, x)^T\phi_{i_k}(t, x) \leq g_{i_k}x^T x$.

Assumption 3: $\|w(t)\|^2 \leq \alpha\|x(t)\|^2$ for all $t \in [\tau_0, \infty)$, where α is a positive constant.

From Assumption 3, we see that any bounded measurable function w from $[\tau_0, \infty)$ into R^p satisfying $\|w(t)\|^2 \leq \alpha\|x(t)\|^2$ is an admissible disturbance. In fact, for the case of parameter uncertainties, it is commonly assumed that $w(t) = \Delta Ax(t) = GF(t)Hx(t)$, where $F^T(t)F(t) \leq I$, while G and H are appropriate known matrices. In this case, it can be shown that Assumption 3 is satisfied.

Definition 1: Suppose that $x(0) = 0$. Then, the uncertain nonlinear impulsive switched system (1) is said to be exponentially stable with the L_2-gain condition of disturbance attenuation if the following conditions are satisfied under any switching law: 1) The nonlinear impulsive switched system (1) is exponentially stable with $u(t) = 0$; 2) If a positive constant γ is specified as the value of the performance index, then $\int_0^T \|z(t)\|^2 dt \leq \gamma^2 \int_0^T \|w(t)\|^2 dt$ for any $T > 0$.

For exponential stabilization, we consider a class of switched linear feedback controllers

$$u(t) = F_{i_k}x(t) \tag{2}$$

where $F_{i_k} \in R^{r\times n}$ is a constant matrix. Then, we obtain the following impulsive switched closed-loop system

$$\begin{cases} \dot{x}(t) = (A_{i_k} + C_{i_k}F_{i_k})\,x(t) + B_{i_k}w(t) \\ \quad +\phi_{i_k}(t, x(t)), & t \neq \tau_k, \\ \Delta x(t) = D_k x(t) + \psi_k(t, x(t)), & t = \tau_k, \\ z(t) = E_{i_k}x(t), \\ x(\tau_0^+) = x_0. \end{cases} \tag{3}$$

Sufficient conditions for exponential stability of the impulsive switched closed-loop system (3) will be derived, and then be used to construct feedback gain matrices F_{i_k}, $i_k = 1, 2, \ldots, m$, $m \in N$, such that the impulsive switched closed-loop system is exponentially stable and L_2-gain conditions are satisfied. A computational algorithm, in the form of solving algebraic inequality constraints and linear matrix inequalities, will be devised for the construction of such a stabilizing state feedback controller.

III. Main Results

To present our main results, the following two lemmas are needed.

Lemma 1: ([4]) Let $\varepsilon > 0$ be a given scalar and let $\Xi \in R^{p\times q}$ be a matrix such that $\Xi^T\Xi \leq I$, where I is an identity matrix with appropriate dimension. Then

$$2x^T\Xi y \leq \varepsilon x^T x + \varepsilon^{-1}y^T y \tag{4}$$

for all $x \in R^p$ and $y \in R^q$.

Lemma 2: ([7]) Let $P \in R^{n\times n}$ be a given symmetric positive definite matrix and let $Q \in R^{n\times n}$ be a given symmetric matrix. Then

$$\lambda_{\min}(P^{-1}Q)\Omega(t) \leq x(t)^T Qx(t) \leq \lambda_{\max}(P^{-1}Q)\Omega(t) \tag{5}$$

for all $x(t) \in R^n$, where $\Omega(t) = x(t)^T Px(t)$, while $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ denote, respectively, the largest and the smallest eigenvalues of the matrix inside the brackets.

For brevity, we introduce the following notations:

$$\begin{aligned} Q_{i_k} &= P_{i_k}^2 + A_{i_k}^T P_{i_k} + P_{i_k}A_{i_k} + \gamma^{-2}P_{i_k}B_{i_k}B_{i_k}^T P_{i_k} \\ &\quad + \left(\gamma^2\alpha + g_{i_k}\right)I; \\ \hat{\lambda}_k &= \lambda_{\max}\left(P_{i_{k-1}}^{-1}(I + D_k)^T P_{i_k}(I + D_k)\right), \\ &\quad i_k = 1, 2, \ldots, m, m \in N; \\ \eta_{i_k} &= \lambda_{\max}\left(P_{i_k}^{-1}Q_{i_k}\right); \\ \beta_k &= \hat{\lambda}_k + \frac{\lambda_{\max}\left(P_{i_k}\right)}{\lambda_{\min}\left(P_{i_{k-1}}\right)}\rho_k^2 + 2\sqrt{\hat{\lambda}_k\frac{\lambda_{\max}\left(P_{i_k}\right)}{\lambda_{\min}\left(P_{i_{k-1}}\right)}\rho_k^2}. \end{aligned}$$

We are now in a position to present sufficient conditions to ensure that the nonlinear impulsive switched system (1) is exponentially stable.

Theorem 1: Let Assumptions 1–3 be satisfied and $u(t) = 0$. Furthermore, suppose that there exists a symmetric positive definite matrix $P_{i_k} \in R^{n\times n}$ such that the following conditions are satisfied:

(a) There exists a constant $0 < \mu < \min_{i_k\in\{1,\ldots,m\}}\{-\eta_{i_k}\}$ such that

$$\ln\beta_k - \mu(\tau_k - \tau_{k-1}) \leq 0 \tag{6}$$

(b)

$$\begin{bmatrix} \Omega_{i_k} & \gamma^{-1}P_{i_k}B_{i_k} & P_{i_k} & \sqrt{\alpha}\gamma I & \sqrt{g_{i_k}}I \\ * & -I & 0 & 0 & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -I \end{bmatrix} < 0 \quad (7)$$

where $\Omega_{i_k} = A_{i_k}^T P_{i_k} + P_{i_k} A_{i_k}$ and the symbol $*$ represents the matrix's symmetric part. Then, the nonlinear impulsive switched system (1) is exponentially stable.

Proof: Define

$$V(t) = x(t)^T P_{i_k} x(t). \quad (8)$$

Then, when $t \in (\tau_k, \tau_{k+1}]$, evaluating the time derivative of $V(t)$ along the trajectory of the nonlinear impulsive switched system (1) with $u(t) = 0$ gives

$$\dot{V}(t) = x(t)^T \left(A_{i_k}^T P_{i_k} + P_{i_k} A_{i_k}\right) x(t) + 2x(t)^T P_{i_k} B_{i_k} w(t) + 2x(t)^T P_{i_k} \phi_{i_k}(t,x). \quad (9)$$

By Lemma 1, it is clear that

$$2x(t)^T P_{i_k} B_{i_k} w(t) \leq \gamma^{-2} x(t)^T P_{i_k} B_{i_k} B_{i_k}^T P_{i_k} x(t) + \gamma^2 w(t)^T w(t) \quad (10)$$

and

$$2x(t)^T P_{i_k} \phi_{i_k}(t,x) \leq \phi_{i_k}(t,x)^T \phi_{i_k}(t,x) + x(t)^T P_{i_k}^2 x(t). \quad (11)$$

Thus, by Assumptions 2 and 3, substituting (10) and (11) into (9) yields

$$\begin{aligned} \dot{V}(t) &\leq x(t)^T \left(P_{i_k}^2 + A_{i_k}^T P_{i_k} + P_{i_k} A_{i_k} + \gamma^{-2} P_{i_k} B_{i_k} B_{i_k}^T P_{i_k} + \left(\gamma^2 \alpha + g_{i_k}\right) I\right) x(t) \\ &= x(t)^T Q_{i_k} x(t) \end{aligned} \quad (12)$$

where $Q_{i_k} < 0$ which, by using Schur complement theorem [7], is equivalent to (7). By Lemma 2 and (12), we obtain

$$\dot{V}(t) \leq \eta_{i_k} V(t) \quad (13)$$

where $\eta_{i_k} = \lambda_{\max}(P_{i_k}^{-1} Q_{i_k})$. This leads to

$$V(t) \leq V\left(\tau_k^+\right) \exp\left[\int_{\tau_k}^{t} \eta_{i_k}\right] d\tau, \quad t \in (\tau_k, \tau_{k+1}], \quad k = 0, 1, 2, \ldots. \quad (14)$$

At the impulsive switching time point τ_k, by using the Cauchy-Schwarz inequality [15], we have

$$\begin{aligned} V\left(\tau_k^+\right) \leq\ & \hat{\lambda}_k x(\tau_k)^T P_{i_{k-1}} x(\tau_k) + \lambda_{\max}\left(P_{i_k}\right) \|\psi_{i_k}(\tau_k, x)\|^2 \\ & + 2\sqrt{[(I + D_k)x(\tau_k)]^T P_{i_k} [(I + D_k)x(\tau_k)]} \\ & \times \sqrt{\psi_{i_k}(\tau_k, x)^T P_{i_k} \psi_{i_k}(\tau_k, x)}. \end{aligned} \quad (15)$$

Applying (8) and Assumption 1 to (15) yields

$$\begin{aligned} V\left(\tau_k^+\right) &\leq \hat{\lambda}_k V(\tau_k) + \lambda_{\max}\left(P_{i_k}\right) \rho_k^2 x(\tau_k)^T x(\tau_k) \\ &\quad + 2\sqrt{\hat{\lambda}_k V(\tau_k)} \sqrt{\lambda_{\max}\left(P_{i_k}\right) \rho_k^2 x(\tau_k)^T x(\tau_k)} \\ &\leq \beta_k V(\tau_k), \quad k = 1, 2, \ldots \end{aligned} \quad (16)$$

where

$$\beta_k = \hat{\lambda}_k + \frac{\lambda_{\max}\left(P_{i_k}\right)}{\lambda_{\min}\left(P_{i_{k-1}}\right)} \rho_k^2 + 2\sqrt{\hat{\lambda}_k \frac{\lambda_{\max}\left(P_{i_k}\right)}{\lambda_{\min}\left(P_{i_{k-1}}\right)} \rho_k^2}.$$

For $t \in (\tau_0, \tau_1]$, it follows from (14) that

$$V(t) \leq V\left(\tau_0^+\right) \exp\left[\int_{\tau_0}^{t} \eta_{i_0} d\tau\right].$$

Thus, by (16), we get

$$V\left(\tau_1^+\right) \leq \beta_1 V(\tau_1) \leq \beta_1 V\left(\tau_0^+\right) \exp\left[\int_{\tau_0}^{\tau_1} \eta_{i_0} d\tau\right]. \quad (17)$$

For $t \in (\tau_1, \tau_2]$, it follows from (14) and (17) that:

$$\begin{aligned} V(t) &\leq V\left(\tau_1^+\right) \exp\left[\int_{\tau_1}^{t} \eta_{i_1} d\tau\right] \\ &\leq \beta_1 V\left(\tau_0^+\right) \exp\left[\int_{\tau_0}^{\tau_1} \eta_{i_0} d\tau + \int_{\tau_1}^{t} \eta_{i_1} d\tau\right]. \end{aligned}$$

Similarly, for $t \in (\tau_k, \tau_{k+1}]$

$$V(t) \leq V\left(\tau_0^+\right) \beta_1 \ldots \beta_k \exp\left[\int_{\tau_0}^{\tau_1} \eta_{i_0} d\tau + \int_{\tau_1}^{\tau_2} \eta_{i_1} d\tau + \ldots + \int_{\tau_k}^{t} \eta_{i_k} d\tau\right].$$

Hence, we have

$$\begin{aligned} V(t) &\leq V\left(\tau_0^+\right) \beta_1 \ldots \beta_k \exp\left[\eta_{i_0}(\tau_1 - \tau_0) + \eta_{i_1}(\tau_2 - \tau_1) + \ldots + \eta_{i_k}(t - \tau_k)\right] \\ &\leq V\left(\tau_0^+\right) \beta_1 \ldots \beta_k \exp\left[-\hat{\eta}(t - \tau_0)\right] \end{aligned}$$

where $\hat{\eta} = \min_{i_k \in \{1,\ldots,m\}} \{-\eta_{i_k}\} > 0$. Thus

$$\begin{aligned} V(t) &\leq V\left(\tau_0^+\right) \beta_1 \ldots \beta_k \exp\left[-\mu(t - \tau_0)\right] \\ &\quad \times \exp\left[-(\hat{\eta} - \mu)(t - \tau_0)\right] \\ &\leq V\left(\tau_0^+\right) \beta_1 \exp\left[-\mu(\tau_1 - \tau_0)\right] \ldots \\ &\quad \beta_k \exp\left[-\mu(\tau_k - \tau_{k-1})\right] \exp\left[-(\hat{\eta} - \mu)(t - \tau_0)\right]. \end{aligned} \quad (18)$$

Since (6) holds, i.e., there exists a constant $0 < \mu < \min_{i_k \in \{1,\ldots,m\}} \{-\eta_{i_k}\}$ such that $\ln \beta_k - \mu(\tau_k - \tau_{k-1}) \leq 0$, it follows from (18) that

$$V(t) \leq V\left(\tau_0^+\right) \exp\left[-(\hat{\eta} - \mu)(t - \tau_0)\right], \quad t \geq \tau_0.$$

Thus, for all $t \geq \tau_0$, by Lemma 2, we have

$$\|x(t, \tau_0, x_0)\| \leq \Gamma_{i_k} \|x_0\| \exp\left[-\frac{1}{2}(\hat{\eta} - \mu)(t - \tau_0)\right].$$

where $\Gamma_{i_k} = \lambda_{\min}(P_{i_k})^{-(1/2)} \lambda_{\max}(P_{i_0})^{1/2}$. Therefore, the nonlinear impulsive switched system (1) is exponentially stable with the convergence rate $(1/2)[\min_{i_k \in \{1,\ldots,m\}} \{-\eta_{i_k}\} - \mu]$ under any switching law. This completes the proof. ■

Remark 1: The exponential stability results obtained above are derived by using switched Lyapunov functions [3]. These Lyapunov functions are not required to decrease at the impulsive switching time points as required in previous works reported in [7] and [8]. Our results allow the switched Lyapunov functions to decrease during the continuous portion of the trajectory and can experience a jump increase at the impulses.

The following result establishes sufficient conditions for exponential stability with the L_2-gain condition of the nonlinear impulsive switched system (1).

Theorem 2: Let Assumptions 1–3 be satisfied, $u(t) = 0$, and $x(0) = 0$. Furthermore, for a prescribed $\gamma > 0$, suppose that

$$\max_{i_k \in \{1,\ldots,m\}} \left\{ \lambda_{\max} \left(E_{i_k}^T E_{i_k} \right) \right\} \leq \gamma^2 \alpha \tag{19}$$

and that there exists a symmetric positive definite matrix $P_{i_k} \in R^{n \times n}$ such that the following conditions are satisfied:

(a)

$$0 < \beta_k \leq 1 \tag{20}$$

(b)

$$\begin{bmatrix} \Omega_{i_k} & \gamma^{-1} P_{i_k} B_{i_k} & P_{i_k} & \sqrt{\alpha}\gamma I & \sqrt{g_{i_k}} I \\ * & -I & 0 & 0 & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -I \end{bmatrix} < 0 \tag{21}$$

where $\Omega_{i_k} = A_{i_k}^T P_{i_k} + P_{i_k} A_{i_k}$. Then, the nonlinear impulsive switched system (1) is exponentially stable with the L_2-gain condition of disturbance attenuation.

Proof: By (9) and Lemma 1, we obtain

$$\begin{aligned} \dot{V}(t) \leq & x(t)^T \left(A_{i_k}^T P + P A_{i_k} + \gamma^{-2} P_{i_k} B_{i_k} B_{i_k}^T P_{i_k} \right) x(t) \\ & + \gamma^2 w(t)^T w(t) + \phi_{i_k}(t,x)^T \phi_{i_k}(t,x) + x(t)^T P_{i_k}^2 x(t) \\ \leq & x(t)^T \left(P_{i_k}^2 + A_{i_k}^T P_{i_k} + P_{i_k} A_{i_k} + \gamma^{-2} P_{i_k} B_{i_k} B_{i_k}^T P_{i_k} \right. \\ & \left. + E_{i_k}^T E_{i_k} + g_{i_k} I \right) x(t) - \| E_{i_k} x(t) \|^2 \\ & + \gamma^2 \| w(t) \|^2 . \end{aligned} \tag{22}$$

It follows from (21) and Schur complement theorem [7] that:

$$P_{i_k}^2 + A_{i_k}^T P_{i_k} + P_{i_k} A_{i_k} + \gamma^{-2} P_{i_k} B_{i_k} B_{i_k}^T P_{i_k} + \left(\gamma^2 \alpha + g_{i_k} \right) I \leq 0. \tag{23}$$

Moreover, (19) and (23) lead to

$$P_{i_k}^2 + A_{i_k}^T P_{i_k} + P_{i_k} A_{i_k} + \gamma^{-2} P_{i_k} B_{i_k} B_{i_k}^T P_{i_k} + E_{i_k}^T E_{i_k} + g_{i_k} I \leq 0. \tag{24}$$

Thus, substituting (24) into (22) yields

$$\dot{V}(t) \leq - \| E_{i_k} x(t) \|^2 + \gamma^2 \| w(t) \|^2 = - \| z(t) \|^2 + \gamma^2 \| w(t) \|^2$$

i.e.

$$\| z(t) \|^2 \leq -\dot{V}(t) + \gamma^2 \| w(t) \|^2 . \tag{25}$$

For any given $T \in (t_k, t_{t+1}]$, integrating both sides of (25) from 0 to T, we obtain

$$\int_0^T \| z(t) \|^2 dt \leq - \int_0^T \dot{V}(t) dt + \gamma^2 \int_0^T \| w(t) \|^2 dt \tag{26}$$

where $T \in (t_k, t_{k+1}]$. From (8) and under zero initial condition, we see that $V(0) = 0, V(T) \geq 0$.

Since $0 < \beta_k \leq 1$, we have

$$\begin{aligned} \int_0^T \dot{V}(t) dt = & \int_0^{\tau_1} \dot{V}(t) dt + \int_{\tau_1}^{\tau_2} \dot{V}(t) dt + \ldots + \int_{\tau_k}^{T} \dot{V}(t) dt \\ \geq & \sum_{i=1}^{k} (1 - \beta_k) V(\tau_k) + V(T) \geq 0. \end{aligned} \tag{27}$$

Thus, from (26) and (27), it follows that:

$$\int_0^T \| z(t) \|^2 dt \leq \gamma^2 \int_0^T \| w(t) \|^2 dt, \quad T \in (\tau_k, \tau_{k+1}].$$

This is the L_2-gain condition of disturbance attenuation. Since (20) holds, i.e., $0 < \beta_k \leq 1$, $k = 1, 2, \ldots$, which can lead to $\ln \beta_k - \mu(\tau_k - \tau_{k-1}) \leq 0, 0 < \mu < \min_{i_k \in \{1,\ldots,m\}} \{-\eta_i\}$. Thus, by (19)–(21), it is clear that all the conditions of Theorem 1 are satisfied. Then, the nonlinear impulsive switched system (1) is also exponentially stable. This completes the proof. ■

In what follows, we devise a switched stabilizing controller to exponentially stabilize the nonlinear impulsive switched system (1) with the L_2-gain condition of disturbance attenuation.

Theorem 3: Let Assumptions 1–3 be satisfied and $x(0) = 0$. Furthermore, for a prescribed $\gamma > 0$, suppose that

$$\max_{i_k \in \{1,\ldots,m\}} \left\{ \lambda_{\max} \left(E_{i_k}^T E_{i_k} \right) \right\} \leq \gamma^2 \alpha \tag{28}$$

and that there exist a symmetric positive definite matrix L_{i_k} and a matrix W_{i_k} such that the following conditions are satisfied:

(a)

$$0 < \beta_k \leq 1 \tag{29}$$

(b)

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & \Pi_{14} & \Pi_{15} \\ * & -I & 0 & 0 & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -I \end{bmatrix} < 0 \tag{30}$$

where $\Pi_{11} = A_{i_k} L_{i_k} + C_{i_k} W_{i_k} + (A_{i_k} L_{i_k} + C_{i_k} W_{i_k})^T$, $\Pi_{12} = \gamma^{-1} B_{i_k}$, $\Pi_{13} = I$, $\Pi_{14} = \sqrt{\alpha}\gamma L_{i_k}$, $\Pi_{15} = \sqrt{g_{i_k}} L_{i_k}$. Then, the nonlinear impulsive switched system (1) is exponentially stabilizable with the L_2-gain condition of disturbance attenuation. Moreover, if (28) holds and there exist feasible matrices L_{i_k} and W_{i_k} such that (29) and (30) are satisfied, then the feedback controller

$$u(t) = F_{i_k} x(t), \quad F_{i_k} = W_{i_k} (L_{i_k})^{-1} \tag{31}$$

exponentially stabilizes the nonlinear impulsive switched system (1) under the L_2-norm condition of disturbance attenuation. In other words, the corresponding impulsive switched closed-loop system (3) is exponentially stable with the L_2-gain conditions of disturbance attenuation.

Proof: Let $L_{i_k} = P_{i_k}^{-1}$ and $W_{i_k} = F_{i_k} P_{i_k}^{-1}$ and then substitute them into (30). Then, we obtain the following nonlinear matrix inequality:

$$\begin{bmatrix} \bar{A_{i_k}} & \gamma^{-1} B_{i_k} & I & \sqrt{\alpha}\gamma P_{i_k}^{-1} & \sqrt{g_{i_k}} P_{i_k}^{-1} \\ * & -I & 0 & 0 & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -I \end{bmatrix} < 0 \tag{32}$$

where

$$\bar{A_{i_k}} = A_{i_k} P_{i_k}^{-1} + C_{i_k} F_{i_k} P_{i_k}^{-1} + \left(A_{i_k} P_{i_k}^{-1} + C_{i_k} F_{i_k} P_{i_k}^{-1} \right)^T .$$

Multiplying $diag\{P_{i_k}, I, I, I\}$ on both sides of the matrix inequality (32) gives

$$\begin{bmatrix} \hat{A_{i_k}} & \gamma^{-1}P_{i_k}B_{i_k} & P_{i_k} & \sqrt{\alpha}\gamma I & \sqrt{g_{i_k}}I \\ * & -I & 0 & 0 & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -I \end{bmatrix} < 0 \qquad (33)$$

where $\hat{A_{i_k}} = (A_{i_k} + C_{i_k}F_{i_k})^T P_{i_k} + P_{i_k}(A_{i_k} + C_{i_k}F_{i_k})$. Thus, in addition to (28) and (29), all the conditions of Theorem 2 are satisfied. Therefore, the nonlinear impulsive switched system (1) is exponentially stable with the L_2-gain condition. Since $W_{i_k} = F_{i_k}P_{i_k}^{-1}$, the linear feedback controller (31) can be constructed. This completes the proof. ∎

We now devise a computational algorithm based on the sufficient conditions of Theorem 3 for constructing a feedback controller which exponentially stabilizes the nonlinear impulsive switched system (1) with the L_2-gain condition.

Algorithm 1:

Step 1) Input the matrices $A_{i_k}, B_{i_k}, C_{i_k}, D_k, E_{i_k}$ and nonnegative scalars g_{i_k}. Set

$$\gamma = \sqrt{\max_{i_k\in\{1,\ldots,m\}} \left\{\lambda_{\max}\left(E_{i_k}^T E_{i_k}\right)\right\}/\alpha}.$$

Step 2) Solve the linear matrix inequality (30) subject to (29) to obtain L_{i_k} and W_{i_k}.

Step 3) Compute F_{i_k} by substituting L_{i_k} and W_{i_k} into (31).

Step 4) Construct the feedback controller $u(t) = F_{i_k}x(t)$, where F_{i_k} are the computed gain matrices.

IV. A Numerical Example

Consider a nonlinear impulsive switched system with arbitrary switching laws and two switching status ($i_k \in \{1,2\}$). When $i_k = 1$ and $i_k = 2$, the subsystems' specifications are: $A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1.5 \end{bmatrix}$, $B_1 = \begin{bmatrix} 2.1 & 0.2 \\ -1.1 & 0 \end{bmatrix}$, $C_1 = \begin{bmatrix} 1.6 & 1 \\ 2 & 1.5 \end{bmatrix}$, $\phi_1(t, x(t)) = \begin{bmatrix} \sin(x_1) \\ 0 \end{bmatrix}$, $E_1 = \begin{bmatrix} 0.3 & 0.2 \\ -1 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$, $B_2 = \begin{bmatrix} 1.5 & 1.3 \\ 1.2 & 0 \end{bmatrix}$, $C_2 = \begin{bmatrix} 1.8 & 0.8 \\ 1 & 1.7 \end{bmatrix}$, $\phi_2(t, x(t)) = \begin{bmatrix} \sin(x_1) \\ \sin(x_2) \end{bmatrix}$, $E_2 = \begin{bmatrix} -1 & 1 \\ 0 & 0.2 \end{bmatrix}$, $D_k = -\begin{bmatrix} 0.58 & 0 \\ 0 & 0.58 \end{bmatrix}$, $\psi_k(\tau_k, x(\tau_k)) = \begin{bmatrix} 0.3sin(x_1(\tau_k)) \\ 0.3sin(x_2(\tau_k)) \end{bmatrix}$, $w_1(t) = w_2(t) = \begin{bmatrix} sin(20\pi t)x_1 \\ cos(20\pi t)x_2 \end{bmatrix}$. From Assumptions 1–3, we can choose $\alpha = g_1 = g_2 = 1$. By Algorithm 1, we obtain: $\gamma = 1.4213$, $L_1 = \begin{bmatrix} 0.9372 & 0.0018 \\ 0.0018 & 0.9139 \end{bmatrix}$, $L_2 = \begin{bmatrix} 0.9291 & 0.0081 \\ 0.0081 & 0.9208 \end{bmatrix}$, $W_1 = \begin{bmatrix} 7.3452 & 57.2860 \\ -18.0648 & -80.2996 \end{bmatrix}$, $W_2 = \begin{bmatrix} 43.0927 & 102.7580 \\ -105.6336 & -64.8821 \end{bmatrix}$.

Then, the required state feedback gain matrices are

$$F_1 = \begin{bmatrix} 7.7202 & 62.6696 \\ -19.1111 & -87.8301 \end{bmatrix},$$
$$F_2 = \begin{bmatrix} 45.4049 & 111.1982 \\ -113.0824 & -69.4645 \end{bmatrix}.$$

We also compute $0 < \beta_1 = 0.7179 \leq 1$ and $0 < \beta_2 = 0.7170 \leq 1$. By Theorem 3, the nonlinear impulsive switched system is exponentially stable with the L_2-gain condition.

V. Conclusion

We have developed a set of readily computable conditions in terms of linear matrix inequalities for exponential stability with the L_2-gain condition of nonlinear impulsive switched systems. We devised a computational method for the construction of a switched linear feedback controller, which not only exponentially stabilizes the nonlinear impulsive switched systems but also ensures the satisfaction of the L_2-gain condition.

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