School of Mathematics and Statistics

Parametric Estimation For Randomly Censored Autocorrelated Data

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Certification

I hereby certify that the work presented in this thesis is my own work and that all references are duly acknowledged. This work has not been submitted previously, in whole or in part, in respect of any other academic award at this University or elsewhere.

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Abstract

This thesis is mainly concerned with the estimation of parameters in autoregressive models with censored data. For convenience, attention is restricted to the first-order stationary autoregressive (AR(1)) model in which the response random variables are subject to right-censoring. In their present form, currently available methods of estimation in regression analysis with censored autocorrelated data, which includes the MLE, are applicable only if the errors of the AR component of the model are Gaussian. Use of these methods in AR processes with non-Gaussian errors requires, essentially, rederivations of the estimators. Hence, in this thesis, we propose new estimators which are robust in the sense that they can be applied with minor or no modifications to AR models with non-Gaussian. We propose three estimators, two of which the form of the distribution of the errors needs to be specified. The third estimator is a distribution-free estimator. As the reference to this estimator suggests, it is free from distributional assumptions in the sense that the error distribution is calculated from the observed data. Hence, it can be used in a wide variety of applications.

In the first part of the thesis, we present a summary of the various currently available estimators for the linear regression model with censored independent and identically distributed (i.i.d.) data. In our review of these estimators, we note that the linear regression model with censored i.i.d. data has been studied quite extensively. Yet, use of autoregressive models with censored data has received very little attention. Hence, the remainder of the thesis focuses on the estimation of parameters for censored autocorrelated data. First, as part of the study, we review currently available estimators in regression with censored autocorrelated data. Then we present descriptions of the new estimators for censored autocorrelated data. With the view that extensions to the AR(p), model, $p > 1$, and to left-censored data can be easily achieved, all the esti-

mators, both currently available and new, are discussed in the context of the AR(1) model. Next, we establish some asymptotic results for the estimators in which specification of the form of the error distribution is necessary. This is followed by a simulation study based on Monte Carlo experiments in which we evaluate and compare the performances of the new and currently available estimators among themselves and with the least-squares estimator for the uncensored case. The performances of the asymptotic variance estimators of the parameter estimators are also evaluated.

In summary, we establish that for each of the two new estimators for which the distribution of the errors is assumed known, under suitable conditions on the moments of the error distribution function, if the estimator is consistent, then it is also asymptotically normally distributed. For one of these estimators, if the errors are Gaussian and alternate observations are censored, then the estimator is consistent. Hence, for this special case, the estimator is consistent and asymptotically normal. The simulation results suggest that this estimator is comparable with the distribution-free estimator and a currently available pseudolikelihood (PL) estimator. All three estimators perform worse than the least squares estimator for the uncensored case. The MLE and another currently available PL estimator perform comparably not only with the least squares estimator for the uncensored case but also with estimators from the above-mentioned group of three estimators, which includes the distribution-free estimator. The other new estimator for which the form of the error distribution is assumed known compares favourably with the least-squares estimator for the uncensored case and better than the rest of the estimators when the true value of the autoregression parameter is 0.2. When the true value of the parameter is 0.5, this estimator performs comparably with the rest of the estimators and worse when the true value of the parameter is 0.8. The simulation results of the asymptotic variance estimators suggest that for each estimator and for a fixed value of the true autoregression parameter, if the error distribution is fixed and the censoring rate is constant, the asymptotic formulas lead to values which are asymptotically insensitive to the censoring pattern. Also, the estimated asymptotic variances decrease as the sample size increases and their behaviour, with respect to changes in the true value of autoregression parameter, is consistent with the behaviour of the asymptotic variance of the least-squares estimator for the uncensored case.

Some suggestions for possible extensions conclude the thesis.

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Preface

This thesis includes the following paper:

Vasudaven, M. Nair, M.G. and Sithole, M.M. (1996). On estimation for censored autoregressive data. *Statistics and Probability Letters.* **31**, 97-105.

The paper is extracted mainly from section 3.3 of Chapter 3. Apart from the joint work represented by the paper, the rest of the thesis has been done by myself.

Contents

List of Tables

List of Figures

Chapter 1

Preliminaries

1.1 Introduction

Censoring arises in many applications in the physical, medical and environmental sciences and in business and economics. Reference for examples of applications in engineering and medical studies can be made to Miller (1981). Following is an example of an application in engineering. Suppose we put to test at time $t = 0$ a batch of light bulbs or transistors and record their time to failure. Some light bulbs may take a long time to burn out, and it may not be feasible to wait until all of them have failed before we end the experiment. In such cases, we may stop the experiment at a pre-specified time (say, t_c). The lifetimes of all the units that have not failed by the time t_c are censored. This form of censoring is known as type I censoring. Sometimes we may not know a good value of t_c. Therefore, we may decide on a pre-specified fraction of the units that must burn out before we end the experiment. This is type II censoring.

Random censoring often occurs in medical applications involving animal studies or clinical trials. In clinical trials, patients may enter the study at different times and receive one of several available therapies. We may be interested in their lifetimes, but censoring occurs through loss to follow up, drop out or termination of study. Loss to follow up occurs if a patient decides to move elsewhere and we never see him or her again. Drop out can occur if the therapy has bad side effects making it necessary to discontinue the treatment. Alternatively, the patient may still be in contact, but refuses to continue the treatment. If it seems sensible to assume independence among the lifetimes of patients, these lifetimes may be modelled with independent errors, as it is the case in linear regression with censored independent and identically distributed (i.i.d.) data.

Examples of censoring in environmental studies and business and economics typically involve recording data sequentially in time. In such cases, often the sensitivity of the measurement may be limited and an exact value can be recorded only if it falls within specified range. This gives rise to time series censored on the right if the range is defined by an upper limit of detection, left censored if the range is defined by a lower limit and double-censoring if the range is defined by both upper and lower limits of detection. Examples of censored time series may be found in Wecker (1974, 1978), Robinson (1980) and Zeger and Brookmeyer (1986). Wecker (1974, 1978) considered forecasting and estimation for sales of a product which are subject to stockouts, and thus differ from true demand. The following are examples in Robinson (1980). Measurement of rainfall may be limited by the size of the rain gauge, and subject to evaporation. Boiling water provides a safety feature in nuclear reactors by limiting the power of the reactor. Signals may be quantized or limited for ease of storage or processing. Many econometric models have mixed probability distributions, with both a discrete and continuous component. These may be conveniently modelled in terms of a continuous variable that is censored when it crosses certain thresholds, even if there is no physical meaning attached to this random variable. The examples in Zeger and Brookmeyer (1986) illustrate how censored time series may arise in environmental and medical studies: there may be an upper or lower limit of detection when one is monitoring levels of an airborne contaminant or recording daily bioassays of hormone levels in a patient. We may fit autoregressive models to account for the time dependence.

Having referred the reader to some examples of how censoring may arise, we now turn to the motivation for the current investigation and put our objectives in perspective. This thesis is mainly concerned with the estimation of parameters in stationary first-order autoregressive (AR(1)) models with possibly censored response variables, about which little at present is known. In the context of regression with censored autocorrelated data, we are only able to refer the reader to Robinson (1982), Dagenais (1982,1986,1989), Bussiére (1983) and Zeger and Brookmeyer (1986). In their present form, currently available estimators for these regression models, which includes the maximum likelihood estimator (MLE), are based on the assumption that the errors of the autoregressive component are i.i.d. Gaussian random variables (rv's). However, the Gaussian assumption may not be satisfied in practical applications. If, indeed, this is the case, these estimators need to be suitably modified for the specified (non-Gaussian) error distribution. In some cases, the modification is essentially a rederivation of the estimator. This means that extensive preparatory calculations may have to be carried out before one can apply the currently available estimators to non-Gaussian error distributions. Further, the MLE can be computationally intensive if high dimensional integrals must be evaluated to 'correct' for the bias due to the censored values, as noted by Zeger and Brookmeyer (1986). These authors also noted that problems of non-convergence have been experienced with the Newton-Raphson procedure in the case of the MLE for censored i.i.d. data (see, e.g. Sampford and Taylor, 1959 and Lawless, 1982). In computing the MLE in regression with censored autocorrelated data, Zeger and Brookmeyer (1986) avoided problems of non-convergence by using an EM algorithm or a quasi-Newton procedure which chooses between a Newton-Raphson step and a steepest-descent step. This procedure was developed by Dennis and Mei (1979) and is available with the 'S' statistical software. Zeger and Brookmeyer (1986) proposed a pseudolikelihood (PL) estimator to avoid the computational difficulties experienced with the MLE. Dagenais (1986) proposed another estimator. However, consistent estimators of the asymptotic variances of these PL estimators are difficult to obtain. This is because in each case, the contributions of the score function are not independent and the mean of each contribution is not zero. Hence, the sum of squares of these contributions is not an unbiased estimator of the variance of the score function.

In view of the above-mentioned limitations of the currently available estimators, we are prompted to propose new estimators for stationary autoregressive models with censored data. We derive these estimators in the context of the AR(1) model. The reason for this is that the autoregressive component of the regression model with censored autocorrelated errors is the only distinguishing feature between this model and the linear regression model with censored i.i.d. data. Therefore, any results obtained for the autoregressive component can be easily incorporated into the former model, which has the regression component as well. Also, the results can be easily extended to the AR(p) model, $p > 1$. Further, the regression model without the autoregressive component has been studied extensively by many authors, as will be seen in Chapter 2. Thus, it suffices to restrict attention to the AR(1) model. A major contribution of this thesis is that with the new estimators, the error distribution may be non-Gaussian.

We propose three estimators. In two of these estimators, the form of the distribution of the errors must be specified. Although iteration is necessary to compute the two estimators, we have, so far, not experienced problems of non-convergence. This is attributable to the fact that we use an EM algorithm and this is known to converge slowly but more surely (see, e.g., Dempster, Laird and Rubin, 1977 or McLaughlan and Krishanan, 1997). Further, even though evaluation of high dimensional integrals may be necessary, the new estimators require fewer computations than the MLE. This is because with the MLE, in a 'block' of r consecutive censored observations, the integrals that need to be evaluated for each of the observations are all r-dimensional. For the new estimators, however, the integral corresponding to the first observation is *one*-dimensional, the one for the second observation is *two*-dimensional, and so on. Also, the problems of asymptotic variance estimation experienced in the case of the PL estimators are not experienced with these new estimators. The third estimator we propose is a distribution-free estimator and as the reference to it suggests, the distribution of the errors need not be known. This means that the estimator can be applied in a wide variety of situations.

The thesis is organized as follows. The present chapter closes with a summary of notation and conventions which will be used in subsequent chapters without comment. In chapter 2, we first review the currently existing estimators in linear regression with independent censored data. Then we review available estimators in regression with censored autocorrelated data. In the latter review, we restrict our discussion to only the autoregressive component of the regression model and the estimators are described in the context of the AR(1) model. The reason is the same as the one above, put forward for the new estimators, that it is sufficient to develop the ideas within the framework of the AR(1) model, since they can then be extended to suit the full model.

In chapter 3, the new estimators are described along with the motivation and justification for them. We also state the main differences in the principles of obtaining these estimators. Chapter 4 investigates the asymptotic normality of the estimating functions for the two new estimators for which the form of the distribution of the errors in the AR(1) model must be specified. We also investigate conditions for the consistency and asymptotic normality of the estimators.

Chapter 5 contains a simulation study in which the performance of the estimators (new and currently available), in finite samples, is evaluated and the estimators compared among themselves. This simulation study includes the MLE. Therefore, in part, it addresses a recommendation by Dagenais (1982), who suggested an investigation of the large sample as well as small sample properties of the MLE. There has been no previous attempt to address this recommendation. We consider three error distributions in our simulations, the Gaussian, the double exponential (also known as the Laplace distribution) and the gamma distribution. For the Gaussian error distribution, we consider two censoring distributions, Laplace and Gaussian. For the Laplace error distribution, we consider the Gaussian distribution as the error distribution. For the gamma error distribution, we consider another gamma distribution as the censor distribution. For the Gaussian error distribution, we compare all the estimators (both new and currently available) among themselves and with the least-squares estimator corresponding to the uncensored case. For the non-

Gaussian error distributions, however, we compare only the new estimators among themselves and with the least-squares estimator for the uncensored case. The reason for not including the currently existing estimators in the study involving the non-Gaussian error distributions is found in the argument given earlier that, in their present form, the currently available estimators are not suitable for non-Gaussian error distributions.

Overall conclusions of the current research study are presented in Chapter 6. Some general comments on the estimation methods discussed in this thesis and suggestions for possible extensions and future developments are also outlined in Chapter 6 to conclude the thesis.

1.2 Notation

The following notation will be used without comment in the sequel.

a.s.	almost surely (i.e., with probability one)
a.e.	almost exactly
i.i.d.	independent and identically distributed
r.v.	random variable
n	sample size
$\Re$	set of real numbers
$\Re^k$	set of real numbers on a k-dimensional space
$\mathcal{Z}$	set of integers
$\in$	is a member of (belongs to)
$\ni$	such that
$\exists$	there exists
$\forall$	for all
$\Longleftrightarrow$	if and only if
$\sim$	has the same distribution as
$\approx$	approximately equal to
A'	transpose of the matrix A
$f^{(1)}$	first derivative of f

CLT	central limit theorem
ML	maximum likelihood
MLE	maximum likelihood estimator
PL	Pseudolikelihood

Almost sure convergence, convergence in probability, convergence in L^p and convergence in distribution are denoted by $\stackrel{a.s.}{\rightarrow}$, $\stackrel{P}{\rightarrow}$, $\stackrel{L^p}{\rightarrow}$ and $\stackrel{\mathcal{D}}{\rightarrow}$, respectively.

Let $X = \{..., X_{-1}, X_0, X_1, ...\}$ denote a sequence of possibly dependent random variables defined on a probability space $(\Re^{Z}, \mathcal{B}^{Z}, P_\theta)$, θ being an unknown parameter taking values in $\Theta \subset \Re$. Denote the sample vector of n rv's $\{X_{-1}, X_0, ..., X_{n-2}\}$ or $\{X_0, X_1, ..., X_{n-1}\}$ or $X_1, X_2, ..., X_n$, etc., by $X(n)$. Then we will denote by $\sigma\{X(n)\}$, the sigma-field generated by $X(n)$.

We will denote the indicator function of an event E by $I(E)$, where

$$I(E)(\omega) = \begin{cases} 1 & \text{if } \omega \in E, \\ 0 & \text{otherwise.} \end{cases}$$

$N(\mu, \sigma^2)$ is the normal distribution with mean μ and variance σ^2. The distribution of a double exponential (also known as Laplace) rv X with density function

$$f_X(x) = \frac{1}{2\beta}\exp(-\frac{1}{\beta}|x - A|), \quad -\infty < x < \infty,$$

for some $\beta > 0$, $-\infty < A < \infty$, will be denoted by Laplace (β,A). Similarly, if X is a gamma rv with density function

$$g_X(x) = \begin{cases} \frac{(x-A)^{\alpha-1}}{\beta^\alpha\Gamma(\alpha)}\exp(-\frac{(x-A)}{\beta}), & \text{if } A \leq x < \infty, \\ 0 & \text{otherwise,} \end{cases}$$

for some $\alpha > 0$, $\beta > 0$, $-\infty < A < \infty$, then its distribution will be denoted by gamma (α,β,A).

Chapter 2

Existing Estimators in Regression with Censored Data

2.1 Introduction

A variety of methods have been developed for linear regression problems in which the dependent variable is subject to censoring and the errors are independent and identically distributed (i.i.d.) [See, e.g., Miller (1976), Schmee and Hahn (1979), Buckley and James (1979), Koul, Susarla and Van Ryzin (1981), Bennet (1983), Sweeting (1987), Leurgans (1987), Wei and Tanner (1991), Zhou (1992), Fygenson and Zhou (1992, 1994), Breiman, Tsur and Zemel (1993), Fan and Gijbels (1994) and Lai and Ying (1994), among others]. The methods proposed by Schmee and Hahn (1979), Bennet (1983), Sweeting (1987) and Breiman, Tsur and Zemel (1993) assume particular families of survival distributions, whereas the others avoid this requirement. All of these methods are for the following censored linear model. Let $Y_1, ..., Y_n$ be n independent random variables (rv's) satisfying

$$Y_i = \alpha + \beta x_i + \varepsilon_i, \;\; i = 1, ..., n, \tag{2.1}$$

where, x_i's are known explanatory variables and ε_i's are i.i.d. rv's with distribution F(.), zero mean and finite variance σ^2. The parameters of interest are the intercept, α, and the vector of regression coefficients, β. Some of the response variables, Y_i's, may be right censored and thus one observes

$$Z_i = \min(Y_i, T_i) \text{ and } \delta_i = I(Y_i \le T_i), \tag{2.2}$$

where, given the x_i's, $T_1, ..., T_n$ are i.i.d. rv's with distribution $G(.; x_i)$ independent of $\varepsilon_1, ..., \varepsilon_n$. $I(A)$ is the indicator function of the event A. The rv's $T_1, ..., T_n$ are called censoring variables. The Y_i's could just as well be left censored with $Z_i = \max(Y_i, T_i)$ and $\delta_i = I(Y_i > T_i)$. If $\beta = 0$, then (2.1.1) corresponds to the location model with location parameter α. James (1986) has proposed censored data estimating equations for various models including the location model. These equations require full distributional assumptions, and James suggests a distribution-free modification based on the product limit (Kaplan and Meier, 1958) estimator. However, the problem considered here is the estimation of (α, β) based on $(Z_1, \delta_1), ..., (Z_n, \delta_n)$.

Miller (1976) proposed an estimator of (α, β) obtained by minimizing the weighted sum of squares of the residuals with the weights computed from the Kaplan-Meier estimator of the error distribution based on the residuals. Buckley and James (1979) suggested another estimator of (α, β) which utilizes an expectation identity and substitutes Y_i into the usual least squares normal equations if Y_i is uncensored, or an estimate of it based on the Kaplan-Meier estimator of the error distribution, if censored. Both these estimators require the use of iteration methods. In both cases, as pointed out by the authors, the iterations can settle down to oscillating between two values. Buckley and James argue that, for their estimator, the values are closer to each other than for the Miller estimator and suggest taking the average and using it as an estimate. The first study to investigate the consistency of the Buckley-James estimator is due to James and Smith (1984). A slight modification of the estimator has led Ritov (1990) and Lai and Ying (1991) to establish the asymptotic normality of this type of estimators. A family of asymptotically equivalent estimators, based on linear rank tests for the slope in the linear model, has been recently introduced by Tsiatis (1990). Other methods of making inference about coefficients in the censored linear regression model have been studied by Wei, Ying and Lin (1990) and Lin and Wei (1992).

Koul, Susarla and Van Ryzin (1981) proposed an estimator of (α, β) which is easy to compute and requires no iteration. Like the Buckley-James estimator, this estimator is also based on an expectation identity. It differs from the Buckley-James estimator in that the Y_i's in the normal equations are replaced by pseudo rv's computed from the Kaplan-Meier estimator of the censoring distribution based on the censoring rv's, T_i's. Koul *et al.* (1981) conducted a complete investigation of the consistency and asymptotic normality of their estimator. Recently, Srinivasan and Zhou (1994) have used another approach based on counting processes and martingale techniques to prove the asymptotic normality of the estimator. Leurgans (1987) proposed another non-iterative estimator based on 'synthetic data' (pseudo rv's) in a similar way that the Koul-Susarla-Van Ryzin estimator is based on pseudo rv's.

Miller and Halpern (1982) compared the Koul-Susarla-Van Ryzin estimator with that of Miller (1976) and the Buckley-James estimator using the Stanford Heart Transplant Data. They recommended the use of the Buckley-James estimator. Leurgans (1987) compared her estimator with the Koul-Susarla-Van Ryzin estimator and concluded that her estimator performs better. Her conclusion is based on the performances of the two estimators on the Stanford Heart Transplant Data and the Leukemia Data that appears in Freireich, *et al* (1963). Recently, Heller and Simonoff (1990) compared several estimators using Monte Carlo experiments and concluded that the Buckley-James estimator is preferred. However, more recently, Fygenson and Zhou (1992) have suggested a slight modification of the Koul-Susarla-Van Ryzin estimator and have demonstrated using simulations, the Stanford Heart Transplant Data and the Leukemia Data from Freireich, *et al* (1963) that the modified Koul-Susarla-Van Ryzin estimator compares favourably with the Buckley-James estimator than the Leurgans estimator. Given the performance of the modified Koul-

Susarla-Van Ryzin estimator, it is also important to point out that, in the location model, the original Koul-Susarla-Van Ryzin estimator has been shown at a theoretical level by Tsai, Susarla and Van Ryzin (1984) to be identical to the Buckley-James estimator.

The modified Koul-Susarla-Van Ryzin estimator is almost identical to the original estimator. The difference is that in the original estimator, the censoring rv's are assumed to be i.i.d., whereas in the modified estimator, these rv's are assumed to be i.i.d. only within strata, as it often happens in practice (e.g., the Stanford Heart Transplant Data and the Leukemia Data). Therefore, in deriving the modified estimator, within each strata the Kaplan-Meier estimator of the censoring distribution is computed and used in the computation of the pseudo rv's that are substituted into the usual normal equations. If the sample consists of only a single stratum, then the two estimators differ only in the definition of the largest observation. In the original Koul-Susarla-Van Ryzin estimator, zero weight is assigned to a censored observation and the following uncensored observation is inflated by pre-multiplying it with the inverse of one minus the corresponding value of the censoring distribution based on the Kaplan-Meier estimator. In the modified Koul-Susarla-Van Ryzin estimator, the largest observation is redefined in the spirit of the 'redistribution-to-the-right' algorithm of Efron (1967). The largest observation is defined as uncensored even if it is censored because there is no larger observation on which to distribute its weight.

The Miller (1976), Buckley-James estimator, original Koul-Susarla-Van Ryzin and Leurgans (1987) estimators differ with respect to their assumption that the censoring variables are i.i.d. and the requirement that the errors, ε_i's should be i.i.d. With respect to the assumption about the censoring distribution, the estimators range from highly restrictive (the Miller estimator) to least restrictive (the Buckley-James estimator). However, the Buckley-James estimator relies on the assumption about the error distribution, whereas the

Koul-Susarla-Van Ryzin and Leurgans estimators do not. From applications point of view, this means that the Koul-Susarla-Van Ryzin and Leurgans estimators can be applied to survival data even if the error distribution varies from patient to patient. Also, the Miller and Buckley-James estimators need special programming, iterative computation and have convergence problems as mentioned earlier. On the other hand, the Koul-Susarla-Van Ryzin and Leurgans estimators can be easily incorporated into a regression package and the estimates obtained quickly without iteration. These advantages are shared by the modified Koul-Susarla-Van Ryzin estimator and become more appreciable when one considers a multiple multivariate model. However, when one considers a time series model, as it is done in this thesis, the use of the modified Koul-Susarla-Van Ryzin estimator seems impractical since by dividing the observations into strata according to the censoring variables, one loses the underlying time series structure. Hence, the distribution-free estimator for censored autocorrelated data proposed in this thesis is based on a suitably modified version of the Buckley-James estimator for the linear regression set-up.

As seen above, linear regression with i.i.d. censored data has been well studied. However, despite the fact that autocorrelation among errors is known to be a major problem in regression analysis (see, e.g., Dagenais 1982), estimation for regression with censored autocorrelated data has received very little attention. Wecker (1974) and Robinson (1980) considered prediction and estimation methods for censored time series data. A formal introduction of autocorrelation in the censored regression model was considered by Robinson (1982). Further research has been limited to the works of Dagenais (1982,1986,1989), Bussiére (1983) in a université de Montréal M.Sc. thesis and Zeger and Brookmeyer (1986). We present a brief account of these studies below. A detailed account is presented in section 2.3.

Robinson (1982) proved the consistency and asymptotic normality of the pseudolikelihood (PL) estimator obtained by maximizing the likelihood function which ignores the autocorrelation among the observations. Dagenais (1982) derived the full likelihood function of the AR(1) model with normal errors. This function involves multivariate normal integrals, the dimensions of which equal the number of observations in a sequence of consecutive censored observations. Bussiére (1983) computed exact maximum likelihood (ML) estimates for samples with at most five consecutive censored observations. Zeger and Brookmeyer (1986) derived the score equations corresponding to the full likelihood function of the AR(p) model with normal errors. Then they considered two numerical procedures to solve these equations: A modified Newton-Raphson routine and an EM algorithm. The details of each of these procedures are given in section 2.3.1 of this thesis. Like the likelihood function of Dagenais (1982), Zeger and Brookmeyer's likelihood also involves multivariate normal integrals. The dimensions of these integrals dimensions equal the number of censored observations in a 'censored string'. A censored string is defined by these as follows: Begin with the first observation and work forward in time. The first censored observation encountered begins the first censored string. Then a censored string begins with a censored observation and ends immediately after the next set of p censored observations. The use of censored strings in deriving the maximum likelihood estimator (MLE) will become clear in our description of the estimator in section 2.3.1

Zeger and Brookmeyer (1986) argued that the MLE can be 'computationally intensive' when high dimensional normal integrals must be evaluated and they gave an alternative PL approach. They showed that their PL estimator is consistent, however, Dagenais (1991) has challenged their proof and has given a counter-example †. They also illustrated the use of the MLE with a single simulated data set of length 50 and air pollution data subject to left censoring.

†Dagenais (1991) notes that the problem in Zeger and Brookmeyer's proof may have arisen because of the ambiguity in the notation used by the authors in their article where they used $E(Y|X)$ while, in fact, they meant $E(Y|X)$ evaluated at $X = Z$, where X, Y and Z are stochastic variables - without clarifying the implications for the proof, of the difference between the usual meaning of $E(Y|X)$ and that of $E(Y|X)_{X=Z}$.

The simulated sample was generated from a Gaussian AR(1) process with the first-lag autocorrelation set equal to 0.5 and the white noise variance set equal to 1. They concluded that the MLE performs better than the estimator based on ignoring the autocorrelation or one that would be obtained by using the censoring points as though they were the actual values taken by the underlying time series rv's and fitting an AR model to estimate the parameters. The air pollution data are time series data on the chemical composition of atmospheric deposition as measured at Lawrence Livermore, California, site (see Toonkel, 1981). The main objective was to study geographical differences and time trends in precipitation chemistry and concentration of pollutants in deposition. The results from the analysis using the ML method led to the conclusion that there is very little evidence of a trend at the site. The PL gave similar results. Dagenais (1989) has also suggested an alternative PL estimator and has compared it with the PL estimators of Zeger and Brookmeyer (1986) and Robinson (1982) in small samples by means of Monte Carlo experiments. He concludes that, although there is no clear-cut comparison between his PL estimator and that of Zeger and Brookmeyer (1986), the two estimators perform well and markedly better than the PL estimator of Robinson (1982) obtained by simply ignoring the autocorrelation.

A major contribution of the current thesis is to provide estimators of the autoregression parameter in censored AR(1) models which are easy to compute, perform well numerically in small and large samples and have desirable theoretical properties such as consistency and asymptotic normality. Thus the remainder of the current chapter mainly focuses on regression with censored autocorrelated data. However, we give a brief description of the Buckley-James estimator for linear regression in section 2.2. The reason for this is that the approach utilised in the distribution-free estimator proposed in the next chapter is to find parameter estimates that maximize a pseudolikelihood based on the Kaplan-Meier estimator of the error distribution. This is equivalent to replacing the censored observations by their corresponding Kaplan-Meier conditional means in the usual least-squares estimator which would be obtained if the data had not been censored - an idea utilised in the Buckley-James estimator. Thus, the description of the Buckley-James method will shed some light into the derivation of our distribution-free estimator. In section 2.3 we review existing estimators in regression with censored autocorrelated data. This is accomplished by first giving a formulation of the MLE for an AR model with normal errors, derived for example, in Zeger and Brookmeyer (1986). Then we present descriptions of the PL estimators of Robinson (1982), Dagenais (1986) and Zeger and Brookmeyer (1986). For convenience in subsequent discussions, all these estimators, including the MLE, are described in the context of this thesis, i.e., with the regression parameter being assumed to be zero, the time series to follow a zero-mean AR(1) model and the data are subject to random right censorship. The Buckley-James estimator is also described in the context of the random censorship model.

2.2 The Buckley-James estimator

Let $Y_1, ..., Y_n$ follow the model (2.1.1) and suppose we have observed $(Z_1, \delta_1), ...,$ (Z_n, δ_n). Suppose we have a single explanatory variable and consider the model

$$Y_i = \alpha + \beta x_i + \varepsilon_i, \ \ i = 1, ..., n, \tag{2.1}$$

where the independent partial residuals $r_i = Y_i - \beta x_i$ have the distribution function $F_\beta(\cdot)$ and survival function $S_\beta(\cdot) = 1 - F_\beta(\cdot)$. Note that $F_\beta(\cdot)$ has mean α and variance σ^2. If we let

$$\begin{aligned} \psi_i(t;\beta) &= E_\beta(Y_i|Y_i > t, x_i) \\ &= \beta x_i + \frac{\int_{(t-\beta x_i)}^{\infty} s\, dF_\beta(s)}{1 - F_\beta(t - \beta x_i)}, \end{aligned} \tag{2.2}$$

then the method of Buckley and James (1979) is motivated by the expectation identity

$$E_\beta[\delta_i Y_i + (1-\delta_i)\psi_i(T_i;\beta)|x_i] = E_\beta(Y_i|x_i) = \alpha + \beta x_i, \tag{2.3}$$

and replaces the censored observations in the usual least-squares normal equations by their estimated conditional expectations in the following manner. Let

$$e_i(b) = Z_i - bx_i, \quad i = 1, ..., n, \tag{2.4}$$

and let

$$\tilde{F}_b(s) = 1 - \prod_{i: e_{(i)}(b) \leq s} \left(\frac{n-i}{n-i+1}\right)^{\delta_{(i)}} \tag{2.5}$$

denote the Kaplan-Meier product limit estimator calculated from $e_i(b)$. In this formula $e_{(i)}(b)$ is the ith ordered observed residual and $\delta_{(i)}$ its associated indicator. As noted by Buckley and James (1979), $\tilde{F}_b(s)$ is discrete, with jumps only at the values of uncensored residuals. If the largest residual, $e_{(n)}(b)$, is censored, $\tilde{F}_b(s)$ does not approach 1 as $s \to \infty$ and the estimates of the integrals in (2.2.3) based on $\tilde{F}_b(s)$ are infinite. To overcome this problem, the convention adopted here is always to redefine the largest residual as uncensored (Efron, 1967; Meier, 1975; Miller 1976, 1981).

Let

$$\begin{aligned} \hat{\psi}(t; b) &= \hat{E}_b(Y_i | Y_i > t) \\ &= bx_i + \frac{\int_{(t-bx_i)}^{\infty} s \, d\tilde{F}_b(s)}{1 - \tilde{F}_b(t - bx_i)}, \quad \text{if} \quad t - bx_i < e_{(n)}(b), \\ &= t, \quad \text{if} \quad t - bx_i \geq e_{(n)}(b), \end{aligned} \tag{2.6}$$

and define for each $i = 1, ..., n$,

$$\begin{aligned} \hat{Y}_i(b) &= Y_i, \quad \text{if} \quad \delta_i = 1, \\ &= \hat{\psi}(T_i; b), \quad \text{if} \quad \delta_i = 0. \end{aligned} \tag{2.7}$$

Thus $\hat{Y}_i(b)$ is the observed response Y_i if uncensored, or an estimate of it, based on the $e_i(b)$, if censored. One then attempts to find estimator $\tilde{\beta}$ such that

$$\tilde{\beta} = \frac{\sum_{i=1}^{n} \hat{Y}_i(\tilde{\beta})(x_i - \overline{x})}{\sum_{i=1}^{n}(x_i - \overline{x})^2}. \tag{2.8}$$

If we denote the right hand side of the above equation by $\gamma_n(\tilde{\beta})$, then we try to solve $\gamma_n(\tilde{\beta}) = \tilde{\beta}$. A natural way of solving this is to use an iterative scheme which starts with an initial estimate of $\tilde{\beta}$ and successively updates it by $\gamma_n(\tilde{\beta})$. However, as mentioned in section 2.1 above, in common with the estimation function of Miller (1976), $\gamma_n(b)$ is discontinuous and piecewise linear in b. Therefore, an exact solution need not exist and if it exists, it need not be unique. If no solution exists, the iterations can settle down to oscillating between two values. According to Buckley and James, the two values are closer to each other for their estimator than for the Miller estimator. They suggest taking the average of the two values and using it as an estimate of β .

2.3 Estimators in regression with censored autocorrelated data

2.3.1 The maximum likelihood estimator

In the context of regression with censored autocorrelated data, the MLE described in Zeger and Brookmeyer (1986) is for the following model. Let Y_1, ..., Y_n be rv's satisfying

$$Y_i = x_i'\beta + u_i, \;\; i = 1, ..., n, \tag{2.1}$$

where x_i is the $m \times 1$ vector of known covariates and β is an $m \times 1$ vector of unknown regression coefficients. The errors, u_i, are assumed to arise from a stationary AR(p) process satisfying

$$u_i = \theta_1 u_{i-1} + ... + \theta_p u_{i-p} + \varepsilon_i, \tag{2.2}$$

where, ε_i are i.i.d. normal random variables with mean zero and variance σ^2. One observes not $\{Y_i\}$ but

$$Z_i = \min(Y_i, T_i) \;\text{ and }\; \delta_i = I(Y_i \leq T_i), \tag{2.3}$$

where, given the x_i's, $T_1, ..., T_n$ are i.i.d. rv's with distribution $G(.; x_i)$ independent of $\varepsilon_1, ..., \varepsilon_n$. Henceforth, we shall refer to $\{Y_i\}$ and $\{Z_i\}$ as the underlying and observed time series, respectively.

An obvious extension in assuming the above model rather than the linear regression model with i.i.d. errors given by equation (2.1.1) in section 2.1 is that here the Z_i's form a dependent sequence. It is the autoregression component in (2.3.2) that causes the dependency. In the absence of this component, the model here reduces to the one in (2.1.1) which has been studied extensively as seen section 2.1. Therefore, it is sufficient to study only the autoregression component of the extended model. Further, any results obtained for the AR(1) model with p set equal to 1 in (2.3.2) can be easily extended to the AR(p) model, $p > 1$. In turn, these results can be easily extended to the general linear regression model with autocorrelated errors given by (2.3.1). Thus, our description of the MLE and indeed the rest of the estimators is for the pure AR(1) model with β set equal to zero in (2.3.1) and (2.3.2) replaced by

$$u_i = \theta u_{i-1} + \varepsilon_i. \tag{2.4}$$

For the new estimators described in Chapter 3, however, the ε_i's are assumed to be i.i.d. F, not necessarily normal. Before we describe the MLE, we first look at the effect of censoring on the standard Markov property, utilised in the construction of the likelihood for uncensored data. Then we state and prove a Markov result proposed by Zeger and Brookmeyer (1986) which allows the likelihood for censored data to be constructed.

The Markov property enables us to express the likelihood function for uncensored data from a stationary AR(1) process as a contribution of each observation given the preceding value. With censored data, part of the problem is that the preceding observation may be censored and hence conditioning on it is not equivalent to conditioning on the entire past. Thus, the standard Markov property does not apply for censored data. Zeger and Brookmeyer (1986) noted this and proposed a Markov result suitably modified for the censored data problem. Although these authors did not give a proof for this result, they noted that it follows from first principles. In the sequel, the Markov result is stated, proved and used in the development of the likelihood for randomly censored data from a stationary Gaussian AR(1) process.

Lemma 2.3.1 *Let Z_i be a possibly censored observation from a stationary AR(1) process with $f_{Z_i|Z_{i-1},Z_{i-2},\ldots}(z|z_{i-1},z_{i-2},\ldots)$ being the conditional density function given the past. Then,*

$$f_{Z_i|Z_{i-1},Z_{i-2},\ldots}(z|z_{i-1},z_{i-2},\ldots)$$
$$= f_{Z_i|\omega_i,Z_{i-1},\ldots,Y_{i-k-1}}(z|k,z_{i-1},\ldots,z_{i-k-1}), \quad \textit{if } k \geq 0, \tag{2.5}$$

where ω_i is the number of consecutive censored observations preceding Z_i.

Proof

$$\begin{aligned}
&P(Z_i \leq z|Z_{i-1} = z_{i-1}, Z_{i-2} = z_{i-2}, \ldots, Z_1 = z_1, \ldots) \\
&= \sum_{k=0}^{\infty} P(Z_i \leq z|Z_{i-1} = z_{i-1}, \ldots, Z_1 = z_1, \ldots, \omega_i = k)P(\omega_i = k).
\end{aligned}$$

Now,

$$\begin{aligned}
&P(Z_i \leq z|Z_{i-1} = z_{i-1}, \ldots, Z_1 = z_1, \omega_i = 0) \\
&= P(Z_i \leq z|Z_{i-0-1} = z_{i-0-1}, Z_{i-0-2} = z_{i-0-2}, \ldots, Z_1 = z_1, \ldots, \omega_i = 0) \\
&= P(Z_i \leq z|Y_{i-1} = z_{i-1}, \ldots, Z_1 = z_1, \ldots, \omega_i = 0).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&P(Z_i \leq z|Z_{i-1} = z_{i-1}, \ldots, Z_1 = z_1, \omega_i = 1) \\
&= P(Z_i \leq z|Z_{i-1} = z_{i-1}, Z_{i-2} = z_{i-2}, \ldots, Z_1 = z_1, \ldots, \omega_i = 1) \\
&= P(Z_i \leq z|Z_{i-1} = z_{i-1}, Y_{i-2} = z_{i-2}, \ldots, Z_1 = z_1, \ldots, \omega_i = 1).
\end{aligned}$$

In general,

$$\begin{aligned}
&P(Z_i \leq z|Z_{i-1} = z_{i-1}, \ldots, Z_1 = z_1, \ldots, \omega_i = k) \\
&= P(Z_i \leq z|Z_{i-1} = z_{i-1}, \ldots, Z_{i-k} = z_{i-k}, Y_{i-k-1} = z_{i-k-1}, \\
&\quad \ldots, Z_1 = z_1, \ldots, \omega_i = k),
\end{aligned}$$

since $\omega_i = k$ implies Z_{i-k-1} is uncensored. Therefore,

$$
\begin{aligned}
&P(Z_i \le z | Z_{i-1} = z_{i-1}, ..., Z_1 = z_1, ...) \\
&= \sum_{k=0}^{\infty} P(Z_i \le z | Z_{i-1} = z_{i-1}, ..., Z_1 = z_1, ..., \omega_i = k) P(\omega_i = k) \\
&= \sum_{k=0}^{\infty} P(Z_i \le z | Z_{i-1} = z_{i-1}, ..., Z_{i-k} = z_{i-k}, Y_{i-k-1} = z_{i-k-1}, \\
&\quad ..., Z_1 = z_1, ..., \omega_i = k) P(\omega_i = k) \\
&= P(Z_i \le z | Z_{i-1} = z_{i-1}, ..., Z_{i-k} = z_{i-k}, Y_{i-k-1} = z_{i-k-1}, ..., Z_1 = z_1, ...) \\
&= P(Z_i \le z | Z_{i-1} = z_{i-1}, ..., Z_{i-k} = z_{i-k}, Y_{i-k-1} = z_{i-k-1}),
\end{aligned}
$$

by the Markov property. □

The lemma states that to condition on the entire past, it suffices to condition back to the last uncensored observation. Note that, unlike in the uncensored case, the conditional expectation of an observation given its past is not linear in the preceding value. For this reason, Zeger and Brookmeyer (1986) factorise the likelihood into two components. One component contains contributions of uncensored observations that are immediately preceded by an uncensored observation. The other component contains contributions of *censored strings* - that is, contributions of all censored observations and of uncensored observations for which the preceding value is censored. In order to write an expression for the likelihood as a product of these components, we slightly modify the notation of Zeger and Brookmeyer (1986) in the following manner.

Let U be the index set of times i for which both Z_i and Z_{i-1} are uncensored. Apply the definition of a censored string for an AR(p) process given earlier in section 2.1 to define a censored string for an AR(1) process. Here, the definition translates as follows. Start with the first observation and work forward in time. The first censored observation encountered begins the first censored string, and the string ends immediately after the first uncensored observation. In general, a censored string starts with a censored observation and ends immediately after the next uncensored observation. Let $\underline{Z}_j$ denote the jth censored string. Let $\underline{Z}_j^c$ be the ν_j^c consecutive censored values in $\underline{Z}_j$ and let $\underline{Y}_j^c$ be the corresponding rv's of the underlying time series. Denote by Y_j^u the only uncensored value in $\underline{Z}_j$. Let X_j be the uncensored observation that appears immediately before the first (censored) value in $\underline{Z}_j$. Thus, X_j is either the last observation in string $j-1$ or it belongs to the set of observations with indices in the index set U.

Now, assuming that the realization $(Z_1, ..., Z_n)$ is such that Z_1 is uncensored, application of proposition 2.2 of Zeger and Brookmeyer (1986), which is based on the Markov result for censored data in lemma 2.3.1, leads to the likelihood,

$$l(\theta, \sigma^2) = \prod_{i \in U} f_{Y_i|Y_{i-1}}(Z_i|Z_{i-1}) \prod_{j=1}^{K} f_{Y_j^u|X_j}(Y_j^u|X_j) \tilde{F}_{j,\nu_j^c}(\underline{Z}_{j,1}^c, ..., \underline{Z}_{j,\nu_j^c}^c, X_j, Y_j^u), \tag{2.6}$$

where K is the number of censored strings and

$$\tilde{F}_{j,m}(t_1, ..., t_m, X_j, Y_j^u) = \int_{t_m}^{\infty} \cdots \int_{t_1}^{\infty} f_{\underline{Y}_{j,1},...,\underline{Y}_{j,m}|\nu_j^c, X_j, Y_j^u}(s_1, ..., s_m|m, X_j, Y_j^u) \Pi_{k=1}^m ds_k, \tag{2.7}$$

with $f_{\underline{Y}_{j,1},...,\underline{Y}_{j,m}|\nu_j^c, X_j, Y_j^u}(s_1, ..., s_m|m, X_j, Y_j^u)$ being the conditional density of the underlying time series rv's, $\underline{Y}_j^c$, corresponding to $\underline{Z}_j^c$, given that $\nu_j^c = m > 0$ and given the uncensored value, Y_j^u at the end of the censored string and the uncensored value, X_j preceding the string.

Inspection of (2.3.6) reveals that each observation in the index set makes a contribution to the likelihood in a similar way as in the uncensored case - that is, conditional on the preceding observation. The uncensored observations in a censored string, however, contribute conditionally on the most recent uncensored observation, whereas the contributions of censored observations is conditional on the surrounding uncensored values - the most recent uncensored value and the last (uncensored) observation in the string.

Zeger and Brookmeyer (1986) simplify the likelihood by making use of the following notation for conditional means and covariances. For $i \in U$, let

$$\mu_i = E(Y_i | Y_{i-1}), \tag{2.8}$$

and note that $var(Y_i|Y_{i-1}) = \sigma^2$. For each of the K censored strings, let

$$\begin{aligned} \eta_j^u &= E(Y_j^u | X_j, \nu_j^c), \qquad \sigma_j^u = var(Y_j^u | X_j, \nu_j^c), \\ \eta_j^c &= E(\underline{Y}_j^c | Y_j^u, X_j, \nu_j^c), \qquad \Sigma_j^c = cov(\underline{Y}_j^c | Y_j^u, X_j, \nu_j^c). \end{aligned} \tag{2.9}$$

Note that the conditional expectations, μ_i, η_j^c and η_j^c are all linear functions of the conditioning rv's. For example, $\mu_i = \theta Y_{i-1}$, and $\eta_j^u = \theta^{\nu_j^c+1} X_j$.

We note that there is an error (most likely typographical) in the expression of Zeger and Brookmeyer (1986) for $\eta_{j,k}^c$, the conditional expectation of the underlying time series random variable corresponding to the kth censored value in string j. This error is corrected by simply replacing ν_j^c by $\nu_j^c + 1$ in Zeger and Brookmeyer's expression and this yields,

$$\eta_{j,k}^c = [1/(1-\theta^{2(\nu_j^c+1)})][\theta^k(1-\theta^{2(\nu_j^c-k+1)})X_j + \theta^{(\nu_j^c-k+1)}(1-\theta^{2k})Y_j^u]. \tag{2.10}$$

As an example, consider the case when $\nu_j^c = 1$. Then the expression in (2.3.10) gives

$$\eta_{j,1}^c = [\theta/(1+\theta^2)][X_j + Y_j^u]. \tag{2.11}$$

We confirmed this expression using the software package, *mathematica, 2.2.* Expressions corresponding to varying values of ν_j^c and k were also confirmed similarly. Inspection of (2.3.10) reveals that larger weight is placed on the uncensored value X_j in calculating $\eta_{j,k}^c$ for observations near the beginning of the string and Y_j^u receives more weight in calculating the conditional expectation for observations near the end of the string.

For higher-order AR models, Zeger and Brookmeyer suggested evaluating σ_j^u and Σ_j^c numerically using expressions for conditional means and variances given, for example, in Dempster (1969). However, we have obtained direct expressions for the AR(1) model. The expression for σ_j^u is given by

$$\sigma_j^u = \sigma^2 \sum_{k=0}^{\nu_j^c} \theta^{2k}, \tag{2.12}$$

and the entries of Σ_j^c are given by

$$\begin{aligned} \Sigma_{j,(k,l)}^c &= cov(\underline{Y}_{j,k}^c, \underline{Y}_{j,l}^c | X_j, Y_j^u) \\ &= \frac{\sigma^2 \theta^{(k-l)} (\sum_{r=0}^{\nu_j^c - k} \theta^{2r})(\sum_{s=0}^{l-1} \theta^{2s})}{\sum_{r=0}^{\nu_j^c} \theta^{2r}}, \quad k = 1, ..., \nu_j^c, \ l \le k, \end{aligned} \tag{2.13}$$

where, $\underline{Y}_{j,k}^c$ is the underlying random variable corresponding to kth censored value in the jth string, $\underline{Z}_j$. Once again, we confirmed the expression in (2.3.13) using *Mathematica, 2.2.*

Using the preceding definitions, the likelihood given in (2.3.6) for the censored Gaussian AR(1) model can be written as

$$l(\theta, \sigma^2) = \prod_{i \in U} \phi(\frac{Z_i - \theta Z_{i-1}}{\sigma}) \prod_{j=1}^{K} \phi(\frac{Y_j^u - \eta_j^u}{\sqrt{\sigma_j^u}}) \tilde{\Phi}_{j,\nu_j^c}(\underline{Z}_{j,1}^c, ..., \underline{Z}_{j,\nu_j^c}^c, X_j, Y_j^u), \tag{2.14}$$

where ϕ is the univariate standard normal density and

$$\begin{aligned} &\tilde{\Phi}_{j,\nu_j^c}(t_1, ..., t_m, X_j, Y_j^u) \\ &= \int_{t_m}^{\infty} ... \int_{t_1}^{\infty} \phi_{j,\nu_j^c}[(\Sigma_j^c)^{-1}(\underline{s} - \eta_j^c)] \Pi_{k=1}^m ds_k, \end{aligned} \tag{2.15}$$

with $\nu_j^c = m$, $\underline{s}$ being the $m \times 1$ vector of integration variables, $(s_1, ..., s_m)$, and ϕ_{j,ν_j^c}, the ν_j^c-dimensional standard normal density.

A further simplification is accomplished by defining, for each string, conditional means of the underlying rv's $\underline{Y}_j^c$, given the censoring at $\underline{Z}_j^c$ and the uncensored values X_j and Y_j^u. That is, for the kth censored observation in string j, define

$$\hat{\underline{Y}}_{j,k}^c = \psi_{j,k,\nu_j^c}(\underline{Z}_{j,1}^c, ..., \underline{Z}_{j,\nu_j^c}^c, X_j, Y_j^u), \tag{2.16}$$

where,

$$\begin{aligned} &\psi_{j,k,m}(t_1, ..., t_m, X_j, Y_j^u) \\ &= \frac{\int_{t_m}^{\infty} ... \int_{t_1}^{\infty} s_k \phi_{j,\nu_j^c}[(\Sigma_j^c)^{-1}(\underline{s} - \eta_j^c)] \Pi_{k=1}^m ds_k}{\int_{t_m}^{\infty} ... \int_{t_1}^{\infty} \phi_{j,\nu_j^c}[(\Sigma_j^c)^{-1}(\underline{s} - \eta_j^c)] \Pi_{k=1}^m ds_k}. \end{aligned} \tag{2.17}$$

Then the likelihood score function for σ^2 takes the form

$$\begin{aligned}\partial ln\ l/\partial\sigma^2 &= \sum_{i\in U}\{[(-1)/2\sigma^2] + (1/2\sigma^4)(Z_i - \theta Z_{i-1})^2\} \\ &+\sum_{j=1}^{K}\{[(-1)/2\sigma^2] + (1/2\sigma^2)[\frac{(Y_j^u - \eta_j^u)^2}{\sigma_j^u}] + [(-\nu_j^c)/2\sigma^2] \\ &+(1/2\sigma^2)[(\underline{\hat{Y}}_j^c - \eta_j^c)'(\Sigma_j^c)^{-1}(\underline{\hat{Y}}_j^c - \eta_j^c)] \\ &+(1/2\sigma^2)tr((\Sigma_j^c)^{-1}V_j^c)\}. \end{aligned} \tag{2.18}$$

This can also be written as

$$\begin{aligned}\partial ln\ l/\partial\sigma^2 &= [-(n-1)/2\sigma^2] + \sum_{i\in U}(1/2\sigma^4)(Z_i - \theta Z_{i-1})^2 \\ &+\sum_{j=1}^{K}(1/2\sigma^2)\{[\frac{(Y_j^u - \eta_j^u)^2}{\sigma_j^u}] + [(\underline{\hat{Y}}_j^c - \eta_j^c)'(\Sigma_j^c)^{-1}(\underline{\hat{Y}}_j^c - \eta_j^c)] \\ &+tr((\Sigma_j^c)^{-1}V_j^c)\}. \end{aligned} \tag{2.19}$$

Here, V_j^c is the $\nu_j^c \times \nu_j^c$ conditional covariance matrix of the underlying time series rv's, $\underline{Y}_j^c$, given the censoring at $\underline{Z}_j^c$ and the uncensored values X_j and Y_j^u. Let

$$\begin{aligned}&\psi_{j,k,l,m}(t_1,...,t_m,X_j,Y_j^u) \\ &= \frac{\int_{t_m}^{\infty}\cdots\int_{t_1}^{\infty} s_k s_l \phi_{j,\nu_j^c}[(\Sigma_j^c)^{-1}(\underline{s} - \eta_j^c)]\Pi_{k=1}^{m} ds_k}{\int_{t_m}^{\infty}\cdots\int_{t_1}^{\infty} \phi_{j,\nu_j^c}[(\Sigma_j^c)^{-1}(\underline{s} - \eta_j^c)]\Pi_{k=1}^{m} ds_k}.\end{aligned} \tag{2.20}$$

Then, the conditional covariance between $\underline{Y}_{j,k}^c$ and $\underline{Y}_{j,l}^c$ is given by

$$V_{j,(k,l)}^c = \psi_{j,k,l,\nu_j^c}(\underline{Z}_{j,1}^c,...,\underline{Z}_{j,\nu_j^c}^c,X_j,Y_j^u) - (\underline{\hat{Y}}_{j,k}^c)(\underline{\hat{Y}}_{j,l}^c). \tag{2.21}$$

Here $\underline{Y}_{j,k}^c$ is the underlying time series random variable corresponding to the kth censored value in the string, $k = 1,...,\nu_j^c$, $l \leq k$.

The presence of information about θ in the conditional means, η_j^u and η_j^c and the conditional variances, σ_j^u and Σ_j^c makes the score function for θ, $\partial ln\ l/\partial\theta$, more complicated for censored data than it is in the uncensored case. Denote this function by $S_n(\theta)$. Then $S_n(\theta)$ can be written as a sum of three terms. The first term,

$$\sum_{i\in U}(1/\sigma^2)Z_{i-1}(Z_i - \theta Z_{i-1}), \tag{2.3.22-a}$$

is the contribution of uncensored observations preceded by an uncensored value. The second term,

$$\sum_{j=1}^{K}\left\{\frac{-1}{2\sigma_j^u}\frac{\partial\sigma_j^u}{\partial\theta}+\frac{1}{2(\sigma_j^u)^2}\frac{\partial\sigma_j^u}{\partial\theta}(Y_j^u-\eta_j^u)^2+\frac{\partial\eta_j^u}{\partial\theta}\frac{(Y_j^u-\eta_j^u)}{\sigma_j^u}\right\}, \qquad (2.3.22\text{-b})$$

is the contribution of Y_j^u and the third term,

$$\sum_{j=1}^{K}\{\frac{-1}{2|\Sigma_j^c|}\frac{\partial|\Sigma_j^c|}{\partial\theta}-\frac{1}{2}(\underline{\hat{Y}}_j^c-\eta_j^c)'\frac{\partial}{\partial\theta}[(\Sigma_j^c)^{-1}](\underline{\hat{Y}}_j^c-\eta_j^c)$$
$$+(\frac{\partial\eta_j^c}{\partial\theta})'(\Sigma_j^c)^{-1}(\underline{\hat{Y}}_j^c-\eta_j^c)-\frac{1}{2}tr\left(\frac{\partial}{\partial\theta}[(\Sigma_j^c)^{-1}]V_j^c\right)\}, \qquad (2.3.22\text{-c})$$

is the contribution of each $\underline{Y}_j^c$. Zeger and Brookmeyer (1986) noted that the trace term in this expression as well as the one in the score function for σ^2 can be thought of as a correction for the estimated $\underline{\hat{Y}}_j^c$ being closer to η_j^c than one would expect for the underlying time series.

We note that there is an error (possibly typographical) in Zeger and Brookmeyer's expression for the third term of (2.3.22), the contribution of each $\underline{Y}_j^c$. Their expression differs from (2.3.22-c) only in the trace term. Unlike (2.3.22-c), the trace in their expression is multiplied by +1 rather than −0.5. We have shown our derivation of (2.3.22-c) in the proof of lemma A.1.1 in the appendix.

Zeger and Brookmeyer (1986) suggested two numerical procedures to solve the corresponding score equations: a modified Newton-Raphson procedure and an EM algorithm. At each iteration the modified Newton-Raphson procedure which is available with the 'S' statistical software, chooses between a Newton-Raphson step and a steepest-descent step depending on current estimates of the gradient and Hessian matrix. Zeger and Brookmeyer (1986) also note that for the data sets they have analysed, this approach has avoided problems of non-convergence sometimes experienced with strict Newton-Raphson procedures in regression with i.i.d. censored data. The EM algorithm (Dempster, Laird, and Rubin 1977 and McLaughlan and Krishnan, 1997), which has also been used for i.i.d. censored data, is outlined by Zeger and Brookmeyer (1986) as follows.

Let $\hat{\theta}^{(0)}$ and $\hat{\sigma}^{2(0)}$ be initial estimates of θ and σ^2, respectively. Then the EM algorithm consists of the following steps:

1. *E step*: Estimate $\hat{\underline{Y}}_j^c$ and V_j^c for each string of censored values, using (2.3.16) and (2.3.21), respectively, for $j = 1, ..., K$,

2. *M step*: Calculate updated estimates, $\hat{\theta}$ and $\hat{\sigma}^2$, by solving the score equations corresponding to (2.3.22) and (2.3.19), respectively.

3. *Iteration*: Iterate steps 1 and 2 until succesive parameter estimates do not change within the required error bound.

The Newton-Raphson procedure requires the computation of the Hessian of the likelihood function. An advantage of this numerical procedure over the EM algorithm is that the Hessian is available upon convergence for use in calculating the variances of the parameter estimates, whereas, with the EM algorithm an extra step is required to calculate the Hessian after convergence has been reached (see, e.g., Meng and Rubin, 1991). The Hessian for σ^2 is given by

$$\begin{aligned}\partial^2 ln\ l/\partial(\sigma^2)^2 &= [(n-1)/2\sigma^4] - \sum_{i\in U}(1/2\sigma^6)(Z_i - \theta Z_{i-1})^2 \\ &- \sum_{j=1}^{K}(1/\sigma^4)\{[\frac{(Y_j^u - \eta_j^u)^2}{\sigma_j^u}] + [(\hat{\underline{Y}}_j^c - \eta_j^c)'(\Sigma_j^c)^{-1}(\hat{\underline{Y}}_j^c - \eta_j^c)] \\ &+ tr((\Sigma_j^c)^{-1}V_j^c)\}. \end{aligned} \qquad (2.3.23)$$

Let $J_n(\theta)$ denote the Hessian for θ, $\partial^2 ln\ l/\partial\theta^2$. Then $J_n(\theta)$ is the sum of three components. The first component,

$$\sum_{i\in U}(-1/\sigma^2)Z_{i-1}^2, \qquad (2.3.24\text{-a})$$

is the contribution of uncensored observations preceded by uncensored observations. The second component,

$$\sum_{j=1}^{K}\{\frac{-1}{2\sigma_j^u}\frac{\partial^2\sigma_j^u}{\partial\theta^2} + \frac{1}{2(\sigma_j^u)^2}(\frac{\partial\sigma_j^u}{\partial\theta})^2 - \frac{2}{(\sigma_j^u)^2}\frac{\partial\sigma_j^u}{\partial\theta}\frac{\partial\eta_j^u}{\partial\theta}(Y_j^u - \eta_j^u)$$

$$+\frac{1}{2(\sigma_j^u)^2}\frac{\partial^2\sigma_j^u}{\partial\theta^2}(Y_j^u-\eta_j^u)^2-\frac{1}{(\sigma_j^u)^3}(\frac{\partial\sigma_j^u}{\partial\theta})^2(Y_j^u-\eta_j^u)^2$$
$$+\frac{\partial^2\eta_j^u}{\partial\theta^2}\frac{(Y_j^u-\eta_j^u)}{\sigma_j^u}-\frac{1}{\sigma_j^u}(\frac{\partial\eta_j^u}{\partial\theta})^2\}, \qquad (2.3.24\text{-b})$$

is the contribution of each Y_j^u and the third component is the contribution of each $\underline{Y}_j^c$. Since the $\hat{\underline{Y}}_j^c$'s depend on θ, an exact expression for this component is hard to obtain. An estimate

$$\sum_{j=1}^{K}\{\frac{-1}{2|\Sigma_j^c|}\frac{\partial^2|\Sigma_j^c|}{\partial\theta^2}+\frac{1}{2|\Sigma_j^c|^2}(\frac{\partial|\Sigma_j^c|}{\partial\theta})^2$$
$$-\frac{1}{2}(\hat{\underline{Y}}_j^c-\eta_j^c)'\frac{\partial^2}{\partial\theta^2}[(\Sigma_j^c)^{-1}](\hat{\underline{Y}}_j^c-\eta_j^c)$$
$$+2(\frac{\partial\eta_j^c}{\partial\theta})'\frac{\partial}{\partial\theta}[(\Sigma_j^c)^{-1}](\hat{\underline{Y}}_j^c-\eta_j^c)-(\frac{\partial\eta_j^c}{\partial\theta})'(\Sigma_j^c)^{-1}\frac{\partial\eta_j^c}{\partial\theta}$$
$$+(\frac{\partial^2\eta_j^c}{\partial\theta^2})'(\Sigma_j^c)^{-1}(\hat{\underline{Y}}_j^c-\eta_j^c)-\frac{1}{2}tr[\frac{\partial^2}{\partial\theta^2}[(\Sigma_j^c)^{-1}]V_j^c]\}, \qquad (2.3.24\text{-c})$$

is obtained by treating the $\hat{\underline{Y}}_j^c$'s as if they do not depend on θ.

In Chapter 5, we use simulations to evaluate the performance of the MLE for θ currently under discussion and compare it with the performances of the pseudolikelihood estimators described in section 2.3.2 and the new estimators described in Chapter 3. One of the main criteria used in this comparative study is the estimated asymptotic variances of the estimators. Following is the description of the asymptotic variance estimator of the MLE. Let $\hat{\theta}_n^{mle}$ be the estimator and denote the true value of θ by θ_o. Then under suitable regularity conditions,

$$n^{\frac{1}{2}}(\hat{\theta}_n^{mle}-\theta_o)\xrightarrow{\mathcal{D}}N(0,V_{\theta_o}), \qquad (2.3.25)$$

where,

$$V_\theta=\lim_{n\to\infty}\hat{V}_\theta=\lim_{n\to\infty}\frac{n\,var_\theta\{S_n(\theta)\}}{E_\theta^2\{S_n^{(1)}(\theta)\}} \qquad (2.3.26)$$

(see, e.g., Hall and Heyde, 1980 or Godambe, 1985).

Theorem 4.1 of Robinson (1980) states that if the autoregressive process $\{Y_i, i\in\mathcal{Z}\}$ is stationary, then conditional on the uncensored observations, X_j

and Y_j^u, the $\underline{Y}_j^c$'s are mutually independent and hence uncorrelated. Therefore, we propose the asymptotic variance estimator,

$$a\widehat{svar}(\hat{\theta}_n^{mle}) = \frac{I_n(\hat{\theta}_n^{mle})}{J_n^2(\hat{\theta}_n^{mle})} \tag{2.3.27}$$

Here, $I_n(\theta)$ is the sum in the components,

$$\sum_{i \in U}(1/\sigma^4) Z_{i-1}^2 (Z_i - \theta Z_{i-1})^2, \tag{2.3.28-a}$$

$$\sum_{j=1}^{K}\{\frac{-1}{2\sigma_j^u}\frac{\partial \sigma_j^u}{\partial \theta} + \frac{1}{2(\sigma_j^u)^2}\frac{\partial \sigma_j^u}{\partial \theta}(Y_j^u - \eta_j^u)^2 + \frac{\partial \eta_j^u}{\partial \theta}\frac{(Y_j^u - \eta_j^u)}{\sigma_j^u}\}^2, \tag{2.3.28-b}$$

and

$$\sum_{j=1}^{K}\{\frac{-1}{2|\Sigma_j^c|}\frac{\partial |\Sigma_j^c|}{\partial \theta} - \frac{1}{2}(\hat{\underline{Y}}_j^c - \eta_j^c)'\frac{\partial}{\partial \theta}[(\Sigma_j^c)^{-1}](\hat{\underline{Y}}_j^c - \eta_j^c)$$
$$+(\frac{\partial \eta_j^c}{\partial \theta})'(\Sigma_j^c)^{-1}(\hat{\underline{Y}}_j^c - \eta_j^c) - \frac{1}{2}tr[\frac{\partial}{\partial \theta}[(\Sigma_j^c)^{-1}]V_j^c]\}^2, \tag{2.3.28-c}$$

derived from the score function in (2.3.22-a), (2.3.22-b) and (2.3.22-c). To overcome the problem encountered as a result of the dependence on θ of the $\hat{\underline{Y}}_j^c$'s, we use the EM aided differentiation technique described of Meilijson (1989) to numerically determine the third component of $J_n(\hat{\theta}_n^{mle})$, i.e., the contribution for each $\underline{Y}_j^c$. For a general θ, denote this component by $\tilde{J}(\theta)$. Let $\tilde{S}(\theta)$ be the corresponding component of the score function $S_n(\theta)$ given by (2.3.22-c). Then the EM aided differentiation can be outlined as follows: Choose ε sufficiently small. Expand $\tilde{S}(\hat{\theta}_n^{mle})$ about $\hat{\theta}_n^{mle} + \varepsilon$. Then

$$\tilde{J}(\hat{\theta}_n^{mle}) \approx \tilde{S}^{(1)}(\hat{\theta}_n^{mle} + \delta) = \frac{\tilde{S}(\hat{\theta}_n^{mle} + \varepsilon)}{\varepsilon}, \quad 0 < \delta < \varepsilon. \tag{2.3.29}$$

$\tilde{J}(\hat{\theta}_n^{mle})$ is then added to the sum of the first and second components of $J_n(\theta)$ given by (2.3.24-a) and (2.3.24-b), evaluated at $\theta = \hat{\theta}_n^{mle}$. This technique is applied only once on convergence of the EM algorithm described above. A similar EM algorithm has been proposed by Meng and Rubin (1991) for multiparameter problems where only a subset of the parameters are affected by missing information.

We note that the asymptotic variance estimator given by (2.3.27) is analogous to the one given for the PL estimator of Dagenais (1986) in the case of regression with autocorrelated censored data. In the next section, we give descriptions of this PL estimator and the one proposed by Zeger and Brookmeyer (1986).

2.3.2 Pseudolikelihood estimators

The PL estimators of Robinson (1982), Zeger and Brookmeyer (1986) and Dagenais (1986) are all approximations to the MLE. In the context of regression with censored autocorrelated data, they are for the model in section 2.3.1. However, in the descriptions of the estimators of Robinson (1982) and Dagenais (1986), (2.3.2) is replaced by (2.3.4), i.e. they are for the AR(1) model, although they can be extended to the AR(p) model. In this thesis, however, all the estimators will be discussed in the context of the stationary AR(1) model. Further, the PL estimator of Robinson (1982) does not apply in the context of this thesis. This is because, as mentioned earlier, this thesis is concerned with estimation for pure autoregressive processes, i.e., with $\beta = 0$ in (2.3.1). On the other hand, the Robinson estimator is for the parameters, β and σ^2, with the autoregression parameter, θ, being assumed to be zero. Hence, the description of this estimator is omitted and the PL estimator of Zeger and Brookmeyer (1986) hereafter denoted by $\hat{\theta}_n^{zb}$ is described next. Let

$$\begin{aligned} \tilde{\varphi}(t,u) &= E(Y_i|Y_i > t, Y_{i-1} = u) \\ &= \frac{\int_t^\infty s f_{Y_i|Y_{i-1}}(s|u)ds}{\int_t^\infty f_{Y_i|Y_{i-1}}(s|u)ds}, \end{aligned} \tag{2.3.30}$$

Assume the first observation, Z_1, of the realization $(Z_1, ..., Z_n)$ of the AR(1) process is uncensored and define sequentially,

$$\begin{aligned} \tilde{Y}_i &= Z_i, \text{ if } \delta_i = 1, \\ &= \tilde{\varphi}(T_i, \tilde{Y}_{i-1}; \theta), \text{ if } \delta_i = 0. \end{aligned} \tag{2.3.31}$$

Then the estimator of Zeger and Brookmeyer (1986) is obtained by maximizing the pseudolikelihood,

$$pl_{zb}(\theta,\sigma^2) = \prod_{i=2}^{n} [f_{Y_i|Y_{i-1}}(Z_i|\tilde{Y}_{i-1})]^{\delta_i}[\tilde{F}(T_i,\tilde{Y}_{i-1})]^{1-\delta_i}, \quad (2.3.32)$$

where,

$$\tilde{F}(t,u) = \int_t^{\infty} f_{Y_i|Y_{i-1}}(s|u)ds. \quad (2.3.33)$$

In the context of Zeger and Brookmeyer's original derivation of the estimator, i.e., for Gaussian AR processes, a simplification is achieved by letting

$$h(t) = \frac{\phi(t)}{1-\Phi(t)}, \quad (2.3.34)$$

the hazard function for a standard normal variate, where, ϕ and Φ are the standard univariate normal density and distribution functions, respectively. Then (2.3.30) becomes

$$\begin{aligned} \tilde{\varphi}(t,u) &= E(Y_i|Y_i > t, Y_{i-1} = u) \\ &= \theta u + \sigma h(\frac{t-\theta u}{\sigma}), \end{aligned} \quad (2.3.35)$$

and the pseudolikelihood, pl_1, takes the form,

$$pl_{zd} = \prod_{i=2}^{n} [\phi_1(\frac{Z_i - \theta\tilde{Y}_{i-1}}{\sigma})]^{\delta_i}[1-\Phi_1(\frac{T_i-\theta\tilde{Y}_{i-1}}{\sigma})]^{1-\delta_i}. \quad (2.3.36)$$

The corresponding score function for σ^2 is given by

$$\begin{aligned} \partial ln\, pl_1/\partial\sigma^2 = \sum_{i=2}^{n}\{\delta_i[(-1/2\sigma^2) + (1/2\sigma^4)(Z_i - \theta\tilde{Y}_{i-1})^2] \\ +(1-\delta_i)[(1/2\sigma^3)(T_i - \theta\tilde{Y}_{i-1})h(\frac{T_i-\theta\tilde{Y}_{i-1}}{\sigma})]\}, \end{aligned} \quad (2.3.37)$$

and the score function for θ takes the form,

$$\begin{aligned} \partial ln\, pl_1/\partial\theta = (1/\sigma^2)\sum_{i=2}^{n}\{\delta_i[\tilde{Y}_{i-1}(Z_i - \theta\tilde{Y}_{i-1})] \\ +(1-\delta_i)[\tilde{Y}_{i-1}(\tilde{Y}_i - \theta\tilde{Y}_{i-1})]\} \\ = (1/\sigma^2)\sum_{i=2}^{n}\tilde{Y}_{i-1}(\tilde{Y}_i - \theta\tilde{Y}_{i-1}). \end{aligned} \quad (2.3.38)$$

Note that the equation corresponding to the score function for θ is identical to the one obtained for the AR(1) model without censoring, except that, corresponding to censored observations, the 'filled-in' estimates, $\tilde{Y}_i$, are used in place of the Y_i. Therefore, the score equations can be solved using the following iterative procedure:

1. Given the estimates, $\hat{\theta}^{(m)}$ and $\hat{\sigma}^{2(m)}$, from the mth iteration, use (2.3.31) to obtain $\tilde{Y}_i$, the estimates of the censored values.

2. Estimate $\hat{\theta}^{(m+1)}$ using the standard AR(1) fitting techniques on the pseudo-scores, $\tilde{Y}_i$. Obtain $\hat{\sigma}^{2(m+1)}$ by solving the equation corresponding to the score function in (2.3.37).

3. Iterate steps 1 and 2 until succesive parameter estimates converge.

Zeger and Brookmeyer (1986) noted that a consistent variance estimator for this PL estimator is difficult to obtain because the contributions of the likelihood are not independent. However, they also noted that if the censor rate is not too high, the variance estimator obtained by assuming independence of the contributions can be used. Therefore, in the simulations of Chapter 5, we have used an estimator of the asymptotic variance which is similar to the one for the MLE described above. Following the above suggestion of Zeger and Brookmeyer, an equivalent for $I(\theta)$ in the asymptotic variance expression for the MLE, (2.3.27), is the sum of squares of the individual scores of the score function for θ in (2.3.38). An equivalent for $J(\hat{\theta}_n^{zb})$ is obtained by using the EM aided differentiation utilized in the case of the MLE.

We conclude this chapter by giving a description of the PL estimator of Dagenais (1986) (see Dagenais 1989). We describe the estimator by making use of the modified notation of Zeger and Brookmeyer (1986) used in the description of the MLE in section 2.3.1. As with the 'true' likelihood, the likelihood of Dagenais (1986) can also be decomposed into two components: one for the contributions of uncensored observations for which the preceding observation is uncensored, i.e., observations in the index set U, and the other component for the contributions of observations in each censored string.

Recall that X_j and Y_j^u are the uncensored values preceding and at the end of the jth string, $\underline{Z}_j$, respectively, and that $\underline{T}_{j,k}^c$ is the censoring point for the kth censored value in the string. Then the pseudolikelihood of Dagenais (1986) has the form,

$$pl_d(\theta,\sigma^2) = \prod_{i\in U} f_{Y_i|Y_{i-1}}(Z_i|Z_{i-1}) \prod_{j=1}^{K} f_{Y_j^u|X_j}(Y_j^u|X_j) \prod_{j=1}^{K}\prod_{k=1}^{\nu_j^c} \tilde{F}_{1,k}(\underline{T}_{j,k},X_j,Y_j^u). \tag{2.3.39}$$

Here,

$$\tilde{F}_{1,k}(t,X_j,Y_j^u) = \int_t^\infty f_{\underline{Y}_{j,k}^c|X_j,Y_j^u}(s|X_j,Y_j^u)ds, \tag{2.3.40}$$

where, $\underline{Y}_{j,k}^c$ is the underlying observation corresponding to the k censored value in the string. For convenience, we use the same notation given in (2.3.1) given for the conditional means and covariances for Gaussian AR(1) processes. Given these definitions, the likelihood becomes

$$pl_d = \prod_{i\in U} \phi_1(\frac{Z_i-\theta Z_{i-1}}{\sigma}) \prod_{j=1}^{K} \phi_1(\frac{Y_j^u-\eta_j^u}{\sqrt{\sigma_j^u}}) \prod_{j=1}^{K}\prod_{k=1}^{\nu_j^c} \tilde{\Phi}_k(\underline{T}_{j,k},X_j,Y_j^u), \tag{2.3.41}$$

where,

$$\tilde{\Phi}_k(t,X_j,Y_j^u) = \int_t^\infty \phi_1(\frac{s-\eta_{j,k}^c}{\sqrt{\sigma_{j,k}^c}})ds, \tag{2.3.42}$$

$\eta_{j,k}^c$ and $\sigma_{j,k}^c$ being the conditional mean and variance, respectively, of the underlying observation, $\underline{Y}_{j,k}^c$, corresponding to the kth censored value, $\underline{Z}_{j,k}^c$, in string j, given the uncensored values, X_j and Y_j^u.

We found that, as with the MLE, a further simplification of the pl_d results by defining the following conditional means and variances for each censored observation in the jth string, $j = 1, ..., K$. Let

$$\tilde{\psi}_k(t,X_j,Y_j^u) = \frac{\int_t^\infty s\phi_1[(\sigma_{j,k}^c)^{-1}(s-\eta_{j,k}^c)]ds}{\int_t^\infty \phi_1[(\sigma_{j,k}^c)^{-1}(s-\eta_{j,k}^c)]ds}, \tag{2.3.43}$$

and

$$\hat{\psi}_k(t, X_j, Y_j^u) = \frac{\int_t^\infty s^2 \phi_1[(\sigma_{j,k}^c)^{-1}(s - \eta_{j,k}^c)]ds}{\int_t^\infty \phi_1[(\sigma_{j,k}^c)^{-1}(s - \eta_{j,k}^c)]ds}. \tag{2.3.44}$$

Then,

$$\underline{\tilde{Y}}_{j,k}^c = \tilde{\psi}_k(\underline{T}_{j,k}, X_j, Y_j^u) \tag{2.3.45}$$

is the conditional expectation of the underlying observation, $\underline{Y}_{j,k}^c$, corresponding to the kth censored value in the string, given the censoring at $\underline{T}_{j,k}$ and the uncensored observations, X_j and Y_j^u. The corresponding conditional variance is given by

$$V_{j,k}^c = \hat{\psi}_k(\underline{T}_{j,k}, X_j, Y_j^u) - (\underline{\tilde{Y}}_{j,k}^c)^2. \tag{2.3.46}$$

The likelihood score function for σ^2 can now be written as

$$\begin{aligned}\partial ln\ pl_2/\partial\sigma^2 &= \sum_{i\in U}\{[(-1)/2\sigma^2] + (1/2\sigma^4)(Z_i - \theta Z_{i-1})^2\} \\ &+\sum_{j=1}^{K}\{[(-1)/2\sigma^2] + (1/2\sigma^2)[\frac{(Y_j^u - \eta_j^u)^2}{\sigma_j^u}]\} + \sum_{j=1}^{K}\sum_{k=1}^{\nu_j^c}\{[(-1)/2\sigma^2] \\ &+(1/2\sigma^2)[\frac{(\underline{\tilde{Y}}_{j,k}^c - \eta_{j,k}^c)^2}{\sigma_{j,k}^c}] + (1/2\sigma^2)(\frac{V_{j,k}^c}{\sigma_{j,k}^c})\},\end{aligned} \tag{2.3.47}$$

or

$$\begin{aligned}\partial ln\ pl_2/\partial\sigma^2 &= [-(n-1)/2\sigma^2] \\ &+\sum_{i\in U}(1/2\sigma^4)(Z_i - \theta Z_{i-1})^2 + \sum_{j=1}^{K}(1/2\sigma^2)[\frac{(Y_j^u - \eta_j^u)^2}{\sigma_j^u}] \\ &+\sum_{j=1}^{K}\sum_{k=1}^{\nu_j^c}(1/2\sigma^2)\{[\frac{(\underline{\tilde{Y}}_{j,k}^c - \eta_{j,k}^c)^2}{\sigma_{j,k}^c}] + (\frac{v_{j,k}^c}{\sigma_{j,k}^c})\}.\end{aligned} \tag{2.3.48}$$

As with the MLE, the likelihood score function for θ is a sum of three terms. The first term is the contribution of uncensored observations for which the preceding value is uncensored. This term is identical to the corresponding term for the MLE given by equation (2.3.22-a). The second term, which identical to the term in (2.3.22-b), is the contribution of the uncensored values, Y_j^u. The third term,

$$\sum_{j=1}^{K}\sum_{k=1}^{\nu_j^c}\{\frac{-1}{2\sigma_{j,k}^c}\frac{\partial\sigma_{j,k}^c}{\partial\theta} + \frac{1}{2(\sigma_{j,k}^c)^2}\frac{\partial\sigma_{j,k}^c}{\partial\theta}(\underline{\tilde{Y}}_{j,k}^c - \eta_{j,k}^c)^2$$

$$+\frac{\partial \eta^c_{j,k}}{\partial \theta}\frac{(\tilde{\underline{Y}}^c_{j,k} - \eta^c_{j,k})}{\sigma^c_{j,k}} + \frac{1}{2(\sigma^c_{j,k})^2}\frac{\partial \sigma^c_{j,k}}{\partial \theta}V^c_{j,k}\}, \qquad (2.3.49)$$

is the contribution of censored observations, $\underline{Z}^c_j$, in each string. The last term involving the conditional variance, $V^c_{j,k}$, in this expression and in the score function for σ^2 can be interpreted as a correction for the $\tilde{\underline{Y}}^c_{j,k}$ being closer to $\eta^c_{j,k}$ than it would be expected for the underlying data. A similar interpretation was given by Zeger and Brookmeyer (1986) about an analogous term in the case of the MLE.

Notice that, unlike with the true likelihood, the contribution of the censored values is written as a quadratic form that ignores the dependence among the censored observations. However, only univariate normal integrals need to be evaluated and hence, the estimator is not as computationally intensive as the MLE.

The EM algorithm or the modified Newton-Raphson routine used by Zeger and Brookmeyer for the MLE can be used here to solve the likelihood score equations. The EM algorithm in this case can be suitably modified as follows:

1. *E step*: Given the estimates, $\hat{\theta}^{(m)}$ and $\hat{\sigma}^{2(m)}$, from the mth iteration, use (2.3.45) to obtain $\tilde{\underline{Y}}^c_{j,k}$, the estimates of the censored values, and use (2.3.46) to calculate the conditional variances $v^c_{j,k}$.

2. *M step*: Estimate $\hat{\theta}^{(m+1)}$ by solving the the estimating equation defined by (2.3.22-a), (2.3.22-b) and (2.3.49). Obtain $\hat{\sigma}^{2(m+1)}$ by solving the equation corresponding to the score function in (2.3.48).

3. *Iteration*: Iterate steps 1 and 2 until parameter estimates converge.

As with the MLE, the Hessian is required to implement the Newton-Rapson procedure and to calculate the asymptotic variances of the parameter estimates. Here, the Hessian for σ^2 is given by

$$\partial^2 ln\ l/\partial(\sigma^2)^2 = [(n-1)/2\sigma^4]$$

$$+\sum_{i\in U}(-1/2\sigma^6)(Z_i-\theta Z_{i-1})^2+\sum_{j=1}^{K}(-1/\sigma^4)\{[\frac{(Y_j^u-\eta_j^u)^2}{\sigma_j^u}]\}$$

$$+\sum_{j=1}^{K}\sum_{k=1}^{\nu_j^c}(-1/\sigma^4)\{\frac{(\tilde{\underline{Y}}_{j,k}^c-\eta_{j,k}^c)^2}{\sigma_{j,k}^c}+\frac{V_{j,k}^c}{\sigma_{j,k}^c}\}, \qquad (2.3.50)$$

while the Hessian for θ is the sum of the components (2.3.29-a), (2.3.29-b) and the derivative of the term in (2.3.49). As in the case of the MLE, an exact value of this derivative is difficult to obtain because of the dependence on θ of the $\tilde{\underline{Y}}_{j,k}^c$'s. One estimate of it,

$$\sum_{j=1}^{K}\sum_{j,k}^{\nu_j^c}\{\frac{-1}{2\sigma_{j,k}^c}\frac{\partial^2\sigma_{j,k}^c}{\partial\theta^2}+\frac{1}{2(\sigma_{j,k}^c)^2}(\frac{\partial\sigma_{j,k}^c}{\partial\theta})^2-\frac{2}{(\sigma_{j,k}^c)^2}\frac{\partial\sigma_{j,k}^c}{\partial\theta}\frac{\partial\eta_{j,k}^c}{\partial\theta}(\tilde{\underline{Y}}_{j,k}^c-\eta_{j,k}^c)$$
$$+\frac{1}{2(\sigma_{j,k}^c)^2}\frac{\partial^2\sigma_{j,k}^c}{\partial\theta^2}(\tilde{\underline{Y}}_{j,k}^c-\eta_{j,k}^c)^2-\frac{1}{(\sigma_{j,k}^c)^3}(\frac{\partial\sigma_{j,k}^c}{\partial\theta})^2(\tilde{\underline{Y}}_{j,k}^c-\eta_{j,k}^c)^2$$
$$+\frac{\partial^2\eta_{j,k}^c}{\partial\theta^2}\frac{(\tilde{\underline{Y}}_{j,k}^c-\eta_{j,k}^c)}{\sigma_{j,k}^c}-\frac{1}{\sigma_{j,k}^c}(\frac{\partial\eta_{j,k}^c}{\partial\theta})^2$$
$$+\frac{1}{2(\sigma_j^c)^2}\frac{\partial^2\sigma_j^c}{\partial\theta^2}V_j^c-\frac{1}{(\sigma_j^c)^3}(\frac{\partial\sigma_j^c}{\partial\theta})^2V_j^c\}, \qquad (2.3.51)$$

is obtained by ignoring the dependence on θ of the $\tilde{\underline{Y}}_{j,k}^c$'s. However, an estimator of the Hessian which does not require ignoring this dependence can be obtained numerically by applying the EM aided differentiation technique utilized in the case of the MLE to the third component of the pseudolikelihood score function given by (2.3.49). We have used the latter estimator of the Hessian to estimate the asymptotic variance in the simulations of Chapter 5. The asymptotic variance estimator of θ takes a form similar to the one for the MLE. Let $\hat{\theta}_n^{dag}$ be the solution of the score equation for θ. Then the asymptotic variance estimator is given by

$$a\widehat{sv}ar(\hat{\theta}_n^{dag})=\frac{I_n^{dag}(\hat{\theta}_n^{dag})}{(J^{dag})_n^2(\hat{\theta}_n^{dag})}, \qquad (2.3.52)$$

where, $J_n^{dag}(\theta)$ is the Hessian for θ given above and $I_n^{dag}(\theta)$ is the sum of the terms, (2.3.29-a), (2.3.29-b) and

$$\sum_{j=1}^{K}\{\sum_{k=1}^{\nu_j^c}\{\frac{-1}{2\sigma_{j,k}^c}\frac{\partial\sigma_{j,k}^c}{\partial\theta}+\frac{1}{2(\sigma_{j,k}^c)^2}\frac{\partial\sigma_{j,k}^c}{\partial\theta}(\tilde{\underline{Y}}_{j,k}^c-\eta_{j,k}^c)^2$$

$$+\frac{\partial\eta^c_{j,k}}{\partial\theta}\frac{(\underline{\tilde{Y}}^c_{j,k}-\eta^c_{j,k})}{\sigma^c_{j,k}}+\frac{1}{2(\sigma^c_{j,k})^2}\frac{\partial\sigma^c_{j,k}}{\partial\theta}v^c_{j,k}\}\}^2. \qquad (2.3.53)$$

This expression is arrived at by assuming that the mean vector of $\underline{\tilde{Y}}^c_j$ is approximately zero (Dagenais, 1989). In case of the MLE, the mean vector of $\underline{\hat{Y}}^c_j$ is zero and hence the theory behind the expression for the asymptotic variance is exact.

In this chapter, we have reviewed the existing estimators in regression with censored data. We have noted that linear regression with censored i.i.d. data has been studied extensively while very little work has been done for regression with censored autocorrelated data. We have argued that it is sufficient to look only at the autoregression component of the model assumed in the latter problem as the results can be easily extended to the case where the regression component is incorporated. Thus, we have reviewed the estimators in regression with censored autocorrelated data in the context of the pure AR(1) model.

The existing estimators in regression with censored autocorrelated data reviewed in this chapter need to be re-derived for each error distribution considered. This is a very tedious task. Therefore, we have proposed two estimators which have the same form (in the least squares sense) regardless of the error distribution. In this sense these estimators are less restrictive in comparison with the existing estimators. However, since these estimators involve conditional expectations, knowledge of the form of the error distribution is required. Thus, we have also proposed a distribution-free estimator based on the Kaplan-Meier estimator of the error distribution. These new estimators are described in the next chapter. In Chapter 4, We study some asymptotic properties of the two new estimators for which the form of the distribution is assumed to be known. These new estimators are then compared among themselves and with the existing estimators in Chapter 5.

Chapter 3

New Estimators for Censored Autocorrelated Data

3.1 Introduction

In this chapter, we introduce new estimators of parameters in autoregressive (AR) models with possibly censored response rv's. The reason for these new estimators is that, in their present form, the currently available estimators are only suitable for AR models in which the errors are Gaussian. However, the Gaussian assumption may not be satisfied in practical applications. If, indeed, this is the case, the currently available estimators need to be suitably modified for the specified error distribution. In some cases, the modification is essentially a re-derivation of the estimator. Hence, extensive preparatory calculations are inevitable if these estimators are to be applied for error distributions other than the Gaussian distribution. Further, while problems of non-convergence in the case of the MLE can be avoided by using the EM-algorithm or the quasi-Newton algorithm suggested by Zeger and Brookmeyer (1986), the MLE can be computationaly intensive if high dimensional integrals must be evaluated, as seen in the preceding chapter. The PL estimators of Zeger and Brookmeyer (1986) and Dagenais (1986) were introduced as alternatives to avoid the computational difficulties encountered with the MLE. However, consistent estimators of the variances of these estimators are difficult to obtain. This is because the contributions of the respective likelihoods are not independent. Hence, the sums of squares of the individual scores of the score functions lead to biased variance estimators. Thus, we were prompted to propose two estimators which not only avoid this variance estimation problem but also can be applied to AR models with any specified error distribution. Both these estimators are based on the least-squares estimating function that would be obtained in the absence of censoring. In one of the estimators, each underlying time series random variable in the estimating function (one corresponding to index time i, $1 \leq i \leq n$, say) is replaced by its conditional expectation given the sigma-field generated by the observed (censored) time series rv's with index times $j \leq i$. In the second estimator, the ith summand in the estimating function is replaced by its conditional mean given the censoring at the index times $j \leq i$. We also introduce a distribution-free estimator in which the underlying time series random variable with index time i is replaced by its corresponding conditional mean given the censoring. This conditional expectation is computed conditional on the censoring at the index time i and given that the underlying time series random variable at index time $i-1$ is equal to its corresponding conditional expectation which has also been computed similarly. This sequential computational scheme is applied for all n time series rv's, where n is the sample size. The conditional expectations are computed with the error distribution replaced by its Kaplan-Meier estimator based on the residuals. It is our thesis that these new estimators can be applied in situations where the autoregressive process is non-Gaussian and hence are more flexible than their competitors. The distribution-free estimator has an additional advantage because the error distribution is computed from the observed time series data. Thus, this estimator can be applied in a wide variety of applications.

Like the currently available estimators described in the preceding chapter, the new estimators are for the following model. Define the stationary, ergodic autoregressive process, $\{Y_i, i \in \mathcal{Z}\}$, where $\mathcal{Z}$ is the set of integers, by

$$Y_i = \theta Y_{i-1} + \varepsilon_i, \tag{3.1}$$

where the errors, ε_i's, are i.i.d. $\sim F$, assumed to be known, with mean zero and unknown variance σ^2. Further, ε_i is assumed to be independent of the sigma-field generated by $Y_j,\ j \leq i-1$. Let T_i be a sequence of i.i.d. random variables, independent of ε_i's. We observe $\{(Z_i, \delta_i), i \leq n\}$, where $Z_i = Y_i \wedge T_i$, $\delta_i = I(Y_i \leq T_i)$.

Our interest lies, primarily, with the estimation of the autoregression parameter, θ, from a single realization, $\{(Z_i, \delta_i), i \leq n\}$. However, estimation of the variance of the errors, σ^2, is implicit in the estimation process. While focus in this thesis is restricted to right-censored data from an AR(1) model, our proposed estimators can be easily extended to the AR(p) model and can be suitably modified for left-censored data.

The chapter is organised as follows. We begin by describing, in section 3.2, two estimators for which F is assumed to be known. Then in section 3.3, we present a description of the distribution-free estimator. The descriptions are presented along with the motivation and justification for these estimators.

3.2 Estimators with error distribution assumed known

3.2.1 Estimator based on conditional means of individual time series rv's

Let $\mathcal{F}_i = \sigma\{(Z_j, \delta_j), j \leq i\}$. Define new pseudo rv's, $Y_i^*(\theta)$'s, corresponding to the underlying time series rv's, Y_i's, by

$$Y_i^*(\theta) = E_\theta(Y_i|\mathcal{F}_i). \tag{3.1}$$

Let

$$X_i(\theta) = Y_{i-1}^*(\theta)\{Y_i^*(\theta) - \theta Y_{i-1}^*(\theta)\} \tag{3.2}$$

Define an estimating function $M_n(\theta)$ by

$$M_n(\theta) = \sum_{i=1}^{n} X_i(\theta). \tag{3.3}$$

Then the first estimator of θ we consider, hereafter referred to as $\hat{\theta}_n^a$, is the solution of the estimating equation,

$$M_n(\theta) = 0. \tag{3.4}$$

The following result establishes a desirable and useful property of the estimating function $M_n(\theta)$. This property also means that $M_n(\theta)$ is unbiased and forms part of the motivation behind the estimator $\hat{\theta}_n^a$.

Lemma 3.2.1 *Let the estimating function $M_n(\theta)$ be as defined in equation (3.2.3). Let $\bar{\mathcal{F}}_i = \sigma\{(Y_j, T_j), j \leq i\}$. Suppose the AR(1) model in section 3.1 is such that ε_i in (3.1.1) are independent of $\bar{\mathcal{F}}_{i-1}$. Then $\{M_n(\theta), \mathcal{F}_n\}$ is a zero mean martingale under the probability measure P_θ.*

Proof

Note that,

$$\begin{aligned} E\left(X_i(\theta)|\mathcal{F}_{i-1}\right) &= E_\theta\{Y_{i-1}^*(\theta)[Y_i^*(\theta) - \theta Y_{i-1}^*(\theta)]|\mathcal{F}_{i-1}\} \\ &= Y_{i-1}^* E_\theta(Y_i^*|\mathcal{F}_{i-1}) - \theta Y_{i-1}^{*^2}, \end{aligned}$$

since Y_{i-1}^* is measurable with respect to the sigma field $\mathcal{F}_{i-1}$. Here we are suppressing the dependence of Y_i^* on θ. But

$$\begin{aligned} E_\theta(Y_i^*|\mathcal{F}_{i-1}) &= E_\theta\left(E_\theta(Y_i|\mathcal{F}_i)|\mathcal{F}_{i-1}\right) \\ &= E_\theta(Y_i|\mathcal{F}_{i-1}) \\ &= E_\theta(\theta Y_{i-1} + \varepsilon_i|\mathcal{F}_{i-1}) \\ &= E\{E_\theta(\theta Y_{i-1} + \varepsilon_i|\mathcal{F}_{i-1})|\bar{\mathcal{F}}_{i-1}\} \\ &= E_\theta(\theta Y_{i-1}|\mathcal{F}_{i-1}) = \theta Y_{i-1}^*, \end{aligned}$$

since, $\mathcal{F}_{i-1} \subseteq \mathcal{F}_i$, $\mathcal{F}_{i-1} \subseteq \bar{\mathcal{F}}_{i-1}$ and ε_i is independent of $\bar{\mathcal{F}}_{i-1}$. Therefore, $X_i(\theta)$ is a martingale difference sequence with respect to $\mathcal{F}_i$ and hence $M_n(\theta)$ is a martingale estimating function. □

The preceding result implies that we can use martingale convergence results which includes the central limit theorem for martingales (see, e.g., Hall and Heyde 1980) to study large sample properties of the corresponding estimator, $\hat{\theta}_n^a$. This is done in chapter 4.

At this point, having defined the estimator, it seems appropriate to give its justification before showing how it is computed. Note that in the uncensored case, the least squares estimator of θ is the solution of the estimating equation

$$g^*(\theta) = \sum_{i=1}^{n} Y_{i-1}(Y_i - \theta Y_{i-1}) = 0. \tag{3.5}$$

The estimating function $g_n^*(\theta)$ is also the optimal estimating function in the class of estimating functions

$$\mathcal{G}_1 = \left\{ g : g = \sum_{i=1}^{n} a_{i-1}(Y_i - \theta Y_{i-1}) \right\} \tag{3.6}$$

with respect to the Godambe (1985) optimality criterion, where $E_\theta(g) = 0$ and the coefficients a_{i-1} are functions of $Y_1, ..., Y_{i-1}$ and θ. According to this criterion, in the class of unbiased estimating functions, $\mathcal{G}_1$, the one which minimizes

$$\lambda_{g(\theta)} = \frac{E_\theta(g_n^2(\theta))}{\left[E_\theta\left(\frac{\partial g(\theta)}{\partial \theta}\right)\right]^2} \tag{3.7}$$

is the optimal estimating function. By Theorem 1 of Godambe (1985),

$$g^*(\theta) = \sum_{i=1}^{n} a_{i-1}^*(Y_i - \theta Y_{i-1}) \tag{3.8}$$

is optimal according to criterion (3.2.7), within the class $\mathcal{G}_1$ for the choice

$$a_{i-1}^* = \frac{E_\theta\left(\partial(Y_i - \theta Y_{i-1})/\partial\theta | \bar{\mathcal{F}}_{i-1}\right)}{E_\theta\left((Y_i - \theta Y_{i-1})^2 | \bar{\mathcal{F}}_{i-1}\right)}, \tag{3.9}$$

where, $\bar{\mathcal{F}}_i = \sigma\{(Y_j, T_j), j \le i\}$. But $E_\theta((Y_i - \theta Y_{i-1})^2 | \bar{\mathcal{F}}_{i-1}) = var(\varepsilon_i) = \sigma^2$, constant, this yields

$$a_{i-1}^* = \frac{-Y_{i-1}}{\sigma^2}, \tag{3.10}$$

and hence,

$$g_n^*(\theta) = \sum_{i=1}^{n} Y_{i-1}(Y_i - \theta Y_{i-1}). \tag{3.11}$$

Note that, $g_n^*(\theta)$ is also the maximum likelihood score function if the distribution of the errors, F, in the underlying time series model defined by equation (3.1.1) is Gaussian.

Our estimating function in (3.2.3) is a modification of $g_n^*(\theta)$ for censored data. It is easy to see that the censored data estimating function is obtained by replacing Y_i's in $g_n^*(\theta)$ by their corresponding conditional means, $Y_i^*(\theta)$'s, calculated from the observed realization, $((Z_i, \delta_i), i \leq n)$. Next we describe a method of computing the $Y_i^*(\theta)$'s and thus the estimator.

In order to compute $Y_i^*(\theta)$ we make use of Lemma 2.3.1 which states that to condition on the entire past, it is enough to condition back to the last uncensored observation. Suppose the realization, $((Z_i, \delta_i), i \leq n)$, is such that Z_1 is uncensored. Let

$$p_{io} = \delta_i, \varphi_{io} = Y_i,$$
$$p_{ij} = (1-\delta_i)(1-\delta_{i-1})...(1-\delta_{i-j+1})\delta_{i-j}, \ \ j = 1, ..., i-1, \qquad (3.12)$$

and

$$\begin{aligned} &\varphi_{i,j}(t_i, t_{i-1}, ..., t_{i-j+1}, y; \theta) \\ &= E_\theta(Y_i | Y_i > t_i, Y_{i-1} > t_{i-1}, ..., Y_{i-j+1} > t_{i-j+1}, Y_{i-j} = y) \\ &= \frac{\int_{t_i}^{\infty} \cdots \int_{t_{i-j+1}}^{\infty} s_i f_{Y_i,...,Y_{i-j+1}|Y_{i-j}}(s_i, ..., s_{i-j+1}|y;\theta)\Pi_{m=1}^{j} ds_{i-j+m}}{\int_{t_i}^{\infty} \cdots \int_{t_{i-j+1}}^{\infty} f_{Y_i,...,Y_{i-j+1}|Y_{i-j}}(s_i, ..., s_{i-j+1}|y;\theta)\Pi_{m=1}^{j} ds_{i-j+m}}, \end{aligned} \qquad (3.13)$$

where, $f_{Y_i,...,Y_{i-j+1}|Y_{i-j}}$ is the joint density of $Y_i, ..., Y_{i-j+1}$ given Y_{i-j}. Then

$$\begin{aligned} Y_i^*(\theta) &= \delta_i Y_i + ... + (1-\delta_i)(1-\delta_{i-1})...(1-\delta_2)\delta_1 \varphi_{i\ i-1}(Z_i, Z_{i-1}, ..., Z_1; \theta) \\ &= \sum_{j=0}^{i-1} p_{ij}\varphi_{ij}. \end{aligned} \qquad (3.14)$$

Notice that $Y_i^*(\theta)$ is the observed response Y_i if uncensored, or an estimate of it if censored. If the underlying autoregressive process, $\{Y_i, i \in \mathcal{Z}\}$, is Gaussian, then

$$\varphi_{i,j}(t_i, t_{i-1}, ..., t_{i-j+1}, y; \theta)$$

$$= \frac{\int_{t_i}^{\infty} \cdots \int_{t_{i-j+1}}^{\infty} s_i \phi_j \left[(\Sigma_j)^{-1}(\underline{s} - \eta_{ij})\right] \Pi_{m=1}^{j} ds_{i-j+m}}{\int_{t_i}^{\infty} \cdots \int_{t_{i-j+1}}^{\infty} \phi_j \left[(\Sigma_j)^{-1}(\underline{s} - \eta_{ij})\right] \Pi_{m=1}^{j} ds_{i-j+m}}. \tag{3.15}$$

Here, ϕ_j is the j-dimensional standard normal density function, $\underline{s} = (s_{i-j+1}, ..., s_i)$ is the $j \times 1$ vector of variables of integration and η_{ij} is the $j \times 1$ conditional mean vector of the censored observations, up to the ith, given the most recent uncensored observation, Y_{i-j}. The kth entry of η_{ij} is given by

$$\eta_{ijk} = E_\theta(Y_{i-j+k}|Y_{i-j}) = \theta^k Y_{i-j}, \ \ k = 1, ...j. \tag{3.16}$$

Σ_j is the corresponding conditional covariance matrix whose entries are given by

$$\sigma_{ij(k,l)} = cov(Y_{i-j+k}, Y_{i-j+l}|Y_{i-j}) = \sigma^2 \sum_{s=0}^{k-1} \theta^{2s+(l-k)}, \ \ l > k. \tag{3.17}$$

Notice that, through conditioning, we can write the joint density, ϕ_j, as a product of univariate densities, i.e.,

$$\begin{aligned} &\phi_j \left[(\Sigma_j)^{-1}(\underline{s} - \eta_{ij})\right] \\ = \ &\phi_1(\frac{s_i - \theta s_{i-1}}{\sigma}) ... \phi_1(\frac{s_{i-j+2} - \theta s_{i-j+1}}{\sigma}) \phi_1(\frac{s_{i-j+1} - \theta y}{\sigma}), \end{aligned} \tag{3.18}$$

where, ϕ_1 is the univariate standard normal density function. This way, we do not need to compute the conditional mean vectors, η_{ij}'s, and the corresponding conditional covariance matrices, Σ_j's. Hence, the computation of $Y_i^*(\theta)$'s is made much easier and quicker. The estimator of σ^2 is obtained as the solution of the equation

$$\sigma^2 = \frac{\sum_{i=1}^{n} \left(Y_i^*(\theta) - \theta Y_{i-1}^*(\theta)\right)^2}{n-1}. \tag{3.19}$$

Once again this is obtained by replacing Y_i's by the corresponding $Y_i^*(\theta)$'s in the least-squares equation obtained in the uncensored case. Note that $Y_i^*(\theta)$'s are functions of σ^2 since they are computed using conditional distributions of the underlying time series rv's, Y_i's which are themselves functions of σ^2. Hence, the equation in (3.2.20) is solved iteratively.

Having defined the estimator and demonstrated a way of computing the $Y_i^*(\theta)$'s, we are now ready to describe an algorithm that can be used to solve the estimating equations (3.2.4) and (3.2.20). In order to do this, we first note that (3.2.4) can be re-written to give the form

$$\theta = \frac{\sum_{i=1}^{n} Y_i^*(\theta) Y_{i-1}^*(\theta)}{\sum_{i=1}^{n} Y_{i-1}^{*2}(\theta)}$$

Then we can use an EM-type algorithm which consists of the following steps:

1. *E step*: Given the estimates, $\hat{\theta}^{(m)}$ and $\hat{\sigma}^{2(m)}$, from the mth iteration, use (3.2.15) to obtain $Y_i^*(\theta^{(m)})$'s, the estimates of the censored values.

2. *M step*: Estimate $\hat{\theta}^{(m+1)}$ by plugging in $Y_i^*(\theta^{(m)})$'s on the right hand side of (3.2.21). Obtain $\hat{\sigma}^{2(m+1)}$ by solving the equation (3.2.20)

3. *Iteration*: Iterate steps 1 and 2 until succesive parameter estimates do not change.

As with the MLE, the asymptotic variance estimator of $\hat{\theta}_n^a$ we consider, makes use of its asymptotic results. The asymptotic results for this estimator are discussed in detail in the next chapter. Under suitable conditions (see Chapter 4),

$$n^{\frac{1}{2}}(\hat{\theta}_n^a - \theta_o) \xrightarrow{\mathcal{D}} N\left(0, \frac{E_{\theta_o}\left(X_1^2(\theta_o)\right)}{\left(E_{\theta_o}\left(X_1^{(1)}(\theta_o)\right)\right)^2}\right), \quad \text{under } P_{\theta_o},$$

where $X_i(\theta)$ is given by (3.2.2) and $X_1^{(1)}(\theta)$ is the derivative of $X_i(\theta)$ with respect to θ. Hence,

$$a\widehat{svar}(\hat{\theta}_n^a) = \frac{I_n^a(\hat{\theta}_n^a)}{\left(J_n^a(\hat{\theta}_n^a)\right)^2}, \tag{3.20}$$

where,

$$\begin{aligned} I_n^a(\theta) &= \sum_{i=1}^{n} X_i(\theta) \\ &= \sum_{i=1}^{n} Y_{i-1}^{*2}(\theta)\left(Y_i^*(\theta) - \theta Y_{i-1}^*(\theta)\right)^2, \end{aligned} \tag{3.21}$$

and

$$J_n^a(\theta) = \sum_{i=1}^{n} X_i^{(1)}(\theta). \tag{3.22}$$

We approximate $J_n^a(\hat{\theta}_n^a)$ by using the *EM aided differentiation* technique mentioned in Meilijson (1989) and used in sections 2.3.1 and 2.3.2 of this thesis to compute $J_n(\hat{\theta}_n^{mle})$ and $J_n^d(\hat{\theta}_n^{dag})$, respectively. This technique uses the truncated Taylor expansion in the following manner: Choose $\varepsilon > 0$ sufficiently small. Expand $M_n(\hat{\theta}_n^a)$ in the neighbourhood of $\hat{\theta}_n^a + \varepsilon$. Then

$$J_n^a(\hat{\theta}_n^a) \approx M_n^{(1)}(\hat{\theta}_n^a + \delta) = \frac{M_n(\hat{\theta}_n^a + \varepsilon)}{\varepsilon}, \quad 0 < \delta < \varepsilon. \tag{3.23}$$

In the next section, we describe the second estimator of (θ, σ^2) in the censored AR(1) model defined in section 3.1 with the error distribution, F, is assumed known but not necessarily Gaussian.

3.2.2 Estimator based on a missing information principle

As mentioned in section 3.1, the estimator described in this section differs from the one described in section 3.2.1 above in that, the summands rather than the individual underlying time series rv's Y_i are replaced by their conditional means in the least-squares estimating function of the uncensored case. The principle of obtaining an estimating function in this manner is referred to as the missing information principle by Lai and Ying (1994). The estimator, henceforth referred to as $\hat{\theta}_n^b$, is the solution of the estimating equation

$$Q_n(\theta) = 0. \tag{3.24}$$

Here,

$$Q_n(\theta) = \sum_{i=1}^{n} D_i(\theta), \tag{3.25}$$

where,

$$D_i(\theta) = E_\theta\left(Y_{i-1}(Y_i - \theta Y_{i-1}) | \mathcal{F}_i\right) \tag{3.26}$$

and $\mathcal{F}_i = \sigma\{(Z_i, \delta_i), j \leq i\}$ as in section 3.2.1 above. As with the estimating $M_n(\theta)$, another motivation for $Q_n(\theta)$ is the martingale property which is summarized in the following lemma.

Lemma 3.2.2 *Let the estimating function $Q_n(\theta)$ be as defined in equation (3.2.28). Let $\bar{\mathcal{F}}_i = \sigma\{(Y_j, T_j), j \leq i\}$. Suppose the AR(1) model in section 3.1 is such that ε_i in (3.1.1) are independent of $\bar{\mathcal{F}}_{i-1}$. Then $\{Q_n(\theta), \mathcal{F}_n\}$ is a zero mean martingale under the probability measure P_θ.*

Proof

Note that,

$$\begin{aligned}
E_\theta\left(D_i(\theta)|\mathcal{F}_{i-1}\right) &= E_\theta\left(E_\theta\left(Y_{i-1}(Y_i - \theta Y_{i-1})|\mathcal{F}_i\right)|\mathcal{F}_{i-1}\right) \\
&= E_\theta\left(Y_{i-1}(Y_i - \theta Y_{i-1})|\mathcal{F}_{i-1}\right) \\
&= E_\theta\left(Y_{i-1}\varepsilon_i|\mathcal{F}_{i-1}\right) \\
&= E_\theta\left(E_\theta\left(Y_{i-1}\varepsilon_i|\mathcal{F}_{i-1}\right)|\bar{\mathcal{F}}_{i-1}\right) \\
&= E_\theta\left(Y_{i-1}E_\theta\left(\varepsilon_i|\bar{\mathcal{F}}_{i-1}\right)|\mathcal{F}_{i-1}\right) = 0, \qquad (3.27)
\end{aligned}$$

The step from the first to the second line in (3.2.30) follows from the fact that $\mathcal{F}_{i-1} \subseteq \mathcal{F}_i$ as mentioned earlier. The steps from the third to the fourth and from the fourth to the fifth lines follows from the fact that $\mathcal{F}_i \subseteq \bar{\mathcal{F}}_i$.

As with the estimator discussed in section 3.2.1 above, the estimator, $\hat{\theta}_n^b$ currently under discussion, is motivated by the uncensored least-squares estimating function, the optimal estimating function (in the sense of Godambe, 1960) or the likelihood score function when the underlying autoregressive process, $\{Y_i, i \in \mathcal{Z}\}$, is Gaussian, all of which lead to the estimating equation (3.2.5). The estimator can be viewed as an extension of least squares estimation in the censored linear regression set-up which uses a 'missing information principle' of Lai and Ying (1994). In section 2 of their paper, Lai and Ying (1994) modified the least squares normal equations by replacing the summands in the estimating functions by their conditional means given the censoring at the corresponding data points. Extending this modification to censored time series requires taking into account the dependence among the observations which requires conditioning on the past as well. Our estimating equation in (3.2.27) can be viewed as a result of the modification of Lai and Ying (1994) suitably adapted for the time series model defined by equation (3.1.1). The corresponding estimator for σ^2 is obtained as a solution of the estimating equation

$$\sigma^2 = \frac{\sum_{i=1}^{n} E_\theta\left((Y_i - \theta Y_{i-1})^2|\mathcal{F}_i\right)}{n-1}. \tag{3.28}$$

Note that, as with $Y_i'(\theta)$'s, the summands in this equation are themselves functions of σ^2 and hence the equation is solved iteratively. In order to compute the parameter estimates, we need to calculate the rv's, $E_\theta(Y_iY_{i-1}|\mathcal{F}_i)$, $E_\theta(Y_{i-1}^2|\mathcal{F}_i)$ and $E_\theta(Y_i^2|\mathcal{F}_i)$. This is done in a way similar to that of calculating $Y_i^*(\theta)$ for the estimator, $\hat{\theta}_n^b$, discussed in the previous section. For two consecutive uncensored observations,

$$E_\theta(Y_iY_{i-1}|\mathcal{F}_i) = Y_iY_{i-1}. \tag{3.29}$$

If Y_{i-1} is uncensored and Y_i is censored, then

$$E_\theta(Y_iY_{i-1}|\mathcal{F}_i) = Y_i^*(\theta)Y_{i-1}, \tag{3.30}$$

and if Y_{i-1} is censored and Y_i uncensored,

$$E_\theta(Y_iY_{i-1}|\mathcal{F}_i) = Y_iY_{i-1}^*(\theta). \tag{3.31}$$

If both Y_i and Y_{i-1} are censored then $E_\theta(Y_iY_{i-1}|\mathcal{F}_i)$ is calculated as follows. Let

$$\begin{aligned}
&\hat{\psi}_{i,j}(t_i, t_{i-1}, ..., t_{i-j+1}, y; \theta) \\
&\quad = E_\theta(Y_iY_{i-1}|Y_i > t_i, Y_{i-1} > t_{i-1}, ..., Y_{i-j+1} > t_{i-j+1}, Y_{i-j} = y) \\
&\quad = \frac{\int_{t_i}^{\infty} \cdots \int_{t_{i-j+1}}^{\infty} s_i s_{i-1} f_{Y_i,...,Y_{i-j+1}|Y_{i-j}}(s_i, ..., s_{i-j+1}|y;\theta)\Pi_{m=1}^{j} ds_{i-j+m}}{\int_{t_i}^{\infty} \cdots \int_{t_{i-j+1}}^{\infty} f_{Y_i,...,Y_{i-j+1}|Y_{i-j}}(s_i, ..., s_{i-j+1}|y;\theta)\Pi_{m=1}^{j} ds_{i-j+m}},
\end{aligned} \tag{3.32}$$

Then

$$E_\theta(Y_iY_{i-1}|\mathcal{F}_i) = \hat{\psi}_{i,j}(Z_i, Z_{i-1}, ..., Z_{i-j+1}, Z_{i-j}; \theta). \tag{3.33}$$

$E_\theta(Y_{i-1}^2|\mathcal{F}_i)$ and $E_\theta(Y_i^2|\mathcal{F}_i)$ are calculated similarly by using the functions,

$$\tilde{\psi}_{i,j}(t_i, t_{i-1}, ..., t_{i-j+1}, y; \theta)$$

$$= E_\theta(Y_{i-1}^2 | Y_i > t_i, Y_{i-1} > t_{i-1}, ..., Y_{i-j+1} > t_{i-j+1}, Y_{i-j} = y)$$
$$= \frac{\int_{t_i}^{\infty} \cdots \int_{t_{i-j+1}}^{\infty} s_{i-1}^2 f_{Y_i,...,Y_{i-j+1}|Y_{i-j}}(s_i, ..., s_{i-j+1}|y;\theta)\Pi_{m=1}^{j} ds_{i-j+m}}{\int_{t_i}^{\infty} \cdots \int_{t_{i-j+1}}^{\infty} f_{Y_i,...,Y_{i-j+1}|Y_{i-j}}(s_i, ..., s_{i-j+1}|y;\theta)\Pi_{m=1}^{j} ds_{i-j+m}}, \tag{3.34}$$

and

$$\bar{\psi}_{i,j}(t_i, t_{i-1}, ..., t_{i-j+1}, y; \theta)$$
$$= E_\theta(Y_i^2 | Y_i > t_i, Y_{i-1} > t_{i-1}, ..., Y_{i-j+1} > t_{i-j+1}, Y_{i-j} = y)$$
$$= \frac{\int_{t_i}^{\infty} \cdots \int_{t_{i-j+1}}^{\infty} s_i^2 f_{Y_i,...,Y_{i-j+1}|Y_{i-j}}(s_i, ..., s_{i-j+1}|y;\theta)\Pi_{m=1}^{j} ds_{i-j+m}}{\int_{t_i}^{\infty} \cdots \int_{t_{i-j+1}}^{\infty} f_{Y_i,...,Y_{i-j+1}|Y_{i-j}}(s_i, ..., s_{i-j+1}|y;\theta)\Pi_{m=1}^{j} ds_{i-j+m}}, \tag{3.35}$$

As seen before, if the underlying autoregressive process, $\{Y_i, i \in \mathcal{Z}\}$, is Gaussian, then,

$$f_{Y_i,...,Y_{i-j+1}|Y_{i-j}}(s_i, ..., s_{i-j+1}|y;\theta)$$
$$= \phi_1(\frac{s_i - \theta s_{i-1}}{\sigma}) ... \phi_1(\frac{s_{i-j+2} - \theta s_{i-j+1}}{\sigma})\phi_1(\frac{s_{i-j+1} - \theta y}{\sigma}), \tag{3.36}$$

where ϕ_1 is the univariate standard normal density function.

The parameter estimates can be computed iteratively using the following EM type algorithm.

1. *E step*: Given the estimates, $\hat{\theta}^{(m)}$ and $\hat{\sigma}^{2(m)}$, from the mth iteration, use (3.2.32), (3.2.33), (3.2.34) and (3.2.36) to compute $E_\theta(Y_i Y_{i-1}|\mathcal{F}_i)$. Use (3.2.37) and (3.2.38) to compute $E_\theta(Y_{i-1}^2|\mathcal{F}_i)$ and $E_\theta(Y_i^2|\mathcal{F}_i)$, respectively.

2. *M step*: Estimate $\hat{\theta}^{(m+1)}$ by solving the the estimating equation defined by (3.2.27). Obtain $\hat{\sigma}^{2(m+1)}$ by solving the equation (3.2.31)

3. *Iteration*: Iterate steps 1 and 2 until parameter estimates converge.

The asymptotic variance estimator for $\hat{\theta}_n^b$ is analogous to the one for $\hat{\theta}_n^a$. It is given by

$$as\widehat{var}(\hat{\theta}_n^b) = \frac{I_n^b(\hat{\theta}_n^b)}{\left(J_n^b(\hat{\theta}_n^b)\right)^2}, \tag{3.37}$$

where,

$$I_n^b(\theta) = \sum_{i=1}^{n} \left(E_\theta \left(Y_{i-1}(Y_i - \theta Y_{i-1})|\mathcal{F}_i\right)\right)^2 \tag{3.38}$$

and

$$J_n^b(\theta) = \sum_{i=1}^{n} D_i^{(1)}(\theta). \tag{3.39}$$

$J_n^b(\hat{\theta}_n^b)$ is calculated using the *EM aided differentiation* technique used to calculate $J_n^a(\hat{\theta}_n^a)$ in section 3.2.1. We describe the distribution-free estimator in the next section.

3.3 A distribution-free estimator

We derive our distribution-free estimator, henceforth referred to as $\hat{\theta}_n^c$, by modifying the log-likelihood function obtained from pseudolikelihood of Zeger and Brookmeyer (1986), pl_{zb}. This modification yields an estimating function that would be optimal in the least-squares sense had there been no censoring. Therefore, the resulting estimating function applies for AR(1) processes with any error distribution, $F = F_{Y_i|Y_{i-1}}$. To illustrate the modification, note that the score function for θ derived from pl_1 is given by

$$\partial ln\, pl_1/\partial\theta = \sum_{i=2}^{n} \delta_i \frac{\partial}{\partial\theta} ln f_{Y_i|Y_{i-1}}(Z_i|\tilde{Y}_{i-1}) + (1-\delta_i)\frac{\partial}{\partial\theta} ln \tilde{F}_1(Z_i, \tilde{Y}_{i-1}).$$

Here, the differentiation is performed first and the evaluation at a point(s) next and $\tilde{F}_1(t,u)$ is given by (2.3.33) in section 2.3.2. That is,

$$\tilde{F}_1(t,u) = \int_t^\infty f_{Y_i|Y_{i-1}}(s|u)ds.$$

Let

$$\zeta(t,y;\theta) = E_\theta\left(\frac{\partial}{\partial\theta} ln f_{Y_i|Y_{i-1}}(Y_i|Y_{i-1})|Y_i > t, Y_{i-1} = y\right).$$

Then

$$\frac{\partial}{\partial\theta} ln \tilde{F}_1(Z_i, \tilde{Y}_{i-1}) = \zeta(Z_i, \tilde{Y}_{i-1}; \theta). \tag{3.1}$$

A similar observation was made by James (1986) for censored i.i.d. data. Note that if the underlying AR process, $Y_i, i \in \mathcal{Z}$, is Gaussian, then

$$\frac{\partial}{\partial\theta} ln f_{Y_i|Y_{i-1}}(Y_i|Y_{i-1}) = 1/\sigma^2 Y_{i-1}(Y_i - \theta Y_{i-1}), \tag{3.2}$$

and as a result,

$$\zeta(Z_i, \tilde{Y}_{i-1}; \theta) = 1/\sigma^2 \tilde{Y}_{i-1}(\tilde{Y}_i - \theta \tilde{Y}_{i-1}), \tag{3.3}$$

and hence the score function (2.3.36). This suggests that a least-squares analogue of the score function (3.3.1) which applies for an AR(1) process with any distribution is given by the same form (3.3.1) but with $ln f_{Y_i|Y_{i-1}}(Y_i|Y_{i-1})$ replaced by $(Y_i - \theta Y_{i-1})^2$. As with the PL estimator of Zeger and Brookmeyer (1986) where the AR process is assumed to be Gaussian, estimating θ using this estimating function is equivalent to using the standard AR(1) fitting techniques (e.g., Box and Jenkins 1970) on the filled-in (pseudo scores), $\tilde{Y}_i$. Hence, in a similar way that the Buckley-James method is motivated by the expectation identity in (2.2.4), our method is motivated by the following result.

Lemma 3.3.1 *Let $\{Z_i, i \in \mathcal{Z}\}$ be a possibly censored AR(1). Denote the underlying AR process, by $\{Y_i, i \in \mathcal{Z}\}$. Let the $F = F_{Y_i|Y_{i-1}}$ be the error distribution of the underlying process. Let $G(\cdot)$ be the censoring distribution. Define a function $\tilde{\varphi}_i$ such that*

$$\tilde{\varphi}_i(t, u) = \frac{\int_t^\infty s dF_{Y_i|Y_{i-1}}(s|u)}{1 - F_{Y_i|Y_{i-1}}(t|u)}.$$

Then

$$E_\theta\left(\delta_i Y_i + (1 - \delta_i)\tilde{\varphi}_i(T_i, Y_{i-1}|Y_{i-1})\right) = E_\theta(Y_i|Y_{i-1}) = \theta Y_{i-1}.$$

Proof

$$\begin{aligned} & E_\theta\left(\delta_i Y_i + (1 - \delta_i)\tilde{\varphi}_i(T_i, Y_{i-1}|Y_{i-1})\right) \\ = & \int_{-\infty}^{\infty} E_\theta\left[\delta_i Y_i|Y_{i-1}, Y_i = s\right] dF_{Y_i|Y_{i-1}}(s|Y_{i-1}) \\ & + \int_{-\infty}^{\infty} E_\theta\left[(1 - \delta_i)\varphi(T_i, Y_{i-1})|Y_{i-1}, T_i = t\right] dG(t). \end{aligned}$$

But

$$\begin{aligned} & E_\theta\left[\delta_i Y_i|Y_{i-1}, Y_i = s\right] \\ = & \; s E_\theta\left[I(Y_i \leq T_i)|Y_{i-1}, Y_i = s\right] \\ = & \; s[1 - G(s)] \end{aligned}$$

and

$$
\begin{aligned}
& E_\theta\left[(1-\delta_i)\tilde{\varphi}_i(T_i, Y_{i-1})|Y_{i-1}, T_i = t\right] \\
& \tilde{\varphi}_i(t, Y_{i-1}) E_\theta\left[I(Y_i > T_i)|Y_{i-1}, T_i = t\right] \\
& \tilde{\varphi}_i(t, Y_{i-1})[1 - FY_i|Y_{i-1}(t|Y_{i-1})]
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& E_\theta\left(\delta_i Y_i + (1-\delta_i)\tilde{\varphi}_i(T_i, Y_{i-1}|Y_{i-1})\right) \\
= & \int_{-\infty}^{\infty} s[1-G(s)]dF_{Y_i|Y_{i-1}}(s|Y_{i-1}) \\
& + \int_{-\infty}^{\infty}\left[\frac{\int_t^\infty s dF_{Y_i|Y_{i-1}}(s|Y_{i-1})}{1-F_{Y_i|Y_{i-1}}(t|Y_{i-1})}\right][1-F_{Y_i|Y_{i-1}}(t|Y_{i-1})]dG(t) \\
= & \int_{-\infty}^{\infty} s[1-G(s)]dF_{Y_i|Y_{i-1}}(s|Y_{i-1}) \\
& + \int_{-\infty}^{\infty}\left[\int_{-\infty}^{s} dG(t)\right] s dF_{Y_i|Y_{i-1}}(s|Y_{i-1}) \\
= & \int_{-\infty}^{\infty} s[1-G(s)]dF_{Y_i|Y_{i-1}}(s|Y_{i-1}) \\
& + \int_{-\infty}^{\infty} sG(s) dF_{Y_i|Y_{i-1}}(s|Y_{i-1}) \\
= & \int_{-\infty}^{\infty} s dF_{Y_i|Y_{i-1}}(s|Y_{i-1}) \\
= & E_\theta(Y_i|Y_{i-1}). \ \square
\end{aligned}
$$

Since $F(\cdot)$ is unknown, the method replaces the censored observations in the standard AR(1) fitting techniques by their estimated conditional expectations calculated using the Kaplan-Meier product limit estimator of $F(\cdot)$ (Kaplan and Meier, 1958, Efron, 1967, Miller, 1981) in the manner described next.

Assume the first observation, Z_1, is uncensored and define sequentially,

$$
\begin{aligned}
\tilde{Y}_i^*(\theta) &= Y_i, \quad \text{if} \quad \delta_i = 1, \\
&= \hat{\varphi}(T_i, \tilde{Y}_{i-1}^*(\theta); \theta), \quad \text{if} \quad \delta_i = 0,
\end{aligned} \tag{3.4}
$$

where $\hat{\varphi}(t, u; \theta)$ is an estimate of the conditional expectation $E_\theta(Y_i|Y_i > t, Y_{i-1} = u)$, calculated using the Kaplan-Meier product limit estimator of $F_b(\cdot)$ in the following fashion.

Let

$$e_i(u;\theta) = Z_i - \theta u, \;\; i = 2, \ldots, n, \tag{3.5}$$

and let

$$\hat{F}_{u,\theta}(s) = 1 - \prod_{i:e_{(i)}(u;\theta)\leq s} \left(\frac{n-i}{n-i+1}\right)^{\delta_{(i)}}, \tag{3.6}$$

denote the Kaplan-Meier product limit estimator calculated from the $e_j(u;\theta)$. Here, $e_{(i)}(\theta;u)$ is the ith ordered observed residual and $\delta_{(i)}$ its associated indicator. $\hat{F}_{u,\theta}(s)$ is defined in such a way that if $e_{(n)}(u;\theta)$ is censored, it is changed to be uncensored.

The function $\hat{\varphi}(t,u;\theta)$ is given by

$$\begin{aligned} \hat{\varphi}(t,u;\theta) &= \hat{E}_\theta(Y_i|Y_i > t, Y_{i-1} = u) \\ &= \theta u + \frac{\int_{(t-\theta u)}^{\infty} s d\hat{F}_{u,\theta}(s)}{1-\hat{F}_{u,\theta}(t-\theta u)}, \;\; \text{if} \;\; t - \theta u < e_{(n)}(u;\theta), \\ &= t, \;\; \text{if} \;\; t-\theta u \geq e_{(n)}(u;\theta). \end{aligned} \tag{3.7}$$

Note that $\tilde{Y}_i^*(\theta)$ is the observed response Y_i if uncensored, or an estimate of it, based on the Kaplan-Meier product limit estimator of $F_b(\cdot)$ calculated from the residuals $e_{(i)}(u;\theta)$ if censored. The estimator $\hat{\theta}_n^c$ is obtained by attempting to solve the estimating equation

$$\sum_{i=1}^{n} \tilde{Y}_{i-1}^*(\theta)\left(\tilde{Y}_i^*(\theta) - \theta\tilde{Y}_{i-1}^*(\theta)\right) = 0. \tag{3.8}$$

If we define

$$\omega_n(\theta) = \sum_{i=1}^{n} \tilde{Y}_{i-1}^*(\theta)\left(\tilde{Y}_i^*(\theta) - \theta\tilde{Y}_{i-1}^*(\theta)\right), \tag{3.9}$$

then (3.3.10) is equivalent to

$$\omega_n(\theta) = 0. \tag{3.10}$$

The corresponding estimator of σ^2 we consider here is the solution of the estimating equation

$$\sigma^2 = \frac{\sum_{i=1}^{n}(\tilde{Y}_i^* - \theta\tilde{Y}_{i-1}^*)^2}{n-1}. \tag{3.11}$$

This equation is obtained by replacing the Y_i's in the least-squares estimating equation for σ^2 by their corresponding conditional means, $\tilde{Y}_i^*$'s, based on the

Kaplan-Meier estimator of the error distribution $F(\cdot)$. Note that, $\tilde{Y}_i^*$'s are functions of σ^2 as it is the variance of $F(\cdot)$.

The parameter estimates of θ and σ^2 can be obtained by using the following EM type algorithm.

1. *E step*: Given the estimates, $\hat{\theta}^{(m)}$ and $\hat{\sigma}^{2(m)}$, from the mth iteration, use (3.3.7) to obtain $\tilde{Y}_i^*(\theta^{(m)})$'s, the estimates of the censored values.

2. *M step*: Estimate $\hat{\theta}^{(m+1)}$ by substituting $\tilde{Y}_i^*(\theta^{(m)})$'s into (3.3.12). Obtain $\hat{\sigma}^{2(m+1)}$ by solving the equation (3.3.13)

3. *Iteration*: Iterate steps 1 and 2 until succesive parameter estimates do not change within a specified error bound.

Note that to start the iterative procedure, $\tilde{Y}_i^*$ is set equal to Z_i, initially. As with the Buckley-James estimator, convergence is not guaranteed. Since $\omega_n(\theta)$ is discontinuous in θ, an exact solution need not exist, and even if it exists, it need not be unique. It was observed in a preliminary Monte Carlo simulation study conducted with this estimator that, sometimes the iteration settles down to oscillation between two values. However, these values stay very close to each other, with the maximum difference being in the order of 10^{-3}. When this happens, we take the average of these values.

The asymptotic variance estimator of $\hat{\theta}_n^c$ used in the simulation study in Chapter 5 is analogous to the one for the modified Buckley-James estimator proposed by Lai and Ying (1991). It is given by

$$a\widehat{svar}(\hat{\theta}_n^c) = \frac{I_n^c(\hat{\theta}_n^c)}{\left(J_n^c(\hat{\theta}_n^c)\right)^2}, \tag{3.12}$$

where,

$$I_n^b(\theta) = \sum_{i=1}^{n} \left(\tilde{Y}_{i-1}^*(\theta)\right)^2 \left(\tilde{Y}_i^*(\theta) - \theta \tilde{Y}_{i-1}^*(\theta)\right)^2 \tag{3.13}$$

and

$$J_n^b(\theta) = \omega_n^{(1)}(\theta). \tag{3.14}$$

$J_n^b(\hat{\theta}_n^b)$ is calculated using the *EM aided differentiation* technique used to calculate $J_n^a(\hat{\theta}_n^a)$ in section 3.2.1.

To conclude the chapter, we compare the three new estimators, $\hat{\theta}_n^a$, $\hat{\theta}_n^b$ and $\hat{\theta}_n^c$ on the basis of the principles used in their derivation. The main difference between $\hat{\theta}_n^a$ and $\hat{\theta}_n^b$ was given in the introduction of this chapter (section 3.1). The estimator $\hat{\theta}_n^a$ is obtained by replacing each of the time series rv's Y_i's in the least-squares estimating function obtained in the uncensored case by its conditional expectation given the sigma-field generated by the censored time series $(Z_j, \delta_j), j \leq i$. On the other hand, $\hat{\theta}_n^b$ is obtained by replacing each of the summands (ith, say) in the estimating function for the uncensored case by its conditional expectation given the sigma-field generated by $(Z_j, \delta_j), j \leq i$. The distribution-free estimator, $\hat{\theta}_n^c$, is similar to $\hat{\theta}_n^a$ in that each Y_i in the least-squares estimating function obtained in the uncensored case is replaced by its conditional expectation given the censored data. The difference is that in obtaining $\hat{\theta}_n^c$, the pseudo random variable which replaces Y_i's is calculated conditional on the censoring at the index time i and given that the time series random variable at index time $i-1$ is equal to the corresponding pseudo random variable. Another difference is that to obtain $\hat{\theta}_n^c$, the pseudo scores are calculated with the error distribution function replaced by its Kaplan-Meier estimator. This means that $\hat{\theta}_n^c$ can be applied in a wide variety of practical applications.

Chapter 4

Some Asymptotic Results

4.1 Introduction

In this chapter we establish the large sample properties of the two new estimators, described in Chapter 3, for which the form of the error distribution of the AR(1) process is assumed to be known. These estimators are $\hat{\theta}_n^a$ and $\hat{\theta}_n^b$ and their descriptions are given in sections 3.2.1 and 3.2.2, respectively.. As in section 3.2.1, $\hat{\theta}_n^a$ is defined as the solution of the estimating equation

$$M_n(\theta) = \sum_{i=1}^{n} X_i(\theta) = 0, \tag{4.1}$$

where

$$X_i(\theta) = Y_{i-1}^*(\theta)\left(Y_i^*(\theta) - \theta Y_{i-1}^*(\theta)\right), \tag{4.2}$$

$Y_i^*(\theta) = E_\theta(Y_i|\mathcal{F}_i)$, $\mathcal{F}_i = \sigma\{(Z_j, \delta_j),\ j \leq i\}$ and the Y_i's are the underlying time series rv's of the AR(1) model (see equation 3.1.1). We show that if $\exists\ \gamma > 0 \ni E_{\theta_o}|Y_i|^{4(1+\gamma)} < \infty$, then $n^{-\frac{1}{2}}M_n(\theta_o)$ is asymptotically normally distributed. A corollary to this, making use of a kind of Taylor expansion, is that if the corresponding estimator, $\hat{\theta}_n^a$, is consistent then $n^{\frac{1}{2}}(\hat{\theta}_n^a - \theta_o)$ is asymptotically normally distributed, where θ_o is the true value of θ. We also establish and discuss conditions under which $\hat{\theta}_n^a$ is consistent. As in section 3.2.2, the estimator $\hat{\theta}_n^b$, is defined as the solution of the estimation equation

$$Q_n(\theta) = \sum_{i=1}^{n} D_i(\theta) = 0, \tag{4.3}$$

where,

$$D_i(\theta) = E_\theta\left(Y_{i-1}(Y_i - \theta Y_{i-1})|\mathcal{F}_i\right). \tag{4.4}$$

We show that if the same condition, $E_{\theta_o}|Y_i|^{4(1+\gamma)} < \infty$, for some $\gamma > 0$ stated above holds, then $n^{-\frac{1}{2}}\ Q_n(\theta_o)$ has a normal distribution, asymptotically. By a similar Taylor expansion as in the case of $\hat{\theta}_n^a$, a corollary to this result is that if $\hat{\theta}_n^b$ is consistent, then $n^{\frac{1}{2}}(\hat{\theta}_n^b - \theta_o)$ asymptotically normally distributed. Note that the condition $E_{\theta_o}|Y_i|^{4(1+\gamma)} < \infty$ for some $\gamma > 0$ is equivalent to the condition that $\exists\ \gamma > \ni E_{\theta_o}|\varepsilon_i|^{4(1+\gamma)}$. To see this, one uses the infinite moving average representation of the AR(1) process to obtain Y_i as

$$Y_i = \sum_{j=0}^{\infty} \theta^j \varepsilon_{i-j}, \tag{4.5}$$

and the stationarity of the process, $|\theta| < 1$ to reach the conclusion $E_\theta|\varepsilon_i|^p < \infty \iff E_\theta|Y_i|^p < \infty$. All of the asymptotic results for $\hat{\theta}_n^a$ are presented in section 4.2 while section 4.3 discusses the corresponding results for $\hat{\theta}_n^b$.

4.2 Estimator based on conditional means of time series rv's

4.2.1 Asymptotic normality

The proof of the asymptotic normality of $n^{-\frac{1}{2}}\ M_n(\theta_o)$ is divided into several lemmas. In the sequel, we state and prove a lemma that establishes the stationarity and ergodicity of the process $\{Y_i^*(\theta), \in \mathcal{Z}\}$.

Lemma 4.2.1 *Let $\{Y_i, i \in \mathcal{Z}\}$ and $\{T_i, i \in \mathcal{Z}\}$ be two stationary, ergodic processes, each on $(\Re^{\mathcal{Z}}, \mathcal{B}^{\mathcal{Z}})$. Define a new bivariate process,*

$$\underline{C}_i = ((Z_i, \delta_i), (Z_{i+1}, (\delta_{i+1})), \ldots),\ \ i \in \mathcal{Z},$$

where

$$Z_i = min(Y_i, T_i),\ \ \delta_i = I(Y_i \leq T_i).$$

Let Ψ_1 be the measurable function on $(\Re^2)^\infty \longrightarrow \Re$ defined by

$$\begin{aligned}
&\Psi_1((x_i, y_i), (x_{i-1}, y_{i-1}), \ldots) \\
&\quad = \ x_i \ \text{if}\ y_i = 1, \\
&\quad = \ \int_{-\infty}^{\infty} u f_{Y_i|(Z_i,\delta_i),(Z_{i-1},\delta_{i-1}),\ldots}(u|(x_i, y_i), (x_{i-1}, y_{i-1}), \ldots)du, \ \ \text{if}\ y_i = 0, \\
&\quad = \ 0, \ \text{otherwise}.
\end{aligned}$$

Then

(a) $\{\underline{C}_i\}$ *is stationary and ergodic.*

(b) the process, $\{Y_i^*(\theta), i \in \mathcal{Z}\}\}$, *defined by*

$$Y_i^* = \Psi_1((Z_i, \delta_i), (Z_{i-1}, \delta_{i-1}), \ldots),$$

is stationary and ergodic.

Proof of Lemma 4.2.1 (a)

The proof makes use of the ideas of Propositions 6.6 and 6.11 of Breiman (1968) (see Propositions A.2.1 and A.2.2 for details). Let φ_1 and φ_2 be two functions such that each is $(\Re^2)^\infty \longrightarrow \Re$ measurable and

$$\varphi_1((x_i, y_i), (x_{i+1}, y_{i+1}), \ldots) = \min(x_i, y_i),$$

$$\begin{aligned} \varphi_2((x_i, y_i), (x_{i+1}, y_{i+1}), \ldots) &= 0 \text{ if } x_i \le y_i, \\ &= 1 \text{ if } x_i > y_i. \end{aligned}$$

Then,

$$\varphi_1((Y_i, T_i), (Y_{i+1}, T_{i+1}), \ldots) = \min(Y_i, T_i) = Z_i,$$

$$\begin{aligned} \delta_i = \varphi_2((Y_i, T_i), (Y_{i+1}, T_{i+1}), \ldots) &= 0 \text{ if } Y_i \le T_i, \\ &= 1 \text{ if } Y_i > T_i, \;\; i \in \mathcal{Z}, \end{aligned}$$

$$(\varphi_1((x_i, y_i), (x_{i+1}, y_{i+1}), \ldots), \varphi_2((x_i, y_i), (x_{i+1}, y_{i+1}), \ldots))$$

is $(\Re^2)^\infty \longrightarrow \Re^2$ measurable. Let

$$\begin{aligned} \varphi = ((&\varphi_1((x_1, y_1), (x_2, y_2), \ldots), \varphi_2((x_1, y_1), (x_2, y_2), \ldots)), \\ &(\varphi_1((x_2, y_2), (x_3, y_3), \ldots), \varphi_2((x_2, y_2), (x_3, y_3), \ldots)), \ldots). \end{aligned}$$

Then φ is $(\Re^2)^\infty \longrightarrow (\Re^2)^\infty$ measurable. Let

$$\underline{U}_i = ((Y_i, T_i), (Y_{i+1}, T_{i+1}), \ldots),$$

and note from the statement of the lemma that

$$\underline{C}_i = ((Z_i, \delta_i), (Z_{i+1}, \delta_{i+1}), \ldots).$$

Then $\varphi(\underline{U}_i) = \underline{C}_i$. Let A be an invariant set of

$$((Z_i, \delta_i), (Z_{i+1}, \delta_{i+1}), ...).$$

Then $\exists\ B \in (\mathcal{B})^\infty \ni$

$$A = \underline{C}_k^{-1}(B),\ \ \forall k \geq i.$$

Let

$$D = \varphi^{-1}(B).$$

Then D is $(\mathcal{B}^2)^\infty$ measurable by the measurability of φ. But $\underline{C}_k = \varphi(\underline{U}_k)$ implies

$$A = \underline{C}_k^{-1}(B) = \underline{U}_k^{-1}(\varphi^{-1}(B)) = \underline{U}_k^{-1}(D),\ \ \forall k \geq i.$$

This implies that every invariant set of $\underline{C}_i$ is an invariant set of $\underline{U}_i$, $i \in \mathcal{Z}$. Now, since $\underline{U}_i$, $i \in \mathcal{Z}$ is stationary and ergodic, this implies that $\underline{C}_i$, $i \in \mathcal{Z}$ is also stationary and ergodic. □

Proof of Lemma 4.2.1 (b)

Let

$$\Psi = (\Psi_1((x_1, y_1), (x_2, y_2), ...), \Psi_1((x_2, y_2), (x_3, y_3), ...), ...).$$

Then Ψ is $(\Re^2)^\infty \longrightarrow \Re^\infty$ measurable. Let

$$\underline{C}_i = ((Z_i, \delta_i), (Z_{i-1}, \delta_{i-1}), ...)$$

and

$$\underline{Y}_i^* = (Y_i^*, Y_{i-1}^*, ...).$$

Then

$$\Psi(\underline{C}_i) = \underline{Y}_i^*.$$

Let A be an invariant set of $\underline{Y}_i^*$. Then $\exists\ B \in \mathcal{B}^\infty \ni$

$$A = \underline{Y}_k^{*^{-1}}(B),\ \ \forall k \geq i.$$

Let

$$D = \Psi^{-1}(B).$$

Then D is $(\mathcal{B}^2)^\infty$ measurable by the measurability of Ψ. To proceed,

$$\underline{Y}_k^* = \Psi(\underline{C}_k)$$

implies

$$A = \underline{Y}_k^{*^{-1}}(B) = \underline{C}_k^{-1}(\Psi^{-1}(B)) = \underline{C}_k^{-1}(D), \ \forall k \leq i.$$

This implies that every invariant set of $\underline{Y}_i^*$ is an invariant set of $\underline{C}_i$. Since by lemma 4.2.1 (a), $\underline{C}_i$ is stationary and ergodic, this implies that $\underline{Y}_i^*$ is also stationary and ergodic. □

In the next lemma, we establish that if $M_n(\theta)$ is the estimating function in equation (4.1.1) then $n^{-1}M_n(\theta_o)$ converges to zero a.s. under P_{θ_o}.

Lemma 4.2.2 *Let $\{Y_i, i \in \mathcal{Z}\}$ be a stationary and ergodic AR(1) process. Let*

$$M_n(\theta) = \sum_{i=1}^{n} X_i(\theta)$$

where

$$X_i(\theta) = Y_{i-1}^*(\theta)\left(Y_i^*(\theta) - \theta Y_{i-1}^*(\theta)\right),$$

$Y_i^*(\theta) = E_\theta(Y_i|\mathcal{F}_i)$, $\mathcal{F}_i = \sigma\{(Z_j, \delta_j),\ j \leq i\}$. *Suppose* $E_{\theta_o}|\varepsilon_i|^{2p} \leq k < \infty$, *for* $1 < p \leq 2$. *Then*

$$n^{-1}M_n(\theta_o) \xrightarrow{a.s.} E_{\theta_o}\{X_1(\theta_o)\} = 0, \ \textit{under } P_{\theta_o}.$$

Proof

For convenience, let E and X_i denote E_{θ_o} and $X_i(\theta_o)$, respectively. Since $\{M_n(\theta_o), \mathcal{F}_n, n \geq 1\}$ is a martingale by lemma 3.2.1, we have by Theorem 2.18 of Hall and Heyde (1980) (see Theorem A.2.1 in the appendix for details), that for $1 < p \leq 2$,

$$\lim_{n\to\infty} n^{-1}M_n(\theta_o) = 0, \ \text{a.s., under } P_{\theta_o},$$

on the set $\{\sum_{i=1}^{\infty} i^{-p}E(|X_i|^p|\mathcal{F}_{i-1}) < \infty\}$. Therefore, we only need to show that the condition, $\{\sum_{i=1}^{\infty} i^{-p}E(|X_i|^p|\mathcal{F}_{i-1}) < \infty\}$, holds a.e. We show this next.

It suffices to show that

$$E\left(\sum_{i=1}^{\infty} i^{-p} E\left(|X_i|^p|\mathcal{F}_{i-1}\right)\right) = \sum_{i=1}^{\infty} i^{-p} E|X_i|^p < \infty.$$

Now,

$$\begin{aligned} E|X_i|^p &= E|Y_{i-1}^*(Y_{i-1}^* - \theta_o Y_{i-1}^*)|^p \\ &= E\left|Y_i^* Y_{i-1}^* - E(Y_i^* Y_{i-1}^*|\mathcal{F}_{i-1})\right|^p \\ &\leq \left(E^{\frac{1}{p}}|Y_i^* Y_{i-1}^*|^p + E^{\frac{1}{p}}|E(Y_i^* Y_{i-1}^*|\mathcal{F}_{i-1})|^p\right)^p, \end{aligned}$$

by Minkowski's inequality. But

$$E\left|Y_i^* Y_{i-1}^*\right|^p \leq E^{\frac{1}{2}}\left|Y_i^*\right|^{2p} E^{\frac{1}{2}}\left|Y_{i-1}^*\right|^{2p},$$

by the Cauchy-Schwartz inequality. Furthermore,

$$E\left|Y_i^*\right|^{2p} = E\left|E(Y_i|\mathcal{F}_i)\right|^{2p} \leq E\left(E\left(|Y_i|^{2p}|\mathcal{F}_i\right)\right) = E|Y_i|^{2p} \leq k,$$

by Jensen's inequality and the condition of the lemma. Therefore,

$$E\left|Y_i^* Y_{i-1}^*\right|^p \leq k.$$

Similarly,

$$E\left|E\left(Y_i^* Y_{i-1}^*|\mathcal{F}_{i-1}\right)\right|^p \leq E\left(E\left(\left|Y_i^* Y_{i-1}^*\right|^p|\mathcal{F}_{i-1}\right)\right) = E\left|Y_i^* Y_{i-1}^*\right|^p \leq k,$$

by Jensen's inequality and the result above. This gives

$$E\left|X_i\right|^p \leq \left(k^{\frac{1}{p}} + k^{\frac{1}{p}}\right)^p = 2^p k < \infty.$$

Hence,

$$\sum_{i=1}^{\infty} i^{-p} E\left|X_i\right|^p \leq \sum_{i=1}^{\infty} i^{-p} 2^p k = 2^p k \sum_{i=1}^{\infty} i^{-p} < \infty, \text{ for } p > 1. \quad \square$$

Alternative proof of Lemma 4.2.2

Note that by Jensen's inequality,

$$E|Y_o^*|^2 = E|E(Y_o|\mathcal{F}_o)|^2 \leq E\{E(|Y_o|^2|\mathcal{F}_o)\} = E|Y_o|^2,$$

and by Minkowski's inequality,

$$\begin{aligned} E^{\frac{1}{2}}|(Y_1^* - \theta_o Y_o^*)|^2 &\leq E^{\frac{1}{2}}|Y_1^*|^2 + E^{\frac{1}{2}}\,|E(Y_1^*|\mathcal{F}_l)|^2 \\ &\leq \; E^{\frac{1}{2}}|Y_1^*|^2 + E^{\frac{1}{2}}\left\{E\left(|Y_1^*|^2|\mathcal{F}_o\right)\right\} \\ &= \; E^{\frac{1}{2}}|Y_1^*|^2 + E^{\frac{1}{2}}|Y_1^*|^2 \leq 2E^{\frac{1}{2}}|Y_1|^2 = 2\frac{\sigma}{\sqrt{1-\theta_o^2}}. \end{aligned}$$

Therefore, by the Cauchy-Schwartz inequality,

$$\begin{aligned} E|X_1| &= E|Y_o^*(Y_1^* - \theta_o Y_o^*)| \\ &\leq \; E^{\frac{1}{2}}|Y_o^*|^2 E^{\frac{1}{2}}|(Y_1^* - \theta_o Y_o^*)|^2 \leq 2\frac{\sigma^2}{1-\theta_o^2} < \infty. \end{aligned}$$

Using the preceding result, $E|X_1| < \infty$, the lemma follows by the ergodic theorem (see Theorem A.2.3 and Corollary A.2.3 in the appendix for details) since $\{Y_i^*(\theta_o), i \in \mathcal{Z}\}\}$ is stationary and ergodic by Lemma 4.2.1 (a). □

Since $\{M_n(\theta),\, \mathcal{F}_n\}$ is a martingale, Theorem 3.2 of Hall and Heyde (1980) (a CLT for martingales, see Theorem A.2.2 in the appendix for details), provides general conditions under which the asymptotic normality of $n^{-\frac{1}{2}}M_n(\theta_o)$ holds. A corollary to the theorem (Corollary 3.1 of Hall and Heyde 1980, see Corollary A.2.1 in the appendix for details), replaces the conditions by a conditional Lindeberg condition and a condition on the conditional variance. Hence, by the Corollary, if the conditional Lindeberg condition,

$$\forall \varepsilon > 0, \quad n^{-1}\sum_{i=1}^{n} E\left[X_i^2 I(|X_i| > \varepsilon\sqrt{n})|\mathcal{F}_{i-1}\right] \xrightarrow{P} 0, \tag{4.1}$$

and a condition on the conditional variance,

$$n^{-1}\sum_{i=1}^{n} E(X_i^2|\mathcal{F}_{i-1}) \xrightarrow{P} E(X_1^2), \tag{4.2}$$

are both satisfied, then the asymptotic normality of $n^{-\frac{1}{2}}M_n(\theta_o)$ follows. In the following proposition, an equivalent condition to the condition (4.2.1) is given. This equivalent condition will be proved rather than (4.2.1).

Proposition 4.2.1 *Let $\{Y_i, i \in \mathcal{Z}\}$ be a stationary AR(1) process. Let*

$$X_i(\theta) = Y_{i-1}^*(\theta)\left(Y_i^*(\theta) - \theta Y_{i-1}^*(\theta)\right),$$

where $Y_i^*(\theta) = E_\theta(Y_i|\mathcal{F}_i)$, $\mathcal{F}_i = \sigma\{(Z_j,\delta_j),\ j \le i\}$. *Then the conditional Lindeberg condition*

$$\forall\varepsilon > 0,\quad n^{-1}\sum_{i=1}^{n} E\left[X_i^2 I(|X_i| > \varepsilon\sqrt{n})|\mathcal{F}_{i-1}\right] \xrightarrow{P} 0,$$

is equivalent to the condition

$$\forall\varepsilon > 0,\quad n^{-1}\sum_{i=1}^{n} E\left[X_i^2 I(|X_i| > \varepsilon\sqrt{n})\right] \longrightarrow 0. \tag{4.3}$$

Proof

Note that if

$$E\left(n^{-1}\sum_{i=1}^{n} E\left[X_i^2 I(|X_i| > \varepsilon\sqrt{n})|\mathcal{F}_{)-\infty}\right]\right)$$
$$= n^{-1}\sum_{i=1}^{n} E\left[X_i^2 I(|X_i| > \varepsilon\sqrt{n})\right] \longrightarrow 0,$$

Then

$$n^{-1}\sum_{i=1}^{n} E\left[X_i^2 I(|X_i| > \varepsilon\sqrt{n})|\mathcal{F}_{i-1}\right] \xrightarrow{L^1} 0,$$

which implies

$$n^{-1}\sum_{i=1}^{n} E\left[X_i^2 I(|X_i| > \varepsilon\sqrt{n})|\mathcal{F}_{i-1}\right] \xrightarrow{P} 0.\ \Box$$

Before we give an equivalent condition to (4.2.2), which we shall prove instead of (4.2.2), we first state and prove a lemma that will be used in the proof of the equivalence of the two conditions.

Lemma 4.2.3 *Let* $\{Y_i, i \in \mathcal{Z}\}$ *be a stationary and ergodic AR(1) process. Let*

$$X_i(\theta) = Y_{i-1}^*(\theta)\left(Y_i^*(\theta) - \theta Y_{i-1}^*(\theta)\right),$$

where $Y_i^*(\theta) = E_\theta(Y_i|\mathcal{F}_i)$, $\mathcal{F}_i = \sigma\{(Z_j,\delta_j),\ j \le i\}$. *Suppose* $E|\varepsilon_i|^{4(1+\gamma)} \le k < \infty$, *for some* $\gamma > 0$, k *constant. Then* $E\left|X_i^2\right|^{(1+\gamma)} \le 2^{2(1+\gamma)}k < \infty$.

Proof

For convenience, denote $Y_i^*(\theta_o)$ by Y_i^*. Then

$$E\left|X_i^2\right|^{(1+\gamma)} = E\left|Y_{i-1}^*\left(Y_i^* - E(Y_i^*|\mathcal{F}_{i-1})\right)\right|^{2(1+\gamma)}$$
$$= E\left|Y_i^*Y_{i-1}^* - E(Y_i^*Y_{i-1}^*|\mathcal{F}_{i-1})\right|^{2(1+\gamma)}$$
$$\le \left\{E^{\frac{1}{2(1+\gamma)}}|Y_i^*Y_{i-1}^*|^{2(1+\gamma)} + E^{\frac{1}{2(1+\gamma)}}\left|E(Y_i^*Y_{i-1}^*|\mathcal{F}_{i-1})\right|^{2(1+\gamma)}\right\}^{2(1+\gamma)},$$

by Minkowski's inequality. But

$$\begin{aligned} E\left|Y_i^* Y_{i-1}^*\right|^{2(1+\gamma)} &= E\left|Y_i^{*2(1+\gamma)} Y_{i-1}^{*2(1+\gamma)}\right| \\ &\leq E^{\frac{1}{2}}\left|Y_i^*\right|^{4(1+\gamma)} E^{\frac{1}{2}}\left|Y_{i-1}^*\right|^{4(1+\gamma)}, \end{aligned}$$

by the Cauchy-Schwartz inequality. Further,

$$\left|Y_i^*\right|^{4(1+\gamma)} = \left|E(Y_i|\mathcal{F}_i)\right|^{4(1+\gamma)} \leq E\left(\left|Y_i\right|^{4(1+\gamma)}|\mathcal{F}_i\right),$$

by Jensen's inequality. This implies

$$E\left|Y_i^*\right|^{4(1+\gamma)} \leq E\left|Y_i\right|^{4(1+\gamma)} \leq k,$$

by the condition of the lemma. This in turn implies that

$$E\left|Y_i^* Y_{i-1}^*\right|^{2(1+\gamma)} \leq k.$$

Similarly,

$$\left|E\left(Y_i^* Y_{i-1}^*|\mathcal{F}_{i-1}\right)\right|^{2(1+\gamma)} \leq E\left(\left|Y_i^* Y_{i-1}^*\right|^{2(1+\gamma)}|\mathcal{F}_{i-1}\right),$$

by Jensen's inequality. This gives

$$E\left|E\left(Y_i^* Y_{i-1}^*|\mathcal{F}_{i-1}\right)\right|^{2(1+\gamma)} \leq E\left|Y_i^* Y_{i-1}^*\right|^{2(1+\gamma)} \leq k.$$

Therefore,

$$\begin{aligned} E\left|X_i^2\right|^{(1+\gamma)} &\leq \left\{k^{\frac{1}{2(1+\gamma)}} + k^{\frac{1}{2(1+\gamma)}}\right\}^{2(1+\gamma)} \\ &= 2^{2(1+\gamma)}k. \quad \square \end{aligned}$$

Next, we give the equivalent condition to (4.2.2).

Proposition 4.2.2 *Let $\{Y_i, i \in \mathcal{Z}\}$ be a stationary and ergodic AR(1) process. Let*

$$X_i(\theta) = Y_{i-1}^*(\theta)\left(Y_i^*(\theta) - \theta Y_{i-1}^*(\theta)\right),$$

where $Y_i^*(\theta) = E_\theta(Y_i|\mathcal{F}_i)$, $\mathcal{F}_i = \sigma\{(Z_j, \delta_j),\ j \leq i\}$. Suppose $E|\varepsilon_i|^{4(1+\gamma)} \leq k$, for some $\gamma > 0$, k constant. Then the condition on the conditional variance,

$$n^{-1}\sum_{i=1}^{n} E(X_i^2|\mathcal{F}_{i-1}) \xrightarrow{P} E(X_1^2),$$

is equivalent to the condition

$$n^{-1}\sum_{i=1}^{n} X_i^2 \xrightarrow{P} E\left(X_1^2\right). \tag{4.4}$$

Proof

We need only to prove that

$$n^{-1}\sum_{i=1}^{n}\left(X_i^2 - E(X_i^2|\mathcal{F}_{i-1})\right) \xrightarrow{P} 0.$$

Note that

$$\left\{\sum_{i=1}^{n}\left(X_i^2 - E(X_i^2|\mathcal{F}_{i-1})\right), \mathcal{F}_n, n \geq 1\right\}$$

is a martingale. Therefore by Theorem 2.18 of Hall and Heyde (1980) (see section A 2 of the appendix),

$$n^{-1}\sum_{i=1}^{n}\left(X_i^2 - E(X_i^2|\mathcal{F}_{i-1})\right) \xrightarrow{\text{a.s.}} 0, \text{ under } P_{\theta_o}$$

on the set

$$\sum_{i=1}^{\infty} i^{-(1+\gamma)} E\left(\left|X_i^2 - E(X_i^2|\mathcal{F}_{i-1})\right|^{1+\gamma} |\mathcal{F}_{i-1}\right) < \infty.$$

Now,

$$\begin{aligned} &E\left|X_i^2 - E(X_i^2|\mathcal{F}_{i-1})\right|^{1+\gamma} \\ &\quad \leq \left\{E^{\frac{1}{1+\gamma}}\left|X_i^2\right|^{1+\gamma} E^{\frac{1}{1+\gamma}}\left|E(X_i^2|\mathcal{F}_{i-1})\right|^{1+\gamma}\right\}^{1+\gamma}, \end{aligned}$$

by Minkowski's inequality. But

$$E\left|X_i^2\right|^{(1+\gamma)} \leq 2^{2(1+\gamma)}k,$$

by lemma 4.2.3. Similarly,

$$E\left|E(X_i^2|\mathcal{F}_{i-1})\right|^{1+\gamma} \leq E\left(E(\left|X_i^2\right|^{1+\gamma}|\mathcal{F}_{i-1})\right) = E\left|X_i^2\right|^{(1+\gamma)} \leq 2^{2(1+\gamma)}k,$$

by Jensen's inequality and lemma 4.2.3. Therefore,

$$\begin{aligned} &E\left|X_i^2 - E(X_i^2|\mathcal{F}_{i-1})\right|^{1+\gamma} \\ &\quad \leq \left\{\left(2^{2(1+\gamma)}k\right)^{\frac{1}{1+\gamma}} + \left(2^{2(1+\gamma)}k\right)^{\frac{1}{1+\gamma}}\right\}^{1+\gamma} \\ &\quad = \left\{2\left(2^{2(1+\gamma)}k\right)^{\frac{1}{1+\gamma}}\right\}^{1+\gamma} = 2^{1+\gamma}2^{2(1+\alpha)}k = 2^{3(1+\gamma)}k. \end{aligned}$$

Hence,

$$\begin{aligned} &\sum_{i=1}^{\infty} i^{-(1+\gamma)} E\left(\left|X_i^2 - E(X_i^2|\mathcal{F}_{i-1})\right|^{1+\gamma} |\mathcal{F}_{i-1}\right) \\ &\quad \leq 2^{3(1+\gamma)}k\sum_{i=1}^{\infty} i^{-(1+\gamma)} < \infty. \ \square \end{aligned}$$

The following proposition establishes that the conditions (4.2.3) and (4.2.4) hold for the martingale difference sequence $\{X_n, n \geq 1\}$ given by equation (4.1.2).

Proposition 4.2.3 *Let $\{Y_i, i \in \mathcal{Z}\}$ be a stationary and ergodic AR(1) process. Let*

$$X_i(\theta) = Y^*_{i-1}(\theta)\left(Y^*_i(\theta) - \theta Y^*_{i-1}(\theta)\right),$$

where $Y^*_i(\theta) = E_\theta(Y_i|\mathcal{F}_i)$, $\mathcal{F}_i = \sigma\{(Z_j, \delta_j),\ j \leq i\}$. Suppose $E|\varepsilon_i|^{4(1+\gamma)} \leq k < \infty$, for some $\gamma > 0$, k constant. Then

(a) $\forall \varepsilon > 0,\quad n^{-1}\sum_{i=1}^n E\left[X_i^2 I(|X_i| > \varepsilon\sqrt{n})\right] \longrightarrow 0.$

(b) $n^{-1}\sum_{i=1}^n X_i^2 \xrightarrow{P} E\left(X_i^2\right).$

Proof of Proposition 4.2.3 (a)

$$\begin{aligned}
&E\left(X_i^2 I\left(|X_i| < \varepsilon\sqrt{n}\right)\right) \\
&\quad\leq E^{\frac{1}{1+\gamma}}|X_i|^{2(1+\gamma)} E^{\frac{\gamma}{1+\gamma}}\left(I\left(|X_i| > \varepsilon\sqrt{n}\right)\right) \\
&\quad\leq 4k^{\frac{1}{1+\gamma}}\left(\max_i P\left(|X_i| > \varepsilon\sqrt{n}\right)\right),
\end{aligned}$$

by Holder's inequality and lemma 4.2.3. But

$$\max_i P\left(|X_i| > \varepsilon\sqrt{n}\right) \longrightarrow 0 \text{ as } n \to \infty,$$

since

$$\max_i P\left(|X_i| > \varepsilon\sqrt{n}\right) = o\left(n^{-2(1+\gamma)}\right)$$

by the corollary in section 1.14 of Serfling (1980), p.47 (see Corollary A.2.2 in the appendix for details), since $E|X_i^2| < \infty$ by lemma 4.2.3. Therefore,

$$n^{-1}\sum_{i=1}^n E\left(X_i^2 I\left(|X_i| > \varepsilon\sqrt{n}\right)\right) \xrightarrow{P} 0. \ \square$$

Proof of Proposition 4.2.3 (b)

Note that the process, $\{\underline{Y}_i^*, i \in \mathcal{Z}\}$ and hence $\{X_i, i \in \mathcal{Z}\}$ is stationary and ergodic by lemma 4.2.1. Also by lemma 4.2.3, $E(X_i^2) = E(X_1^2) < \infty$. Therefore the condition holds by application of the ergodic theorem (see Theorem A.2.3 and Corollary 4.2.3 in the appendix). □

Theorem 4.2.1 *Let $\{Y_i, i \in \mathcal{Z}\}$ be a stationary and ergodic AR(1) process. Let*

$$M_n(\theta) = \sum_{i=1}^{n} X_i(\theta),$$

where,

$$X_i(\theta) = Y_{i-1}^*(\theta)\left(Y_i^*(\theta) - \theta Y_{i-1}^*(\theta)\right),$$

$Y_i^*(\theta) = E_\theta(Y_i|\mathcal{F}_i)$, $\mathcal{F}_i = \sigma\{(Z_j, \delta_j), j \le i\}$. Suppose $E_{\theta_o}|\varepsilon_i|^{4(1+\gamma)} \le k$, $k \le \infty$, for some $\gamma > 0$, k constant . Let $\hat{\theta}_n^a$ be a consistent solution of the estimating equation $M_n(\theta) = 0$. Then

(a) $n^{-\frac{1}{2}} M_n(\theta_o)$ is asymptotically normal, i.e.,

$$n^{-\frac{1}{2}} M_n(\theta_o) \xrightarrow{\mathcal{D}} N(0, E_{\theta_o}(X_1^2(\theta_o))), \;\; \textit{under } P_{\theta_o},$$

(b) The asymptotic distribution of $\hat{\theta}_n^a$ is the normal distribution, i.e.,

$$n^{\frac{1}{2}}(\hat{\theta}_n^a - \theta_o) \xrightarrow{\mathcal{D}} N(0, \frac{E(X_1^2)}{E^2(X_1^{(1)})}), \;\; \textit{a.s. under } P_{\theta_o},$$

where X_i is an abbreviation for $X_i(\theta_o)$, $X_i^{(1)}$ for $X_i^{(1)}(\theta_o)$ and E for E_{θ_o}.

Proof of Theorem 4.2.1 (a)

Since $\{M_n(\theta), \mathcal{F}_n, n \ge 1\}$ is a martingale by Lemma 3.2.1, the result follows by application of Corollary 3.1 of Hall and Heyde (1980) (Corollary A.2.1 in the appendix for details). As noted earlier, this requires the verification of the conditional Lindeberg condition (4.2.1) and the condition (4.2.2) on the conditional variance, which are equivalent to (4.2.3) and (4.2.4), respectively. The latter conditions are verified in Proposition 4.2.3 under the condition of the Theorem, $E_{\theta_o}|\varepsilon_i|^{4(1+\gamma)} \leq k$, for some $\gamma > 0$ and k constant. Hence, the result. □

Proof of Theorem 4.2.1 (b)

We obtain the Taylor expansion of $M_n(\hat{\theta}_n^a)$ about θ_o as follows.

$$\begin{aligned}0 = n^{-\frac{1}{2}} M_n(\hat{\theta}_n^a) &= n^{-\frac{1}{2}} M_n(\theta_o) + n^{\frac{1}{2}}(\hat{\theta}_n^a - \theta_o) n^{-1} \sum_{i=1}^{n} X_i^{(1)}(\theta_n^*) \\ &- n^{\frac{1}{2}}(\hat{\theta}_n^a - \theta_o) E\left(X_1^{(1)}(\theta_o)\right) + n^{\frac{1}{2}}(\hat{\theta}_n^a - \theta_o) E\left(X_1^{(1)}(\theta_o)\right),\end{aligned}$$

where, $\theta_o < \theta_n^* < \hat{\theta}_n^a$. Then

$$\begin{aligned}&-n^{-\frac{1}{2}} M_n(\theta_o) \\ &\quad = n^{\frac{1}{2}}(\hat{\theta}_n^a - \theta_o)\left\{n^{-1}\sum_{i=1}^{n}\left[X_i^{(1)}(\theta_n^*) - E\left(X_1^{(1)}(\theta_o)\right)\right] + E\left(X_1^{(1)}(\theta_o)\right)\right\}\end{aligned}$$

or

$$\begin{aligned}&-n^{-\frac{1}{2}} M_n(\theta_o) \\ &\quad = n^{\frac{1}{2}}(\hat{\theta}_n^a - \theta_o)\left\{\frac{n^{-1}\sum_{i=1}^{n}\left[X_i'(\theta_n^*) - E\left(X_1^{(1)}(\theta_o)\right)\right]}{E\left(X_1^{(1)}(\theta_o)\right)} + 1\right\} E\left(X_1^{(1)}(\theta_o)\right).\end{aligned}$$

Letting

$$R_n = \frac{n^{-1}\sum_{i=1}^{n}\left[X_i^{(1)}(\theta_n^*) - E\left(X_1^{(1)}(\theta_o)\right)\right]}{E\left(X_1^{(1)}(\theta_o)\right)},$$

we have

$$n^{\frac{1}{2}}(\hat{\theta}_n^a - \theta_o) = \frac{-n^{-\frac{1}{2}} M_n(\theta_o)}{(R_n + 1)\, E\left(X_1^{(1)}(\theta_o)\right)}.$$

If $\hat{\theta}_n^a$ is consistent, then for large n, $n^{-1}\sum_{i=1}^{n} X_i^{(1)}(\theta_n^*)$ behaves like $n^{-1}\sum_{i=1}^{n} X_i^{(1)}(\theta_o)$ since $\theta_o < \theta_n^* < \hat{\theta}_n^a$ and $\hat{\theta}_n^a$ tends to θ_o in probability or a.s. under P_{θ_o}. But

$$n^{-1}\sum_{i=1}^{n} X_i^{(1)}(\theta_o) \longrightarrow E\left(X_1^{(1)}(\theta_o)\right), \text{ a.s. under } P_{\theta_o},$$

by the ergodic theorem. Therefore,

$$R_n \xrightarrow{P} 0, \text{ under } P_{\theta_o}.$$

Hence, by the asymptotic normality of $n^{-\frac{1}{2}}M_n(\theta_o)$ (Theorem 4.2.1 (a)) and the convergence of $n^{-1}\sum_{i=1}^{n} X_i^{(1)}(\theta_o)$ to a constant as indicated above, the result is established. □

4.2.2 Consistency

In this section, we investigate conditions under which the estimator $\hat{\theta}_n^a$, which is the solution of the estimating equation $M_n(\theta) = 0$, is consistent. Here $M_n(\theta)$ is given by equation (4.1.1). Hutton, Ogunyemi and Nelson (1991) have used a condition (see section A 2 of the Appendix) for the a.s. existence of a consistent solution of an estimating equation similar to the one discussed in this section. This condition is based on the Brouwer fixed-point theorem and was proved by Aitchson and Silvey (1958). In the context of the estimating function $M_n(\theta)$ with θ being *one*-dimensional, the condition can be written as follows. If $M_n(\theta)$ is continuous a.s. in θ and if for all sufficiently small $\delta > 0$,

$$\begin{aligned}\limsup_{n\to\infty} M_n(\theta) &< 0, \text{ for } \theta > \theta_o, \quad \theta_o - \delta < \theta < \theta_o + \delta, \\ \liminf_{n\to\infty} M_n(\theta) &> 0, \text{ for } \theta < \theta_o, \quad \theta_o - \delta < \theta < \theta_o + \delta, \end{aligned} \tag{4.5}$$

P_{θ_o}, a.s., then $M_n(\theta) = 0$ has a consistent root, $\hat{\theta}_n^a$, in $\{\theta : |\theta - \theta_o| \leq \delta\}$.

Since the process, $\{Y_i^*(\theta), i \in \mathcal{Z}\}$ is a stationary and ergodic process, we have, by the ergodic theorem,

$$\lim_{n\to\infty} n^{-1}M_n(\theta) = E_{\theta_o}(X_1(\theta)), \tag{4.6}$$

P_{θ_o}, a.s. Therefore, the condition (4.2.5) translates to

$$\begin{aligned}E_{\theta_o}(X_1(\theta)) &< 0 \text{ for } \theta > \theta_o, \\ E_{\theta_o}(X_1(\theta)) &> 0 \text{ for } \theta < \theta_o. \end{aligned} \tag{4.7}$$

Let $\psi(\theta) = E_{\theta_o}(X_1(\theta))$. Then by Taylor's expansion of $\psi(\theta_o)$ about θ_o, we have

$$\psi(\theta) = \psi(\theta_o) + (\theta - \theta_o)\psi^{(1)}(\theta^*), \tag{4.8}$$

where $\theta^* \in \{\theta : |\theta - \theta_o| < \delta\}$. Since $X_1(\theta_o)$ is a P_{θ_o} martingale difference, $\psi(\theta_o) = 0$. Hence, by (4.2.8) the condition in (4.2.7) translates to

$$\psi^{(1)}(\theta^*) < 0, \tag{4.9}$$

and since we are interested in the behaviour of $\psi(\theta)$ in a small neighbourhood of θ_o,

$$\psi^{(1)}(\theta_o) < 0, \tag{4.10}$$

is a sufficient condition for the consistency of $\hat{\theta}_n^a$.

Note that

$$\begin{aligned}\psi^{(1)}(\theta) &= E_{\theta_o}\{X_1^{(1)}(\theta)\} \\ &= E_{\theta_o}\{-Y_0^{*^2}(\theta) + Y_0^*(\theta)[Y_1^{*^{(1)}}(\theta) - \theta Y_0^{*^{(1)}}(\theta)] \\ &\quad + Y_0^{*^{(1)}}(\theta)[Y_1^*(\theta) - \theta Y_0^*(\theta)]\}.\end{aligned} \tag{4.11}$$

If we let $E = E_{\theta_0}$, $Y_i^* = Y_i^*(\theta_0)$ and $Y_i^{*^{(1)}} = Y_i^{*^{(1)}}(\theta_o)$, then

$$\begin{aligned}\psi^{(1)}(\theta_o) &= E\{-Y_0^{*^2}\} + E\{Y_0^*[Y_1^{*^{(1)}} - \theta_o Y_0^{*^{(1)}}]\} + E\{Y_0^{*^{(1)}}[Y_1^* - \theta_o Y_0^*]\} \\ &= E\{-Y_0^{*^2}\} + E\{Y_0^* Y_1^{*^{(1)}}\} - E\{E[Y_0^{*^{(1)}} Y_1^* | \mathcal{F}_0]\} \\ &\quad + E\{Y_0^{*^{(1)}}[E(Y_1^*|\mathcal{F}_0) - \theta_o Y_0^*]\} \\ &= E\{-Y_0^{*^2} + Y_0^* Y_1^{*^{(1)}} - Y_0^{*^{(1)}} Y_1^*\}.\end{aligned} \tag{4.12}$$

Using the preceding simplification, we consider two approaches in verifying the sufficient condition in (4.2.10). The first approach is outlined below.

We examine the distributions of the random variables, $Y_0^* Y_1^{*^{(1)}}$ and $Y_0^{*^{(1)}} Y_1^*$. If these random variables have identical distributions, then the condition is satisfied since $E[Y_0^* Y_1^{*^{(1)}} - Y_0^{*^{(1)}} Y_1^*] = 0$ and $\psi^{(1)}(\theta_o) = -E(Y_0^{*^2}) < 0$, as desired.

A property that holds for some stationary process that could be utilised in this approach is time reversibility and this is defined next.

Definition 4.2.1 *(Weiss, 1975) A stationary process, $\{X_i, i \in \mathcal{Z}\}$, is time reversible if $(X_{i_1}, ..., X_{i_n}) \sim (X_{-i_1}, ..., X_{-i_n})$, for every n.*

All stationary Gaussian processes are time reversible (see, e.g., Weiss, 1975; Rao, Johnson and Becker, 1992 or Cambanis and Fakhre-Zakeri, 1996). Therefore, if the AR process, $\{Y_i, i \in \mathcal{Z}\}$, is Gaussian, then it is also time reversible. If the process, $\{Y_i, i \in \mathcal{Z}\}$, is time reversible, then

$$E(Y_i|Y_{i-1}, Y_{i-2}, ...) = E(Y_i|Y_{i+1}, Y_{i+2}, ...). \tag{4.13}$$

A stationary process for which (4.2.13) holds may not be time reversible. First-order time reversibility, $(Y_{i-1}, Y_i) \sim (Y_i, Y_{i-1})$, implies

$$E(Y_2|Y_1) = E(Y_1|Y_2). \tag{4.14}$$

For our process, $\{Y_i^*, i \in \mathcal{Z}\}$,

$$\begin{aligned} E(Y_1^*|Y_0^*) &= E\{E[Y_1^*|Y_0^*]|\mathcal{F}_0\} \\ &= E\{E[Y_1^*|\mathcal{F}_0]|Y_0^*\} \\ &= E\{\theta_o Y_0^*|Y_0^*\} \\ &= \theta_o Y_0^*, \end{aligned} \tag{4.15}$$

since, $\sigma\{Y_0^*\} \subseteq \mathcal{F}_0$. It is not obvious whether first-order time reversibility holds for the process, $\{Y_i^*, i \in \mathcal{Z}\}$. If it holds, then

$$E\{Y_0^*|Y_1^*\} = \theta_o Y_1^*, \tag{4.16}$$

and

$$\begin{aligned} \psi'(\theta_o) &= -E\{Y_0^{*^2}\} + E\{E[Y_0^* Y_1^{*^{(1)}}|Y_1^*]\} + E\{E[Y_0^{*^{(1)}} Y_1^*|Y_0^*]\} \\ &= -E\{Y_0^{*^2}\} + E\{Y_1^{*^{(1)}} E[Y_0^*|Y_1^*]\} + E\{Y_0^{*^{(1)}} E[Y_1^*|Y_0^*]\} \\ &= -E\{Y_0^{*^2}\} + E\{\theta_o Y_1^{*^{(1)}} Y_1^*\} + E\{\theta_o Y_0^{*^{(1)}} Y_0^*\} \\ &= -E\{Y_0^{*^2}\} < 0, \end{aligned} \tag{4.17}$$

by stationarity, verifying the sufficient condition for consistency of $\hat{\theta}_n^a$. However, as noted earlier, the sufficient condition in (4.2.16) could still hold even if the process is not first-order time reversible.

The second approach in verifying the sufficient condition in (4.2.10) is outlined in the sequel. To condition on the entire past, we need only to condition back to the last uncensored observation, by the Markov property. Hence,

$$Y_0^*(\theta) = \sum_{K=0}^{\infty} p_{0,K}\, \varphi_{0,K}(T_0, T_{-1}, ..., T_{-K+1}, Y_{-K}; \theta), \tag{4.18}$$

where, K is the number of steps to go back to find an uncensored observation, $p_{0,0} = \delta_0$, $\varphi_{0,0} = Y_0$, and $\forall K > 0$,

$$p_{0,K} = (1-\delta_0)(1-\delta_{-1})...(1-\delta_{-K+1})\delta_{-K}, \tag{4.19}$$

and

$$\begin{aligned}&\varphi_{0,K}(t_0, ..., t_{-K+1}, s_{-K}; \theta)\\ &= E_\theta(Y_0 | Y_0 > t_0, Y_{-1} > t_{-1}, ..., Y_{-K+1} > t_{-K+1}, Y_{-K} = s_{-K}).\end{aligned} \tag{4.20}$$

Consider the special case where the ε_i's are normal with unit variance and let

$$\begin{aligned}&N_{0,K}(t_0, ..., t_{-K+1}, s_{-K}; \theta)\\ &= \int_{t_0}^{\infty} ... \int_{t_{-K+1}}^{\infty} s_0 \prod_{l=1}^{K} \phi(s_{-K+l} - \theta s_{-K+l-1}) \prod_{l=1}^{K} ds_{-K+l}\end{aligned} \tag{4.21}$$

and

$$\begin{aligned}&D_{0,K}(t_0, ..., t_{-K+1}, s_{-K}; \theta)\\ &= \int_{t_0}^{\infty} ... \int_{t_{-K+1}}^{\infty} \prod_{l=1}^{K} \phi(s_{-K+l} - \theta s_{-K+l-1}) \prod_{l=1}^{K} ds_{-K+l},\end{aligned} \tag{4.22}$$

where, ϕ is the univariate standard normal density. Then,

$$\varphi_{0,K}(t_0, ..., t_{-K+1}, s_{-K}; \theta) = \frac{N_{0,K}(t_0, ..., t_{-K+1}, s_{-K}; \theta)}{D_{0,K}(t_0, ..., t_{-K+1}, s_{-K}; \theta)}. \tag{4.23}$$

Further, let

$$W = -Y_0^{*^2} + Y_0^* Y_1^{*(1)} - Y_0^{*(1)} Y_1^*, \tag{4.24}$$

then,

$$\begin{aligned}\psi^{(1)}(\theta_o) &= E(W) = \sum_{l=0}^{\infty} E(W|K=l)P(K=l)\\ &= \sum_{l=0}^{\infty}(-\varphi_{0,l-1}^2 + \varphi_{0,l-1}\varphi_{1,l}^{(1)} - \varphi_{0,l-1}^{(1)}\varphi_{1,l})P(K=l).\end{aligned} \tag{4.25}$$

Here,

$$\varphi_{0,k}^{(1)} = \frac{D_{0,k}N_{0,k}^{(1)} - D_{0,k}^{(1)}N_{0,k}}{D_{0,k}^2}, \ \forall k > 0, \tag{4.26}$$

where,

$$\begin{aligned}N_{0,k}^{(1)}(t_0, ..., t_{-k+1}, s_{-k}; \theta_o) &= \int_{t_0}^{\infty} ... \int_{t_{-k+1}}^{\infty} \sum_{l=1}^{k}\{s_0 s_{-k+l-1}(s_{-k+l} - \theta_o s_{-k+l-1})\\ &\prod_{l=1}^{k} \phi(s_{-k+l} - \theta_o s_{-k+l-1}) \prod_{l=1}^{k} ds_{-k+l}\},\end{aligned} \tag{4.27}$$

and

$$D^{(1)}_{0,k}(t_0,...,t_{-k+1},s_{-k};\theta_o) = \int_{t_0}^{\infty}...\int_{t_{-k+1}}^{\infty}\sum_{l=1}^{k}\{s_{-k+l-1}(s_{-k+l}-\theta_o s_{-k+l-1})$$

$$\prod_{l=1}^{k}\phi(s_{-k+l}-\theta_o s_{-k+l-1})\prod_{l=1}^{k}ds_{-k+l}\}, \tag{4.28}$$

are obtained by differentiating inside the integral. Using the preceding notation, we have

$$\begin{aligned} &E(W|K=1) \\ &= Y_0^2\{(t_1-\theta_o Y_0)\frac{\phi(t_1-\theta_o Y_0)}{\bar\Phi_1(t_1-\theta_o Y_0)} - (\frac{\phi_1(t_1-\theta_o Y_0)}{\bar\Phi_1(t_1-\theta_o Y_0)})^2\}, \end{aligned} \tag{4.29}$$

where, $\bar\Phi = 1-\Phi$, Φ being the standard univariate normal distribution function. Note that

$$x\frac{\phi(x)}{\bar\Phi(x)} - (\frac{\phi(x)}{\bar\Phi(x)})^2 < 0, \ \ \forall x. \tag{4.30}$$

To see this, note that since $\bar\Phi(x) > 0 \ \forall x$, then showing the above is equivalent to showing that

$$x\phi(x)\frac{\bar\Phi(x)}{\bar\Phi^2(x)} - \frac{\phi^2(x)}{\bar\Phi^2(x)} < 0,$$

or that

$$x\phi(x)\bar\Phi(x) - \phi^2(x) > 0,$$

or that

$$x\bar\Phi(x) - \phi(x) > 0.$$

In order to show that (4.2.33) holds $\forall x$, let

$$g(x) = x\bar\Phi(x) - \phi(x).$$

Then

$$g^{(1)}(x) = -x\phi(x) + \bar\Phi(x) + x\phi(x) = \bar\Phi(x) > 0$$

But

$$\lim_{x\to\infty} g(x) = 0.$$

Therefore,

$$g(x) \leq 0, \ \forall x.$$

Hence, we have shown that for $K = 1$, the condition the sufficient condition for consistency, (4.2.10), holds.

Remark: As noted by Robinson (1980), the sampling scheme may be 'periodic', so that the same missing-observed regime is repeated. Suppose that the time series is censored rather than missed and there is at most one observation in each censored component of the censored-observed regime. Then for such a scheme, the argument presented above, based on the second approach in verifying the sufficient condition for the consistency of $\hat{\theta}_n^a$ in the normal case, is adequate. Thus, for this special case, $\hat{\theta}_n^a$ is consistent by the above argument and asymptotically normal by Theorem 4.2.1.

4.3 Estimator based on a missing information principle

4.3.1 Asymptotic normality

In this section, we prove the asymptotic normality of $n^{-\frac{1}{2}}\ Q_n(\theta_o)$, where $Q_n(\theta) = 0$ (see equations (4.1.3) and (4.1.4)) is the estimating equation for which $\hat{\theta}_n^b$ is the induced estimator. As a consequence of this result, we show that if $\hat{\theta}_n^b$, then $n^{\frac{1}{2}}(\hat{\theta}_n^b - \theta_o)$ is also asymptotically normally distributed. As with the proof of Theorem 4.2.1, the proof of the asymptotic normality of $n^{-\frac{1}{2}}$ $Q_n(\theta_o)$ is also divided into propositions and lemmas. We have already established in Chapter 3 (see Lemma 3.2.2) that $\{Q_n(\theta), \mathcal{F}_n, n \geq 1\}$ is a martingale under the measure P_θ. Therefore, our proof makes use of martingale convergence results (see Theorem A.2.2 of the Appendix for a CLT result of Hall and Heyde, 1980). More specifically, we apply Corollary 3.1 of Hall and Heyde (1980) which requires the verification of the conditional Lindeberg condition and a condition on the conditional variance. A simplification in verifying the condition on the conditional variance is accomplished by showing that $\{D_i(\theta),$ $i \in \mathcal{Z}\}$ is stationary and ergodic. Then using this result and the ergodic theorem to show that the condition holds. The stationarity, ergodicity result and the ergodic theorem also enable us to prove the convergence to zero a.s. and hence in probability of the normed martingale $\{n^{-1}Q_n(\theta_o),\ \mathcal{F}_n,\ n \geq 1\}$, under P_{θ_o}. This would otherwise be proved by applying Theorem 2.18 of Hall and Heyde (1980) as we have done for $\{n^{-1}M_n(\theta_o),\ \mathcal{F}_n,\ n \geq 1\}$, in the previous section. First, we state and prove the stationarity, ergodicity result.

Lemma 4.3.1 *Let* $\{Y_i, i \in \mathcal{Z}\}$ *and* $\{T_i, i \in \mathcal{Z}\}$ *be two stationary, ergodic processes, each on* $(\Re^{\mathcal{Z}}, \mathcal{B}^{\mathcal{Z}})$. *Define a new bivariate process,*

$$\underline{C}_i = ((Z_i, \delta_i), (Z_{i+1}, \delta_{i+1}), \ldots),\ \ i \in \mathcal{Z},$$

where

$$Z_i = min(Y_i, T_i),\ \ \delta_i = I(Y_i \leq T_i).$$

Let ζ_1 *be an* $(\Re^2)^\infty \longrightarrow \Re$ *measurable function defined by*

$$\begin{aligned}&\zeta_1((x_i, y_i), (x_{i-1}, y_{i-1}), \ldots)\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{v(u-\theta v)\\ &\qquad f_{Y_i,Y_{i-1}|(Z_i,\delta_i),(Z_{i-1},\delta_{i-1}),\ldots}(u, v|(x_i, y_i), (x_{i-1}, y_{i-1}), \ldots)\}dudv,\end{aligned}$$

Then the process, $\{D_i(\theta), i \in \mathcal{Z}\}$, *defined by*

$$D_i = \zeta_1((Z_i, \delta_i), (Z_{i-1}, \delta_{i-1}), \ldots),$$

is stationary and ergodic.

Proof

The proof is similar to the proof of lemma 4.2.1 (b) in the previous section. □

The next lemma is analogous to Lemma 4.2.2 in the previous section and it establishes that $n^{-1}Q_n(\theta_o)$ converges to zero with probability 1 under P_{θ_o}.

Lemma 4.3.2 *Let* $\{Y_i, i \in \mathcal{Z}\}$ *be a stationary and ergodic AR(1) process. Let*

$$Q_n(\theta) = \sum_{i=1}^{n} D_i(\theta)$$

where

$$D_i(\theta) = E_\theta\left(Y_{i-1}(Y_i - \theta Y_{i-1})|\mathcal{F}_i\right),$$

$\mathcal{F}_i = \sigma\{(Z_j, \delta_j),\ j \leq i\}$. *Suppose* $E_{\theta_o}|\varepsilon_i|^{2p} \leq k < \infty$, *for* $1 < p \leq 2$. *Then*

$$n^{-1}Q_n(\theta_o) \longrightarrow E_{\theta_o}\{D_1(\theta_o)\} = 0, \ \ a.s. \ under \ P_{\theta_o}.$$

Proof

Let E and D_i denote E_{θ_o} and $D_i(\theta_o)$, respectively. Since $\{Q_n(\theta_o),\ \mathcal{F}_n, n \geq 1\}$ is a martingale and $\{D_i, i \in \mathcal{Z}\}$ is stationary and ergodic, we have

$$E(D_i) = E(D_1) = E\left\{E\left(Y_o(Y_1 - \theta_o Y_o)|\mathcal{F}_1\right)\right\} = 0.$$

Further,

$$\begin{aligned} E|D_i| = E|D_1| &= E\left|E\left(Y_o(Y_1 - \theta_o Y_o)|\mathcal{F}_1\right)\right| \\ &\leq E\left\{E\left(|Y_o(Y_1 - \theta_o Y_o)|\,|\mathcal{F}_1\right)\right\} \\ &= E\left|Y_o(Y_1 - \theta_o Y_o)\right| \\ &= E\left|Y_1 Y_o - E(Y_1 Y_o|\bar{\mathcal{F}}_o)\right|, \end{aligned}$$

by Jensen's inequality. $\bar{\mathcal{F}}_i = \sigma\{(Y_j, T_j), j \leq i\}$. But

$$E\left|Y_1 Y_o - E(Y_1 Y_o|\bar{\mathcal{F}}_o)\right| \leq E|Y_1 Y_o| + E\left|E(Y_1 Y_o|\bar{\mathcal{F}}_o)\right|,$$

by Minkowski's inequality. Also,

$$E|Y_1 Y_o| \leq E^{\frac{1}{2}}|Y_1|^2 E^{\frac{1}{2}}|Y_o|^2 = \frac{\sigma^2}{1 - \theta_o^2},$$

by the Cauchy-Schwartz inequality and similarly,

$$E\left|E(Y_1 Y_o|\bar{\mathcal{F}}_o)\right| \leq E\left(E\left(|Y_1 Y_o||\bar{\mathcal{F}}_o\right)\right) = E|Y_1 Y_o| \leq \frac{\sigma^2}{1 - \theta_o^2}.$$

So that,

$$E|D_i| = E|D_1| \leq 2\frac{\sigma^2}{1 - \theta_o^2}.$$

Therefore the lemma follows by lemma 4.3.1 (stationarity and ergodicity of $\{D_i, i \in \mathcal{Z}\}$) and application of the ergodic theorem. Alternatively, the lemma can be proved by verifying the conditions of Theorem 2.18 of Hall and Heyde (1980) (see section A 2 of the appendix for details) as we have done for $n^{-1}M_n(\theta_o),\ \mathcal{F}_n\}$, $n \geq 1$ in the previous section. □

By Corollary 3.1 of Hall and Heyde (1980) (see section A 2 of the appendix for details), if the conditional Lindeberg condition,

$$\forall \varepsilon > 0, \quad n^{-1} \sum_{i=1}^{n} E\left[D_i^2 I(|D_i| > \varepsilon\sqrt{n})|\mathcal{F}_{i-1}\right] \xrightarrow{P} 0, \tag{4.1}$$

and a condition on the conditional variance,

$$n^{-1} \sum_{i=1}^{n} E(D_i^2|\mathcal{F}_{i-1}) \xrightarrow{P} E(D_1^2), \tag{4.2}$$

are satisfied, then the asymptotic normality of $n^{-\frac{1}{2}}Q_n(\theta_o)$ follows. To verify the conditional Lindeberg condition (4.3.1), we give an equivalent condition in the following proposition. For convenience, it is this equivalent condition, instead of (4.3.1) that is proved later.

Proposition 4.3.1 *Let $\{Y_i, i \in \mathcal{Z}\}$ be a stationary AR(1) process. Let*

$$D_i(\theta) = E_\theta\left(Y_{i-1}(Y_i - \theta Y_{i-1})|\mathcal{F}_i\right),$$

where $\mathcal{F}_i = \sigma\{(Z_j, \delta_j),\ j \le i\}$. Then the conditional Lindeberg condition

$$\forall \varepsilon > 0, \quad n^{-1} \sum_{i=1}^{n} E\left[D_i^2 I(|D_i| > \varepsilon\sqrt{n})|\mathcal{F}_{i-1}\right] \xrightarrow{P} 0,$$

is equivalent to the condition

$$\forall \varepsilon > 0, \quad n^{-1} \sum_{i=1}^{n} E\left[D_i^2 I(|D_i| > \varepsilon\sqrt{n})\right] \longrightarrow 0. \tag{4.3}$$

Proof

The proof is similar to the proof Proposition 4.2.1. □

The following lemma is utilized in the verification of not only the sufficient condition (4.3.3) but also the condition on the conditional variance (4.3.2).

Lemma 4.3.3 *Let $\{Y_i, i \in \mathcal{Z}\}$ be an AR(1) process. Let*

$$D_i(\theta) = E_\theta\left(Y_{i-1}(Y_i - \theta Y_{i-1})|\mathcal{F}_i\right).$$

where $\mathcal{F}_i = \sigma\{(Z_j, \delta_j),\ j \le i\}$. Suppose $E|\varepsilon_i|^{4(1+\gamma)} \le k < \infty$, for some $\gamma > 0$, k constant. Then

$$E\left|D_i\right|^{2(1+\gamma)} \le 2^{2(1+\gamma)}k < \infty.$$

Proof

$$
\begin{aligned}
E\left|D_i\right|^{2(1+\gamma)} &= E\left|E\left(Y_{i-1}(Y_i-\theta_o Y_{i-1})|\mathcal{F}_i\right)\right|^{2(1+\gamma)} \\
&\le E\left(E\left(|Y_{i-1}(Y_i-\theta_o Y_{i-1})|^{2(1+\gamma)}|\mathcal{F}_i\right)\right) \\
&= E\left|Y_{i-1}(Y_i-\theta_o Y_{i-1})\right|^{2(1+\gamma)},
\end{aligned}
$$

by Jensen's inequality. But

$$
\begin{aligned}
&E\left|Y_{i-1}(Y_i-\theta_o Y_{i-1})\right|^{2(1+\gamma)} = E\left|Y_iY_{i-1} - E(Y_iY_{i-1}|\bar{\mathcal{F}}_{i-1})\right|^{2(1+\gamma)} \\
&\le \left\{E^{\frac{1}{2(1+\gamma)}}\left|Y_iY_{i-1}\right|^{2(1+\gamma)} + E^{\frac{1}{2(1+\gamma)}}\left|E(Y_iY_{i-1}|\bar{\mathcal{F}}_{i-1})\right|^{2(1+\gamma)}\right\}^{2(1+\gamma)}
\end{aligned}
$$

by Minkowski's inequality. Further,

$$
\begin{aligned}
E|Y_iY_{i-1}|^{2(1+\gamma)} &\le E^{\frac{1}{2}}\left|Y_i^2\right|^{2(1+\gamma)} E^{\frac{1}{2}}\left|Y_{i-1}^2\right|^{2(1+\gamma)} \\
&= E^{\frac{1}{2}}|Y_i|^{4(1+\gamma)}E^{\frac{1}{2}}|Y_{i-1}|^{4(1+\gamma)} \\
&= k,
\end{aligned}
$$

by the Cauchy-Schwartz inequality and the condition, $E|Y_i|^{4(1+\gamma)} \le k < \infty$. Similarly,

$$
\begin{aligned}
E\left|E(Y_iY_{i-1}|\bar{\mathcal{F}}_{i-1})\right|^{2(1+\gamma)} &\le E\left(E\left(|Y_iY_{i-1}|^{2(1+\gamma)}|\bar{\mathcal{F}}_{i-1}\right)\right) \\
&= E|Y_iY_{i-1}|^{2(1+\gamma)} \le k.
\end{aligned}
$$

Therefore,

$$
E|D_i|^{2(1+\gamma)} \le \left\{k^{\frac{1}{2(1+\gamma)}} + k^{\frac{1}{2(1+\gamma)}}\right\}^{2(1+\gamma)} = 2^{2(1+\gamma)}k < \infty. \ \square
$$

In the following proposition, we give an equivalent to the condition on the conditional variance, (4.3.2)

Proposition 4.3.2 *Let $\{Y_i, i \in \mathcal{Z}\}$ be a stationary and ergodic AR(1) process. Let*

$$
D_i(\theta) = E_\theta\left(Y_{i-1}(Y_i-\theta Y_{i-1})|\mathcal{F}_i\right),
$$

where $\mathcal{F}_i = \sigma\{(Z_j,\delta_j),\ j \le i\}$. Suppose $E|\varepsilon_i|^{4(1+\gamma)} \le k$, for some $\gamma > 0$, k constant. Then the condition on the conditional variance,

$$
n^{-1}\sum_{i=1}^{n} E(D_i^2|\mathcal{F}_{i-1}) \xrightarrow{P} E(X_1^2),
$$

is equivalent to the condition

$$n^{-1}\sum_{i=1}^{n} D_i^2 \xrightarrow{P} E\left(D_1^2\right). \tag{4.4}$$

Proof

We need to prove that

$$n^{-1}\sum_{i=1}^{n}\left(D_i^2 - E(D_i^2|\mathcal{F}_{i-1})\right) \xrightarrow{P} 0.$$

Note that

$$\left\{\sum_{i=1}^{n}\left(D_i^2 - E(D_i^2|\mathcal{F}_{i-1})\right), \mathcal{F}_n, n \geq 1\right\}$$

is a martingale. Therefore by Theorem 2.18 of Hall and Heyde (1980) (see section A 2 of the appendix),

$$n^{-1}\sum_{i=1}^{n}\left(D_i^2 - E(D_i^2|\mathcal{F}_{i-1})\right) \longrightarrow 0, \text{ a.s. under } P_{\theta_o}$$

on the set

$$\sum_{i=1}^{\infty} i^{-(1+\gamma)} E\left(\left|D_i^2 - E(D_i^2|\mathcal{F}_{i-1})\right|^{1+\gamma}|\mathcal{F}_{i-1}\right) < \infty.$$

Since, $E|D_i|^{2(1+\gamma)} \leq 2^{2(1+\gamma)}k < \infty$ by lemma 4.3.3, following the steps of the proof of Proposition 4.2.2, one obtains

$$\begin{aligned} E&\left|D_i^2 - E(D_i^2|\mathcal{F}_{i-1})\right|^{1+\gamma} \\ &\leq \left\{\left(2^{2(1+\gamma)}k\right)^{\frac{1}{1+\gamma}} + \left(2^{2(1+\gamma)}k\right)^{\frac{1}{1+\gamma}}\right\}^{1+\gamma} \\ &= \left\{2\left(2^{2(1+\gamma)}k\right)^{\frac{1}{1+\gamma}}\right\}^{1+\gamma} = 2^{1+\gamma}2^{2(1+\gamma)}k = 2^{3(1+\gamma)}k. \end{aligned}$$

Therefore,

$$\begin{aligned} &\sum_{i=1}^{\infty} i^{-(1+\gamma)} E\left(\left|D_i^2 - E(D_i^2|\mathcal{F}_{i-1})\right|^{1+\gamma}|\mathcal{F}_{i-1}\right) \\ &\qquad \leq 2^{3(1+\gamma)}k\sum_{i=1}^{\infty} i^{-(1+\gamma)} < \infty. \ \square \end{aligned}$$

In the following proposition, we establish that the conditions (4.3.3) and (4.3.4) hold a.e. for the martingale difference $\{D_n, n \geq n\}$ defined by equation (4.1.4).

Proposition 4.3.3 *Let $\{Y_i, i \in \mathcal{Z}\}$ be a stationary and ergodic AR(1) process. Let*

$$D_i(\theta) = E_\theta\left(Y_{i-1}(Y_i - \theta Y_{i-1})|\mathcal{F}_i\right),$$

where $\mathcal{F}_i = \sigma\{(Z_j, \delta_j),\, j \leq i\}$. Suppose $E|\varepsilon_i|^{4(1+\gamma)} \leq k < \infty$, for some $\gamma > 0$, k constant. Then

(a) $\forall \varepsilon > 0, \quad n^{-1}\sum_{i=1}^{n} E\left[D_i^2 I(|X_i| > \varepsilon\sqrt{n})\right] \longrightarrow 0.$

(b) $n^{-1}\sum_{i=1}^{n} X_i^2 \xrightarrow{P} E\left(D_i^2\right).$

Proof of Proposition 4.3.3 (a)

The proof is similar to the proof of Proposition 4.2.3 (a), which is for the martingale $\{M_n(\theta_o),\ \mathcal{F}_n,\ n \geq 1\}$ in the previous section and the details are presented below.

$$\begin{aligned} &E\left(D_i^2 I\left(|D_i| < \varepsilon\sqrt{n}\right)\right) \\ &\quad \leq E^{\frac{1}{1+\gamma}}|D_i|^{2(1+\gamma)}E^{\frac{\gamma}{1+\gamma}}\left(I\left(|D_i| > \varepsilon\sqrt{n}\right)\right) \\ &\quad \leq 4k^{\frac{1}{1+\gamma}}\left(\max_i P\left(|D_i| > \varepsilon\sqrt{n}\right)\right), \end{aligned}$$

by Holder's inequality and lemma 4.3.3. But

$$\max_i P\left(|D_i| > \varepsilon\sqrt{n}\right) \longrightarrow 0 \text{ as } n \to \infty,$$

since

$$\max_i P\left(|X_i| > \varepsilon\sqrt{n}\right) = o\left(n^{-2(1+\gamma)}\right)$$

by the corollary of Serfling (1980) (see Corollary A.2.2 in the appendix) used in the proof of Proposition 4.2.3 (a) since $E|D_i^2| < \infty$ by lemma 4.3.3. Therefore,

$$n^{-1}\sum_{i=1}^{n} E\left(D_i^2 I\left(|D_i| > \varepsilon\sqrt{n}\right)\right) \xrightarrow{P} 0. \square$$

Proof of Proposition 4.3.3 (b)

We have already established through lemma 4.3.1 that the process $\{D_i, i \in \mathcal{Z}\}$

is stationary and ergodic. Also by lemma 4.3.3, $E(D_i^2) = E(D_1^2) < \infty$. Therefore the condition holds by application of the ergodic theorem. □

We are now ready to prove the asymptotic normality of $n^{-\frac{1}{2}}Q_n(\theta_o)$. We also show that this result, in turn, implies that if $\hat{\theta}_n^b$ is a consistent solution of the estimating equation $Q_n(\theta) = 0$, then $n^{\frac{1}{2}}(\hat{\theta}_n^b - \theta_o)$ is asymptotically normal.

Theorem 4.3.1 *Let $\{Y_i, i \in \mathcal{Z}\}$ be a stationary and ergodic AR(1) process. Let*

$$Q_n(\theta) = \sum_{i=1}^{n} D_i(\theta),$$

where,

$$D_i(\theta) = E_\theta\left(Y_{i-1}\left(Y_i - \theta Y_{i-1}\right)|\mathcal{F}_i\right),$$

$\mathcal{F}_i = \sigma\{(Z_j, \delta_j), j \leq i\}$. Suppose $E_{\theta_o}|varepsilon_i|^{4(1+\gamma)} \leq k$, $k \leq \infty$, for some $\gamma > 0$, k constant. Let $\hat{\theta}_n^b$ be a consistent solution of the estimating equation $Q_n(\theta) = 0$. Then

(a) $n^{-\frac{1}{2}}Q_n(\theta_o)$ is asymptotically normal, i.e.,

$$n^{-\frac{1}{2}}Q_n(\theta_o) \xrightarrow{\mathcal{D}} N(0, E_{\theta_o}(D_1^2(\theta_o))), \quad \text{under } P_{\theta_o},$$

(b) The asymptotic distribution of $\hat{\theta}_n^b$ is the normal distribution, i.e.,

$$n^{\frac{1}{2}}(\hat{\theta}_n^b - \theta_o) \xrightarrow{\mathcal{D}} N(0, \frac{E(D_1^2)}{E^2(D_1^{(1)})}), \quad a.s. \text{ under } P_{\theta_o},$$

where D_i is an abbreviation for $D_i(\theta_o)$, $D_i^{(1)}$ for $D_i^{(1)}(\theta_o)$ and E for E_{θ_o}.

Proof of Theorem 4.3.1 (a)

Since $\{Q_n(\theta), \mathcal{F}_n, n \geq 1\}$ is a P_θ martingale, the theorem is proved by verifying the conditional Lindeberg condition (4.3.1) and the condition (4.3.2) on the conditional variance by Corollary 3.1 of Hall and Heyde (1980) (see Corollary A.2.1 in the appendix for details). These conditions are equivalent to (4.3.3) and (4.3.4), respectively by Propositions 4.3.1 and 4.3.2. Therefore, the proof of the present result follows by Proposition 4.3.3, which was proved under the condition that $E_{\theta_o}|Y_i|^{4(1+\gamma)} \leq k$, for some $\gamma > 0$ and k constant. □

Proof of Theorem 4.3.1 (b)

The proof is similar to the proof of Theorem 4.2.1 (b). □

We have not investigated the consistency of $\hat{\theta}_n^b$ or the large sample properties of $\hat{\theta}_n^c$ or the currently available estimators. However, simulations in Chapter 5 of this thesis suggest that $\hat{\theta}_n^a$ is comparable with $\hat{\theta}_n^c$ and the currently available estimators, which perform comparably among themselves. Hence, intuitively, under possibly different conditions to those established for $\hat{\theta}_n^a$, the currently available estimators have similar asymptotic behaviour to the behaviour of $\hat{\theta}_n^a$. However, a theoretical investigation is needed to establish the large sample properties of not only the currently available estimators of $\hat{\theta}_n^a$ as well. Also, the consistency of $\hat{\theta}_n^b$ could be investigated and that of $\hat{\theta}_n^a$ could be extended to general error distributions and censoring patterns.

Chapter 5

Comparative Simulation Studies About the Estimators

5.1 Introduction

In this chapter, we present simulation studies in which the performances of the new estimators for censored autocorrelated data are evaluated and compared with the performances of the currently available estimators. The currently available estimators were described in Chapter 2 and they are the maximum likelihood estimator (MLE), $\hat{\theta}_n^{mle}$ (see section 2.3.1), the Pseudolikelihood (PL) estimator of Zeger and Brookmeyer (1986), $\hat{\theta}_n^{zb}$ and the PL estimator of Dagenais (1986) (see section 2.3.2). The new estimators are described in Chapter 3. One of these is $\hat{\theta}_n^a$ (see section 3.2.2). This is the estimator based on replacing each of the underlying time series rv's Y_i's in the least-squares estimating function for the uncensored case by its conditional mean, given the sigma-field generated by the obeserved censored time series for the index times j, $j \leq i$. The second estimator among the new estimators is $\hat{\theta}_n^b$ (see section 3.2.3). This is based on a missing information principle and it is obtained by replacing the differences between successive sums in the least-squares estimating function for the uncensored case by their conditional means, given the corresponding sigma fields used in the case of $\hat{\theta}_n^a$. The third estimator among the new estimators is the distribution-free estimator, $\hat{\theta}_n^c$ (see section 3.2.4). All the estimators, both new and currently available, are also compared with the least-squares estimator for the uncensored case, $\hat{\theta}_n^{ls}$.

We consider three error distributions of the underlying time series rv's, the Gaussian (N) distribution, the double exponential (DE) distribution (also known as the Laplace distribution) and the Gamma distribution. Note that for Gaussian errors we compare all the estimators among themselves, both new and currently existing. For errors drawn from the Laplace and the Gamma distributions, we only compare the least-squares estimator with the new estimators, which are also compared among themselves. The reason is that, as noted in the introduction of Chapter 3, the currently available estimators were derived under the assumption that the errors are Gaussian. Hence, in their present form these estimators cannot be applied if the model deviates from the Gaussian assumption. Nevertheless, provided suitable modifications are implemented for each distribution considered, the currently available estimators can be applied for error distributions other than the Gaussian distribution. However, these modifications are essentially re-derivations of the estimators and these can be quite tedious as demonstrated in section 2.3.1 for the MLE in the Gaussian case.

Three censoring distributions are considered in this simulation study, the Gaussian, the Laplace and the Gamma distributions. The Gaussian censoring distribution is used when the errors are themselves Gaussian or Laplace. The Laplace censoring distribution is used when the errors are Gaussian and the Gamma distribution is used when the errors are themselves Gamma. Details of the values of the mean and shape parameters used in these censoring distributions and the error distributions will follow in the next section.

The chapter is organised as follows. In section 5.2, we describe the design of our Monte Carlo experiments. In section 5.3, we present the results of the simulation study. Section 5.4 concludes the chapter.

5.2 Design of Monte Carlo experiments

We carried out designed experiments to compare the new estimators, $\hat{\theta}_n^a$, $\hat{\theta}_n^b$ and the distribution-free estimator, $\hat{\theta}_n^c$ with the least squares estimator, $\hat{\theta}_n^{ls}$, which corresponds to the uncensored case, the mle, $\hat{\theta}_n^{mle}$, the PL estimator of Dagenais (1986), $\hat{\theta}_n^{dag}$ and the PL estimator of Zeger and Brookmeyer (1986), $\hat{\theta}_n^{zb}$. The matrix langauge programming software GAUSS (1988) was used throughout the Monte Carlo study. The results are summarized in Tables 1-72. In each table the results arose from samples generated according to the following model. Define the stationary, ergodic AR process, $\{Y_i, i \in \mathcal{Z}\}$, where $\mathcal{Z}$ the integer set, by

$$Y_i = \theta Y_{i-1} + \varepsilon_i, \tag{5.2.1}$$

where the errors, ε_i's, are i.i.d. F with mean zero and variance σ^2. Further, assume that the ε_i's are independent of $\sigma\{Y_j, j \le i-1\}$. Let T_i be a sequence of i.i.d. censoring rv's, independent of ε_i's. We observe $\{(Z_i, \delta_i), i \le n\}$, where $Z_i = \min(Y_i, T_i)$, $\delta_i = I(Y_i \le T_i)$. $I(A)$ is the indicator function of the set A.

As mentioned earlier, three error distributions were considered in the simulation experiments, the Gaussian distribution (N), the Laplace distribution and the gamma distribution shifted to have mean zero. For the Gaussian distribution, the variance, σ^2, was fixed at two values, 1 and 2. For the Laplace and gamma distributions, however, σ^2 was set equal to 1 in each case. For each combination of the error distribution F and the value of σ^2, three sample sizes were considered, small samples ($n = 25$), moderate samples ($n = 50$) and large samples ($n = 100$). For each sample size the true value of θ, θ_o, was fixed at six values, ± 0.2, ± 0.5 and ± 0.8. This was to ensure that the estimators are assessed and compared for values of θ_o which are representative of the interval $|\theta_o| < 1$ which means stationarity of the AR process. For a fixed value of θ_o, three censoring patterns were considered. This ensured that the estimators are compared under varying censoring patterns. The results in Tables 1-18 were obtained by using Gaussian (normal) errors with σ^2 set equal to 1 and for each value of θ_o, the censoring rv's, T_i's, were drawn from the Laplace distribution with mean 2 and the shape parameter fixed at three values, 1, 2 and 3. The results in Tables 19-36 were also obtained with the errors drawn from the Gaussian distribution but this time σ^2 was set equal to 2 and the censoring rv's were again drawn from the Gaussian distribution with mean 2.5 and variance fixed at three values, 4, 6 and 8. The results in Tables 37-54 were obtained with the errors drawn from the Laplace distribution with mean zero and σ^2 set equal to 1 and the T_i's were drawn from the Gaussian distribution with mean 2 and variance variance fixed at 1, 2 and 4. The results in Tables 55-72 were obtained with the errors drawn from gamma $(3,\frac{1}{\sqrt{3}},-\sqrt{3})$ so that they had mean zero and $\sigma^2 = 1$. The censoring rv's were drawn from gamma (3,1,-0.5), gamma (3,1,-0.75) and gamma (3,1,-1) for a fixed value of θ_o. This information about the error and censoring distributions is summarized in Table 5.0 below

Table 5.0
Parameters of error and censoring distributions

Table	Figure	Error distribution	Censoring distribution	Sample size
1-6	1-6	Gaussian (0,1)	Laplace	25
7-12	7-12	Gaussian (0,1)	Laplace	50
13-18	13-18	Gaussian (0,1)	Laplace	100
19-24	19-24	Gaussian (0,2)	Gaussian	25
25-30	25-30	Gaussian (0,2)	Gaussian	50
31-36	31-36	Gaussian (0,2)	Gaussian	100
37-42	37-42	Laplace ($\frac{1}{\sqrt{5}}$,0)	Gaussian	25
43-48	43-48	Laplace ($\frac{1}{\sqrt{5}}$,0)	Gaussian	50
49-54	49-54	Laplace ($\frac{1}{\sqrt{5}}$,0)	Gaussian	100
55-60	55-60	Gamma (3,$\frac{1}{\sqrt{3}}$,-$\sqrt{3}$)	Gamma	25
61-66	61-66	Gamma (3,$\frac{1}{\sqrt{3}}$,-$\sqrt{3}$)	Gamma	50
67-72	67-72	Gamma (3, $\frac{1}{\sqrt{3}}$,-$\sqrt{3}$)	Gamma	100

Note: *Each of the six tables in each group corresponds to one of the following values of* θ_o*:* ± 0.2, ± 0.5, ± 0.8. *For each table, the parameter values of the censoring distribution are varied to yield three censoring patterns.*

In Tables (1-18), corresponding to each estimator and for each combination of σ^2, n, θ_o and the censoring pattern, the first tabulated value is the mean estimate of θ calculated from 50 replicates with the corresponding sampling variance ($\times 10^2$) given in parentheses immediately below the mean. The second value is the mean square error (MSE) ($\times 10^2$) of the estimates, also calculated from the 50 replicates. The third value is the mean estimate of the asymptotic variance of θ ($\times 10^2$) calculated from the 50 repetitions. Once again, the associated sampling variance ($\times 10^2$) is given in parentheses immediately below the mean. The fourth value is the mean estimate of σ^2 with its sampling variance

($\times 10^2$) given in parentheses immediately below. The fifth value is the average CPU time (in seconds) it takes to compute an estimate of the parameters and the estimated asymptotic variance for the given estimator on the *pentium*, 64 *Mb Ram*, 200 *Mhz* personal computer using the matrix language programming software GAUSS (1988). In each case the average was calculated from the 50 repetitions. Notice that no computational time is given for $\hat{\theta}_n^{ls}$ since it is obtained in closed form and hence does not require any iterations. The information given by Tables 19-72 for each combination of σ^2, n, θ_o and the censoring pattern is similar to the information provided by Tables 1-18 except that in Tables 19-72 the MSE's are not given. The reason for not including the MSE's in Tables 1-72 is that we found that comparisons among the estimators based on the MSE's is not as interesting as comparisons made according to the asymptotic variance estimates. More will be said on this subject later. To compute parameter estimates for the rest of the estimators, the maximum number of iterations was set equal to 20 and estimates were considered to have converged if successive estimates differed by a value no more than 10^{-8}. To ensure a fair comparison among the estimators, for each estimator, for each combination of σ^2, n, θ_o and the censoring pattern and for each of the 50 repetitions the estimates were computed using a fixed sample. Hence, for each combination the average percentage of censored observations appearing under the caption '% cens.' in Tables 1-72 is the same for all estimators. The corresponding censoring pattern is given alongside these values under the caption 'T_i'.

As mentioned above, the main criterion utilized to assess and compare the performance of the estimators is the asymptotic variance. The bias and the MSE criteria are also considered, but the comparisons they provide are not as interesting as those obtained from the asymptotic variance criterion. In each case, the smaller the value of the criterion, the better the estimator. To compare the performance of the estimators using the asymptotic variance and MSE criteria, for each n, θ_o combination we carried out seperate randomized block analyses of variance on the estimated asymptotic variance of the $\hat{\theta}_n$'s using the values of the variance of T_i as 'blocks' and the seven estimators as treatments. Using the asymptotic variance estimates, there were 72 analyses of variance, one for each of Tables 1-72 and hence 6 for each σ^2, n combination. For each of the analyses of variance which showed a significant estimator effect at the 0.05 level we carried out a Fisher's Least Significant Difference (LSD) analysis (see, e.g., Ott, 1988, page 441). As usual, for the analyses of variance with a statistically insignificant estimator effect the estimators were considered not different among themselves and no LSD analyses were required. Otherwise, a Fisher's LSD was computed at the 0.05 level and estimators for which the corresponding mean (over the 3 censoring patterns) asymptotic variance estimates differed by more than the value of the LSD were declared significantly different from each other. The results of applying this standard LSD procedure are summarized in Figures 1-72. Figure 1 summarizes the means of the estimated asymptotic variances in Table 1, Figure 2 summarizes the corresponding results in Table 2, and so on. In each of these figures, the estimators are arranged in order of increasing mean (over the 3 censoring patterns) estimated asymptotic variance. Estimators underlined by a common line are not significantly different among themselves, while estimators not underlined by a common line are declared significantly different from each other. A summary of the description of the figures (Figures 1-72) is given in Table 5.0. The same comparison procedure was applied with the MSE's as the criterion and there were 18 analyses of variance, one for each of Tables 1-18. The results of the corresponding LSD anayses are summarized in Figures A1-A18 in section A.3 of the appendix

5.3 Simulation results

5.3.1 Errors from the Gaussian distribution with unit variance

The results considered in this section are for Tables 1-18 obtained by setting σ^2 equal to 1 and drawing the censoring samples from the Laplace distribution with mean 2 and the shape parameter fixed at 1, 2 and 3. We begin by studying the bahaviour of the estimators in accordance with the bias criterion. For all sample sizes, all the estimators behave similarly to each other when $1 - \theta_o^2$ is close to 1. On the other hand, for small values of $1 - \theta_o^2$, $\hat{\theta}_n^b$ tends to have larger bias than the other estimators, particularly as the percentange of censored observations increases. In this case the other estimators perform comparably among themselves. Next, we examine the behaviour of the estimators according to the MSE criterion. The results suggest that the performance of the estimators improves as sample size increases. They perform better for small values of $1 - \theta_o^2$ than they do for values closer to unity. Their performance deteriorates as the the proportion of censored observations (censoring rate) increases. This conclusion is reached by using the LSD analyses described earlier but with the seven estimators regarded as 'blocks' and the three censoring patterns as treatments. Let us now compare the estimators among themselves according to this criterion (MSE). As noted earlier, the results of the LSD analyses are summarized in Figures A1-A18 in section A.3 of the appendix. The results suggest that when $\theta_o = \pm 0.2$, $\hat{\theta}_n^{ls}$ and $\hat{\theta}_n^b$ behave similarly and perform better than the rest of the estimators, which are not significantly different among themselves. When $\theta_o = \pm 0.5$, all the estimators tend to perform comparably among themselves. When $\theta_o = \pm 0.8$, $\hat{\theta}_n^b$ tends to perform worse than the rest of the estimators, which are not significantly different among themselves.

We now turn to the behaviour of the estimators on the basis of the asymptotic variance criterion. As one would expect, the performance of the estimators improves as the sample size increases. In particular, the estimators perform best for large values of n and small values of $1-\theta_o^2$ and worst if n is small and $1-\theta_o^2$ is large. To monitor the behaviour of the estimators as the censoring rate increases, we make use of the randomized block analyses of variance described in the previous section. As noted therein, these analyses were carried out on the estimated asymptotic variances of the estimators using the censoring rates as blocks and the 7 estimators as treatments. In comparing the censoring rates, however, we use the same analyses with the 7 estimators taken as blocks and the 3 censoring patterns as treatments.

Eighteen of the 36 analyses of variance (for Tables 1-18) are for σ^2 set equal to 1. Twelve of these 18 analyses show a statistically significant effect due to censoring pattern at the 0.05 level. In 7 of these 12 cases, the estimators exhibit a significantly better performance for the censoring pattern Laplace (1,2) than they do for Laplace (2,2) and Laplace (3,2) at the 0.05 level using LSD analyses. In 3 of the remaining 5 cases, the estimators perform significantly worse for Laplace (3,2) than they do for Laplace (1,2) and Laplace (2,2). In one of the remaining 2 cases, the estimators perform worse for Laplace (3,2) than when the T_i's are distributed as Laplace (1,2). However, there is no significant difference in the performance of the estimators due to changing the censoring pattern from Laplace (1,2) to Laplace (2,2) or from Laplace (2,2) to Laplace (3,2). In the second case, the behaviour of the estimators is significantly different from one censoring pattern to another with Laplace (2,2) showing the best behaviour, Laplace (1,2) the second best and Laplace (3,2) the worst behaviour. Overall, the performance of the estimators tends to deteriorate with increasing proportion of censored observations

Now, to compare the estimators we revert to the original set-up where the 3 censoring rates are used as blocks and the 7 estimators as treatments in the analyses of variance. Twelve of 18 analyses show a statistically significant estimator effect at the 0.05 level. Two of these 12 cases are for small samples, 5 are for moderate samples and the remaining 5 correspond to large samples. For small samples, in one of the 2 analyses which show a significant estimator effect (Figure 5.2), $\hat{\theta}_n^{ls}$, $\hat{\theta}_n^{b}$ and $\hat{\theta}_n^{mle}$ perform significantly better than $\hat{\theta}_n^{c}$, $\hat{\theta}_n^{zb}$ and $\hat{\theta}_n^{a}$ but with no significant difference among the estimators in each group at the 0.05 level according to Fisher's LSD analysis. The estimators $\hat{\theta}_n^{ls}$ and $\hat{\theta}_n^{b}$ perform better than $\hat{\theta}_n^{dag}$ whilst $\hat{\theta}_n^{dag}$ perform better than $\hat{\theta}_n^{a}$. In the second analysis (Figure 5.3), $\hat{\theta}_n^{b}$ and $\hat{\theta}_n^{ls}$ perform significantly better than the rest of the estimators.

For moderate samples, in 2 of the 5 analyses of variance which exhibit a statistically significant estimator effect (Figures 5.8 and 5.9), $\hat{\theta}_n^{ls}$ and $\hat{\theta}_n^{b}$ perform significantly better than $\hat{\theta}_n^{zb}$, $\hat{\theta}_n^{a}$, $\hat{\theta}_n^{dag}$ and $\hat{\theta}_n^{c}$ at the 0.05 level according to Fisher's LSD analysis. The MLE, $\hat{\theta}_n^{mle}$, performs comparably with estimators in both groups in the first of these two analyses (Figure 5.8), while in the second, it performs worse than $\hat{\theta}_n^{ls}$ and $\hat{\theta}_n^{b}$ and comparably with the other estimators. In another 2 of the 5 analyses showing a significant estimator effect (Figures 5.11 and 5.12), $\hat{\theta}_n^{ls}$, $\hat{\theta}_n^{b}$, $\hat{\theta}_n^{dag}$ and $\hat{\theta}_n^{mle}$ perform comparably among themselves. In the first of the two cases (Figure 5.11), estimators in this group perform significantly better than $\hat{\theta}_n^{zb}$, $\hat{\theta}_n^{a}$ and $\hat{\theta}_n^{c}$. In the second case, however, $\hat{\theta}_n^{c}$ performs comparably with estimators in both groups. In the fifth of the 5 analyses of variance with a significant estimator effect (Figure 5.10), $\hat{\theta}_n^{c}$ performs worse than the other estimators which are not significantly different among themselves at the 0.05 level.

For large samples, in 2 of the analyses of variance with a significant estimator effect (Figures 5.13 and 5.14), $\hat{\theta}_n^{b}$ and $\hat{\theta}_n^{ls}$ exhibit a better performance than $\hat{\theta}_n^{c}$, $\hat{\theta}_n^{a}$, $\hat{\theta}_n^{dag}$ and $\hat{\theta}_n^{zb}$ at the 0.05 level. In one of these two cases (Figure 5.13), $\hat{\theta}_n^{mle}$ performs comparably with $\hat{\theta}_n^{b}$ and $\hat{\theta}_n^{ls}$ and better than the other estimators. In the second case, $\hat{\theta}_n^{mle}$ performs worse than $\hat{\theta}_n^{ls}$ and $\hat{\theta}_n^{b}$ and comparably with the other estimators. In 2 of the remaining 3 analyses (Figures 5.16 and 5.17), $\hat{\theta}_n^{ls}$ and $\hat{\theta}_n^{b}$ perform significantly better than $\hat{\theta}_n^{a}$, $\hat{\theta}_n^{zb}$ and $\hat{\theta}_n^{c}$. In one of these 2 cases

(Figure 5.16), $\hat{\theta}_n^{mle}$ and $\hat{\theta}_n^{dag}$ perform comparably with $\hat{\theta}_n^{a}$, $\hat{\theta}_n^{zb}$ and better than $\hat{\theta}_n^{c}$. In the second case, $\hat{\theta}_n^{mle}$ and $\hat{\theta}_n^{dag}$ perform comparably with $\hat{\theta}_n^{ls}$ and $\hat{\theta}_n^{b}$ and better than $\hat{\theta}_n^{c}$. In the remaining analysis (Figure 5.18), $\hat{\theta}_n^{ls}$ performs better than $\hat{\theta}_n^{zb}$, $\hat{\theta}_n^{mle}$, $\hat{\theta}_n^{a}$ and $\hat{\theta}_n^{c}$. The estimators $\hat{\theta}_n^{b}$ and $\hat{\theta}_n^{dag}$ perform comparably with estimators from both groups but better than $\hat{\theta}_n^{c}$.

Overall, about one-third of the time, the estimators still perform comparably among themselves with respect to the asymptotic variance criterion. In the remaining two-thirds of the time, $\hat{\theta}_n^{ls}$ and $\hat{\theta}_n^{b}$ behave similarly to each other and better than $\hat{\theta}_n^{a}$, $\hat{\theta}_n^{zb}$ and $\hat{\theta}_n^{c}$ which are not significantly different among themselves, while $\hat{\theta}_n^{mle}$ and $\hat{\theta}_n^{dag}$ perform comparably with estimators in both groups of estimators.

Table 5.1

Performance of the new estimators in 50 simulations. Errors are from Gaussian (0,1), $\theta_o = 0.8$ and $n = 25$. DE refers to the double exponential distribution

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
DE(1,2)	15.68	0.7608	0.7623	0.7636	0.7673	0.7613	0.7399	0.7621[*]
		(1.2053)	(1.2524)	(1.2937)	(1.3679)	(1.3368)	(1.3516)	(1.2876)[¶]
		1.3108	1.3444	1.3745	1.4201	1.4331	1.6587	1.3797[‖]
		1.5480	1.4048	1.5502	1.5586	1.5632	1.4611	1.5889[†]
		(0.0068)	(0.0128)	(0.0074)	(0.0204)	(0.0076)	(0.0070)	(0.0097)
		0.9659	0.9293	0.8214	0.9764	0.9186	1.0170	0.9246[‡]
		(8.4000)	(9.3157)	(8.1372)	(12.351)	(9.7028)	(12.531)	(8.3788)
			42.715	0.0372	0.2494	1.4840	4.2312	0.0856[§]
DE(2,2)	20.00	0.7158	0.7225	0.7222	0.7242	0.7214	0.6824	0.7132
		(2.4247)	(2.4956)	(2.5361)	(2.6016)	(2.5517)	(2.6149)	(2.7309)
		3.0367	2.9964	3.0399	2.6257	3.0674	3.8933	3.3751
		1.7159	1.6548	1.6760	1.7806	1.6713	1.4755	1.5736
		(0.0116)	(0.0122)	(0.0062)	(0.0135)	(0.0061)	(0.0035)	(0.0098)
		0.9572	0.9272	0.7775	0.9478	0.8734	1.0576	1.0061
		(7.8643)	(9.2756)	(9.4667)	(8.7218)	(8.8986)	(10.205)	(13.394)
			46.962	0.0538	0.2846	1.4972	6.1474	0.1296
DE(3,2)	28.08	0.7255	0.7302	0.7328	0.7201	0.7293	0.6548	0.7385
		(2.7052)	(2.9353)	(3.2384)	(4.0611)	(3.1307)	(2.8826)	(2.8800)
		3.1520	3.3051	3.6425	4.5371	3.5053	4.8756	3.1430
		1.6261	1.6296	2.3679	1.9863	1.9278	1.6049	1.9420
		(0.0074)	(0.0116)	(0.1836)	(0.0188)	(0.0205)	(0.0087)	(0.0296)
		0.9321	0.8651	0.6714	0.9171	0.7862	1.1251	0.8364
		(7.2005)	(9.5184)	(7.5007)	(8.4209)	(6.8692)	(12.560)	(15.074)
			85.680	0.0822	0.3912	4.4666	17.422	0.1986

[*]Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

[†]The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

[‖]The third value is the MSE.

[‡]The fourth value is mean estimate of σ^2.

[§]The fifth value is the average CPU time in seconds needed to compute the estimates.

[¶]The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.2

Performance of the new estimators in 50 simulations. Errors are from Gaussian (0,1), $\theta_o = 0.5$ and $n = 25$. DE refers to the double exponential distribution

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
DE(1,2)	11.20	0.4549	0.4419	0.4520	0.4338	0.4518	0.4355	0.4562*
		(2.9024)	(3.7102)	(3.9370)	(3.9164)	(3.8972)	(3.5246)	(3.8468)¶
		2.9897	3.8994	4.010	4.1980	3.9736	3.7996	3.8848‖
		2.8524	2.6989	3.1175	3.0046	3.1404	2.8580	3.1948†
		(0.0127)	(0.0223)	(0.0152)	(0.0319)	(0.0160)	(0.0127)	(0.0195)
		0.9193	0.9324	0.8439	0.9586	0.8852	0.9477	0.8737‡
		(5.0075)	(6.4109)	(5.8581)	(8.0387)	(5.2200)	(6.5415)	(5.4211)
			32.462	0.0342	0.2350	0.4310	1.0072	0.0552§
DE(2,2)	21.92	0.4796	0.4802	0.4824	0.4741	0.4836	0.4383	0.4904
		(4.2041)	(4.8035)	(4.6824)	(5.2557)	(4.7156)	(4.0925)	(4.5394)
		4.0776	4.6506	4.5261	5.1126	4.5539	4.3095	4.3670
		2.5494	2.7471	3.1731	2.8304	3.2099	2.5929	3.0929
		(0.0112)	(0.0243)	(0.0231)	(0.0221)	(0.0228)	(0.0114)	(0.0463)
		0.9158	0.9301	0.7415	0.9347	0.8259	1.0081	0.8476
		(5.5150)	(10.615)	(7.4708)	(11.170)	(9.0421)	(12.333)	(8.6558)
			50.147	0.0606	0.3502	1.2852	3.5930	0.1392
DE(3,2)	24.80	0.5036	0.5035	0.4945	0.4976	0.4919	0.4207	0.5044
		(2.8812)	(3.3150)	(3.5835)	(3.3698)	(3.4515)	(3.0037)	(3.2353)
		2.7660	3.1836	3.4432	3.3256	3.2200	3.5124	3.1078
		2.4430	3.0889	3.3527	3.1311	3.4847	2.5383	3.1449
		(0.0093)	(0.0425)	(0.0652)	(0.0362)	(0.0532)	(0.0153)	(0.0497)
		0.9770	0.9720	0.6888	0.9744	0.8289	1.1019	0.8835
		(12.044)	(13.149)	(8.6854)	(13.050)	(9.4336)	(18.293)	(18.949)
			59.855	0.0758	0.4118	1.3180	4.4324	0.1726

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‖The third value is the MSE.

‡The fourth value is mean estimate of σ^2.

§The fifth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.3

Performance of the new estimators in 50 simulations. Errors are from Gaussian (0,1), $\theta_o = 0.2$ and $n = 25$. DE refers to the double exponential distribution

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
DE(1,2)	10.64	0.1773	0.1658	0.1690	0.1612	0.1695	0.1653	0.1818[*]
		(4.6601)	(5.2696)	(5.4902)	(5.2237)	(5.4639)	(5.1176)	(5.2756)[¶]
		4.5252	5.1758	5.3667	5.1653	5.3384	5.0333	5.0977[‖]
		3.6524	3.5205	3.7838	3.6031	3.7980	3.5365	3.9924[†]
		(0.0300)	(0.0256)	(0.0239)	(0.0272)	(0.0242)	(0.0197)	(0.0492)
		0.9462	0.9544	0.8661	0.9550	0.9066	0.9600	0.8872[‡]
		(8.5002)	(10.008)	(9.0216)	(10.027)	(9.1188)	(10.141)	(8.3431)
			31.873	0.0374	0.2364	0.0648	0.1530	0.0622[§]
DE(2,2)	18.88	0.2285	0.2493	0.2500	0.2285	0.2495	0.2248	0.2554
		(2.9247)	(3.8304)	(3.9783)	(4.3475)	(3.9811)	(3.2803)	(4.2531)
		2.8890	3.9202	4.0692	4.2548	4.0669	3.2106	4.3899
		3.4189	4.2517	4.3325	4.2546	4.3863	3.4860	4.2533
		(0.0152)	(0.0586)	(0.0496)	(0.0762)	(0.0465)	(0.0250)	(0.0878)
		0.9158	0.9121	0.7529	0.9140	0.7929	0.9287	0.7911
		(9.2836)	(10.825)	(8.7897)	(10.825)	(7.7143)	(10.794)	(7.5787)
			48.928	0.0548	0.3536	0.5370	1.3732	0.1252
DE(3,2)	26.56	0.1735	0.1196	0.1254	0.1099	0.1250	0.1086	0.1598
		(4.8944)	(5.5571)	(6.5685)	(5.4990)	(6.4243)	(4.2111)	(5.5098)
		4.7688	5.9812	6.8623	6.0908	6.7298	4.8781	5.4510
		3.3539	4.7316	4.4285	4.4907	4.6003	3.2396	4.3766
		(0.0112)	(0.3287)	(0.1326)	(0.2942)	(0.0638)	(0.0264)	(0.2135)
		0.9179	0.9126	0.6609	0.9149	0.7272	0.9356	0.7476
		(8.0288)	(11.722)	(13.568)	(12.042)	(8.5385)	(11.593)	(8.5087)
			60.472	0.0836	0.4626	0.8658	3.6868	0.2000

[*]Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

[†]The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

[‖]The third value is the MSE.

[‡]The fourth value is mean estimate of σ^2.

[§]The fifth value is the average CPU time in seconds needed to compute the estimates.

[¶]The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.4

Performance of the new estimators in 50 simulations. Errors are from Gaussian (0,1), θ_o = -0.2 and n = 25. DE refers to the double exponential distribution

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
DE(1,2)	10.48	-0.1985	-0.1989	-0.1988	-0.2101	-0.1988	-0.1920	-0.1865[*]
		(3.1876)	(3.3731)	(3.4132)	(3.3049)	(3.4274)	(3.2105)	(3.5284)[¶]
		3.0603	3.2383	3.2768	3.1829	3.2904	3.0884	3.4055[‖]
		3.4701	3.7144	3.8934	3.6004	3.9373	3.6492	3.9260[†]
		(0.0124)	(0.0228)	(0.0186)	(0.0266)	(0.0190)	(0.0160)	(0.0443)
		1.0025	1.0069	0.8990	1.0029	0.9561	1.0087	0.9283[‡]
		(12.150)	(13.122)	(10.927)	(12.930)	(11.291)	(12.906)	(9.1515)
			31.199	0.0340	0.2348	0.0568	0.1296	0.0640[§]
DE(2,2)	17.76	-0.1927	-0.2070	-0.2143	-0.2142	-0.2145	-0.1948	-0.2078
		(3.9372)	(4.5062)	(4.5405)	(4.3570)	(4.5465)	(3.8964)	(4.7467)
		3.7850	4.3309	4.3793	4.2029	4.3857	3.7432	4.5629
		3.3456	3.7003	4.0502	3.6244	4.0511	3.3661	4.4435
		(0.0260)	(0.0326)	(0.0290)	(0.0322)	(0.0300)	(0.0167)	(0.0709)
		0.9773	0.9640	0.8302	0.9636	0.8475	0.9760	0.8466
		(7.6122)	(12.407)	(8.5713)	(12.239)	(9.4158)	(12.441)	(9.3545)
			41.755	0.0560	0.3230	0.2560	0.7044	0.1308
DE(3,2)	26.80	-0.1645	-0.1966	-0.1955	-0.1931	-0.2008	-0.1626	-0.1561
		(3.9335)	(5.0088)	(5.1972)	(4.6750)	(5.1832)	(3.4078)	(5.1181)
		3.9022	4.8096	4.9913	4.4998	4.9760	3.4114	5.1061
		3.4495	4.5504	4.4284	4.7423	4.7183	3.2979	4.2021
		(0.0163)	(0.0618)	(0.0614)	(0.0623)	(0.0462)	(0.0133)	(0.1024)
		0.9399	0.9330	0.6857	0.9411	0.7320	0.9688	0.7596
		(6.1506)	(9.5595)	(9.8705)	(9.6162)	(5.4397)	(9.7128)	(5.7686)
			73.972	0.0858	0.4530	1.9388	7.2282	0.2342

[*]Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

[†]The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

[‖]The third value is the MSE.

[‡]The fourth value is mean estimate of σ^2.

[§]The fifth value is the average CPU time in seconds needed to compute the estimates.

[¶]The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.5

Performance of the new estimators in 50 simulations. Errors are from Gaussian (0,1), θ_o = -0.5 and n = 25. DE refers to the double exponential distribution

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
DE(1,2)	10.80	-0.3920	-0.3941	-0.3944	-0.3979	-0.3932	-0.3794	-0.3910*
		(3.1551)	(3.2991)	(3.2611)	(3.2505)	(3.2803)	(3.0481)	(3.3042)¶
		4.1953	4.2886	4.2458	4.1629	4.2897	4.3806	4.3601‖
		2.8025	2.5621	2.8686	2.5495	2.8785	2.6944	3.1347†
		(0.0144)	(0.0096)	(0.0101)	(0.0098)	(0.0101)	(0.0090)	(0.0181)
		0.9132	0.8992	0.7981	0.8988	0.8631	0.9200	0.8522‡
		(5.9434)	(6.4037)	(4.8391)	(6.4227)	(5.3743)	(6.8824)	(4.9233)
			30.254	0.0354	0.2242	0.0560	0.1310	0.0560§
DE(2,2)	19.92	-0.4843	-0.4827	-0.4718	-0.4859	-0.4731	-0.4395	-0.4603
		(3.0430)	(4.1849)	(4.3535)	(4.1659)	(4.3429)	(3.8516)	(4.7178)
		2.9459	4.0474	4.2589	4.0191	4.2415	4.0636	4.6867
		2.8623	2.8692	3.1735	2.7641	3.1810	2.6877	5.0738
		(0.0203)	(0.0340)	(0.0328)	(0.0293)	(0.0315)	(0.0159)	(1.5880)
		0.9151	0.9076	0.7023	0.9055	0.8245	0.9769	0.8337
		(7.7287)	(10.385)	(10.007)	(10.074)	(9.5783)	(11.532)	(10.934)
			40.769	0.0540	0.3328	0.2370	0.6648	0.1462
DE(3,2)	25.36	-0.4815	-0.4604	-0.4567	-0.4566	-0.4598	-0.3987	-0.4252
		(2.7126)	(3.9150)	(4.2246)	(3.8240)	(3.9996)	(3.2358)	(4.8407)
		2.6383	3.9152	4.2431	3.8594	4.0012	4.1325	5.2066
		2.5972	3.0392	3.1619	2.7766	3.2887	2.6115	3.2376
		(0.0115)	(0.0433)	(0.0360)	(0.0275)	(0.0309)	(0.0016)	(0.1336)
		1.0189	1.0157	0.7391	1.0265	0.8497	1.1172	0.8878
		(12.595)	(19.073)	(15.015)	(19.910)	(13.230)	(21.004)	(13.824)
			65.062	0.0770	0.3900	2.2972	6.3990	0.2240

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‖The third value is the MSE.

‡The fourth value is mean estimate of σ^2.

§The fifth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.6

Performance of the new estimators in 50 simulations. Errors are from Gaussian (0,1), θ_o = -0.8 and n = 25. DE refers to the double exponential distribution

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
DE(1,2)	17.52	-0.7445	-0.7403	-0.7468	-0.7407	-0.7464	-0.7232	-0.7451[*]
		(1.5238)	(1.7898)	(1.7969)	(1.7910)	(1.7711)	(1.6089)	(1.7905)[¶]
		1.7709	2.0746	2.0080	2.0710	1.9876	2.1344	1.7429[‖]
		1.4925	1.3736	1.5892	1.3689	1.5848	1.4936	1.2462[†]
		(0.0067)	(0.0091)	(0.0089)	(0.0091)	(0.0086)	(0.0069)	(0.0096)
		0.9893	0.9969	0.8330	0.9972	0.9533	1.1162	0.9847[‡]
		(8.7699)	(13.000)	(13.288)	(13.068)	(9.3555)	(14.854)	(1.5658)
			30.573	0.0382	0.2472	0.0648	0.1626	0.1296[§]
DE(2,2)	22.32	-0.7073	-0.7064	-0.7042	-0.7070	-0.7020	-0.6538	-0.6911
		(2.4043)	(2.7090)	(2.8407)	(2.7176)	(2.8105)	(2.8986)	(2.9522)
		3.1675	3.4768	3.6448	3.4738	3.6585	4.9201	4.0200
		1.7896	1.7592	2.1021	1.7514	2.1005	1.8113	2.1101
		(0.0116)	(0.0109)	(0.0190)	(0.0109)	(0.0178)	(0.0106)	(0.0308)
		0.9452	0.9203	0.7111	0.9213	0.8427	1.0791	0.8847
		(7.8210)	(7.4958)	(6.8811)	(7.6831)	(5.4758)	(11.144)	(6.1301)
			35.897	0.0538	0.3054	0.1770	0.5656	0.1614
DE(3,2)	26.56	-0.7442	-0.7525	-0.7544	-0.7547	-0.7507	-0.6838	-0.7364
		(1.8070)	(2.0394)	(1.9621)	(2.0392)	(1.9690)	(2.0987)	(2.2535)
		2.0461	2.1835	2.0916	2.1628	2.1333	3.3650	2.5679
		1.5495	1.7752	1.9277	1.8163	1.9070	1.6490	1.9789
		(0.0070)	(0.0186)	(0.0345)	(0.0209)	(0.0289)	(0.0143)	(0.0367)
		1.0184	1.0156	0.7515	1.0181	0.9279	1.3028	1.0141
		(6.1727)	(9.2428)	(7.1636)	(9.2571)	(9.0865)	(19.336)	(12.303)
			41.169	0.0670	0.3374	0.6088	2.5782	0.2044

[*] Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

[†] The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

[‖] The third value is the MSE.

[‡] The fourth value is mean estimate of σ^2.

[§] The fifth value is the average CPU time in seconds needed to compute the estimates.

[¶] The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.7

Performance of the new estimators in 50 simulations. Errors are from Gaussian (0,1), $\theta_o = 0.8$ and $n = 50$. DE refers to the double exponential distribution

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
DE(1,2)	13.68	0.7807	0.7776	0.7784	0.7829	0.7776	0.7572	0.7795[*]
		(0.4640)	(0.4910)	(0.4671)	(0.5600)	(0.4617)	(0.4626)	(0.4611)[¶]
		0.4920	0.5314	0.5044	0.5780	0.5026	0.6365	0.4939[‖]
		0.6634	0.5975	0.6990	0.6435	0.7029	0.6618	0.6788[†]
		(0.0005)	(0.0006)	(0.0006)	(0.0012)	(0.0006)	(0.0005)	(0.0009)
		1.0185	1.0175	0.9445	1.0470	1.0045	1.1056	0.9967[‡]
		(4.6559)	(5.1529)	(5.2258)	(5.2723)	(4.6870)	(5.8693)	(4.9096)
			46.219	0.0590	0.2908	2.0386	7.2962	0.2520[§]
DE(2,2)	23.52	0.7440	0.7456	0.7381	0.7457	0.7380	0.6899	0.7421
		(1.2572)	(1.6001)	(1.7224)	(2.004)	(1.6998)	(1.6791)	(1.6104)
		1.5457	1.8640	2.0711	2.2588	2.0502	2.8577	1.9134
		0.8589	0.8432	0.9844	0.9858	0.9786	0.8395	0.9507
		(0.0014)	(0.0028)	(0.0029)	(0.0044)	(0.0029)	(0.0019)	(0.0043)
		0.9538	0.9370	0.7677	0.9742	0.8757	1.0422	0.8884
		(2.8527)	(3.6980)	(3.0550)	(3.7649)	(2.8125)	(5.4290)	(3.3142)
			81.080	0.0978	0.4364	5.0150	17.981	0.5384
DE(3,2)	27.84	0.7466	0.7450	0.7440	0.7408	0.7445	0.6751	0.7390
		(0.9868)	(0.9509)	(1.0882)	(1.1265)	(1.0300)	(1.2537)	(1.2446)
		1.2522	1.2344	1.3800	1.4544	1.3174	2.7886	1.5918
		0.8272	0.8473	1.0151	1.5292	1.0107	0.8579	1.0802
		(0.0013)	(0.0022)	(0.0047)	(0.1453)	(0.0042)	(0.0020)	(0.0055)
		0.9794	0.9814	0.7308	1.0096	0.8751	1.2137	0.9733
		(4.0523)	(6.5343)	(6.3381)	(6.5581)	(4.9133)	(9.1648)	(17.702)
			75.977	0.1222	0.5176	4.4880	17.472	0.6372

[*] Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

[†] The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

[‖] The third value is the MSE.

[‡] The fourth value is mean estimate of σ^2.

[§] The fifth value is the average CPU time in seconds needed to compute the estimates.

[¶] The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.8

Performance of the new estimators in 50 simulations. Errors are from Gaussian (0,1), $\theta_o = 0.5$ and $n = 50$. DE refers to the double exponential distribution

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
DE(1,2)	12.44	0.5008	0.5077	0.5114	0.5020	0.5112	0.4930	0.5146*
		(1.4526)	(1.4751)	(1.4548)	(1.5389)	(1.4490)	(1.3117)	(1.4251)¶
		1.4236	1.4515	1.4387	1.5085	1.5207	1.2904	1.4179‖
		1.4028	1.3434	1.5218	1.6114	1.5318	1.4131	1.5962†
		(0.0016)	(0.0023)	(0.0021)	(0.0119)	(0.0021)	(0.0017)	(0.0052)
		0.9520	0.9433	0.8369	0.9599	0.9037	0.9710	0.9042‡
		(2.7421)	(4.2261)	(3.3170)	(4.3876)	(4.1926)	(4.4488)	(4.1367)
			35.169	0.0672	0.3186	0.6832	1.9442	0.2308§
DE(2,2)	19.60	0.4825	0.4674	0.4613	0.4543	0.4621	0.4218	0.4695
		(1.1908)	(1.5242)	(1.7799)	(1.8181)	(1.7115)	(1.6037)	(1.6257)
		1.1976	1.6000	1.8941	1.9906	1.8209	2.1832	1.6862
		1.4945	1.8813	1.9075	2.2596	1.9146	1.5702	2.0747
		(0.0022)	(0.0147	(0.0060)	(0.0997)	(0.0061)	(0.0034)	(0.0145)
		0.9293	0.9597	0.7792	0.9675	0.8540	1.0113	0.8612
		(3.1702)	(4.4141)	(3.9351)	(4.4790)	(3.4250)	(4.8444)	(3.6668)
			51.756	0.1008	0.4522	1.7804	5.3642	0.4720
DE(3,2)	28.08	0.4782	0.4892	0.4904	0.4647	0.4900	0.4161	0.4961
		(1.3968)	(1.8254)	(1.7561)	(2.1997)	(1.7404)	(1.5593)	(1.8259)
		1.4164	1.8010	1.7302	2.2803	1.7156	2.2320	1.7909
		1.4820	1.7687	2.0311	1.8487	2.0151	1.5163	2.0942
		(0.0022)	(0.0063)	(0.0074)	(0.0059)	(0.0065)	(0.0027)	(0.0193)
		1.0062	0.9637	0.7410	0.9876	0.7976	1.0591	0.8255
		(4.6043)	(5.8059)	(5.0801)	(6.5221)	(4.1458)	(6.8175)	(4.0788)
			92.926	0.1492	0.6360	4.5444	14.926	0.7612

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‖The third value is the MSE.

‡The fourth value is mean estimate of σ^2.

§The fifth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.9

Performance of the new estimators in 50 simulations. Errors are from Gaussian (0,1), $\theta_o = 0.2$ and $n = 50$. DE refers to the double exponential distribution

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
DE(1,2)	10.96	0.1483	0.1545	0.1561	0.1402	0.1558	0.1497	0.1582[*]
		(2.5230)	(2.7670)	(2.8391)	(2.8386)	(2.8334)	(2.6259)	(2.9075)[¶]
		2.7398	2.9187	2.9750	3.1394	2.9721	2.8264	3.0241[‖]
		1.8487	1.9892	2.0966	2.0413	2.0981	1.9357	2.3030[†]
		(0.0022)	(0.0063)	(0.0036)	(0.0066)	(0.0036)	(0.0032)	(0.0089)
		0.9636	0.9451	0.8682	0.9475	0.8988	0.9498	0.8823[‡]
		(4.8592)	(5.1560)	(4.3121)	(5.1854)	(4.7177)	(5.2225)	(4.2389)
			30.936	0.0616	0.3008	0.1200	0.2726	0.2032[§]
DE(2,2)	20.20	0.1834	0.1728	0.1732	0.1599	0.1728	0.1579	0.1842
		(2.5074)	(3.9961)	(4.0506)	(3.9656)	(4.0122)	(3.1965)	(3.8981)
		2.4848	3.9902	4.0414	4.0471	4.0059	3.3098	3.8451
		1.8136	2.5642	2.2069	2.3025	2.2031	1.7795	2.2263
		(0.0022)	(0.0611)	(0.0078)	(0.0142)	(0.0076)	(0.0035)	(0.0201)
		0.9622	0.9593	0.7771	0.9614	0.8294	0.9725	0.8403
		(5.8490)	(6.8220)	(5.2203)	(6.8302)	(4.9317)	(7.0005)	(4.9428)
			58.822	0.1098	0.4870	1.749	5.0334	0.5182
DE(3,2)	25.80	0.1946	0.1777	0.1818	0.1634	0.1817	0.1536	0.1943
		(2.1747)	(4.0230)	(3.7548)	(3.7725)	(3.7681)	(2.7214)	(3.7247)
		2.1341	3.9923	3.7128	3.8310	3.7262	2.8823	3.6535
		1.8281	2.7984	2.7478	2.6471	2.8245	1.8880	2.9008
		(0.0013)	(0.0176)	(0.0094)	(0.0113)	(0.0127)	(0.0029)	(0.0184)
		0.9733	0.9700	0.7436	0.9737	0.7818	0.9906	0.7969
		(4.6709)	(6.1392)	(6.8721)	(6.1059)	(5.0845)	(6.0242)	(5.1410)
			93.481	0.1528	0.6768	3.8092	11.147	0.7460

[*] Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

[†] The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

[‖] The third value is the MSE.

[‡] The fourth value is mean estimate of σ^2.

[§] The fifth value is the average CPU time in seconds needed to compute the estimates.

[¶] The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.10

Performance of the new estimators in 50 simulations. Errors are from Gaussian (0,1), θ_o = -0.2 and n = 50. DE refers to the double exponential distribution

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
DE(1,2)	11.28	-0.1965	-0.2056	-0.2066	-0.2178	-0.2063	-0.1979	-0.2067*
		(1.7490)	(1.9229)	(1.8882)	(2.0702)	(1.8905)	(1.7460)	(1.8879)¶
		1.7152	1.8876	1.8548	2.0605	1.8567	1.7115	1.8546‖
		1.7252	1.7401	1.8652	1.6466	1.8700	1.7160	2.1273†
		(0.0021)	(0.0027)	(0.0021)	(0.0022)	(0.0021)	(0.0017)	(0.0050)
		0.9417	0.9373	0.8416	0.9339	0.8901	0.9433	0.8840‡
		(4.4357)	(4.5047)	(3.7284)	(4.4073)	(4.0481)	(4.5539)	(3.8845)
			29.959	0.0616	0.2912	0.0968	0.2314	0.2132§
DE(2,2)	20.72	-0.2017	-0.2124	-0.2137	-0.2203	-0.2138	-0.1910	-0.2068
		(2.1202)	(2.5670)	(2.7259)	(2.4902)	(2.7056)	(2.1678)	(2.6994)
		2.0781	2.5310	2.6902	2.4816	2.6705	2.1325	2.6500
		1.8210	2.3309	2.3595	2.2536	2.3617	1.8884	2.8267
		(0.0023)	(0.0076)	(0.0047)	(0.0070)	(0.0047)	(0.0027)	(0.0305)
		0.9585	0.9639	0.7843	0.9637	0.8316	0.9788	0.8400
		(4.0285)	(5.6832)	(4.6418)	(5.7027)	(4.2933)	(5.5457)	(4.3410)
			53.902	0.1068	0.4792	1.3350	4.1896	0.5724
DE(3,2)	26.88	-0.2255	-0.2324	-0.2353	-0.2389	-0.2364	-0.1940	-0.2227
		(2.0448)	(2.3013)	(2.2068)	(2.0757)	(2.2145)	(1.5520)	(2.3301)
		2.0689	2.3603	3.2873	2.1855	2.3027	1.5246	2.3350
		1.8146	2.7892	2.5795	2.3262	2.5917	1.8151	4.5847
		(0.0028)	(0.1110)	(0.0114)	(0.0130)	(0.0116)	(0.0041)	(0.7020)
		0.9878	1.0064	0.7568	1.0077	0.8017	1.0298	0.8185
		(4.6049)	(5.8091)	(4.9753)	(5.8172)	(3.9976)	(5.6292)	(4.1900)
			79.554	0.1450	0.6380	3.0010	8.1468	0.8076

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‖The third value is the MSE.

‡The fourth value is mean estimate of σ^2.

§The fifth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.11

Performance of the new estimators in 50 simulations. Errors are from Gaussian (0,1), θ_o = -0.5 and n = 50. DE refers to the double exponential distribution

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
DE(1,2)	11.92	-0.4810	-0.4761	-0.4799	-0.4786	-0.4798	-0.4620	-0.4783*
		(1.6666)	(1.8782)	(1.9227)	(1.8829)	(1.9194)	(1.7423)	(2.0091)¶
		1.6694	1.8978	1.9246	1.8910	1.9218	1.8519	2.0160‖
		1.5276	1.4949	1.7002	1.4691	1.6977	1.5719	1.7633†
		(0.0031)	(0.0042)	(0.0040)	(0.0041)	(0.0040)	(0.0034)	(0.0097)
		0.9331	0.9392	0.8380	0.9383	0.8968	0.9670	0.9010‡
		(3.2068)	(4.1297)	(3.9026)	(4.1038)	(3.6992)	(4.5303)	(4.1589)
			30.469	0.0638	0.2958	0.1330	0.3574	0.2394§
DE(2,2)	20.68	-0.4860	-0.4962	-0.4984	-0.4925	-0.4971	-0.4528	-0.4897
		(1.0529)	(1.2362)	(1.3951)	(1.3182)	(1.3823)	(1.2632)	(1.4268)
		1.0514	1.2129	1.3675	1.2975	1.3555	1.4607	1.4089
		1.3380	1.4703	1.6860	1.4251	1.6801	1.4161	1.8163
		(0.0016)	(0.0039)	(0.0066)	(0.0036)	(0.0071)	(0.0033)	(0.0132)
		0.9380	0.9443	0.7774	0.9520	0.8432	1.0103	0.8613
		(5.0311)	(5.3358)	(4.3455)	(5.6312)	(4.5580)	(6.4792)	(4.5610)
			59.020	0.1040	0.4482	2.0060	7.1430	0.5152
DE(3,2)	26.36	-0.5004	-0.4955	-0.4907	-0.4948	-0.4905	-0.4199	-0.4821
		(1.4688)	(1.9639)	(2.2511)	(2.0083)	(2.2386)	(1.8322)	(2.2370)
		1.4394	1.9266	2.2049	1.9708	2.2029	2.4372	2.2243
		1.3365	1.6600	1.7662	1.6180	1.7881	1.3754	1.8881
		(0.0020)	(0.0055)	(0.0065)	(0.0039)	(0.0057)	(0.0019)	(0.0129)
		0.9963	0.9945	0.7293	0.9991	0.8303	1.0129	0.8530
		(5.1609)	(7.9177)	(4.7549)	(8.3224)	(5.2633)	(8.8670)	(5.2635)
			64.429	0.1416	0.5832	1.9182	6.1278	0.7732

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‖The third value is the MSE.

‡The fourth value is mean estimate of σ^2.

§The fifth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.12

Performance of the new estimators in 50 simulations. Errors are from Gaussian (0,1), θ_o = -0.8 and n = 50. DE refers to the double exponential distribution

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
DE(1,2)	15.60	-0.7408	-0.7416	-0.7433	-0.7430	-0.7424	-0.7235	-0.7418[*]
		(1.0078)	(0.9957)	(0.9684)	(0.9754)	(0.9493)	(0.9614)	(0.9618)[¶]
		1.3381	1.3168	1.2705	1.2808	1.2621	1.5274	1.2813[‖]
		0.7920	0.7414	0.8431	0.7346	0.8428	0.7873	0.7470[†]
		(0.0008)	(0.0008)	(0.0009)	(0.0008)	(0.0009)	(0.0008)	(0.0024)
		0.9676	0.9649	0.8181	0.9629	0.9380	1.0524	0.9611[‡]
		(4.1241)	(3.9495)	(3.2987)	(3.9003)	(3.9076)	(5.7203)	(4.4191)
			29.738	0.0692	0.31060	1.0720	0.2766	0.3648[§]
DE(2,2)	24.64	-0.7663	-0.7670	-0.7702	-0.7690	-0.7681	-0.7166	-0.7628
		(0.9719)	(1.2179)	(1.3185)	(1.1643)	(1.2851)	(1.3355)	(1.4190)
		1.0660	1.3024	1.3809	1.2371	1.3612	2.0043	1.5290
		0.7394	0.7529	0.8139	0.7513	0.8107	0.7406	0.8324
		(0.0014)	(0.0017)	(0.0028)	(0.0014)	(0.0026)	(0.0014)	(0.0030)
		0.9978	1.0024	0.7611	1.0075	0.9238	1.2239	0.9566
		(2.5821)	(3.9115)	(4.0364)	(4.0247)	(3.3077)	(6.2739)	(3.9795)
			41.416	0.1022	0.4106	1.0328	4.1786	0.6120
DE(3,2)	26.64	-0.7740	-0.7687	-0.7751	-0.7702	-0.7733	-0.7077	-0.7666
		(0.9419)	(1.0376)	(1.0964)	(1.1091)	(1.0830)	(1.3341)	(1.2051)
		0.9907	1.1148	1.1365	1.1757	1.1326	2.1593	1.2926
		0.7082	0.7500	0.8273	0.7828	0.8241	0.7697	0.8006
		(0.0008)	(0.0012)	(0.0018)	(0.0015)	(0.0017)	(0.0012)	(0.0037)
		1.0125	1.0313	0.7518	1.0363	0.9246	1.2850	0.9517
		(4.6431)	(6.2846)	(5.3299)	(6.5664)	(5.9098)	(9.4930)	(6.1695)
			51.276	0.1090	0.4548	1.9824	8.7212	0.7734

[*] Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

[†] The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

[‖] The third value is the MSE.

[‡] The fourth value is mean estimate of σ^2.

[§] The fifth value is the average CPU time in seconds needed to compute the estimates.

[¶] The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.13

Performance of the new estimators in 50 simulations. Errors are from Gaussian (0,1), $\theta_o = 0.8$ and $n = 100$. DE refers to the double exponential distribution

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
DE(1,2)	14.40	0.7728	0.7734	0.7753	0.7799	0.7751	0.7543	0.7753[*]
		(0.3467)	(0.3646)	(3.6530)	(0.3934)	(0.3624)	(0.3712)	(0.3664)[¶]
		0.4172	0.4317	0.4227	0.4299	0.4208	0.5763	0.4237[‖]
		0.3936	0.3627	0.4159	0.4104	0.4162	0.3840	0.4223[†]
		(0.0001)	(0.0001)	(0.0002)	(0.0005)	(0.0002)	(0.0001)	(0.0003)
		0.9939	0.9819	0.9140	1.0161	0.9565	1.0514	0.9579[‡]
		(2.2260)	(2.0641)	(1.8623)	(2.2900)	(1.8383)	(2.5512)	(1.9030)
			66.427	0.1200	0.4394	4.2436	13.942	0.9634[§]
DE(2,2)	20.16	0.7765	0.7782	0.7786	0.7802	0.7767	0.7348	0.7767
		(0.5633)	(0.5534)	(0.5919)	(0.5602)	(0.5849)	(0.6584)	(0.5858)
		0.6129	0.5954	0.6318	0.5938	0.6333	1.0769	0.6342
		0.3799	0.3555	0.4020	0.3840	0.4013	0.3677	0.3646
		(0.0002)	(0.0002)	(0.0003)	(0.0003)	(0.0003)	(0.0002)	(0.0004)
		0.9784	0.9577	0.8088	0.9711	0.9164	1.1281	0.9276
		(2.1703)	(1.7424)	(1.6990)	(1.9812)	(1.9891)	(2.8882)	(2.0985)
			63.643	0.1560	0.5808	3.8898	14.218	1.7610
DE(3,2)	26.80	0.7797	0.7686	0.7732	0.7697	0.7704	0.7000	0.7684
		(0.3148)	(0.5246)	(0.5394)	(0.5764)	(0.5231)	(0.6111)	(0.5733)
		0.3529	0.6180	0.6058	0.6624	0.6055	1.6050	0.6674
		0.3972	0.4085	0.4504	0.4713	0.4446	0.4080	0.4711
		(0.0001)	(0.0002)	(0.0003)	(0.0005)	(0.0002)	(0.0001)	(0.0005)
		0.9900	0.9800	0.7619	1.0023	0.8961	1.2507	0.9239
		(2.0664)	(2.6087)	(3.1845)	(2.4814)	(2.4996)	(5.2425)	(3.8535)
			224.05	0.2192	0.8194	12.505	55.8760	2.5322

[*]Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

[†]The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

[‖]The third value is the MSE.

[‡]The fourth value is mean estimate of σ^2.

[§]The fifth value is the average CPU time in seconds needed to compute the estimates.

[¶]The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.14

Performance of the new estimators in 50 simulations. Errors are from Gaussian (0,1), $\theta_o = 0.5$ and $n = 100$. DE refers to the double exponential distribution

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
DE(1,2)	12.52	0.4897	0.4874	0.4893	0.4829	0.4892	0.4706	0.4897*
		(0.6716)	(0.7981)	(0.8003)	(0.8214)	(0.8035)	(0.7485)	(0.8005)¶
		0.6755	0.8060	0.8037	0.8424	0.8071	0.8275	0.8031‖
		0.6924	0.6881	0.7391	0.7507	0.7412	0.6827	0.7982†
		(0.0002)	(0.0003)	(0.0002)	(0.0006)	(0.0002)	(0.0002)	(0.0006)
		1.0256	1.0415	0.9472	1.0510	0.9949	1.0686	0.9982‡
		(1.1496)	(1.2665)	(1.2034)	(1.2223)	(1.1090)	(1.3207)	(1.0078)
			33.602	0.1244	0.4516	0.6680	1.7026	0.8570§
DE(2,2)	21.92	0.4793	0.4791	0.4802	0.4682	0.4798	0.4317	0.4831
		(1.1735)	(1.2685)	(1.2536)	(1.4390)	(1.2555)	(1.1229)	(1.2438)
		1.2046	1.2995	1.2803	1.5257	1.2837	1.5782	1.2599
		0.7518	0.9019	0.9568	0.9670	0.9595	0.7797	0.9876
		(0.0003)	(0.0011)	(0.0007)	(0.0018)	(0.0007)	(0.0003)	(0.0030)
		1.0074	1.0187	0.8252	1.0268	0.8937	1.0783	0.9050
		(1.6575)	(2.5409)	(2.1341)	(2.6074)	(2.0305)	(2.5722)	(2.1043)
			65.929	0.2076	0.7468	3.1128	9.4844	2.2290
DE(3,2)	26.72	0.4830	0.4837	0.4832	0.4707	0.4816	0.4088	0.4835
		(0.7466)	(1.1200)	(1.0979)	(1.1680)	(1.0687)	(0.7973)	(1.0437)
		0.7680	1.1354	1.1151	1.2422	1.0919	1.6212	1.0605
		0.7456	0.9704	1.0283	0.9991	1.0249	0.7886	0.9781
		(0.0002)	(0.0013)	(0.0009)	(0.0009)	(0.0009)	(0.0004)	(0.0029)
		0.9752	0.9621	0.7142	0.9726	0.8004	1.0580	0.8245
		(2.1583)	(3.6098)	(3.0769)	(3.9916)	(2.3418)	(4.5491)	(2.4417)
			118.85	0.2632	0.9744	7.8940	24.0442	3.1554

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‖The third value is the MSE.

‡The fourth value is mean estimate of σ^2.

§The fifth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.15

Performance of the new estimators in 50 simulations. Errors are from Gaussian (0,1), $\theta_o = 0.2$ and $n = 100$. DE refers to the double exponential distribution

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
DE(1,2)	11.08	0.1865	0.1854	0.1865	0.1788	0.1864	0.1791	0.1877[*]
		(0.8134)	(0.9012)	(0.9161)	(0.9215)	(0.9128)	(0.8579)	(0.8745)[¶]
		0.8235	0.9135	0.9252	0.9572	0.9223	0.8930	0.8809[‖]
		0.9222	0.9726	0.9916	0.9967	0.9931	0.9100	1.0453[†]
		(0.0004)	(0.0005)	(0.0003)	(0.0006)	(0.0003)	(0.0002)	(0.0016)
		0.9877	0.9780	0.8908	0.9788	0.9277	0.9809	0.9260[‡]
		(1.6456)	(1.8292)	(1.6550)	(1.8321)	(1.6642)	(1.8297)	(1.7226)
			30.445	0.1252	0.4278	0.2084	0.4570	0.8484[§]
DE(2,2)	20.54	0.2099	0.1884	0.1881	0.1693	0.1877	0.1669	0.1913
		(0.8972)	(1.2202)	(1.2239)	(1.2995)	(1.2230)	(0.9939)	(1.1982)
		0.8978	1.2215	1.2258	1.3808	1.2259	1.0936	1.1938
		0.9488	1.1937	1.2302	1.2077	1.2306	0.9624	1.3481
		(0.0003)	(0.0011)	(0.0008)	(0.0011)	(0.0008)	(0.0004)	(0.0028)
		0.9920	1.0046	0.8148	1.0071	0.8660	1.0153	0.8722
		(2.1931)	(3.6481)	(2.6038)	(3.7199)	(2.9377)	(3.7250)	(2.9596)
			68.351	0.2020	0.7888	2.6158	7.6148	2.0004
DE(3,2)	27.02	0.1965	0.1961	0.1977	0.1809	0.1974	0.1619	0.2064
		(0.8600)	(1.1644)	(1.1616)	(1.1186)	(1.1593)	(0.7472)	(1.1432)
		0.8526	1.1543	1.1505	1.1440	1.1484	0.8849	1.1359
		0.9406	0.6461	1.4122	1.3277	1.4134	0.9570	1.4777
		(0.0003)	(0.0016)	(0.0012)	(0.0018)	(0.0013)	(0.0004)	(0.0043)
		0.9739	1.3282	0.7503	0.9733	0.7704	0.9879	0.7832
		(1.7150)	(2.1268)	(2.2694)	(2.1313)	(1.6719)	2.0775	(1.5633)
			114.92	0.2836	1.0754	6.1252	16.951	3.1032

[*] Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

[†] The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

[‖] The third value is the MSE.

[‡] The fourth value is mean estimate of σ^2.

[§] The fifth value is the average CPU time in seconds needed to compute the estimates.

[¶] The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.16

Performance of the new estimators in 50 simulations. Errors are from Gaussian (0,1), θ_o = -0.2 and n = 100. DE refers to the double exponential distribution

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
DE(1,2)	10.90	-0.2004	-0.2011	-0.2006	-0.2086	-0.2008	-0.1927	-0.1996*
		(0.9804)	(0.9788)	(0.9791)	(0.9826)	(0.9788)	(0.8979)	(1.0011)¶
		0.9706	0.9691	0.9693	0.9802	0.9691	0.8943	0.9911‖
		0.9485	0.9683	1.0398	0.9524	1.0406	0.9586	1.0468†
		(0.0004)	(0.0007)	(0.0005)	(0.0007)	(0.0005)	(0.0004)	(0.0018)
		0.9819	0.9880	0.8802	0.9867	0.9415	0.9930	0.9402‡
		(2.6710)	(3.2458)	(2.2432)	(3.2291)	(2.9319)	(3.2713)	(2.9579)
			31.590	0.1254	0.4252	0.2108	0.4658	0.8652§
DE(2,2)	19.92	-0.1915	-0.2021	-0.1994	-0.2131	-0.1995	-0.1796	-0.1948
		(0.7442)	(0.7923)	(0.7717)	(0.8197)	(0.7668)	(0.6330)	(0.7945)
		0.7440	0.7848	0.7640	0.8287	0.7592	0.6683	0.7893
		0.9542	1.2921	1.2793	1.2206	1.2756	1.0405	1.4640
		(0.0002)	(0.0023)	(0.0011)	(0.0018)	(0.0011)	(0.0008)	(0.0065)
		0.9942	0.9885	0.8206	0.9874	0.8598	0.9999	0.8708
		(1.8835)	(1.8620)	(1.8048)	(1.8460)	(1.3704)	(1.8538)	(1.4729)
			61.702	0.2076	0.7678	1.7796	4.9996	2.0782
DE(3,2)	27.72	-0.2224	-0.2154	-0.2195	-0.2194	-0.2194	-0.1818	-0.2151
		(0.7955)	(1.3785)	(1.3303)	(1.2057)	(1.3281)	(0.9287)	(1.4417)
		0.8377	1.3884	1.35500	1.2313	1.3525	0.9525	1.4501
		0.9267	1.2313	1.4191	1.1868	1.3955	0.9545	1.4492
		(0.0002)	(0.0016)	(0.0013)	(0.0014)	(0.0012)	(0.0004)	(0.0047)
		1.0201	1.0317	0.7477	1.0339	0.8164	1.0523	0.8351
		(1.3760)	(2.0430)	(2.2401)	(2.0293)	(1.5475)	(2.0902)	(1.6719)
			206.99	0.3010	1.0676	10.381	31.1574	3.3414

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‖The third value is the MSE.

‡The fourth value is mean estimate of σ^2.

§The fifth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.17

Performance of the new estimators in 50 simulations. Errors are from Gaussian (0,1), θ_o = -0.5 and n = 100. DE refers to the double exponential distribution

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
DE(1,2)	12.00	-0.5050	-0.5050	-0.5066	-0.5082	-0.5063	-0.4888	-0.5055*
		(0.7230)	(0.7694)	(0.8038)	(0.7493)	(0.8025)	(0.7547)	(0.8136)¶
		0.7183	0.7642	0.8001	0.7485	0.7984	0.7597	0.8085‖
		0.7392	0.7004	0.7797	0.6897	0.7782	0.7180	0.7128†
		(0.0003)	(0.0004)	(0.0004)	(0.0004)	(0.0004)	(0.0003)	(0.0015)
		1.0109	1.0120	0.8924	1.0104	0.9693	1.0396	0.9754‡
		(2.4124)	(2.3046)	(1.5926)	(2.2812)	(2.0593)	(2.4841)	(2.1339)
			31.319	0.1220	0.4196	0.3754	1.0502	1.0006§
DE(2,2)	21.46	-0.5042	-0.4993	-0.4983	-0.5018	-0.4976	-0.4496	-0.4967
		(0.6498)	(0.7717)	(0.7708)	(0.7561)	(0.7728)	(0.6756)	(0.7945)
		0.6451	0.7640	0.7634	0.7489	0.7656	0.9229	0.7876
		0.6935	0.7726	0.8730	0.7606	0.8662	0.7198	0.8923
		(0.0002)	(0.0006)	(0.0005)	(0.0006)	(0.0005)	(0.0002)	(0.0018)
		0.9797	0.9748	0.7824	0.9774	0.8638	1.0427	0.8717
		(1.8222)	(2.5515)	(2.0109)	(2.6068)	(1.9519)	(2.9917)	(2.0950)
			48.806	0.1976	0.6910	1.4664	4.2548	2.2244
DE(3,2)	27.04	-0.4951	-0.4956	-0.5003	-0.4965	-0.4995	-0.4245	-0.4923
		(0.6128)	(0.6981)	(0.8091)	(0.6118)	(0.8137)	(0.6826)	(0.8816)
		0.6091	0.6931	0.8010	0.6069	0.8056	1.2458	0.8787
		0.7415	0.9299	1.0658	0.9267	1.0495	0.8020	1.1589
		(0.0003)	(0.0010)	(0.0012)	(0.0010)	(0.0012)	(0.0004)	(0.0034)
		0.9892	0.9822	0.4750	0.9875	0.8144	1.0763	0.8401
		(1.9871)	(3.7876)	(2.8591)	(3.9496)	(2.7188)	(4.8549)	(3.1027)
			66.703	0.2658	0.9528	2.6882	9.1188	3.0736

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‖The third value is the MSE.

‡The fourth value is mean estimate of σ^2.

§The fifth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.18

Performance of the new estimators in 50 simulations. Errors are from Gaussian (0,1), θ_o = -0.8 and n = 100. DE refers to the double exponential distribution

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
DE(1,2	17.56	-0.7878	-0.7862	-0.7893	-0.7866	-0.7878	-0.7631	-0.7880[*]
		(0.2970)	(0.3126)	(0.3347)	(0.3085)	(0.3298)	(0.3147)	(0.3405)[¶]
		0.3089	0.3285	0.3428	0.3234	0.3414	0.4477	0.3515[‖]
		0.3639	0.3577	0.3912	0.3557	0.3882	0.3598	0.4426[†]
		(0.0001)	(0.0001)	(0.0002)	(0.0001)	(0.0002)	(0.0001)	(0.0019)
		1.0419	1.0402	0.8480	1.0402	0.9974	1.1547	1.0153[‡]
		(1.8744)	(1.9877)	(1.4738)	(1.9486)	(1.9970)	(3.0006)	(2.2634)
			28.354	0.1418	0.4680	0.2186	0.5706	1.7288[§]
DE(2,2)	22.56	-0.7702	-0.7721	-0.7707	-0.7706	-0.7678	-0.7216	-0.7628
		(0.4329)	(0.4679)	(0.4722)	(0.4414)	(0.4639)	(0.5448)	(0.5051)
		0.5174	0.5411	0.5533	0.5234	0.5629	1.1540	0.6384
		0.3792	0.4257	0.4383	0.4067	0.4349	0.3952	0.4289
		(0.0001)	(0.0004)	(0.0002)	(0.0003)	(0.0002)	(0.0002)	(0.0004)
		1.0199	1.0185	0.7955	1.0278	0.9601	1.2192	1.0123
		(2.1698)	(2.6926)	(2.2595)	(2.5805)	(2.2712)	(3.5742)	(2.6397)
			44.017	0.1722	0.6022	1.4754	5.7394	2.1112
DE(3,2)	27.14	-0.7818	-0.7887	-0.7866	-0.7883	-0.7839	-0.7168	-0.7821
		(0.2978)	(0.3538)	(0.3775)	(0.3233)	(0.3645)	(0.4533)	(0.3824)
	0.3279	0.3630	0.3917	0.3338	0.3868	1.1410	0.4106	
		0.3752	0.4661	0.4176	0.4328	0.4361	0.4009	0.4473
		(0.0007)	(0.0011)	(0.0003)	(0.0003)	(0.0003)	(0002)	(0.0005)
		0.9767	0.9437	0.6640	0.9521	0.8695	1.2245	0.8873
		(2.4405)	(2.8604)	(3.4588)	(2.6627)	(2.3480)	(4.0092)	(2.7070)
			164.24	0.2276	0.7426	6.0738	28.4808	2.7230

[*]Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

[†]The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

[‖]The third value is the MSE.

[‡]The fourth value is mean estimate of σ^2.

[§]The fifth value is the average CPU time in seconds needed to compute the estimates.

[¶]The values in parentheses are the corresponding sampling variances $\times 10^2$.

Figure 5.1
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,1), $\theta_o = 0.8$ and $n = 25$.

Estimator	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{c}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{zb}$
	———	———	———	———	———	———	———

Figure 5.2
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,1), $\theta_o = 0.5$ and $n = 25$.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{c}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{a}$
	———	———	———				
			———	———			
				———	———	———	
					———	———	———

Figure 5.3
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,1), $\theta_o = 0.2$ and $n = 25$.

Estimator	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{c}$	$\hat{\theta}_n^{a}$
	———	———					
			———	———	———	———	———

Figure 5.4
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,1), θ_o = -0.2 and n = 25.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{c}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{dag}$
	—	—	—	—	—	—	—

Figure 5.5
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,1), θ_o = -0.5 and n = 25.

Estimator	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{c}$
	—	—	—	—	—	—	—

Figure 5.6
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,1), θ_o = -0.8 and n = 25.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{zb}$
	—	—	—	—	—		
			—	—	—	—	
				—	—	—	—

Figure 5.7
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,1), $\theta_o = 0.8$ and $n = 50$.

Estimator	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{c}$	$\hat{\theta}_n^{dag}$
	———	———	———	———	———	———	———

Figure 5.8
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,1), $\theta_o = 0.5$ and $n = 50$.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{c}$
	———	———	———				
			———	———	———		
				———	———	———	———

Figure 5.9
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,1), $\theta_o = 0.2$ and $n = 50$.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{c}$
	———	———					
			———	———	———	———	———

Figure 5.10

Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,1), θ_o = -0.2 and n = 50.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{c}$

Figure 5.11

Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,1), θ_o = -0.5 and n = 50.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{c}$

Figure 5.12

Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,1), θ_o = -0.8 and n = 50.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{zb}$

Figure 5.13
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,1), $\theta_o = 0.8$ and $n = 100$.

Estimator	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{c}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{zb}$

Figure 5.14
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,1), $\theta_o = 0.5$ and $n = 100$.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{c}$

Figure 5.15
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,1), $\theta_o = 0.2$ and $n = 100$.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{c}$

Figure 5.16
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,1), θ_o = -0.2 and n = 100.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{c}$
	—	—					
		—	—				
			—	—	—	—	
				—	—	—	—

Figure 5.17
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,1), θ_o = -0.5 and n = 100.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{c}$
	—	—	—	—			
			—	—	—	—	
				—	—	—	—

Figure 5.18
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,1), θ_o = -0.8 and n = 100.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{c}$
	—	—	—				
		—	—	—	—	—	
				—	—	—	—

5.3.2 Errors from the Gaussian distribution with variance two

Next, we consider the results in Tables 19-36 obtained by drawing the time series samples from the normal distribution with mean zero and variance $\sigma^2 = 2$. As mentioned earlier, the censoring rv's in this case are also from the normal distribution but with mean 2.5 and variance fixed at three values, 4, 6 and 8. First, we compare the behaviour of the estimators on the basis of the bias criterion. The behaviour exhibited by the estimators is similar to their behaviour when σ^2. For all the three sample sizes considered in this study, the estimators behave similarly to each other when $1-\theta_o^2$. For large values of $1-\theta_o^2$, however, $\hat{\theta}_n^b$ performs worse than the other estimators which have a similar behaviour.

Let us now consider the behaviour of the estimators using the asymptotic variance criterion. Before we compare the estimators, we shall first study their behaviour as sample size, θ_o, and the censoring rate vary. As in the previous case where $\sigma^2 = 1$, the performance of the estimators improves as sample size increases. It is better for small values of $1 - \theta_o^2$ than it is for values of this quantity close to 1. Of the 72 analyses of variance, 18 (for Tables 19-36) are for normal errors with σ^2 equal to 2. Fourteen of these exhibit a significant effect due to censoring rate. In three of these 14 cases, the performance of the estimators for the censoring rate N(2.5,4) is equivalent to their performance for N(2.5,6) and this is significantly better than their performance for N(2.5,8) at the 0.05 level according to Fisher's LSD analysis. In another one of the 14 cases, the censoring rate N(2.5,4) shows a better performance for the estimators than the rates N(2.5,6) and N(2.5,8) which are not significantly different from each other. In another 4 of the 14 cases, the estimators perform differently for the different censoring rates. Three of these 4 cases show that the performance of the estimators is best for N(2.5,4), second best for N(2.5,6) and worst for N(2.5,8). The fourth case shows that the estimators perform best for N(2.5,8), second best for N(2.5,4) and worst for N(2.5,6). In 3 of the 14 cases, the estimators perform better for N(2.5,8) than they do for N(2.5,4) and $N(2.5,6)$ which are not significantly different from each other. In another 2 of the 14 cases, the performance of the estimators if better for $N(2.5,6)$ than it is for $N(2.5,4)$ and $N(2.5,8)$. In the fourteenth case, the estimators behave similarly for $N(2.5,6)$ and $N(2.5,8)$ and better than they do for $N(2.5,4)$. Overall, there is sufficient evidence to conclude that the performance of the estimators deteriorates as the censoring rate increases.

Once again, 18 analyses of variance are available for comparisons between the estimators. Fifteen of these show a significant estimator effect at the 0.05 level. Four of these 15 cases correspond to small sample sizes, 5 are for moderate samples and the remaining 6 are for large samples. For small samples, in 3 of the 4 analyses of variance which show a significant estimator effect (Figures 5.20, 5.21 and 5.22), $\hat{\theta}_n^{ls}$ and $\hat{\theta}_n^b$ perform better than $\hat{\theta}_n^a$, $\hat{\theta}_n^{zb}$ and $\hat{\theta}_n^c$ at the 0.05 level using Fisher's LSD analysis. In the first of these 3 cases (Figure 5.20), the behaviour of $\hat{\theta}_n^{mle}$ is similar to the behaviour of the estimators from the first group and better than $\hat{\theta}_n^c$'s behaviour. The PL estimator of Dagenais, $\hat{\theta}_n^{dag}$ performs comparably with estimators from the first group. In the second of the 3 analyses (Figure 5.21), $\hat{\theta}_n^{dag}$ and $\hat{\theta}_n^{mle}$ perform comparably with $\hat{\theta}_n^b$, $\hat{\theta}_n^a$ and $\hat{\theta}_n^{zb}$ and better than $\hat{\theta}_n^c$. In the third case (Figure 5.22), $\hat{\theta}_n^{dag}$ behaves similarly to $\hat{\theta}_n^b$ while $\hat{\theta}_n^{mle}$ performs better than $\hat{\theta}_n^c$ and similarly to $\hat{\theta}_n^{dag}$, $\hat{\theta}_n^{zb}$ and $\hat{\theta}_n^a$. In the fourth analysis of variance (Figure 5.23), $\hat{\theta}_n^{ls}$, $\hat{\theta}_n^{dag}$, $\hat{\theta}_n^b$ and $\hat{\theta}_n^{mle}$ are comparable among themselves and perform better than $\hat{\theta}_n^a$, $\hat{\theta}_n^{zb}$ and $\hat{\theta}_n^c$ which are not significantly different from each other.

For moderate samples, in 2 of the 5 analyses of variance with a significant estimator effect (Figures 5.25 and 5.26), $\hat{\theta}_n^{ls}$ and $\hat{\theta}_n^b$ perform significantly better than $\hat{\theta}_n^{dag}$, $\hat{\theta}_n^a$, $\hat{\theta}_n^{zb}$ and $\hat{\theta}_n^c$. In one of these 2 cases (Figure 5.25), $\hat{\theta}_n^{mle}$ behaves similarly to $\hat{\theta}_n^b$ and $\hat{\theta}_n^{ls}$ and better than the other estimators. In the other case (Figure 5.26), $\hat{\theta}_n^{mle}$ performs comparably with estimators from both groups. In another 2 of the 5 analyses (Figures 5.27 and 5.28), $\hat{\theta}_n^{ls}$ and $\hat{\theta}_n^b$ peform better than $\hat{\theta}_n^{mle}$, $\hat{\theta}_n^{zb}$, $\hat{\theta}_n^a$ and $\hat{\theta}_n^c$. In both these cases, $\hat{\theta}_n^{mle}$ behaves similarly to $\hat{\theta}_n^{zb}$, $\hat{\theta}_n^a$

and $\hat{\theta}_n^{dag}$ which are superior to $\hat{\theta}_n^c$. In the the fifth analysis (Figure 5.29), $\hat{\theta}_n^{ls}$ and $\hat{\theta}_n^b$ feature better than $\hat{\theta}_n^a$, $\hat{\theta}_n^{zb}$ and $\hat{\theta}_n^c$ while $\hat{\theta}_n^{dag}$ and $\hat{\theta}_n^{mle}$ perform comparably with estimators from the first group and better than $\hat{\theta}_n^c$ from the second group. The MLE behaves similarly to $\hat{\theta}_n^a$ and $\hat{\theta}_n^{zb}$ while $\hat{\theta}_n^{dag}$ performs better than these two estimators.

For large samples, in 3 of the 6 analyses of variance which show a significant estimator effect at the 0.05 level (Figures 5.31, 5.32 and 5.33), $\hat{\theta}_n^b$ and $\hat{\theta}_n^{ls}$ exhibit a better performance than $\hat{\theta}_n^a$, $\hat{\theta}_n^{zb}$, $\hat{\theta}_n^{dag}$ and $\hat{\theta}_n^c$. In 2 of these 3 cases (Figures 5.32 and 5.33), the behaviour of $\hat{\theta}_n^{mle}$ is similar that of $\hat{\theta}_n^a$, $\hat{\theta}_n^{zb}$ and $\hat{\theta}_n^{dag}$. In the third case (Figure 5.31), $\hat{\theta}_n^{mle}$ behaves similarly to $\hat{\theta}_n^b$. In all the cases, $\hat{\theta}_n^a$ and $\hat{\theta}_n^{zb}$ behave similarly. Their performance is superior to that of $\hat{\theta}_n^c$ in two of these cases (Figures 5.32 and 5.33) and similar in the third case (Figure 5.31). In one of the remaining 3 analyses (Figure 5.35), $\hat{\theta}_n^{dag}$, $\hat{\theta}_n^c$, $\hat{\theta}_n^b$ and $\hat{\theta}_n^{mle}$ behave similarly and better than $\hat{\theta}_n^a$, $\hat{\theta}_n^c$ and $\hat{\theta}_n^{zb}$ which are not significantly different from each other at the 0.05 level. In another one of the 3 cases (Figure 5.36), apart from $\hat{\theta}_n^c$, $\hat{\theta}_n^{ls}$ and $\hat{\theta}_n^b$'s superiority over $\hat{\theta}_n^{mle}$, the estimators compare similarly among themselves. In the third case (Figure 5.34), $\hat{\theta}_n^{ls}$ and $\hat{\theta}_n^b$ perform comparably among themselves and better than $\hat{\theta}_n^a$, $\hat{\theta}_n^{zb}$ and $\hat{\theta}_n^c$ which behave similarly to each other. The MLE behaves similarly to $\hat{\theta}_n^a$ and $\hat{\theta}_n^{zb}$ from the latter group while $\hat{\theta}_n^{dag}$ behaves similarly to $\hat{\theta}_n^{zb}$ former group of estimators. The behaviour of $\hat{\theta}_n^{dag}$ and $\hat{\theta}_n^{mle}$ is similar.

In general, like in the previous case where σ^2 is equal to 1, $\hat{\theta}_n^{ls}$ and $\hat{\theta}_n^b$ perform equivalently to each other and better than $\hat{\theta}_n^a$, $\hat{\theta}_n^{zb}$ and $\hat{\theta}_n^c$ which are not significantly different among themselves. The MLE and $\hat{\theta}_n^{dag}$ perform comparably with estimators in either group with $\hat{\theta}_n^{mle}$'s performance closer to the performance $\hat{\theta}_n^{ls}$ and $\hat{\theta}_n^b$ and $\hat{\theta}_n^{dag}$ performing more like $\hat{\theta}_n^a$ and $\hat{\theta}_n^{zb}$. The difference between this case and the previous one where σ^2 is equal to 1 is that here the superiority of $\hat{\theta}_n^{ls}$ and $\hat{\theta}_n^b$ over $\hat{\theta}_n^a$, $\hat{\theta}_n^{zb}$ and $\hat{\theta}_n^c$ is enhanced, particularly for large samples. In the previous case, in a significant number cases, all the estimators perform equivalently. Further, in the present case, $\hat{\theta}_n^{mle}$ and $\hat{\theta}_n^{dag}$ behave more like $\hat{\theta}_n^{a}$ and $\hat{\theta}_n^{zb}$ than in the previous case.

Table 5.19

Performance of the new estimators in 50 simulations. Errors are from Gaussian (0,2), $\theta_o = 0.8$ and $n = 25$

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
$N(2.5,4)$	15.60	0.7310	0.7307	0.7296	0.7361	0.7290	0.7065	0.7370*
		(1.6236)	(1.7439)	(1.7716)	(2.0783)	(1.7475)	(1.8535)	(1.6362)¶
		1.7959	1.6442	1.8620	2.2893	1.8409	1.6649	1.8940†
		(0.0103)	(0.0116)	(0.0122)	(0.0987)	(0.0122)	(0.0097)	(0.0189)
		1.9579	1.9382	1.7811	1.9936	1.8656	2.0518	1.9713‡
		(52.018)	(57.153)	(61.943)	(59.342)	(52.270)	(56.629)	(72.220)
		2.0212	50.617	0.0398	0.2328	2.0212	6.7382	0.0892§
$N(2.5,6)$	19.76	0.7187	0.7136	0.7190	0.7158	0.7151	0.6901	0.7204
		(2.3620)	(2.7991)	(2.6918)	(2.9307)	(2.7212)	(2.7528)	(2.6460)
		1.6359	1.6445	1.8527	1.9784	1.8172	1.6382	1.6988
		(0.0078)	(0.0093)	(0.0110)	(0.0274)	(0.0107)	(0.0069)	(0.0176)
		1.9589	1.9282	1.7981	2.0135	1.8480	2.1028	2.0444
		(25.214)	(34.765)	(30.792)	(31.014)	(27.777)	(32.770)	(66.971)
		2.0596	59.234	0.0492	0.2640	2.0596	7.0678	0.1260
$N(2.5,8)$	23.36	0.7287	0.7182	0.7229	0.7176	0.7190	0.6796	0.7246
		(2.0602)	(2.4486)	(2.5622)	(2.8782)	(2.5535)	(2.7690)	(2.7322)
		1.6781	1.8777	1.8354	1.6989	1.8558	1.6456	1.6911
		(0.0081)	(0.0402)	(0.0093)	(0.0093)	(0.0109)	(0.0083)	(0.0221)
		1.7749	1.6760	1.3763	1.7280	1.5588	1.8862	1.6311
		(21.815)	(19.591)	(19.196)	(23.717)	(22.187)	(33.083)	(26.278)
		3.1010	58.144	0.0582	0.3190	3.1010	9.2746	0.1628

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.20

Performance of the new estimators in 50 simulations. Errors are from Gaussian (0,2), $\theta_o = 0.5$ and $n = 25$

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
$N(2.5,4)$	12.96	0.4429	0.4325	0.4349	0.4294	0.4332	0.4178	0.4393*
		(2.5474)	(2.8137)	(2.7749)	(3.0200)	(2.7906)	(2.6467)	(2.7946)¶
		2.6861	3.0666	3.1029	3.2556	3.1105	2.8114	3.3729†
		(0.0074)	(0.0245)	(0.0151)	(0.0263)	(0.0152)	(0.0098)	(0.0348)
		1.9076	1.8987	1.6973	1.9089	1.7987	1.9283	1.8062‡
		(32.360)	(29.493)	(21.088)	(28.636)	(26.222)	(30.265)	(24.444)
		0.5042	34.248	0.0408	0.2462	0.5042	1.3358	0.0676§
$N(2.5,6)$	19.52	0.4779	0.4728	0.4797	0.4637	0.4781	0.4492	0.4821
		(4.0503)	(5.4925)	(5.4189)	(5.9326)	(5.3558)	(4.5092)	(5.3960)
		2.5933	2.4326	2.8833	2.5872	2.8954	2.4979	3.1599
		(0.0119)	(0.0135)	(0.0235)	(0.0152)	(0.0217)	(0.0129)	(0.0523)
		1.9164	1.8631	1.5537	1.8797	1.6973	1.9632	1.7722
		(33.867)	(42.672)	(35.325)	(44.419)	(34.693)	(46.445)	(42.240)
		0.8414	42.333	0.0516	0.3028	0.8414	2.2520	0.1264
$N(2.5,8)$	20.00	0.4592	0.4311	0.4350	0.4292	0.4352	0.4030	0.4451
		(3.0699)	(4.3510)	(4.4592)	(4.5573)	(4.3600)	(3.8720)	(4.0165)
		2.7253	3.3639	3.1996	3.4939	3.1970	2.7119	3.3948
		(0.0091)	(0.0440)	(0.0176)	(0.0415)	(0.0180)	(0.0122)	(0.0392)
		1.9213	1.8736	1.5088	1.8877	1.6800	1.9506	1.7022
		(28.947)	(30.840)	(30.703)	(31.385)	(25.116)	(34.618)	(24.411)
		0.7788	47.084	0.0506	0.3374	0.7788	2.5672	0.1206

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.21

Performance of the new estimators in 50 simulations. Errors are from Gaussian (0,2), $\theta_o = 0.2$ and $n = 25$

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
$N(2.5,4)$	15.44	0.1372	0.1349	0.1373	0.1133	0.1364	0.1310	0.1434[*]
		(3.1976)	(3.3400)	(3.4629)	(3.3410)	(3.4229)	(3.0604)	(3.4157)[¶]
		3.3609	3.5056	3.7906	3.7876	3.7745	3.3782	4.3139[†]
		(0.0145)	(0.0446)	(0.0307)	(0.0595)	(0.0294)	(0.0213)	(0.0921)
		2.1305	2.0916	1.8311	2.0986	1.9497	2.1027	1.9282[‡]
		(34.661)	(42.368)	(46.275)	(43.274)	(38.778)	(41.206)	(35.694)
		0.1352	35.780	0.0438	0.2844	0.1352	0.3660	0.0878[§]
$N(2.5,6)$	18.80	0.1239	0.1118	0.1116	0.0928	0.1099	0.1042	0.1166
		(4.1924)	(4.6948)	(4.6212)	(4.7469)	(4.6044)	(3.9512)	(4.5636)
		3.7311	3.9096	4.4425	4.1404	4.4061	3.7517	4.8214
		(0.0247)	(0.0626)	(0.0487)	(0.0812)	(0.0455)	(0.0309)	(0.0766)
		2.0531	2.0549	1.7275	2.0584	1.8640	2.0758	1.9017
		(24.069)	(33.492)	(29.033)	(33.864)	(29.353)	(32.680)	(54.813)
		0.2020	41.233	0.0508	0.3342	0.2020	0.5460	0.1164
$N(2.5,8)$	21.04	0.1542	0.1649	0.1824	0.1522	0.1777	0.1623	0.1916
		(4.4439)	(5.4595)	(4.6195)	(5.0553)	(4.5798)	(3.8702)	(4.4465)
		3.3774	4.4088	4.2105	3.8388	4.2433	3.6723	4.1737
		(0.0183)	(0.1571)	(0.0505)	(0.0470)	(0.0530)	(0.0756)	(0.1474)
		1.8605	1.7983	1.4344	1.8126	1.5928	1.8358	1.5978
		(21.120)	(26.737)	(24.435)	(25.717)	(24.526)	(26.864)	(25.108)
		1.3624	57.808	0.0572	0.3566	1.3624	4.2436	0.1470

[*]Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

[†]The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

[‡]The third value is mean estimate of σ^2.

[§]The fourth value is the average CPU time in seconds needed to compute the estimates.

[¶]The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.22

Performance of the new estimators in 50 simulations. Errors are from Gaussian (0,2), θ_o = -0.2 and n = 25

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
N(2.5,4)	14.64	-0.1745	-0.1804	-0.1768	-0.1938	-0.1784	-0.1723	-0.1726[*]
		(2.7580)	(3.5991)	(3.4994)	(3.3388)	(3.5233)	(3.2377)	(3.4247)[¶]
		3.5581	4.1226	3.9970	3.8227	4.0165	3.6209	4.3912[†]
		(0.0232)	(0.1244)	(0.0508)	(0.0926)	(0.0549)	(0.0349)	(0.1604)
		1.9420	1.9082	1.6214	1.9056	1.7844	1.9173	1.7680[‡]
		(29.293)	(28.475)	(25.418)	(28.716)	(26.176)	(28.303)	(26.313)
		0.1098	35.117	0.0462	0.2668	0.1098	0.2594	0.0890[§]
N(2.5,6)	17.60	-0.1126	-0.0947	-0.0966	-0.1182	-0.0978	-0.0911	-0.0865
		(3.8775)	(4.6446)	(4.7214)	(4.0675)	(4.6788)	(3.9811)	(5.0163)
		3.5955	3.7324	4.3769	3.7600	4.3849	3.6974	4.5809
		(0.0163)	(0.0315)	(0.0315)	(0.0294)	(0.0325)	(0.0178)	(0.1670)
		1.8494	1.8661	1.5619	1.8663	1.6907	1.8754	1.6959
		(27.858)	(32.784)	(22.701)	(32.773)	(28.200)	(33.035)	(27.328)
		0.5996	43.156	0.0506	0.3010	0.5996	1.5214	0.1278
N(2.5,8)	20.96	-0.1793	-0.1793	-0.1844	-0.1862	-0.1843	-0.1713	-0.1744
		(5.0069)	(7.4379)	(7.6157)	(7.1183)	(7.4897)	(6.1426)	(8.1648)
		3.1345	3.7181	3.8012	3.6914	3.7956	3.0905	3.9197
		(0.0102)	(0.0519)	(0.0443)	(0.0685)	(0.0410)	(0.0167)	(0.1103)
		1.9784	1.8264	1.4635	1.8369	1.6204	1.8695	1.6194
		(30.446)	(35.839)	(32.365)	(37.773)	(31.880)	(36.930)	(30.814)
		0.4678	48.263	0.0606	0.3650	0.4678	1.4212	0.1480

[*] Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

[†] The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

[‡] The third value is mean estimate of σ^2.

[§] The fourth value is the average CPU time in seconds needed to compute the estimates.

[¶] The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.23

Performance of the new estimators in 50 simulations. Errors are from Gaussian (0,2), θ_o = -0.5 and $n = 25$

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
N(2.5,4)	16.48	-0.4749	-0.4758	-0.4778	-0.4811	-0.4766	-0.4570	-0.4734[*]
		(2.7853)	(2.9749)	(3.2349)	(2.7580)	(3.1857)	(3.0203)	(3.2430)[¶]
		2.8440	2.8071	3.1956	2.7761	3.1571	2.8654	3.4635[†]
		(0.0155)	(0.0318)	(0.0304)	(0.0312)	(0.0291)	(0.0217)	(0.0782)
		1.9588	1.9385	1.6432	1.9373	1.8266	2.0025	1.8575[‡]
		(26.095)	(30.680)	(22.447)	(30.345)	(26.358)	(31.716)	(36.432)
		0.0886	33.209	0.0458	0.2680	0.0886	0.2384	0.1108[§]
N(2.5,6)	19.12	-0.4667	-0.4682	-0.4741	-0.4688	-0.4740	-0.4439	-0.4623
		(2.7900)	(3.3170)	(3.5017)	(3.2437)	(3.5060)	(3.2182)	(3.7162)
		2.8534	2.9178	3.3809	2.7796	3.3278	2.9003	3.3955
		(0.0151)	(0.0260)	(0.0267)	(0.0194)	(0.0253)	(0.0162)	(0.0901)
		1.9640	1.9539	1.6660	1.9599	1.7741	2.0180	1.8159
		(29.895)	(35.334)	(27.880)	(34.596)	(28.154)	(35.410)	(32.622)
		0.7132	40.481	0.0528	0.2838	0.7132	2.2940	0.1386
N(2.5,8)	21.68	-0.4598	-0.4552	-0.4577	-0.4615	-0.4546	-0.4143	-0.4449
		(3.1906)	(2.8515)	(3.1423)	(2.6170)	(3.1271)	(2.6702)	(3.4198)
		2.6968	3.1486	3.6884	3.0072	3.6626	3.0143	3.7754
		(0.0115)	(0.0265)	(0.0299)	(0.0191)	(0.0311)	(0.0149)	(0.0846)
		1.9355	1.9265	1.5576	1.9267	1.7122	2.0429	1.7564
		(22.758)	(34.586)	(34.516)	(33.702)	(29.957)	(39.351)	(38.336)
		0.2018	39.875	0.0606	0.3306	0.2018	0.5496	0.1714

[*] Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

[†] The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

[‡] The third value is mean estimate of σ^2.

[§] The fourth value is the average CPU time in seconds needed to compute the estimates.

[¶] The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.24

Performance of the new estimators in 50 simulations. Errors are from Gaussian (0,2), θ_o = -0.8 and n = 25

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
N(2.5,4)	20.08	-0.7559	-0.7491	-0.7579	-0.7494	-0.7480	-0.7227	-0.7530*
		(2.2695)	(2.3963)	(2.5590)	(2.3638)	(2.3534)	(2.4267)	(2.8431)¶
		1.6152	1.6157	1.7439	1.5984	1.7952	1.6461	1.6385†
		(0.0079)	(0.0108)	(0.0144)	(0.0102)	(0.0131)	(0.0099)	(0.0306)
		1.8442	1.8132	1.4827	1.8150	1.7328	1.9922	1.7439‡
		(26.344)	(29.221)	(22.779)	(29.212)	(28.762)	(37.383)	(29.393)
		0.0646	29.939	0.0468	0.2514	0.0646	0.1536	0.1382§
N(2.5,6)	22.16	-0.7637	-0.7552	-0.7654	-0.7560	-0.7526	-0.7201	-0.7616
		(2.1644)	(2.2871)	(2.3594)	(2.2811)	(2.1714)	(2.2008)	(2.1612)
		1.4776	1.7283	1.6717	1.6983	1.6756	1.5293	1.6417
		(0.0077)	(0.0181)	(0.0132)	(0.0158)	(0.0097)	(0.0072)	(0.0301)
		2.0652	2.0381	1.5436	2.0398	1.9342	2.3124	2.0009
		(33.552)	(39.799)	(31.776)	(40.001)	(32.793)	(48.834)	(47.976)
		0.1276	31.195	0.0516	0.2646	0.1276	0.3404	0.1626
N(2.5,8)	21.04	-0.7175	-0.7099	-0.7135	-0.7114	-0.7089	-0.6710	-0.7013
		(2.2407)	(2.5317)	(2.6370)	(2.4768)	(2.5416)	(2.5268)	(2.6189)
		1.6697	1.7075	1.9030	1.7079	1.9265	1.6961	2.0800
		(0.0069)	(0.0110)	(0.0147)	(0.0111)	(0.0132)	(0.0082)	(0.0303)
		1.8871	1.8099	1.4420	1.8092	1.6952	2.0669	1.7344
		(30.055)	(29.037)	(31.004)	(28.893)	(28.397)	(43.005)	(30.815)
		0.1220	32.351	0.0476	0.2746	0.1220	0.3208	0.1526

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.25

Performance of the new estimators in 50 simulations. Errors are from Gaussian (0,2), $\theta_o = 0.8$ and $n = 50$

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
$N(2.5,4)$	17.24	0.7475	0.7455	0.7503	0.7464	0.7478	0.7253	0.7523[*]
		(1.4148)	(1.5064)	(1.4854)	(1.7234)	(1.4677)	(1.5537)	(1.4745)[¶]
		0.7981	0.7394	0.8640	0.7975	0.8490	0.7790	0.8760[†]
		(0.0014)	(0.0014)	(0.0024)	(0.0023)	(0.0024)	(0.0017)	(0.0036)
		1.9511	1.8977	1.7461	1.9392	1.8192	2.0241	1.8594[‡]
		(15.276)	(17.024)	(18.373)	(18.114)	(16.549)	(20.522)	(20.318)
			50.170	0.0768	0.3306	2.3992	7.2854	0.3396[§]
$N(2.5,6)$	20.40	0.7462	0.7428	0.7490	0.7469	0.7451	0.7116	0.7523
		(0.7860)	(0.8189)	(0.8268)	(0.8987)	(0.8285)	(0.9358)	(0.7954)
		0.8798	0.8730	0.9574	0.9120	0.9411	0.8531	0.9210
		(0.0014)	(0.0018)	(0.0021)	(0.0020)	(0.0020)	(0.0014)	(0.0039)
		2.0115	1.9550	1.6787	1.9661	1.8423	2.1440	1.9140
		(23.017)	(25.147)	(20.107)	(25.598)	(24.390)	(27.151)	(24.040)
			50.207	0.0876	0.3574	2.3882	7.4700	0.4360
$N(2.5,8)$	21.56	0.7353	0.7326	0.7320	0.7366	0.7297	0.6918	0.7340
		(1.4229)	(1.5386)	(1.7205)	(1.6827)	(1.6312)	(1.6575)	(1.6322)
		0.8854	0.9276	0.9901	1.0021	0.9797	0.8758	1.0291
		(0.0015)	(0.0034)	(0.0029)	(0.0043)	(0.0029)	(0.0018)	(0.0085)
		1.7819	1.7364	1.4149	1.7657	1.6423	1.9665	1.6907
		(16.803)	(18.819)	(16.456)	(21.579)	(17.300)	(22.354)	(19.565)
			64.619	0.0956	0.3854	3.1936	11.291	0.5416

[*]Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

[†]The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

[‡]The third value is mean estimate of σ^2.

[§]The fourth value is the average CPU time in seconds needed to compute the estimates.

[¶]The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.26

Performance of the new estimators in 50 simulations. Errors are from Gaussian (0,2), $\theta_o = 0.5$ and $n = 50$

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
$N(2.5,4)$	15.68	0.5089	0.5082	0.5123	0.4989	0.5116	0.4893	0.5140*
		(1.1575)	(1.4785)	(1.4823)	(1.5718)	(1.4842)	(1.4154)	(1.4551)¶
		1.4164	1.3889	1.5130	1.5215	1.5026	1.3527	1.4075†
		(0.0017)	(0.0022)	(0.0020)	(0.0044)	(0.0019)	(0.0014)	(0.0054)
		1.8382	1.8285	1.6284	1.8594	1.7218	1.8782	1.7590‡
		(16.576)	(19.601)	(17.590)	(21.189)	(17.840)	(19.635)	(17.922)
			41.362	0.0758	0.3450	1.2524	3.4262	0.3326§
$N(2.5,6)$	19.56	0.4787	0.4827	0.4834	0.4703	0.4819	0.4500	0.4863
		(1.1950)	(1.4199)	(1.5450)	(1.6055)	(1.5360)	(1.4475)	(1.5411)
		1.4246	1.4973	1:6434	1.6891	1.6301	1.3949	1.7669
		(0.0014)	(0.0026)	(0.0024)	(0.0046)	(0.0023)	(0.0013)	(0.0070)
		2.0500	2.0302	1.6901	2.0597	1.8649	2.1133	1.8950
		(16.159)	(20.073)	(16.797)	(21.867)	(19.745)	(22.502)	(21.180)
			49.709	0.0906	0.4350	1.7730	4.8214	0.4404
$N(2.5,8)$	20.96	0.4776	0.4706	0.4757	0.4631	0.4723	0.4320	0.4779
		(1.2228)	(1.4947)	(1.4949)	(1.6019)	(1.5283)	(1.4527)	(1.5877)
		1.4410	1.7762	1.7952	1.8929	1.7860	1.4970	1.8930
		(0.0014)	(0.0060)	(0.0040)	(0.0063)	(0.0039)	(0.0019)	(0.0083)
		1.9477	1.9885	1.6390	1.9983	1.7893	2.0917	1.8003
		(11.932)	(13.621)	(11.807)	(13.990)	(13.037)	(16.054)	(13.470)
			44.236	0.0988	0.4526	0.9998	2.8478	0.4722

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.27

Performance of the new estimators in 50 simulations. Errors are from Gaussian (0,2), $\theta_o = 0.2$ and $n = 50$

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
$N(2.5,4)$	14.60	0.1396	0.1428	0.1459	0.1234	0.1460	0.1389	0.1500*
		(2.3961)	(2.2806)	(2.4356)	(2.5882)	(2.4074)	(2.1237)	(2.4205)¶
		1.8115	2.0148	2.0308	2.0602	2.0297	1.8459	2.3458†
		(0.0020)	(0.0040)	(0.0029)	(0.0050)	(0.0028)	(0.0024)	(0.0094)
		1.9826	1.9492	1.7444	1.9559	1.8279	1.9584	1.8178‡
		(17.058)	(19.451)	(21.619)	(20.961)	(19.126)	(19.805)	(17.789)
			38.321	0.0758	0.3552	0.4682	1.1686	0.3000§
$N(2.5,6)$	19.80	0.2121	0.2094	0.2150	0.1847	0.2140	0.1967	0.2174
		(2.0248)	(2.3419)	(2.5640)	(2.5132)	(2.5475)	(2.1510)	(2.5605)
		1.6851	1.9986	1.9983	2.1892	1.9961	1.6916	2.3782
		(0.0014)	(0.0065)	(0.0039)	(0.0133)	(0.0038)	(0.0022)	(0.0180)
		1.9349	1.8774	1.5628	1.8846	1.6772	1.8978	1.7080
		(11.918)	(13.610)	(12.439)	(14.383)	(10.846)	(13.288)	(11.479)
			47.550	0.0966	0.4624	1.1942	3.4820	0.4812
$N(2.5,8)$	22.00	0.2090	0.2135	0.2171	0.1965	0.2156	0.1928	0.2184
		(2.0309)	(2.3142)	(2.4477)	(2.4971)	(2.4281)	(1.8721)	(2.4575)
		1.7619	1.9943	2.1325	2.0173	2.1452	1.7341	2.3172
		(0.0027)	(0.0039)	(0.0038)	(0.0038)	(0.0036)	(0.0023)	(0.0125)
		1.9683	1.9312	1.5823	1.9370	1.6861	1.9632	1.7032
		(16.963)	(22.879)	(19.134)	(23.873)	(19.226)	(23.106)	(19.014)
			55.890	0.1042	0.4800	1.5904	4.6674	0.5462

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.28

Performance of the new estimators in 50 simulations. Errors are from Gaussian (0,2), θ_o = -0.2 and n = 50

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
N(2.5,4)	14.88	-0.1545	-0.1594	-0.1600	-0.1746	-0.1596	-0.1517	-0.1552[*]
		(1.5827)	(1.7875)	(1.7408)	(1.7323)	(1.7291)	(1.5615)	(1.7742)[¶]
		1.7624	2.0074	2.0506	1.9468	2.0431	1.8542	2.5396[†]
		(0.0021)	(0.0058)	(0.0038)	(0.0052)	(0.0037)	(0.0031)	(0.0126)
		1.9819	1.9861	1.7185	1.9839	1.8632	1.9966	1.8543[‡]
		(18.216)	(16.842)	(18.765)	(16.825)	(15.213)	(16.915)	(14.321)
			35.410	0.0770	0.3550	0.1608	0.3610	0.3100[§]
N(2.5,6)	17.56	-0.1702	-0.1852	-0.1852	-0.1977	-0.1854	-0.1738	-0.1838
		(1.9926)	(2.3837)	(2.4683)	(2.4041)	(2.4198)	(2.1166)	(2.5377)
		1.8683	1.9782	2.2109	1.9467	2.2014	1.9117	2.5524
		(0.0024)	(0.0063)	(0.0052)	(0.0069)	(0.0052)	(0.0036)	(0.0151)
		2.0085	2.0107	1.6981	2.0078	1.8355	2.0297	1.8437
		(20.669)	(22.163)	(17.502)	(21.875)	(17.986)	(22.442)	(20.588)
			41.386	0.0888	0.3918	0.5712	1.5268	0.3910
N(2.5,8)	22.00	-0.1566	-0.1482	-0.1472	-0.1668	-0.1481	-0.1334	-0.1357
		(1.4315)	(2.2034)	(1.9994)	(2.0128)	(2.0632)	(1.7007)	(2.0393)
		1.7368	2.1340	2.2843	1.9914	2.2688	1.8040	2.2685
		(0.0020)	(0.0085)	(0.0075)	(0.0074)	(0.0071)	(0.0025)	(0.0222)
		2.0489	2.0291	1.5949	2.0259	1.7688	2.0468	1.7961
		(24.534)	(32.381)	(24.303)	(32.048)	(27.973)	(33.203)	(28.330)
			54.4052	0.1066	0.4768	1.2892	3.6624	0.5932

[*] Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

[†] The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

[‡] The third value is mean estimate of σ^2.

[§] The fourth value is the average CPU time in seconds needed to compute the estimates.

[¶] The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.29

Performance of the new estimators in 50 simulations. Errors are from Gaussian (0,2), θ_o = -0.5 and $n = 50$

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
$N(2.5,4)$	17.32	-0.4735	-0.4820	-0.4794	-0.4852	-0.4789	-0.4556	-0.4768*
		(1.8189)	(1.5682)	(1.6888)	(1.5656)	(1.6606)	(1.5705)	(1.7442)¶
		1.4249	1.3643	1.5581	1.3078	1.5465	1.3830	1.6174†
		(0.0022)	(0.0018)	(0.0019)	(0.0016)	(0.0019)	(0.0014)	(0.0025)
		1.9240	1.9387	1.6186	1.9403	1.8123	2.006	1.8381‡
		(8.9974)	(11.211)	(13.858)	(11.432)	(10.388)	(11.438)	(11.305)
			36.714	0.0814	0.3386	0.7656	2.1938	0.4150§
$N(2.5,6)$	19.04	-0.4754	-0.4825	-0.4788	-0.4859	-0.4774	-0.4461	-0.4723
		(1.7064)	(1.6388)	(1.7410)	(1.5878)	(1.6861)	(1.6000)	(1.8702)
		1.4362	1.7374	1.8095	1.6938	1.7714	1.5269	2.1064
		(0.0012)	(0.0057)	(0.0036)	(0.0053)	(0.0030)	(0.0018)	(0.0128)
		1.9053	1.9176	1.5507	1.9194	1.7742	2.0100	1.8148
		(14.912)	(17.221)	(17.252)	(17.136)	(16.254)	(19.249)	(22.449)
			36.802	0.0882	0.3952	0.3162	0.8622	0.4970
$N(2.5,8)$	20.16	-0.5065	-0.5130	-0.5121	-0.5132	-0.5129	-0.4727	-0.5124
		(1.3947)	(1.5010)	(1.7303)	(1.4881)	(1.6542)	(1.4512)	(1.6981)
		1.3970	1.5043	1.6803	1.4631	1.6527	1.4063	1.8769
		(0.0023)	(0.0051)	(0.0040)	(0.0042)	(0.0037)	(0.0024)	(0.0186)
		1.9358	1.8930	1.4891	1.8970	1.7045	1.9967	1.7135
		(17.768)	(18.241)	(13.897)	(18.475)	(14.354)	(18.732)	(14.336)
			38.988	0.0980	0.3984	0.4896	1.1546	0.5128

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.30

Performance of the new estimators in 50 simulations. Errors are from Gaussian (0,2), θ_o = -0.8 and n = 50

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
$N(2.5,4)$	19.68	-0.7665	-0.7605	-0.7646	-0.7614	-0.7545	-0.7301	-0.7610*
		(1.0429)	(0.9618)	(1.1201)	(0.9428)	(1.0656)	(1.0865)	(1.0585)¶
		0.7371	0.8345	0.8382	0.8146	0.8468	0.7889	0.8734†
		(0.0009)	(0.0024)	(0.0018)	(0.0018)	(0.0012)	(0.0009)	(0.0076)
		1.9614	1.9433	1.5389	1.9411	1.8815	2.1532	1.9059‡
		(22.226)	(20.480)	(15.914)	(20.213)	(21.100)	(25.974)	(21.279)
			30.267	0.0890	0.3374	0.1460	0.3538	0.5052§
$N(2.5,6)$	22.04	-0.7688	-0.7696	-0.7746	-0.7696	-0.7655	-0.7337	-0.7713
		(0.7677)	(0.8009)	(0.8811)	(0.7769)	(0.8160)	(0.8475)	(0.8572)
		0.7710	0.8033	0.8013	0.7929	0.8256	0.7715	0.7552
		(0.0011)	(0.0016)	(0.0014)	(0.0016)	(0.0016)	(0.0013)	(0.0026)
		1.9508	1.9008	1.4879	1.9085	1.8246	2.1712	1.9045
		(18.963)	(22.954)	(17.539)	(21.856)	(20.589)	(27.064)	(23.825)
			36.727	0.0898	0.3648	0.7470	2.0300	0.5172
$N(2.5,8)$	24.64	-0.8009	-0.7937	-0.7966	-0.7931	-0.7841	-0.7430	-0.7875
		(0.4842)	(0.4769)	(0.5389)	(0.4765)	(0.4946)	(0.5879)	(0.5805)
		0.6783	0.7673	0.7706	0.7705	0.7898	0.7294	0.6593
		(0.0007)	(0.0012)	(0.0013)	(0.0012)	(0.0010)	(0.0007)	(0.0029)
		1.9182	1.9114	1.4101	1.9188	1.8648	2.3205	1.9539
		(8.4195)	(11.731)	(9.0219)	(11.718)	(15.673)	(21.255)	(19.152)
			36.167	0.0978	0.3986	0.5578	1.8916	0.6976

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.31

Performance of the new estimators in 50 simulations. Errors are from Gaussian (0,2), $\theta_o = 0.8$ and $n = 100$

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
$N(2.5,4)$	16.88	0.7688	0.7687	0.7690	0.7741	0.7678	0.7444	0.7698*
		(0.5024)	(0.5226)	(0.5324)	(0.5674)	(0.5229)	(0.5936)	(0.5187)¶
		0.3971	0.3709	0.4273	0.4302	0.4211	0.3943	0.4070†
		(0.0001)	(0.0001)	(0.0002)	(0.0005)	(0.0002)	(0.0001)	(0.0004)
		1.9140	1.8254	1.6523	1.8878	1.7814	1.9824	1.8146‡
		(5.9185)	(6.1794)	(5.6811)	(6.0988)	(5.4618)	(6.2046)	(5.3457)
			92.921	0.1416	0.4788	6.9244	21.488	1.2764§
$N(2.5,6)$	20.00	0.7624	0.7629	0.7643	0.7633	0.7606	0.7295	0.7636
		(0.5973)	(0.6639)	(0.6053)	(0.7839)	(0.6088)	(0.6861)	(0.6179)
		0.4231	0.3860	0.4411	0.4382	0.4277	0.3999	0.4334
		(0.0002)	(0.0002)	(0.0003)	(0.0005)	(0.0002)	(0.0002)	(0.0006)
		1.9631	1.8948	1.6743	1.9530	1.8290	2.1065	1.8572
		(8.7660)	(11.470)	(12.571)	(11.515)	(11.011)	(11.966)	(10.610)
			189.65	0.1582	0.5854	11.589	35.525	1.7596
$N(2.5,8)$	22.12	0.7770	0.7793	0.7813	0.7814	0.7783	0.7410	0.7809
		(0.3900)	(0.3965)	(0.4300)	(0.5093)	(0.4300)	(0.5469)	(0.4250)
		0.3731	0.3725	0.4147	0.4230	0.4023	0.3720	0.4134
		(0.0001)	(0.0001)	(0.0002)	(0.0003)	(0.0001)	(0.0001)	(0.0002)
		1.9233	1.8291	1.5826	1.8868	1.7457	2.0881	1.7850
		(8.6542)	(10.283)	(11.899)	(9.8628)	(9.8218)	(11.743)	(10.965)
			104.65	0.1602	0.5940	8.0718	27.149	1.7754

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.32

Performance of the new estimators in 50 simulations. Errors are from Gaussian (0,2), $\theta_o = 0.5$ and $n = 100$

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
$N(2.5,4)$	16.40	0.4708	0.4690	0.4692	0.4505	0.4689	0.4476	0.4714*
		(0.7362)	(0.8663)	(0.9064)	(1.1854)	(0.9081)	(0.8998)	(0.9307)¶
		0.7521	0.8119	0.8324	0.8971	0.8286	0.7438	0.9404†
		(0.0003)	(0.0013)	(0.0005)	(0.0013)	(0.0005)	(0.0003)	(0.0012)
		1.9809	1.9513	1.7047	1.9820	1.8364	2.0021	1.8522‡
		(6.8743)	(8.5006)	(8.0802)	(9.0001)	(7.4736)	(8.7366)	(8.5006)
			59.281	0.1404	0.5416	3.1584	8.2882	1.2106§
$N(2.5,6)$	18.52	0.4727	0.4700	0.4709	0.4486	0.4697	0.4404	0.4739
		(0.6936)	(0.7658)	(0.7919)	(0.9886)	(0.7870)	(0.7312)	(0.7935)
		0.7551	0.8326	0.8705	0.8978	0.8647	0.7577	0.9387
		(0.0002)	(0.0007)	(0.0003)	(0.0007)	(0.0003)	(0.0002)	(0.0012)
		2.0513	2.0455	1.7284	2.0815	1.8896	2.1184	1.8898
		(7.5773)	(8.1580)	(7.9610)	(8.3928)	(7.9723)	(9.2974)	(8.0258)
			66.389	0.1650	0.6096	3.8764	10.866	1.5920
$N(2.5,8)$	21.40	0.5051	0.5000	0.5028	0.4876	0.5011	0.4608	0.5049
		(0.5678)	(0.6232)	(0.6722)	(0.7618)	(0.6607)	(0.6049)	(0.6621)
		0.6987	0.7997	0.8417	0.8445	0.8379	0.7122	0.8375
		(0.0002)	(0.0007)	(0.0004)	(0.0008)	(0.0004)	(0.0002)	(0.0013)
		1.9858	1.9862	1.6218	2.0032	1.7913	2.0934	1.8115
		(7.6480)	(9.9193)	(8.0091)	(10.334)	(7.9034)	(10.917)	(8.3849)
			71.483	0.1828	0.6850	4.0942	11.726	1.9002

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.33

Performance of the new estimators in 50 simulations. Errors are from Gaussian (0,2), $\theta_o = 0.2$ and $n = 100$

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
$N(2.5,4)$	15.26	0.1891	0.1840	0.1852	0.1642	0.1843	0.1744	0.1880*
		(0.9960)	(1.2818)	(1.2893)	(1.4212)	(1.2904)	(1.1436)	(1.2656)¶
		0.9224	1.0611	1.0642	1.0951	1.0635	0.9528	1.2239†
		(0.0004)	(0.0009)	(0.0006)	(0.0009)	(0.0006)	(0.0004)	(0.0024)
		2.0164	2.0093	1.7254	2.0136	1.8732	2.0201	1.8756‡
		(10.971)	(13.868)	(11.012)	(14.019)	(12.505)	(13.796)	(12.791)
			46.155	0.1558	0.5570	1.4514	3.7194	1.3152§
$N(2.5,6)$	18.46	0.2066	0.2108	0.2111	0.1903	0.2111	0.1956	0.2137
		(0.5536)	(0.6276)	(0.6296)	(0.7370)	(0.6272)	(0.5390)	(0.6131)
		0.9028	1.0231	1.0539	1.0999	1.0541	0.9052	1.1338
		(0.0003)	(0.0007)	(0.0004)	(0.0009)	(0.0004)	(0.0003)	(0.0022)
		1.9729	1.9890	1.6543	1.9937	1.7992	2.0042	1.8081
		(5.1091)	(6.6653)	(5.8715)	(6.7517)	(6.0896)	(6.7685)	(6.2724)
			42.343	0.1628	0.6326	0.9840	2.5642	1.6530
$N(2.5,8)$	20.66	0.2293	0.2416	0.2430	0.2181	0.2423	0.2209	0.2477
		(1.1104)	(1.2780)	(1.2782)	(1.3095)	(1.2757)	(1.0587)	(1.2751)
		0.8673	1.0041	1.0484	1.0559	1.0487	0.8708	1.1846
		(0.0004)	(0.0011)	(0.0008)	(0.0011)	(0.0008)	(0.0005)	(0.0015)
		1.9138	1.8849	1.5218	1.8934	1.6672	1.9121	1.6833
		(8.4118)	(9.0349)	(7.7481)	(9.1879)	(7.7968)	(8.9107)	(7.9096)
			64.071	0.1824	0.6986	2.7772	7.6414	1.7432

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.34

Performance of the new estimators in 50 simulations. Errors are from Gaussian (0,2), θ_o = -0.2 and n = 100

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
N(2.5,4)	15.54	-0.2122	-0.1994	-0.1970	-0.2136	-0.1971	-0.1871	-0.1944*
		(0.6680)	(0.7677)	(0.7276)	(0.7116)	(0.7385)	(0.6667)	(0.7267)¶
		0.9472	1.0633	1.0821	1.0167	1.0770	0.9678	1.0781†
		(0.0006)	(0.0011)	(0.0006)	(0.0010)	(0.0006)	(0.0004)	(0.0037)
		1.9731	1.9938	1.6926	1.9907	1.8592	2.0052	1.8731‡
		(4.9387)	(6.3991)	(6.2057)	(6.2188)	(5.8695)	(6.4477)	(7.0018)
			35.988	0.1518	0.5150	0.4740	1.1986	1.4874§
N(2.5,6)	19.44	-0.1972	-0.1959	-0.1947	-0.2119	-0.1952	-0.1797	-0.1960
		(1.2217)	(1.3297)	(1.3439)	(1.2533)	(1.3323)	(1.1420)	(1.3207)
		0.9180	1.0811	1.1382	1.0089	1.1327	0.9551	1.2020
		(0.0002)	(0.0009)	(0.0005)	(0.0008)	(0.0005)	(0.0003)	(0.0036)
		1.9445	1.9137	1.5470	1.9116	1.7224	1.9303	1.7413
		(7.1266)	(8.2407)	(7.1733)	(8.2369)	(7.1440)	(8.8161)	(7.6947)
			54.7073	0.1846	0.6424	1.7300	4.8394	1.9554
N(2.5,8)	21.26	-0.1610	-0.1634	-0.1643	-0.1819	-0.1637	-0.1459	-0.1597
		(0.8915)	(0.9782)	(0.9994)	(0.9525)	(0.9821)	(0.7568)	(0.9465)
		0.9500	1.2025	1.2608	1.1482	1.2481	1.0059	1.4597
		(0.0004)	(0.0008)	(0.0010)	(0.0007)	(0.0009)	(0.0005)	(0.0051)
		2.0556	2.0468	1.6203	2.0439	1.7990	2.0640	1.8248
		(7.8685)	(9.3305)	(7.8885)	(9.3032)	(8.0988)	(9.2116)	(8.2277)
			55.804	0.2120	0.7506	1.4742	4.0428	2.2900

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.35

Performance of the new estimators in 50 simulations. Errors are from Gaussian (0,2), θ_o = -0.5 and n = 100

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
N(2.5,4)	17.12	-0.4884	-0.4880	-0.4886	-0.4925	-0.4870	-0.4633	-0.4885*
		(1.0080)	(0.9644)	(1.0086)	(0.9215)	(0.9772)	(0.9231)	(0.9863)¶
		0.7495	0.7825	0.8642	0.7588	0.8425	0.7550	0.8996†
		(0.0004)	(0.0008)	(0.0005)	(0.0008)	(0.0005)	(0.0003)	(0.0025)
		1.9766	1.9858	1.6510	1.9836	1.8648	2.0546	1.8722‡
		(8.0880)	(8.9127)	(6.3328)	(8.7644)	(7.8734)	(10.166)	(8.0465)
			35.248	0.1646	0.5196	0.5958	1.5674	1.5776§
N(2.5,6)	19.52	-0.4908	-0.4926	-0.4937	-0.4932	-0.4924	-0.4607	-0.4923
		(0.5709)	(0.6510)	(0.6443)	(0.6513)	(0.6236)	(0.5520)	(0.6378)
		0.7132	0.7171	0.8223	0.6983	0.8109	0.7141	0.7268
		(0.0002)	(0.0004)	(0.0004)	(0.0004)	(0.0003)	(0.0002)	(0.0022)
		1.9226	1.8845	1.5056	1.8921	1.7325	1.9700	1.7498
		(9.7855)	(11.442)	(8.7955)	(11.342)	(9.3187)	(12.092)	(9.7650)
			48.998	0.1780	0.5776	1.7860	5.2784	2.0378
N(2.5,8)	21.48	-0.4682	-0.4752	-0.4806	-0.4776	-0.4793	-0.4401	-0.4796
		(1.0016)	(1.0256)	(1.0785)	(0.9927)	(1.0484)	(0.8756)	(1.0575)
		0.7257	0.7532	0.8664	0.7304	0.8507	0.7311	0.8792
		(0.0002)	(0.0005)	(0.0004)	(0.0004)	(0.0004)	(0.0003)	(0.0023)
		1.9446	1.8912	1.5043	1.8956	1.6973	1.9878	1.7201
		(5.9307)	(7.3148)	(6.3167)	(7.2547)	(6.1841)	(8.0622)	(5.7553)
			48.518	0.1932	0.6416	1.6234	5.0840	2.2234

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.36

Performance of the new estimators in 50 simulations. Errors are from Gaussian (0,2), θ_o = -0.8 and n = 100

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
N(2.5,4)	20.68	-0.7866	-0.7836	-0.7889	-0.7837	-0.7799	-0.7535	-0.7864*
		(0.3544)	(0.3825)	(0.4187)	(0.3808)	(0.3644)	(0.3820)	(0.4244)¶
		0.3825	0.4069	0.4117	0.4055	0.4151	0.3878	0.3722†
		(0.0002)	(0.0002)	(0.0002)	(0.0002)	(0.0002)	(0.0002)	(0.0007)
		1.9592	1.8999	1.4712	1.9006	1.8509	2.1404	1.9016‡
		(8.4279)	(8.2270)	(5.2622)	(8.1725)	(8.3815)	(11.056)	(9.8747)
			29.785	0.1644	0.5224	0.2630	0.6590	2.0534§
N(2.5,6)	23.00	-0.7808	-0.7764	-0.7828	-0.7748	-0.7711	-0.7397	-0.7785
		(0.4231)	(0.3808)	(0.4331)	(0.3943)	(0.3878)	(0.4124)	(0.4311)
		0.3932	0.4359	0.4258	0.4282	0.4356	0.4053	0.3625
		(0.0002)	(0.0002)	(0.0003)	(0.0002)	(0.0002)	(0.0002)	(0.0006)
		2.0100	1.9654	1.5303	1.9754	1.8986	2.2530	1.9711
		(6.8640)	(7.3846)	(7.4488)	(7.1114)	(6.7792)	(9.4825)	(10.779)
			37.485	0.1690	0.5928	0.9304	2.7662	2.3362
N(2.5,8)	23.88	-0.7970	-0.7959	-0.7969	-0.7935	-0.7878	-0.7527	-0.7949
		(0.2683)	(0.2491)	(0.3206)	(0.2645)	(0.2965)	(0.3365)	(0.3252)
		0.3739	0.5510	0.4116	0.4260	0.4148	0.3851	0.3696
		(0.0001)	(0.0087)	(0.0002)	(0.0002)	(0.0002)	(0.0001)	(0.0008)
		1.9475	1.9061	1.4706	1.9232	1.8411	2.2399	1.8889
		(7.6617)	(8.4109)	(7.4004)	(8.1529)	(7.4004)	(9.8652)	(8.5692)
			47.340	0.1758	0.5978	1.9614	7.0996	2.4618

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Figure 5.19

Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,2), $\theta_o = 0.8$ and $n = 25$.

Estimator	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{c}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$
	—	—	—	—	—	—	—

Figure 5.20

Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,2), $\theta_o = 0.5$ and $n = 25$.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{c}$
	—	—	—				
			—	—	—	—	
				—	—	—	—

Figure 5.21

Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,2), $\theta_o = 0.2$ and $n = 25$.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{c}$
	—	—					
		—	—				
			—	—	—	—	
					—	—	—

Figure 5.22
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,2), θ_o = -0.2 and $n = 25$.

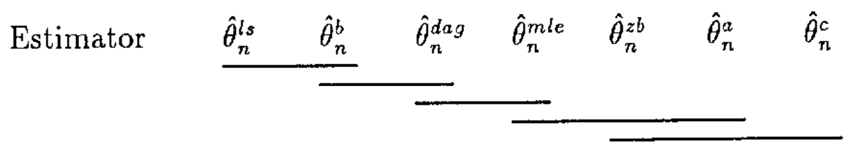

Figure 5.23
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,2), θ_o = -0.5 and $n = 25$.

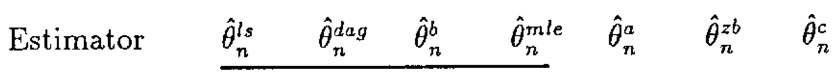

Figure 5.24
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,2), θ_o = -0.8 and $n = 25$.

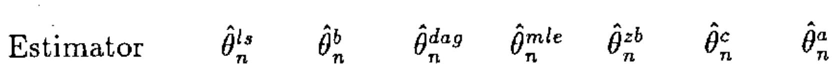

Figure 5.25
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,2), $\theta_o = 0.8$ and $n = 50$.

Estimator	$\hat{\theta}_n^b$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^a$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^c$

Figure 5.26
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,2), $\theta_o = 0.5$ and $n = 50$.

Estimator	$\hat{\theta}_n^b$	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^a$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^c$	$\hat{\theta}_n^{dag}$

Figure 5.27
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,2), $\theta_o = 0.2$ and $n = 50$.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^b$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^a$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^c$

Figure 5.28

Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,2), θ_o = -0.2 and $n = 50$.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{c}$
	—	—	—				
			—	—			
				—	—	—	

Figure 5.29

Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,2), θ_o = -0.5 and $n = 50$.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{c}$
	—	—	—	—			
				—	—	—	
						—	—

Figure 5.30

Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,2), θ_o = -0.8 and $n = 50$.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{c}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{a}$
	—	—	—	—	—	—	—

Figure 5.31
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,2), $\theta_o = 0.8$ and $n = 100$.

Estimator	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{c}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$
	—	—					
		—	—				
				—	—	—	—

Figure 5.32
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,2), $\theta_o = 0.5$ and $n = 100$.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{c}$
	—	—					
			—	—	—		
				—	—	—	
						—	—

Figure 5.33
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,2), $\theta_o = 0.2$ and $n = 100$.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{c}$
	—	—					
			—	—	—		
				—	—	—	

Figure 5.34
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,2), θ_o = -0.2 and n = 100.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{c}$
	———	———					
		———	———				
			———	———	———	———	
					———	———	———

Figure 5.35
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,2), θ_o = -0.5 and n = 100.

Estimator	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{c}$	$\hat{\theta}_n^{zb}$
	———	———	———	———			
					———	———	———

Figure 5.36
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,2), θ_o = -0.8 and n = 100.

Estimator	$\hat{\theta}_n^{c}$	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{mle}$
	———	———	———	———	———	———	
				———	———	———	———

5.3.3 Errors from the Laplace distribution with unit variance

In this section, we discuss the results for which the errors come from the Laplace distribution. The results are summarized in Tables 37-54 and in Figures 37-54. First, we shall consider the behaviour of the estimators with respect to the bias criterion. The results show that for all sample sizes, the estimators behave as in the case discussed earlier, where the errors are i.i.d. normal. They perform comparably among themselves when $1-\theta_o^2$ is close to unity. For small values of this quantity, the estimator $\hat{\theta}_n^b$ performs poorly in comparison with the rest of the estimators which behave similarly among themselves.

Before we compare the estimators on the basis of their estimated asymptotic variances, we first look at their overall behaviour as sample size, θ_o and censoring pattern change. With respect to changes in sample size and θ_o, the behaviour of the estimators is similar to their behaviour in the normal case. They show the best performance for large samples and small values of $1-\theta_o^2$ and the worst behaviour for small samples and large values of $1-\theta_o^2$. To study the behaviour of the estimators as the censor pattern varies, we make use of the analyses of variance which correspond to Tables 37-54 with the estimators regarded as blocks and the censor patterns as treatments.

Of the available 18 analyses of variance, 15 show a significant effect due to censoring pattern at the 0.05 level. In 3 of these 15 cases, the performance of the estimators for $N(2,1)$ is equivalent to their performance for $N(2,2)$ and this is significantly better than their behaviour for $N(2,4)$ at the 0.05 level using Fisher's LSD analyses. Three of the remaining 12 cases show that the estimators perform better for $N(2,1)$ than they do for $N(2,2)$ and $N(2,4)$ which are not significantly different from each other. In one of the remaining 9 cases, the estimators show a deterioration in performance only when the censor pattern is changed from $N(2,1)$ to $N(2,4)$. In 3 of the remaining 8 cases, the estimators show a performance for $N(2,1)$ which is equivalent to their performance for

$N(2,4)$ and better than their performance for $N(2,2)$. One of the remaining 5 cases suggests that the behaviour of the estimators for $N(2,1)$ is similar to their behaviour for $N(2,4)$. They also behave similarly for $N(2,1)$ as they do for $N(2,2)$ but have different bahaviours for $N(2,4)$ and $N(2,2)$, showing a better behaviour for $N(2,4)$. In 2 of the remaining 4 cases, $N(2,1)$ and $N(2,4)$ show a similar behaviour for the estimators but $N(2,1)$ and $N(2,2)$ are significantly different. In one of these 2 cases, the estimators behave better for $N(2,1)$ and in the other, they behave better for $N(2,2)$. In one of the remaining 2 cases, the estimators perform better for $N(2,2)$ than they do for $N(2,1)$ and $N(2,4)$ which are not significantly different from one another. In the last case, the estimators behave differently from one censor pattern to another, showing the best performance for $N(2,4)$, the next best for $N(2,1)$ and the worst behaviour for $N(2,2)$. Overall, there is sufficient evidence to conclude that censor pattern has a significant effect on the performance of the estimators. The results suggest that about one-half of the time, the performance of the estimators become progressively worse as the variance of the censoring distribution increases. In the other one-half of the time, the performance either first improves and then deteriorates or vice versa.

To compare the estimators, we use the same analyses of variance but with the censor patterns taken as blocks and the 4 estimators as treatments. Of the 18 analyses of variance, 10 show a statistically significant estimator effect at the 0.05 level. Three of these 10 cases correspond to small samples, 2 correspond to moderate samples and the remaining 5 are for large samples. For small samples, in all the 3 analyses of variance which show a significant estimator effect, $\hat{\theta}_n^{ls}$ and $\hat{\theta}_n^b$ behave similarly to the other estimators and perform better than $\hat{\theta}_n^c$. In one of these 3 cases (Figure 5.38), $\hat{\theta}_n^a$ performs comparably with $\hat{\theta}_n^c$ and worse than $\hat{\theta}_n^{ls}$ and $\hat{\theta}_n^b$. In one of the remaining 2 cases (Figure 5.39), $\hat{\theta}_n^a$ performs comparably with $\hat{\theta}_n^{ls}$, better than $\hat{\theta}_n^c$ and worse than $\hat{\theta}_n^b$. In the third case (Figure 5.40), $\hat{\theta}_n^a$ behaves similarly to the rest of the esitmators.

For moderate samples, in both the analyses of variance with a significant estimator effect, $\hat{\theta}_n^{ls}$ and $\hat{\theta}_n^b$ exhibit a similar behaviour. Their performance is superior to $\hat{\theta}_n^c$'s. In one of these two cases (Figure 5.45), $\hat{\theta}_n^a$ behaves similarly to $\hat{\theta}_n^{ls}$ and $\hat{\theta}_n^c$ but peforms worse than $\hat{\theta}_n^b$. In the second case (Figure 5.47), $\hat{\theta}_n^a$ behaves like $\hat{\theta}_n^{ls}$ and $\hat{\theta}_n^b$ and performs better than $\hat{\theta}_n^c$.

For large samples, in 4 of the 5 analyses of variance with a significant estimator effect, $\hat{\theta}_n^{ls}$ and $\hat{\theta}_n^b$ have a comparable performance which is superior to $\hat{\theta}_n^c$'s behaviour. In 2 of these 4 cases (Figures 5.50 and 5.53), $\hat{\theta}_n^a$ is comparable with $\hat{\theta}_n^c$ but worse than both $\hat{\theta}_n^b$ and $\hat{\theta}_n^b$. In one of the remaining 2 cases (Figure 5.51), $\hat{\theta}_n^a$ compares favourably with $\hat{\theta}_n^{ls}$ and $\hat{\theta}_n^c$ but it is worse than $\hat{\theta}_n^b$. In the second case (Figure 5.52), $\hat{\theta}_n^a$ is comparable with the rest of the estimators. In the fifth analysis of variance for which estimator effect is significant (Figure 5.54), $\hat{\theta}_n^{ls}$, $\hat{\theta}_n^b$ and $\hat{\theta}_n^c$ are all comparable among themselves and perform better than $\hat{\theta}_n^a$.

Overall, the behaviour of the estimators is similar to their behaviour in the normal case. They have a comparable behaviour among themselves for values of $1 - \theta_o^2$ close to zero. For values of this quantity close to unity, however, $\hat{\theta}_n^{ls}$ and $\hat{\theta}_n^b$ behave similarly and better than $\hat{\theta}_n^c$ while $\hat{\theta}_n^a$ is comparable to estimators in either group. For $1 - \theta_o^2$ in between zero and unity, $\hat{\theta}_n^{ls}$ and $\hat{\theta}_n^b$ have a comparable behaviour. They are superior to $\hat{\theta}_n^a$ and $\hat{\theta}_n^c$ which behave similarly to one another. This pattern of behaviour of the estimators is enhanced as the sample size increases.

Table 5.37

Performance of the new estimators in 50 simulations. Errors are from Laplace ($\frac{1}{\sqrt{2}}$,0), $\sigma^2 = 1$, $\theta_o = 0.8$, and $n = 25$

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^a$	$\hat{\theta}_n^b$	$\hat{\theta}_n^c$
$N(2,1)$	10.32	0.7358 (1.3375)	0.7361 (1.4682)	0.7207 (1.5027)	0.7459* (1.3262)¶
		1.5522 (0.0058)	1.6643 (0.0079)	1.6116 (0.0076)	1.5336† (0.0108)
		1.0738 (28.374)	1.0335 (23.380)	1.1004 (25.556)	0.9996‡ (22.137)
			1.0370	3.3638	0.0530§
$N(2,2)$	14.72	0.7138 (3.1845)	0.7106 (3.2450)	0.6888 (3.0758)	0.7146 (3.3808)
		1.6868 (0.0266)	1.8802 (0.0352)	1.7397 (0.0287)	1.8726 (0.0453)
		0.9329 (23.605)	0.8836 (21.851)	0.9824 (24.778)	0.8937 (26.000)
			0.7910	2.6640	0.0824
$N(2,4)$	18.00	0.7308 (2.1815)	0.7304 (2.2936)	0.7000 (2.3984)	0.7375 (2.1033)
		1.4008 (0.0080)	1.4730 (0.0080)	1.3545 (0.0058)	1.5240 (0.0129)
		0.9301 (21.368)	0.9084 (26.394)	1.0524 (33.423)	0.9068 (28.404)
			0.5908	2.0584	0.1110

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.38

Performance of the new estimators in 50 simulations. Errors are from Laplace $(\frac{1}{\sqrt{2}},0)$, $\sigma^2 = 1$, $\theta_o = 0.5$, and $n = 25$

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^a$	$\hat{\theta}_n^b$	$\hat{\theta}_n^c$
$N(2,1)$	8.96	0.4567 (2.2727)	0.4528 (2.0998)	0.4384 (1.9641)	0.4610[*] (2.1312)[¶]
		2.6036 (0.0252)	2.7237 (0.02325)	2.5565 (0.0209)	2.9211[†] (0.0335)
		0.8455 (10.320)	0.8538 (11.836)	0.9041 (13.792)	0.8258[‡] (11.041)
			0.0796	0.1862	0.0410[§]
$N(2,2)$	12.08	0.5119 (2.0976)	0.5270 (1.9913)	0.5047 (1.9185)	0.5302 (2.0564)
		2.3407 (0.0126)	2.5665 (0.0151)	2.3208 (0.0106)	2.6057 (0.0228)
		0.8869 (13.459)	0.8354 (18.414)	0.9091 (20.792)	0.8308 (17.152)
			0.1164	0.3370	0.0626
$N(2,4)$	15.84	0.5061 (2.0710)	0.4996 (2.5588)	0.4678 (2.3291)	0.5104 (2.6069)
		2.3098 (0.0120)	2.8411 (0.0559)	2.4023 (0.0207)	2.6251 (0.0512)
		0.8718 (14.900)	0.8007 (16.759)	0.9087 (20.105)	0.7989 (16.261)
			0.5868	1.4356	0.1106

[*] Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

[†] The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

[‡] The third value is mean estimate of σ^2.

[§] The fourth value is the average CPU time in seconds needed to compute the estimates.

[¶] The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.39

Performance of the new estimators in 50 simulations. Errors are from Laplace ($\frac{1}{\sqrt{2}}$,0), $\sigma^2 = 1$, $\theta_o = 0.2$, and $n = 25$

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^a$	$\hat{\theta}_n^b$	$\hat{\theta}_n^c$
$N(2,1)$	6.96	0.1869	0.1866	0.1816	0.1936*
		(1.8840)	(2.0273)	(1.9027)	(2.0064)¶
		2.9795	3.1135	2.9230	3.5719†
		(0.0275)	(0.0281)	(0.0237)	(0.0896)
		1.0623	1.0192	1.0564	0.9644‡
		(22.911)	(23.365)	(25.369)	(21.274)
			0.0650	0.1526	0.0330§
$N(2,2)$	12.00	0.1811	0.1762	0.1688	0.1823
		(3.4432)	(4.1700)	(3.7834)	(4.0794)
		2.6999	2.8711	2.5569	3.1127
		(0.0295)	(0.0214)	(0.0159)	(0.0539)
		1.0169	0.9645	1.0344	0.9294
		(14.758)	(15.461)	(16.927)	(13.044)
			0.3550	0.9148	0.0682
$N(2,4)$	16.96	0.1598	0.1808	0.1643	0.1938
		(3.9497)	(5.0753)	(4.3759)	(5.3246)
		2.7497	3.325	2.7866	3.8400
		(0.0241)	(0.0307)	(0.0164)	(0.1344)
		0.9932	0.8084	0.9112	0.7900
		(26.594)	(15.290)	(17.707)	(15.534)
			4.5292	12.421	0.1054

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.40

Performance of the new estimators in 50 simulations. Errors are from Laplace ($\frac{1}{\sqrt{2}}$,0), $\sigma^2 = 1$, $\theta_o = -0.2$, and $n = 25$

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^a$	$\hat{\theta}_n^b$	$\hat{\theta}_n^c$
$N(2,1)$	7.52	-0.2251 (2.5746)	-0.2222 (2.6243)	-0.2159 (2.5191)	-0.2254* (2.9314)¶
		2.7093 (0.0259)	2.7636 (0.0250)	2.6028 (0.0228)	2.8995† (0.0371)
		0.9368 (26.192)	0.8642 (11.796)	0.9005 (12.931)	0.8130‡ (10.068)
			0.0272	0.0450	0.0330§
$N(2,2)$	11.52	-0.1962 (3.0707)	-0.2018 (3.3782)	-0.1887 (2.9769)	-0.1997 (3.3438)
		3.0476 (0.0253)	3.1528 (0.0209)	2.8358 (0.0173)	3.4556 (0.0364)
		0.8668 (9.0343)	0.7911 (9.8039)	0.8544 (11.7570)	0.7635 (8.5161)
			0.0318	0.0640	0.0652
$N(2,4)$	18.4	-0.1730 (4.3453)	-0.1843 (4.3453)	-0.1703 (3.7073)	-0.1806 (4.6649)
		3.2785 (0.0218)	3.8978 (0.0243)	3.2437 (0.0160)	4.3879 (0.1606)
		1.0100 (23.598)	0.8443 (11.350)	0.9617 (13.547)	0.8211 (10.291)
			0.4690	1.5308	0.1174

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.41

Performance of the new estimators in 50 simulations. Errors are from Laplace ($\frac{1}{\sqrt{2}}$,0), $\sigma^2 = 1$, $\theta_o = -0.5$, and $n = 25$

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^a$	$\hat{\theta}_n^b$	$\hat{\theta}_n^c$
$N(2,1)$	9.20	-0.4753	-0.4832	-0.4702	-0.4852*
		(2.6113)	(2.6617)	(2.5279)	(2.6819)¶
		2.5212	2.6279	2.4611	2.5701†
		(0.0169)	(0.0172)	(0.0154)	(0.0358)
		0.9963	0.9358	0.9932	0.9162‡
		(26.700)	(21.459)	(23.3945)	(22.883)
			0.0306	0.0568	0.0560§
$N(2,2)$	14.24	-0.4322	-0.4319	-0.4091	-0.4335
		(3.9651)	(4.5445)	(4.2738)	(4.6195)
		2.8312	3.2510	2.8349	3.5902
		(0.0424)	(0.0481)	(0.0291)	(0.1480)
		0.9552	0.8898	0.9858	0.8722
		(17.209)	(13.228)	(16.464)	(13.115)
			0.1000	0.2504	0.1000
$N(2,4)$	17.84	-0.4725	-0.4627	-0.4260	-0.4492
		(3.0567)	(4.0428)	(3.6140)	(4.5242)
		2.7066	3.3258	2.7671	2.7071
		(0.0168)	(0.0448)	(0.0213)	(0.0574)
		0.81048	0.7472	0.8689	0.7593
		(10.373)	(10.824)	(14.0415)	(12.409)
			0.0764	0.1834	0.1404

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.42

Performance of the new estimators in 50 simulations. Errors are from Laplace ($\frac{1}{\sqrt{2}}$,0), $\sigma^2 = 1$, $\theta_o = -0.8$, and $n = 25$

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^a$	$\hat{\theta}_n^b$	$\hat{\theta}_n^c$
$N(2,1)$	12.88	-0.7565	-0.7559	-0.7387	-0.7551*
		(1.6833)	(1.7851)	(1.7344)	(1.7835)¶
		1.4825	1.5168	1.4526	1.4312†
		(0.0093)	(0.0099)	(0.0082)	(0.0095)
		0.9289	0.8824	0.9807	0.8784‡
		(11.067)	(10.708)	(11.797)	(11.099)
			0.0288	0.0516	0.0770§
$N(2,2)$	17.36	-0.7574	-0.7551	-0.7286	-0.7555
		(1.3484)	(1.3401)	(1.3734)	(1.3409)
		1.5982	1.7905	1.6602	1.5756
		(0.0101)	(0.0144)	(0.0107)	(0.0227)
		0.9537	0.9102	1.0602	0.9331
		(15.097)	(14.836)	(19.243)	(17.093)
			0.0680	0.1824	0.1154
$N(2,4)$	22.32	-0.7684	-0.7608	-0.7264	-0.7634
		(1.6283)	(2.0543)	(2.1249)	(2.2500)
		1.3485	1.5154	1.4063	1.4450
		(0.0110)	(0.0104)	(0.0094)	(0.0307)
		0.9865	0.9314	1.1442	0.9272
		(24.457)	(21.124)	(28.114)	(20.846)
			0.6262	2.1928	0.1758

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.43

Performance of the new estimators in 50 simulations. Errors are from Laplace ($\frac{1}{\sqrt{2}}$,0), $\sigma^2 = 1$, $\theta_o = 0.8$ and $n = 50$

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^a$	$\hat{\theta}_n^b$	$\hat{\theta}_n^c$
$N(2,1)$	10.20	0.7517	0.7506	0.7344	0.7523*
		(0.7952)	(0.8266)	(0.7998)	(0.8484)¶
		0.8278	0.8284	0.7922	0.8015†
		(0.0022)	(0.0022)	(0.0021)	(0.0025)
		0.9043	0.8711	0.9313	0.8654‡
		(8.8941)	(9.5672)	(10.473)	(9.6670)
			0.9008	2.8056	0.1678§
$N(2,2)$	13.80	0.7516	0.7542	0.7311	0.7613
		(0.7578)	(0.8380)	(0.9268)	(0.7386)
		0.7312	0.8208	0.7704	0.7808
		(0.0011)	(0.0018)	(0.0014)	(0.0017)
		0.9991	0.9632	1.0671	0.9511
		(10.094)	(11.097)	(14.093)	(11.255)
			1.6452	4.7432	0.2436
$N(2,4)$	20.84	0.7665	0.7686	0.7299	0.7723
		(0.7592)	(0.8590)	(0.9529)	(0.8226)
		0.7634	0.8654	0.7672	0.8164
		(0.0018)	(0.0027)	(0.0016)	(0.0033)
		0.9565	0.8798	1.0476	0.8967
		(5.8860)	(7.4930)	(9.2743)	(7.7175)
			3.4420	11.259	0.4536

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.44

Performance of the new estimators in 50 simulations. Errors are from Laplace ($\frac{1}{\sqrt{2}}$,0), $\sigma^2 = 1$, $\theta_o = 0.5$ and $n = 50$

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
$N(2,1)$	8.84	0.4850	0.4842	0.4714	0.4906*
		(1.8822)	(2.0314)	(1.9266)	(2.0990)¶
		1.4540	1.4985	1.4130	1.6252†
		(0.0043)	(0.0039)	(0.0035)	(0.0050)
		1.0485	1.0085	1.0569	0.9845‡
		(11.181)	(9.6704)	(10.737)	(10.410)
			0.5416	1.5784	0.1394§
$N(2,2)$	12.48	0.4874	0.4868	0.4642	0.4883
		(1.5846)	(1.4042)	(1.3162)	(1.3885)
		1.2549	1.3928	1.2436	1.3337
		(0.0028)	(0.0034)	(0.0026)	(0.0062)
		0.8687	0.8122	0.8853	0.8220
		(7.5446)	(7.1743)	(8.4002)	(7.9357)
			0.6466	1.6578	0.2538
$N(2,4)$	19.56	0.4918	0.4859	0.4485	0.4916
		(1.5115)	(2.0487)	(1.9371)	(2.0477)
		1.2047	1.4739	1.2464	1.7123
		(0.0022)	(0.0045)	(0.0023)	(0.0089)
		0.9889	0.9010	1.0434	0.8830
		(12.331)	(13.254)	(16.994)	(12.176)
			1.9846	5.8562	0.4294

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.45

Performance of the new estimators in 50 simulations. Errors are from Laplace ($\frac{1}{\sqrt{2}}$,0), $\sigma^2 = 1$, $\theta_o = 0.2$ and $n = 50$

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^a$	$\hat{\theta}_n^b$	$\hat{\theta}_n^c$
$N(2,1)$	6.96	0.1798 (2.4292)	0.1834 (2.3574)	0.1791 (2.2239)	0.1864* (2.4366)¶
		1.6801 (0.0059)	1.6982 (0.0041)	1.6235 (0.0037)	1.8452† (0.0072)
		1.0072 (10.310)	0.9746 (9.925)	1.0100 (10.554)	0.9363‡ (9.1630)
			0.1090	0.2580	0.1020§
$N(2,2)$	13.44	0.2027 (1.6185)	0.2049 (1.8468)	0.1949 (1.6578)	0.2112 (1.8867)
		1.7411 (0.0049)	1.8329 (0.0052)	1.6418 (0.0044)	1.9922 (0.0138)
		0.9234 (6.7838)	0.8677 (5.8861)	0.9410 (6.9687)	0.8539 (6.5717)
			0.4724	1.2796	0.2578
$N(2,4)$	19.48	0.1685 (2.3384)	0.1744 (2.3746)	0.1590 (1.9961)	0.1816 (2.4287)
		1.9405 (0.0164)	2.3478 (0.0151)	1.8604 (0.0067)	2.1679 (0.0242)
		1.0791 (14.833)	0.9669 (17.277)	1.0996 (21.088)	0.9853 (20.253)
			0.7812	2.0268	0.4858

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.46

Performance of the new estimators in 50 simulations. Errors are from Laplace ($\frac{1}{\sqrt{2}}$,0), $\sigma^2 = 1$, $\theta_o = -0.2$ and $n = 50$

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^a$	$\hat{\theta}_n^b$	$\hat{\theta}_n^c$
$N(2,1)$	7.36	-0.2135 (1.4894)	-0.2194 (1.5501)	-0.2132 (1.4721)	-0.2170* (1.6842)¶
		1.7134 (0.0056)	1.7682 (0.0059)	1.6734 (0.0053)	1.6459† (0.0155)
		0.9857 (7.3934)	0.9645 (8.5595)	1.0026 (9.1685)	0.9452‡ (9.2136)
			0.0526	0.0958	0.1408§
$N(2,2)$	12.36	-0.1923 (2.0363)	-0.1943 (2.1270)	-0.1852 (1.9191)	-0.1906 (2.1919)
		1.7209 (0.0065)	1.8286 (0.0044)	1.6380 (0.0036)	2.1221 (0.0097)
		0.9784 (8.4141)	0.9145 (7.5550)	0.9839 (8.5756)	0.9048 (7.7724)
			0.3542	0.8932	0.2426
$N(2,4)$	18.36	-0.2081 (1.3241)	-0.2019 (1.7466)	-0.1848 (1.4855)	-0.1975 (1.8952)
		1.7998 (0.0042)	2.2502 (0.0095)	1.8771 (0.0073)	2.2207 (0.0188)
		1.0271 (10.885)	0.8793 (8.7589)	0.9970 (10.784)	0.8653 (8.8080)
			0.3196	0.8544	0.4856

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.47

Performance of the new estimators in 50 simulations. Errors are from Laplace ($\frac{1}{\sqrt{2}}$,0), $\sigma^2 = 1$, $\theta_o = -0.5$ and $n = 50$

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
$N(2,1)$	9.08	-0.4995	-0.5037	-0.4892	-0.5107*
		(1.2280)	(1.4541)	(1.3904)	(1.4199)¶
		1.2660	1.3505	1.2573	1.5032†
		(0.0046)	(0.0030)	(0.0025)	(0.0076)
		1.0397	0.9649	1.0219	0.9302‡
		(10.237)	(7.2445)	(8.2888)	(6.7801)
			0.0650	0.1284	0.1704§
$N(2,2)$	12.52	-0.4758	-0.4743	-0.4533	-0.4779
		(1.5749)	(1.9447)	(1.7984)	(1.9598)
		1.4682	1.5465	1.3994	1.7021
		(0.0074)	(0.0110)	(0.0080)	(0.0284)
		0.9678	0.9174	0.9973	0.8989
		(8.9509)	(8.9103)	(10.062)	(7.8294)
			0.0680	0.1322	0.2876
$N(2,4)$	19.24	-0.4800	-0.4823	-0.4442	-0.4797
		(1.7476)	(2.0078)	(1.8060)	(2.0371)
		1.3403	1.6127	1.3694	1.8931
		(0.0050)	(0.0069)	(0.0040)	(0.0240)
		1.0273	0.9055	1.0532	0.9147
		(12.218)	(9.5180)	(11.876)	(10.324)
			0.2536	0.6856	0.5140

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.48

Performance of the new estimators in 50 simulations. Errors are from Laplace ($\frac{1}{\sqrt{2}}$,0), $\sigma^2 = 1$, $\theta_o = -0.8$ and $n = 50$

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^a$	$\hat{\theta}_n^b$	$\hat{\theta}_n^c$
$N(2,1)$	14.08	-0.7622	-0.7661	-0.7461	-0.7684*
		(1.0190)	(1.0238)	(1.0156)	(1.1379)¶
		0.7498	0.7544	0.7232	0.6806†
		(0.0015)	(0.0011)	(0.0011)	(0.0022)
		0.9468	0.8954	1.0046	0.8966‡
		(6.5949)	(4.5591)	(6.2611)	(5.2087)
			0.0594	0.1142	0.2932§
$N(2,2)$	15.40	-0.7375	-0.7304	-0.7049	-0.7344
		(1.0861)	(1.2532)	(1.2349)	(1.2934)
		0.8559	0.9851	0.9058	0.9296
		(0.0030)	(0.0062)	(0.0041)	(0.0103)
		1.0178	0.9836	1.1165	0.9680
		(12.222)	(11.856)	(14.255)	(10.760)
			0.0966	0.2594	0.3768
$N(2,4)$	20.96	-0.7812	-0.7865	-0.7503	-0.7861
		(0.7239)	(0.5516)	(0.6212)	(0.6090)
		0.7933	0.7988	0.7561	0.8435
		(0.0046)	(0.0029)	(0.0024)	(0.0067)
		0.9559	0.9126	1.1142	0.9251
		(7.1175)	(8.1060)	(11.706)	(8.5832)
			0.2342	0.7368	0.5120

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.49

Performance of the new estimators in 50 simulations. Errors are from Laplace $(\frac{1}{\sqrt{2}},0)$, $\sigma^2 = 1$, $\theta_o = 0.8$ and $n = 100$

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
$N(2,1)$	11.62	0.7721 (0.3639)	0.7698 (0.4292)	0.7509 (0.4688)	0.7748* (0.3988)¶
		0.3871 (0.0004)	0.4177 (0.0005)	0.3910 (0.0003)	0.3868† (0.0005)
		0.9516 (4.0550)	0.9195 (4.1017)	0.9858 (4.0627)	0.8967‡ (3.9043)
			1.8626	5.4002	0.7480§
$N(2,2)$	15.32	0.7691 (0.5494)	0.7712 (0.5241)	0.7463 (0.5676)	0.7739 (0.5116)
		0.4165 (0.0005)	0.4246 (0.0004)	0.3939 (0.0003)	0.4235 (0.0004)
		1.0048 (6.3391)	0.9587 (5.5451)	1.0712 (6.3967)	0.9516 (5.2957)
			3.1990	10.685	1.0896
$N(2,4)$	19.86	0.7767 (0.3311)	0.7835 (0.3470)	0.7458 (0.4167)	0.7858 (0.3424)
		0.3651 (0.0002)	0.3861 (0.0002)	0.3570 (0.0002)	0.3876 (0.0003)
		0.9464 (3.4227)	0.8834 (3.6314)	1.0522 (4.4469)	0.8789 (3.6957)
			4.9938	17.094	1.5900

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.50

Performance of the new estimators in 50 simulations. Errors are from Laplace ($\frac{1}{\sqrt{2}}$,0), $\sigma^2 = 1$, $\theta_o = 0.5$ and $n = 100$

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^a$	$\hat{\theta}_n^b$	$\hat{\theta}_n^c$
$N(2,1)$	8.32	0.4949 (0.7346)	0.4933 (0.7514)	0.4800 (0.7038)	0.4988* (0.7591)¶
		0.6688 (0.0006)	0.6878 (0.0006)	0.6463 (0.0005)	0.6880† (0.0011)
		1.0069 (5.7453)	0.9696 (6.0922)	1.0142 (6.6133)	0.9433‡ (5.5609)
			0.2438	0.5842	0.5328§
$N(2,2)$	14.30	0.5070 (0.8063)	0.5075 (0.8827)	0.4832 (0.8346)	0.5130 (0.8793)
		0.7359 (0.0007)	0.8178 (0.0009)	0.7235 (0.0007)	0.8453 (0.0023)
		1.0338 (4.5101)	0.9343 (3.5284)	1.0206 (4.0166)	0.9209 (3.3439)
			2.5846	6.8338	1.2050
$N(2,4)$	17.88	0.4861 (0.8834)	0.4877 (0.8859)	0.4522 (0.8626)	0.4915 (0.9089)
		0.7680 (0.0010)	0.8782 (0.0012)	0.7574 (0.0009)	0.8710 (0.0034)
		1.0129 (5.4960)	0.8909 (4.7863)	1.0174 (5.6140)	0.8842 (4.6143)
			2.5224	8.0396	1.7202

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.51

Performance of the new estimators in 50 simulations. Errors are from Laplace ($\frac{1}{\sqrt{2}}$,0), $\sigma^2 = 1$, $\theta_o = 0.2$ and $n = 100$

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
$N(2,1)$	7.80	0.2082	0.2025	0.1966	0.2068*
		(0.8943)	(0.9033)	(0.8613)	(0.8945)¶
		0.8201	0.8547	0.8039	0.8903†
		(0.0007)	(0.0008)	(0.0007)	(0.0012)
		1.0054	0.9528	0.9919	0.9244‡
		(5.5518)	(4.3390)	(4.7890)	(4.2826)
			0.4766	1.2232	0.4576§
$N(2,2)$	11.82	0.1872	0.1820	0.1741	0.1858
		(0.8131)	(0.8514)	(0.7921)	(0.8550)
		0.8593	0.9500	0.8572	1.1234
		(0.0007)	(0.0008)	(0.0006)	(0.0029)
		1.0304	0.9541	1.0212	0.9382
		(6.4220)	(5.3850)	(6.2204)	(5.1711)
			0.5184	1.2346	0.8932
$N(2,4)$	19.08	0.2070	0.2089	0.1906	0.2159
		(0.8592)	(1.1937)	(0.9981)	(1.1690)
		0.8762	1.0411	0.8488	1.0586
		(0.0011)	(0.0018)	(0.0010)	(0.0036)
		0.9714	0.8608	0.9797	0.8651
		(5.3350)	(5.0386)	(6.1284)	(5.7850)
			1.9256	4.6888	1.8270

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.52

Performance of the new estimators in 50 simulations. Errors are from Laplace ($\frac{1}{\sqrt{2}}$,0), $\sigma^2 = 1$, $\theta_o = -0.2$ and $n = 100$

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^a$	$\hat{\theta}_n^b$	$\hat{\theta}_n^c$
$N(2,1)$	7.64	-0.2071	-0.2121	-0.2058	-0.2141*
		(1.2109)	(1.2290)	(1.1515)	(1.3076)¶
		0.8531	0.8964	0.8395	0.9717†
		(0.0010)	(0.0010)	(0.0008)	(0.0018)
		0.9640	0.9249	0.9630	0.8906‡
		(4.9065)	(3.8735)	(4.2819)	(3.8959)
			0.0936	0.1690	0.4780§
$N(2,2)$	12.90	-0.1974	-0.2104	-0.1991	-0.2131
		(1.0582)	(1.0536)	(0.9477)	(1.0860)
		0.9491	1.0087	0.9067	1.1405
		(0.0023)	(0.0020)	(0.0016)	(0.0045)
		1.0004	0.8938	0.9638	0.8802
		(5.1916)	(4.2358)	(4.6254)	(4.5956)
			0.4548	1.1086	1.0818
$N(2,4)$	18.04	-0.2004	-0.2075	-0.1886	-0.2060
		(0.9043)	(1.1001)	(0.8859)	(1.0791)
		0.8387	1.0505	0.8658	1.2357
		(0.0009)	(0.0018)	(0.0011)	(0.0034)
		1.0182	0.9066	1.0260	0.9037
		(3.4645)	(2.5849)	(3.3942)	(2.7785)
			1.3806	3.7566	1.6202

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.53

Performance of the new estimators in 50 simulations. Errors are from Laplace $(\frac{1}{\sqrt{2}},0)$, $\sigma^2 = 1$, $\theta_o = -0.5$ and $n = 100$

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^a$	$\hat{\theta}_n^b$	$\hat{\theta}_n^c$
$N(2,1)$	8.48	-0.4950 (0.6132)	-0.4993 (0.6607)	-0.4856 (0.6391)	-0.5045* (0.6661)¶
		0.6742 (0.0008)	0.7460 (0.0012)	0.7039 (0.0011)	0.7864† (0.0041)
		1.0513 (5.9158)	1.0164 (5.6612)	1.0692 (6.3648)	0.9918‡ (5.6215)
			0.0936	0.1668	0.6998§
$N(2,2)$	12.92	-0.4949 (1.0360)	-0.4939 (0.8552)	-0.4703 (0.7839)	-0.4995 (0.8427)
		0.7626 (0.0010)	0.8304 (0.0013)	0.7610 (0.0011)	0.8007 (0.0029)
		1.0076 (5.5104)	0.9690 (6.4449)	1.0572 (7.5987)	0.9542 (6.2394)
			0.1496	0.3150	1.2224
$N(2,4)$	19.80	-0.4965 (0.6144)	-0.5029 (0.9108)	-0.4645 (0.8105)	-0.5036 (0.9763)
		0.6714 (0.0005)	0.7808 (0.0008)	0.6743 (0.0005)	0.7782 (0.0016)
		0.9968 (4.4654)	0.9014 (3.9653)	1.0504 (5.3619)	0.8998 (3.8536)
			1.7172	5.1224	1.9024

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.54

Performance of the new estimators in 50 simulations. Errors are from Laplace ($\frac{1}{\sqrt{2}}$,0), $\sigma^2 = 1$, $\theta_o = -0.8$ and $n = 100$

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^a$	$\hat{\theta}_n^b$	$\hat{\theta}_n^c$
$N(2,1)$	15.72	-0.7851	-0.7850	-0.7637	-0.7897*
		(0.4516)	(0.4366)	(0.4391)	(0.4280)¶
		0.3624	0.4038	0.3788	0.3349†
		(0.0003)	(0.0004)	(0.0003)	(0.0005)
		1.0351	0.9909	1.1181	0.9897‡
		(4.9756)	(5.4533)	(6.8469)	(6.0898)
			0.1242	0.2346	1.5158§
$N(2,2)$	17.82	-0.7995	-0.7945	-0.7678	-0.8002
		(0.3044)	(0.2994)	(0.3106)	(0.3295)
		0.3594	0.4170	0.3929	0.3167
		(0.0003)	(0.0004)	(0.0004)	(0.0004)
		1.0595	1.0213	1.1851	1.0220
		(6.8139)	(98.202)	(10.675)	(8.7499)
			0.1560	0.3406	1.8126
$N(2,4)$	22.12	-0.7753	-0.7759	-0.7351	-0.7787
		(0.3030)	(0.3303)	(0.3601)	(0.3259)
		0.41581	0.4815	0.4433	0.4207
		(0.0004)	(0.0012)	(0.0008)	(0.0176)
		0.9677	0.8989	1.1077	0.9044
		(3.2251)	(3.1994)	(4.0798)	(3.3648)
			0.7810	2.6846	2.2868

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Figure 5.37
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Laplace $(\frac{1}{\sqrt{2}}, 0)$, $\sigma^2 = 1$, $\theta_o = 0.8$ and $n = 25$.

Estimator $\quad \underline{\hat{\theta}_n^{ls} \quad \hat{\theta}_n^{b} \quad \hat{\theta}_n^{c} \quad \hat{\theta}_n^{a}}$

Figure 5.38
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Laplace $(\frac{1}{\sqrt{2}}, 0)$, $\sigma^2 = 1$, $\theta_o = 0.5$ and $n = 25$.

Estimator $\quad \underline{\hat{\theta}_n^{ls} \quad \hat{\theta}_n^{b}} \quad \underline{\hat{\theta}_n^{a} \quad \hat{\theta}_n^{c}}$

Figure 5.39
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Laplace $(\frac{1}{\sqrt{2}}, 0)$, $\sigma^2 = 1$, $\theta_o = 0.2$ and $n = 25$.

Estimator $\quad \hat{\theta}_n^{b} \quad \hat{\theta}_n^{ls} \quad \hat{\theta}_n^{a} \quad \hat{\theta}_n^{c}$

$\underline{\hat{\theta}_n^{b} \quad \hat{\theta}_n^{ls}}$

$\underline{\hat{\theta}_n^{ls} \quad \hat{\theta}_n^{a}}$

Figure 5.40

Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Laplace $(\frac{1}{\sqrt{2}},0)$, $\sigma^2 = 1$, $\theta_o = -0.2$ and $n = 25$.

Estimator	$\hat{\theta}_n^b$	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^a$	$\hat{\theta}_n^c$
	———	———	———	
			———	———

Figure 5.41

Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Laplace $(\frac{1}{\sqrt{2}},0)$, $\sigma^2 = 1$, $\theta_o = -0.5$ and $n = 25$.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^b$	$\hat{\theta}_n^c$	$\hat{\theta}_n^a$
	———	———	———	———

Figure 5.42

Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Laplace $(\frac{1}{\sqrt{2}},0)$, $\sigma^2 = 1$, $\theta_o = -0.8$ and $n = 25$.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^c$	$\hat{\theta}_n^b$	$\hat{\theta}_n^a$
	———	———	———	———

Figure 5.43
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Laplace ($\frac{1}{\sqrt{2}}$,0), $\sigma^2 = 1$, $\theta_o = 0.8$ and $n = 50$.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$	$\hat{\theta}_n^{a}$

Figure 5.44
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Laplace ($\frac{1}{\sqrt{2}}$,0), $\sigma^2 = 1$, $\theta_o = 0.5$ and $n = 50$.

Estimator	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{c}$

Figure 5.45
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Laplace ($\frac{1}{\sqrt{2}}$,0), $\sigma^2 = 1$, $\theta_o = 0.2$ and $n = 50$.

Estimator	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{c}$

Figure 5.46
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Laplace $(\frac{1}{\sqrt{2}},0)$, $\sigma^2 = 1$, $\theta_o = -0.2$ and $n = 50$.

Estimator $\quad \underline{\hat{\theta}_n^b \quad \hat{\theta}_n^{ls} \quad \hat{\theta}_n^a \quad \hat{\theta}_n^c}$

Figure 5.47
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Laplace $(\frac{1}{\sqrt{2}},0)$, $\sigma^2 = 1$, $\theta_o = -0.5$ and $n = 50$.

Estimator $\quad \underline{\hat{\theta}_n^b \quad \hat{\theta}_n^{ls} \quad \hat{\theta}_n^a} \quad \hat{\theta}_n^c$

Figure 5.48
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Laplace $(\frac{1}{\sqrt{2}},0)$, $\sigma^2 = 1$, $\theta_o = -0.8$ and $n = 50$.

Estimator $\quad \underline{\hat{\theta}_n^b \quad \hat{\theta}_n^{ls} \quad \hat{\theta}_n^c \quad \hat{\theta}_n^a}$

Figure 5.49
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Laplace ($\frac{1}{\sqrt{2}}$,0), $\sigma^2 = 1$, $\theta_o = 0.8$ and $n = 100$.

Estimator $\hat{\theta}_n^b$ $\hat{\theta}_n^{ls}$ $\hat{\theta}_n^c$ $\hat{\theta}_n^a$

Figure 5.50
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Laplace ($\frac{1}{\sqrt{2}}$,0), $\sigma^2 = 1$, $\theta_o = 0.5$ and $n = 100$.

Estimator $\hat{\theta}_n^b$ $\hat{\theta}_n^{ls}$ $\hat{\theta}_n^a$ $\hat{\theta}_n^c$

Figure 5.51
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Laplace ($\frac{1}{\sqrt{2}}$,0), $\sigma^2 = 1$, $\theta_o = 0.2$ and $n = 100$.

Estimator $\hat{\theta}_n^b$ $\hat{\theta}_n^{ls}$ $\hat{\theta}_n^a$ $\hat{\theta}_n^c$

Figure 5.52
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Laplace ($\frac{1}{\sqrt{2}}$,0), $\sigma^2 = 1$, $\theta_o = -0.2$ and $n = 100$.

Estimator	$\hat{\theta}_n^b$	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^a$	$\hat{\theta}_n^c$

Figure 5.53
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Laplace ($\frac{1}{\sqrt{2}}$,0), $\sigma^2 = 1$, $\theta_o = -0.5$ and $n = 100$.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^b$	$\hat{\theta}_n^a$	$\hat{\theta}_n^c$

Figure 5.54
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Laplace ($\frac{1}{\sqrt{2}}$,0), $\sigma^2 = \sqrt{0.5}$, $\theta_o = -0.8$ and $n = 100$.

Estimator	$\hat{\theta}_n^c$	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^b$	$\hat{\theta}_n^a$

5.3.4 Errors from the Gamma distribution with unit variance

Next we shall discuss the results for which the errors are drawn from Gamma $(3,\frac{1}{\sqrt{3}},-\sqrt{3})$. As noted earlier, these results are summarized in Tables 5.55 - 5.72. First, we shall examine the behaviour of the estimators with respect to the bias criterion. Like in the previous cases where the errors are Gaussian or Laplace, the estimators have a similar behaviour for values of $1-\theta_o^2$ near unity. For smaller values of $1-\theta_o^2$, $\hat{\theta}_n^b$ tends to have larger bias than the rest of the estimators which perform comparably among themselves. Once again, the performance of the estimators improves as the sample size increases.

We shall now study the behaviour of the estimators with respect to the asymptotic variance criterion. As in the Gaussian case and indeed the case where the errors come from the Laplace distribution, for all the sample sizes considered in this study, the estimated asymptotic variances of the estimators decrease as $1-\theta_o^2$ decreases. The estimated asymptotic variances of the estimators decrease with increasing sample size. To study the behaviour of the estimators as the censor pattern changes, we make use of the randomized block analyses of variance for Tables 5.55 - 5.72. As noted earlier, these analyses were carried out on the estimated asymptotic variances of the estimators with censor patterns regarded as 'blocks' and the estimators as treatments. For the purpose of monitoring the effect of changes in censor pattern, the same analyses of variance are used, except that the estimators are taken as 'blocks' and the censor patterns as treatments.

As in the previous cases, there are 18 analyses of variance. Six of these 18 cases show a statistically significant effect due to censor pattern at the 0.05 level. In 2 of these 6 cases, the estimators perform better for the censor pattern Gamma (3,1,-1) than they do for Gamma (3,1,-0.5) and Gamma (3,1,-0.75), which are not significantly different from each other using Fisher's LSD analyses. In 2 of the remaining 4 cases, the behaviour of the estimators for gamma

(3,1,-0.5) is similar to their behaviour for Gamma (3,1,-1), which is different from their behaviour for Gamma (3,1,-0.75). In one of these 2 cases, the estimators behave better for Gamma (3,1,-0.75), whereas in the other case they behave worse for this censor pattern. In one of the remaining 2 cases, Gamma (3,1,-0.5) and Gamma (3,1,-1) are not significantly different from each other. Similarly, Gamma (3,1,-1) and Gamma (3,1,-0.75) are not significantly different, but Gamma (3,1,-0.5) is significantly better than Gamma (3,1,-0.75). In the last case, all the censoring patterns are significantly different from one another, with Gamma (3,1,-1) being the best, Gamma (3,1,-0.5) the second best and Gamma (3,1,-0.75) the worst. These results can be further summarized as follows. In the majority of cases (two-thirds), there is no sufficient evidence the performance of the estimators is sensitive to changes in censor pattern. In the remaining one-third of the cases, the performance of the estimators either first remains unaltered or deteriorates and then improves as the censor pattern is successively changed from Gamma (3,1,-0.5) to Gamma (3,1,-0.75) and from the latter censor pattern to Gamma (3,1,-1).

To compare the estimators among themselves, we revert back to the original use of the analyses of variance taking censor patterns as 'blocks' and the 4 estimators as treatments. of the 18 analyses of variance, 10 show a significant estimator effect at the 0.05 level. Three of these 10 cases correspond to small samples, 3 correspond to moderate samples and 4 correspond to large samples. For small samples, in 2 of the 3 analyses of variance wich show a significant estimator effect (Figures 5.57 and 5.59), $\hat{\theta}_n^b$, $\hat{\theta}_n^{ls}$ and $\hat{\theta}_n^a$ are not significantly different among themselves at the 0.05 level using Fisher's LSD analyses. In one of these 2 cases (Figure 5.57), $\hat{\theta}_n^c$ is worse than the rest of the estimators, whereas in the second case (Figure 5.59), $\hat{\theta}_n^c$ compares favourably with $\hat{\theta}_n^a$. In the third analysis of variance (Figure 5.58), $\hat{\theta}_n^{ls}$, $\hat{\theta}_n^a$ and $\hat{\theta}_n^c$ have a similar behaviour. The estimator $\hat{\theta}_n^b$ is superior to $\hat{\theta}_n^a$ and $\hat{\theta}_n^c$ and behaves similarly to $\hat{\theta}_n^{ls}$.

For moderate samples, in one of the 3 analyses of variance with a significant estimator effect (Figure 5.64), $\hat{\theta}_n^{ls}$, $\hat{\theta}_n^b$ and $\hat{\theta}_n^a$ are comparable among themselves and perform better than $\hat{\theta}_n^c$. In the second case (Figure 5.66), $\hat{\theta}_n^{ls}$, $\hat{\theta}_n^b$ and $\hat{\theta}_n^c$ behave similarly to each other and $\hat{\theta}_n^a$ performs as good as $\hat{\theta}_n^b$ but worse than the other two estimators. In the third case (Figure 5.62), $\hat{\theta}_n^{ls}$ and $\hat{\theta}_n^a$ are comparable and their performance is better than the one for $\hat{\theta}_n^c$. The estimator $\hat{\theta}_n^b$ is superior over all the other estimators.

For large samples, in 3 of the 4 analyses of variance which exhibit a significant estimator effect (Figures 5.67, 5.68 and 5.69), $\hat{\theta}_n^a$ performs as good $\hat{\theta}_n^{ls}$. In one of these 3 cases (Figure 5.67), $\hat{\theta}_n^c$ is comparable to $\hat{\theta}_n^a$ while $\hat{\theta}_n^b$ is superior to the rest of the estimators. In the second case (Figure 5.68), $\hat{\theta}_n^{ls}$ and $\hat{\theta}_n^a$ perform better than $\hat{\theta}_n^c$ while $\hat{\theta}_n^b$ performs better than the rest of the estimators. In the third case (Figure 5.69), $\hat{\theta}_n^{ls}$ and $\hat{\theta}_n^a$ are comparable with both $\hat{\theta}_n^b$ and $\hat{\theta}_n^c$ and $\hat{\theta}_n^b$ is better than $\hat{\theta}_n^c$. In the fourth analysis of variance with a significant estimator effect (Figure 5.70), $\hat{\theta}_n^b$ and $\hat{\theta}_n^{ls}$ are comparable to one another and perform better than the rest of the estimators.

Overall, for small and moderate samples, in one-half of the cases, the estimators are comparable among themselves. In the rest of the cases, $\hat{\theta}_n^{ls}$, $\hat{\theta}_n^b$ and $\hat{\theta}_n^a$ tend to have a similar behaviour. Also, for these cases, sometimes $\hat{\theta}_n^c$ is comparable to $\hat{\theta}_n^a$ and other times it is performs worse than the rest of the estimators. For large samples, in one-third of the cases, the estimators behave similarly to one other. In the remaining two-thirds of the cases, $\hat{\theta}_n^a$ compares favourably with $\hat{\theta}_n^{ls}$. In some of these cases, $\hat{\theta}_n^c$ is comparable with $\hat{\theta}_n^a$ or with both $\hat{\theta}_n^a$ and $\hat{\theta}_n^{ls}$. In the rest of the cases, $\hat{\theta}_n^c$ is worse than all the other estimators. In some the cases (one-half), the estimator $\hat{\theta}_n^b$ performs comparably with $\hat{\theta}_n^{ls}$ or with both $\hat{\theta}_n^{ls}$ and $\hat{\theta}_n^a$. In the rest of cases, $\hat{\theta}_n^b$ is superior to the rest of the estimators. This superiority of $\hat{\theta}_n^b$ over the other estimators compansates for its inferior performance with respect to the bias criterion.

Table 5.55

Performance of the new estimators in 50 simulations. Errors are from Gamma $(3,\frac{1}{\sqrt{3}},-\sqrt{3})$, $\sigma^2 = 1$, $\theta_o = 0.8$ and $n = 25$. Ga refers to the Gamma distribution (as the censoring distribution)

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^a$	$\hat{\theta}_n^b$	$\hat{\theta}_n^c$
Ga (3,1,-0.5)	10.08	0.7383	0.7359	0.7196	0.7513*
		(1.3592)	(1.3805)	(1.4467)	(1.3623)¶
		1.6271	1.8026	1.6357	1.6823†
		(0.0097)	(0.0138)	(0.0103)	(0.0126)
		0.9101	0.7912	0.8508	0.7666‡
		(7.5204)	(4.5019)	(5.5108)	(4.4585)
			3.3142	7.8268	0.0482§
Ga (3,1,-0.75)	12.40	0.7617	0.7587	0.7351	0.7802
		(1.6941)	(1.7319)	(1.8420)	(1.6979)
		1.3845	1.3015	1.5808	1.3031
		(0.0088)	(0.0070)	(0.0081)	(0.0063)
		0.8882	0.7837	0.8864	0.7374
		(10.671)	(6.9701)	(8.0528)	(6.1442)
			1.8938	4.9522	0.0682
Ga (3,1,-1)	14.24	0.7269	0.7096	0.6796	0.7309
		(1.3426)	(1.6977)	(1.6372)	(1.6394)
		1.7673	1.8120	1.7845	2.0510
		(0.0078)	(0.0088)	(0.0095)	(0.0160)
		0.8887	0.8355	0.9609	0.8279
		(5.9142)	(6.8024)	(7.7436)	(8.6586)
			10.326	35.983	0.0858

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.56

Performance of the new estimators in 50 simulations. Errors are from Gamma $(3,\frac{1}{\sqrt{3}},-\sqrt{3})$, $\sigma^2 = 1$, $\theta_o = 0.5$ and $n = 25$. Ga refers to the Gamma distribution (as the censoring distribution)

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
Ga (3,1,-0.5)	9.36	0.4355 (3.7604)	0.4528 (3.6966)	0.4394 (3.5478)	0.4651* (3.8878)¶
		2.5737 (0.0109)	2.7618 (0.0104)	2.5708 (0.0090)	2.5629† (0.0215)
		1.0072 (18.140)	0.9081 (12.625)	0.9644 (14.021)	0.8914‡ (19.223)
			0.0798	0.2042	0.0504§
Ga (3,1,-0.75)	9.44	0.4701 (3.9732)	0.4755 (4.2145)	0.4576 (3.9492)	0.4861 (4.5694)
		2.5205 (0.0149)	2.6271 (0.0131)	2.4259 (0.0127)	2.7943 (0.0211)
		0.9670 (14.045)	0.8614 (8.2534)	0.9201 (9.1759)	0.8524 (10.232)
			0.4152	1.0688	0.0424
Ga (3,1,-1)	15.20	0.4738 (1.7811)	0.4840 (1.9136)	0.4563 (1.7096)	0.5024 (2.1981)
		2.4677 (0.0108)	2.4836 (0.0103)	2.1977 (0.0078)	2.5472 (0.0175)
		0.9326 (16.435)	0.8102 (7.8638)	0.9099 (11.417)	0.8295 (18.206)
			0.6546	1.6596	0.0866

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.57

Performance of the new estimators in 50 simulations. Errors are from Gamma $(3,\frac{1}{\sqrt{3}},-\sqrt{3})$, $\sigma^2 = 1$, $\theta_o = 0.2$ and $n = 25$. Ga refers to the Gamma distribution (as the censoring distribution)

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^a$	$\hat{\theta}_n^b$	$\hat{\theta}_n^c$
Ga (3,1,-0.5)	8.08	0.2314 (3.7579)	0.2295 (4.0043)	0.2226 (3.7751)	0.2346* (3.9594)¶
		3.1298 (0.0158)	3.3351 (0.0126)	3.1372 (0.0116)	3.9568† (0.0716)
		0.9259 (15.384)	0.8627 (9.4917)	0.9035 (10.636)	0.8185‡ (8.4945)
			0.0828	0.1964	0.0372§
Ga (3,1,-0.75)	10.40	0.2047 (4.0428)	0.2062 (4.2615)	0.1967 (3.8215)	0.2132 (4.4915)
		3.3177 (0.0160)	3.6693 (0.0249)	3.3134 (0.0202)	4.1696 (0.0570)
		0.9185 (10.636)	0.8564 (11.167)	0.9113 (11.743)	0.8247 (14.449)
			0.4196	1.0384	0.0496
Ga (3,1,-1)	12.56	0.1884 (3.6912)	0.2019 (3.6327)	0.1918 (3.2817)	0.2100 (3.6807)
		3.0000 (0.0165)	3.2217 (0.0117)	2.8746 (0.0091)	3.2793 (0.0681)
		0.9572 (13.424)	0.8054 (7.3391)	0.8830 (9.3250)	0.8008 (10.401)
			0.0780)	0.1806	0.0814

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.58

Performance of the new estimators in 50 simulations. Errors are from Gamma $(3,\frac{1}{\sqrt{3}},-\sqrt{3})$, $\sigma^2 = 1$, θ_o = -0.2 and n = 25. Ga refers to the Gamma distribution (as the censoring distribution)

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^a$	$\hat{\theta}_n^b$	$\hat{\theta}_n^c$
Ga (3,1,-0.5)	6.96	-0.1500 (3.3204)	-0.1548 (3.3824)	-0.1509 (3.2004)	-0.1463* (3.5984)¶
		3.4160 (0.0210)	3.5311 (0.0158)	3.3407 (0.0141)	3.3303† (0.0231)
		0.9891 (20.717)	0.9106 (11.397)	0.9463 (12.2930)	0.8802‡ (12.189)
			0.0242	0.0442	0.0328§
Ga (3,1,-0.75)	10.72	-0.2080 (2.9592)	-0.2145 (3.0627)	-0.2055 (2.8145)	-0.2157 (3.2270)
		3.2850 (0.0192)	3.2949 (0.0176)	3.0458 (0.0170)	3.3782 (0.0443)
		0.9947 (20.665)	0.9298 (15.527)	0.9908 (17.081)	0.9207 (17.804)
			0.0362	0.0714	0.0604
Ga (3,1,-1)	12.08	-0.2040 (4.0397)	-0.1871 (4.5406)	-0.1796 (4.1032)	-0.1883 (4.9208)
		3.3973 (0.0207)	3.5513 (0.0206)	3.2054 (0.0172)	3.5603 (0.0538)
		0.9345 (8.3546)	0.8431 (5.7112)	0.9160 (6.4189)	0.8183 (7.3136)
			0.0848	0.1976	0.0680

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.59

Performance of the new estimators in 50 simulations. Errors are from Gamma $(3, \frac{1}{\sqrt{3}}, -\sqrt{3})$, $\sigma^2 = 1$, θ_o = -0.5 and $n = 25$. Ga refers to the Gamma distribution (as the censoring distribution)

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
Ga (3,1,-0.5)	9.76	-0.4548	-0.4583	-0.4455	-0.4671*
		(3.3563)	(3.5682)	(3.3665)	(3.7427)¶
		2.4084	2.4126	2.2913	2.5010†
		(0.0106)	(0.0112)	(0.0097)	(0.0221)
		0.9487	0.8779	0.9361	0.8443‡
		(15.319)	(10.626)	(12.461)	(10.937)
			0.0330	0.0580	0.0508§
Ga (3,1,-0.75)	12.48	-0.4556	-0.4669	-0.4461	-0.4670
		(4.2589)	(4.1840)	(3.9042)	(4.2332)
		2.7943	2.9176	2.6376	3.3238
		(0.0126)	(0.0141)	(0.0117)	(0.0597)
		0.9522	0.8283	0.9115	0.8171
		(14.093)	(7.1597)	(8.4194)	(10.978)
			0.0450	0.0868	0.0670
Ga (3,1,-1)	14.08	-0.4743	-0.4842	-0.4609	-0.4870
		(4.2495)	(4.2593)	(3.8846)	(4.4246)
		2.3746	2.6571	2.3751	2.8728
		(0.0191)	(0.0194)	(0.0151)	(0.0855)
		0.9382	0.8073	0.9023	0.8066
		(21.967)	(13.115)	(15.300)	(16.742)
			0.0958	0.2286	0.0802

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.60

Performance of the new estimators in 50 simulations. Errors are from Gamma $(3,\frac{1}{\sqrt{3}},-\sqrt{3})$, $\sigma^2 = 1$, θ_o = -0.8 and n = 25. Ga refers to the Gamma distribution (as the censoring distribution)

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
Ga (3,1,-0.5)	14.80	-0.7582 (1.8192)	-0.7560 (1.7338)	-0.7377 (1.7223)	-0.7711* (1.8178)¶
		1.4853 (0.0096)	1.3546 (0.0083)	1.2846 (0.0063)	1.4192† (0.0144)
		0.9221 (10.025)	0.9325 (12.999)	1.0197 (11.437)	0.8530‡ (7.5932)
			0.0496	0.0836	0.1032§
Ga (3,1,-0.75)	16.64	-0.7480 (1.7175)	-0.7355 (1.4812)	-0.71363 (1.4918)	-0.7489 (1.6517)
		1.5653 (0.0082)	1.8331 (0.0102)	1.7203 (0.0081)	1.7621 (0.0152)
		1.0728 (19.984)	1.0805 (30.192)	1.2142 (37.247)	1.0818 (31.072)
			0.0506	0.0900	0.1070
Ga (3,1,-1)	18.24	-0.7776 (1.0609)	-0.7731 (0.8448)	-0.7476 (0.8963)	-0.7882 (1.0251)
		1.4161 (0.0077)	1.2736 (0.0091)	1.2265 (0.0055)	1.4267 (0.0127)
		0.9727 (19.802)	0.9327 (17.326)	1.0706 (20.285)	0.8616 (15.693)
			0.0678	0.1132	0.1320

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.61

Performance of the new estimators in 50 simulations. Errors are from Gamma $(3,\frac{1}{\sqrt{3}},-\sqrt{3})$, $\sigma^2 = 1$, $\theta_o = 0.8$ and $n = 50$. Ga refers to the Gamma distribution (as the censoring distribution)

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
Ga (3,1,-0.5)	12.72	0.7545 (0.9308)	0.7573 (0.9207)	0.7348 (0.9632)	0.7663* (0.9283)¶
		0.7923 (0.0011)	0.7898 (0.0013)	0.7315 (0.0013)	0.7361† (0.0018)
		0.9569 (7.2163)	0.8622 (4.1754)	0.9496 (5.7136)	0.8684‡ (5.4446)
			2.9638	8.7530	0.2490§
Ga (3,1,-0.75)	15.12	0.7603 (0.8407)	0.7529 (0.8880)	0.7260 (0.9600)	0.7662 (0.9093)
		0.8162 (0.0016)	0.8077 (0.0017)	0.7120 (0.0013)	0.8576 (0.0027)
		0.9728 (7.6690)	0.8918 (4.9140)	0.9939 (6.3195)	0.9287 (10.147)
			8.0070	22.967	0.2712
Ga (3,1,-1)	14.48	0.7765 (0.5527)	0.7661 (0.7609)	0.7308 (0.8323)	0.7832 (0.8313)
		0.7797 (0.0009)	0.8670 (0.0073)	0.8618 (0.0165)	0.8652 (0.0283)
		0.9141 (4.8617)	0.8161 (2.7194)	0.9428 (3.7431)	0.8049 (5.0835)
			13.037	37.871	0.3240

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.62

Performance of the new estimators in 50 simulations. Errors are from Gamma $(3,\frac{1}{\sqrt{3}},-\sqrt{3})$, $\sigma^2 = 1$, $\theta_o = 0.5$ and $n = 50$. Ga refers to the Gamma distribution (as the censoring distribution)

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^a$	$\hat{\theta}_n^b$	$\hat{\theta}_n^c$
Ga (3,1,-0.5)	9.60	0.4675	0.4682	0.4529	0.4711*
		(1.8592)	(1.8466)	(1.7212)	(2.0632)¶
		1.4368	1.4963	1.3864	1.6481†
		(0.0025)	(0.0020)	(0.0020)	(0.0023)
		0.9616	0.9093	0.9646	0.9304‡
		(7.5732)	(6.2306)	(7.4137)	(11.424)
			0.5656	1.4762	0.1592§
Ga (3,1,-0.75)	11.44	0.4771	0.4756	0.4558	0.4846
		(1.7677)	(1.7488)	(1.6774)	(1.8099)
		1.3569	1.4206	1.2655	1.4783
		(0.0014)	(0.0016)	(0.0013)	(0.0094)
		0.9688	0.8865	0.9566	0.8789
		(10.576)	(6.8362)	(8.1495)	(8.3714)
			1.1148	3.0980	0.2306
Ga (3,1,-1)	14.76	0.4862	0.4828	0.4563	0.4965
		(1.5845)	(1.5683)	(1.4182)	(1.6599)
		1.4942	1.4643	1.2664	1.7002
		(0.0039)	(0.0025)	(0.0022)	(0.0129)
		0.9192	0.8490	0.9440	0.8509
		(6.5422)	(4.1766)	(4.9525)	(7.4259)
			2.6596	7.0696	0.3492

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.63

Performance of the new estimators in 50 simulations. Errors are from Gamma ($3,\frac{1}{\sqrt{3}},-\sqrt{3}$), $\sigma^2 = 1$, $\theta_o = 0.2$ and $n = 50$. Ga refers to the Gamma distribution (as the censoring distribution)

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
Ga (3,1,-0.5)	8.16	0.2114 (2.3513)	0.2157 (2.3192)	0.2099 (2.2131)	0.2181* (2.4227)¶
		1.7319 (0.0026)	1.7614 (0.0015)	1.6600 (0.0014)	1.7187† (0.0063)
		1.0055 (8.4535)	0.9360 (7.3054)	0.9769 (8.0994)	0.9512‡ (12.131)
			0.4398	0.9734	0.1504§
Ga (3,1,-0.75)	12.04	0.2024 (2.3486)	0.1980 (2.3239)	0.1897 (2.1682)	0.2042 (2.3820)
		1.7808 (0.0046)	1.8674 (0.0018)	1.6861 (0.0017)	2.1056 (0.0087)
		1.0552 (6.4668)	0.9112 (3.8088)	0.9794 (4.3408)	0.9168 (8.1430)
			0.1778	0.4170	0.2244
Ga (3,1,-1)	13.52	0.1633 (2.2036)	0.1709 (2.2716)	0.1617 (2.0547)	0.1759 (2.2996)
		1.6397 (0.0034)	1.8387 (0.0026)	1.6274 (0.0019)	2.0484 (0.0080)
		0.9678 (7.9568)	0.8413 (3.8716)	0.9227 (4.9380)	0.8412 (6.1827)
			0.2900	0.6976	0.2604

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.64

Performance of the new estimators in 50 simulations. Errors are from Gamma $(3,\frac{1}{\sqrt{3}},-\sqrt{3})$, $\sigma^2 = 1$, θ_o = -0.2 and n = 50. Ga refers to the Gamma distribution (as the censoring distribution)

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
Ga (3,1,-0.5)	7.68	-0.2209 (1.7452)	-0.2180 (1.8311)	-0.2122 (1.7491)	-0.2209* (1.7654)¶
		1.7044 (0.0038)	1.8000 (0.0038)	1.6936 (0.0034)	1.8436† (0.0071)
		1.0060 (7.5436)	0.9376 (5.7072)	0.9757 (6.2178)	0.9390‡ (10.821)
			0.0494	0.0986	0.1148§
Ga (3,1,-0.75)	9.60	-0.1685 (1.3965)	-0.1676 (1.4911)	-0.1615 (1.3804)	-0.1676 (1.5333)
		1.7136 (0.0020)	1.8048 (0.0027)	1.6670 (0.0023)	2.0307 (0.0107)
		0.9563 (7.9840)	0.8722 (3.9326)	0.9228 (4.7734)	0.8696 (5.9658)
			0.0726	0.1394	0.1638
Ga (3,1,-1)	12.64	-0.1704 (1.5830)	-0.1817 (1.4120)	-0.1727 (1.2707)	-0.1785 (1.4308)
		1.8398 (0.0053)	1.8635 (0.0023)	1.6544 (0.0019)	2.1719 (0.0127)
		0.9489 (4.6635)	0.8488 (3.6099)	0.9206 (3.8485)	0.8374 (5.1459)
			0.0942	0.1976	0.2328

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.65

Performance of the new estimators in 50 simulations. Errors are from Gamma $(3,\frac{1}{\sqrt{3}},-\sqrt{3})$, $\sigma^2 = 1$, θ_o = -0.5 and n = 50. Ga refers to the Gamma distribution (as the censoring distribution)

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
Ga (3,1,-0.5)	10.00	-0.4642	-0.4632	-0.4481	-0.4678*
		(1.8207)	(1.9430)	(1.8801)	(1.9214)¶
		1.5279	1.6220	1.4949	1.7579†
		(0.0033)	(0.0036)	(0.0029)	(0.0061)
		0.9802	0.9173	0.9768	0.9144‡
		(6.0416)	(4.5937)	(4.9916)	(5.4171)
			0.0606	0.1194	0.1600§
Ga (3,1,-0.75)	10.80	-0.4312	-0.4379	-0.4207	-0.4412
		(1.2599)	(1.3642)	(1.2625)	(1.4515)
		1.5995	1.6582	1.5213	1.6309
		(0.0040)	(0.0034)	(0.0030)	(0.0057)
		0.9351	0.8604	0.9276	0.8884
		(10.935)	(6.2839)	(7.4801)	(20.4250)
			0.0866	0.1844	0.2124
Ga (3,1,-1)	14.20	-0.4836	-0.4837	-0.4594	-0.4860
		(1.4499)	(1.7848)	(1.6330)	(1.9538)
		1.3581	1.4292	1.2827	1.3365
		(0.0015)	(0.0014)	(0.0012)	(0.0059)
		0.9589	0.8832	0.9828	0.9039
		(9.1207)	(5.8883)	(6.8858)	(10.723)
			0.1308	0.2858	0.3122

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.66

Performance of the new estimators in 50 simulations. Errors are from Gamma $(3,\frac{1}{\sqrt{3}},-\sqrt{3})$, $\sigma^2 = 1$, θ_o = -0.8 and n = 50. Ga refers to the Gamma distribution (as the censoring distribution)

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^a$	$\hat{\theta}_n^b$	$\hat{\theta}_n^c$
Ga (3,1,-0.5)	13.52	-0.7635 (1.1471)	-0.7532 (1.2387)	-0.7357 (1.2551)	-0.7611* (1.2424)¶
		0.6632 (0.0008)	0.8162 (0.0024)	0.7656 (0.0020)	0.6843† (0.0020)
		0.9676 (7.4745)	0.9382 (6.6996)	1.0390 (8.2249)	0.9436‡ (7.3663)
			0.0704	0.1168	0.2878§
Ga (3,1,-0.75)	15.08	-0.7617 (0.8574)	-0.7602 (0.7187)	-0.7390 (0.7207)	-0.7745 (0.8522)
		0.6799 (0.0007)	0.7607 (0.0011)	0.6786 (0.0010)	0.6170 (0.0016)
		0.9800 (5.9753)	0.9147 (6.3477)	1.0336 (8.1797)	0.8758 (4.5747)
			0.1088	0.2132	0.3538
Ga (3,1,-1)	20.56	-0.7871 (0.6552)	-0.7601 (0.6288)	-0.7361 (0.6353)	-0.7847 (0.8271)
		0.7393 (0.0023)	0.7969 (0.0042)	0.7202 (0.0030)	0.7537 (0.0065)
		0.9704 (6.7942)	0.9835 (8.9125)	1.1367 (9.5515)	0.9320 (9.4235)
			0.1658	0.3030	0.5658

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.67

Performance of the new estimators in 50 simulations. Errors are from Gamma $(3,\frac{1}{\sqrt{3}},-\sqrt{3})$, $\sigma^2 = 1$, $\theta_o = 0.8$ and $n = 100$. Ga refers to the Gamma distribution (as the censoring distribution)

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^a$	$\hat{\theta}_n^b$	$\hat{\theta}_n^c$
Ga (3,1,-0.5)	11.78	0.7786	0.7721	0.7506	0.7802*
		(0.2520)	(0.2398)	(0.2595)	(0.2801)¶
		0.3735	0.3873	0.3418	0.4154†
		(0.0001)	(0.0001)	(0.0001)	(0.0002)
		0.9374	0.8883	0.9642	0.9024‡
		(2.5631)	(1.3856)	(1.6479)	(2.2767)
			7.5852	22.247	0.7800§
Ga (3,1,-0.75)	14.42	0.7760	0.7652	0.7364	0.7806
		(0.3056)	(0.3446)	(0.3816)	(0.3323)
		0.4016	0.4156	0.3492	0.4256
		(0.0002)	(0.0002)	(0.0002)	(0.0004)
		0.9625	0.8741	0.9791	0.8657
		(3.8770)	(1.9138)	(2.4870)	(2.5365)
			5.6948	13.323	1.0532
Ga (3,1,-1)	15.36	0.7800	0.7686	0.7343	0.7856
		(0.2976)	(0.3105)	(0.3795)	(0.2829)
		0.3640	0.3628	0.2968	0.3746
		(0.0001)	(0.0001)	(0.0001)	(0.0003)
		0.9219	0.8771	1.0014	0.8663
		(2.0523)	(1.9983)	(2.7553)	(2.5330)
			12.175	37.444	1.1922

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.68

Performance of the new estimators in 50 simulations. Errors are from Gamma $(3,\frac{1}{\sqrt{3}},-\sqrt{3})$, $\sigma^2 = 1$, $\theta_o = 0.5$ and $n = 100$. Ga refers to the Gamma distribution (as the censoring distribution)

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^a$	$\hat{\theta}_n^b$	$\hat{\theta}_n^c$
Ga (3,1,-0.5)	8.84	0.4751	0.4755	0.4616	0.4785*
		(0.7899)	(0.8253)	(0.7990)	(0.8543)¶
		0.7459	0.7432	0.6944	0.8397†
		(0.0004)	(0.0004)	(0.0004)	(0.0013)
		0.9496	0.8907	0.9374	0.9146‡
		(3.3380)	(2.6434)	(3.0411)	(4.8294)
			1.5202	3.8362	0.6404§
Ga (3,1,-0.75)	11.84	0.4890	0.4866	0.4658	0.4916
		(0.7186)	(0.7111)	(0.6779)	(0.6963)
		0.7054	0.7374	0.6643	0.8438
		(0.0003)	(0.0003)	(0.0002)	(0.0018)
		0.9870	0.8990	0.9697	0.9210
		(6.2709)	(4.0609)	(4.9296)	(6.0787)
			1.9246	4.6752	0.8578
Ga (3,1,-1)	13.72	0.4736	0.4739	0.4476	0.4794
		(0.7034)	(0.8360)	(0.7557)	(0.8753)
		0.7309	0.7692	0.6738	0.9018
		(0.0003)	(0.0003)	(0.0002)	(0.0019)
		0.9289	0.8229	0.9085	0.8323
		(2.6534)	(1.6578)	(2.1122)	(2.3669)
			2.9184	7.4440	1.0996

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.69

Performance of the new estimators in 50 simulations. Errors are from Gamma $(3,\frac{1}{\sqrt{3}},-\sqrt{3})$, $\sigma^2 = 1$, $\theta_o = 0.2$ and $n = 100$. Ga refers to the Gamma distribution (as the censoring distribution)

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
Ga (3,1,-0.5)	7.80	0.2104 (1.4367)	0.2158 (1.3964)	0.2096 (1.3226)	0.2186* (1.3789)¶
		0.8831 (0.0008)	0.9036 (0.0006)	0.8546 (0.0006)	1.0208† (0.0016)
		0.9990 (3.8919)	0.9244 (2.9102)	0.9613 (3.1447)	0.9295‡ (4.2788)
			0.2064	0.4586	0.4524§
Ga (3,1,-0.75)	10.72	0.1645 (0.8477)	0.1693 (0.9533)	0.1611 (0.8575)	0.1725 (0.9663)
		0.9242 (0.0006)	0.9413 (0.0003)	0.8566 (0.0003)	0.9163 (0.0026)
		1.0015 (3.8375)	0.8999 (2.0372)	0.9590 (2.4822)	0.9264 (5.9803)
			0.2966	0.6578	0.8830
Ga (3,1,-1)	13.52	0.2003 (1.1988)	0.1932 (1.2342)	0.1817 (1.1052)	0.1951 (1.2451)
		0.9190 (0.0007)	0.9277 (0.0002)	0.8162 (0.0002)	1.0549 (0.0021)
		1.0018 (4.3713)	0.8700 (1.9867)	0.9516 (2.6352)	0.8904 (4.1322)
			1.1434	3.0706	1.1340

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.70

Performance of the new estimators in 50 simulations. Errors are from Gamma $(3,\frac{1}{\sqrt{3}},-\sqrt{3})$, $\sigma^2 = 1$, θ_o = -0.2 and n = 100. Ga refers to the Gamma distribution (as the censoring distribution)

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
Ga (3,1,-0.5)	7.68	-0.1859 (1.0580)	-0.1833 (1.1582)	-0.1774 (1.0901)	-0.1824* (1.2001)¶
		0.8773 (0.0004)	0.9403 (0.0004)	0.8812 (0.0004)	1.0206† (0.0013)
		0.9474 (5.7774)	0.8832 (3.1420)	0.9206 (3.4424)	0.8895‡ (4.3564)
			0.1168	0.2350	0.4460§
Ga (3,1,-0.75)	10.58	-0.2209 (1.0528)	-0.2252 (1.0061)	-0.2159 (0.9227)	-0.2274 (1.0711)
		0.8510 (0.0005)	0.9146 (0.0004)	0.8362 (0.0003)	0.9768 (0.0012)
		1.0122 (2.8214)	0.8928 (1.4489)	0.9498 (1.6667)	0.8978 (2.5586)
			0.2028	0.4792	0.7008
Ga (3,1,-1)	13.24	-0.1947 (1.1720)	-0.1998 (1.1511)	-0.1889 (1.02344)	-0.1998 (1.1449)
		0.8886 (0.0005)	0.9203 (0.0003)	0.8188 (0.0002)	0.9529 (0.0019)
		0.9536 (4.1486)	0.8621 (2.2164)	0.9401 (2.6749)	0.8724 (3.0681)
			0.3056	0.7234	1.0172

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.71

Performance of the new estimators in 50 simulations. Errors are from Gamma $(3,\frac{1}{\sqrt{3}},-\sqrt{3})$, $\sigma^2 = 1$, θ_o = -0.5 and n = 100. Ga refers to the Gamma distribution (as the censoring distribution)

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^a$	$\hat{\theta}_n^b$	$\hat{\theta}_n^c$
Ga (3,1,-0.5)	9.10	-0.4815 (0.5410)	-0.4845 (0.5908)	-0.4701 (0.5598)	-0.4868* (0.5872)¶
		0.7184 (0.0006)	0.7200 (0.0003)	0.6673 (0.0003)	0.6421† (0.0012)
		0.9529 (2.6043)	0.9009 (2.4372)	0.9527 (2.7067)	0.9025‡ (3.2434)
			0.1030	0.1826	0.7056§
Ga (3,1,-0.75)	11.54	-0.4812 (0.8607)	-0.4805 (0.9090)	-0.4613 (0.8334)	-0.4838 (0.9007)
		0.7580 (0.0006)	0.8032 (0.0005)	0.7148 (0.0005)	0.8318 (0.0013)
		0.9901 (2.5389)	0.8978 (2.1366)	0.9701 (2.2997)	0.9037 (2.9340)
			0.1366	0.2590	0.8250
Ga (3,1,-1)	14.50	-0.4805 (0.7907)	-0.4879 (0.7077)	-0.4636 (0.6633)	-0.4887 (0.7290)
		0.7350 (0.0005)	0.7807 (0.0004)	0.6900 (0.0003)	0.7327 (0.0020)
		0.9918 (3.2384)	0.8631 (2.2764)	0.9600 (2.7927)	0.8972 (5.4705)
			0.2438	0.5434	1.2986

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Table 5.72

Performance of the new estimators in 50 simulations. Errors are from Gamma $(3,\frac{1}{\sqrt{3}},-\sqrt{3})$, $\sigma^2 = 1$, θ_o = -0.8 and n = 100. Ga refers to the distribution (Gamma as the censoring distribution)

T_i	% cens.	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$
Ga (3,1,-0.5)	13.62	-0.7842 (0.5901)	-0.7723 (0.5244)	-0.7557 (0.5539)	-0.7892* (0.6168)¶
		0.3543 (0.0003)	0.3758 (0.0003)	0.3486 (0.0002)	0.2828† (0.0004)
		0.9432 (3.4311)	0.9281 (3.0294)	1.0236 (3.6279)	0.8882‡ (2.6353)
			0.1546	0.2682	1.2062§
Ga (3,1,-0.75)	16.60	-0.7928 (0.2242)	-0.7802 (0.1973)	-0.7592 (0.2040)	-0.8013 (0.2375)
		0.3455 (0.0002)	0.3445 (0.0002)	0.3152 (0.0001)	0.3311 (0.0002)
		1.0239 (4.5110)	0.9798 (4.9612)	1.1081 (5.5122)	0.9471 (4.9349)
			0.5622	0.8624	1.3492
Ga (3,1,-1)	19.34	-0.7858 (0.4500)	-0.7559 (0.3629)	-0.7301 (0.3969)	-0.7871 (0.4673)
		0.3747 (0.0002)	0.4055 (0.0005)	0.3621 (0.0003)	0.3685 (0.0005)
		0.9800 (4.3226)	0.9730 (4.2600)	1.1284 (5.3473)	0.9219 (4.6662)
			0.3120	0.5326	1.7796

*Throughout the table and for each combination of estimator and T_i, the first tabulated value is the mean estimate of θ.

†The second value is the average estimated asymptotic variance ($\times 10^2$) of the estimator.

‡The third value is mean estimate of σ^2.

§The fourth value is the average CPU time in seconds needed to compute the estimates.

¶The values in parentheses are the corresponding sampling variances $\times 10^2$.

Figure 5.55
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gamma $(3,\frac{1}{\sqrt{3}},-\sqrt{3})$, $\sigma^2 = 1$, $\theta_o = 0.8$ and $n = 25$.

Estimator $\quad \underline{\hat{\theta}_n^{ls} \quad \hat{\theta}_n^{a} \quad \hat{\theta}_n^{b} \quad \hat{\theta}_n^{c}}$

Figure 5.56
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gamma $(3,\frac{1}{\sqrt{3}},-\sqrt{3})$, $\sigma^2 = 1$, $\theta_o = 0.5$ and $n = 25$.

Estimator $\quad \underline{\hat{\theta}_n^{b} \quad \hat{\theta}_n^{ls} \quad \hat{\theta}_n^{a} \quad \hat{\theta}_n^{c}}$

Figure 5.57
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gamma $(3,\frac{1}{\sqrt{3}},-\sqrt{3})$, $\sigma^2 = 1$, $\theta_o = 0.2$ and $n = 25$.

Estimator $\quad \underline{\hat{\theta}_n^{b} \quad \hat{\theta}_n^{ls} \quad \hat{\theta}_n^{a}} \quad \hat{\theta}_n^{c}$

Figure 5.58
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gamma $(3,\frac{1}{\sqrt{3}},-\sqrt{3})$, $\sigma^2 = 1$, θ_o = -0.2 and $n = 25$.

Estimator	$\hat{\theta}_n^b$	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^c$	$\hat{\theta}_n^a$
	———	———		
		———	———	———

Figure 5.59
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gamma $(3,\frac{1}{\sqrt{3}},-\sqrt{3})$, $\sigma^2 = 1$, θ_o = -0.5 and $n = 25$.

Estimator	$\hat{\theta}_n^b$	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^a$	$\hat{\theta}_n^c$
	———	———	———	
			———	———

Figure 5.60
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gamma $(3,\frac{1}{\sqrt{3}},-\sqrt{3})$, $\sigma^2 = 1$, θ_o = -0.8 and $n = 25$.

Estimator	$\hat{\theta}_n^b$	$\hat{\theta}_n^a$	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^c$
	———	———	———	———

Figure 5.61
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gamma $(3,\frac{1}{\sqrt{3}},-\sqrt{3})$, $\sigma^2 = 1$, $\theta_o = 0.8$ and $n = 50$.

Estimator $\underline{\hat{\theta}_n^b \quad \hat{\theta}_n^{ls} \quad \hat{\theta}_n^c \quad \hat{\theta}_n^a}$

Figure 5.62
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gamma $(3,\frac{1}{\sqrt{3}},-\sqrt{3})$, $\sigma^2 = 1$, $\theta_o = 0.5$ and $n = 50$.

Estimator $\hat{\theta}_n^b \quad \underline{\hat{\theta}_n^{ls} \quad \hat{\theta}_n^a} \quad \hat{\theta}_n^c$

Figure 5.63
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gamma $(3,\frac{1}{\sqrt{3}},-\sqrt{3})$, $\sigma^2 = 1$, $\theta_o = 0.2$ and $n = 50$.

Estimator $\underline{\hat{\theta}_n^b \quad \hat{\theta}_n^{ls} \quad \hat{\theta}_n^a \quad \hat{\theta}_n^c}$

Figure 5.64
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gamma $(3, \frac{1}{\sqrt{3}}, -\sqrt{3})$, $\sigma^2 = 1$, θ_o = -0.2 and n = 50.

Estimator $\underline{\hat{\theta}_n^b \quad \hat{\theta}_n^{ls} \quad \hat{\theta}_n^a} \quad \hat{\theta}_n^c$

Figure 5.65
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gamma $(3, \frac{1}{\sqrt{3}}, -\sqrt{3})$, $\sigma^2 = 1$, θ_o = -0.5 and n = 50.

Estimator $\underline{\hat{\theta}_n^b \quad \hat{\theta}_n^{ls} \quad \hat{\theta}_n^a \quad \hat{\theta}_n^c}$

Figure 5.66
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gamma $(3, \frac{1}{\sqrt{3}}, -\sqrt{3})$, $\sigma^2 = 1$, θ_o = -0.8 and n = 50.

Estimator $\underline{\hat{\theta}_n^c \quad \hat{\theta}_n^{ls} \quad \hat{\theta}_n^b} \quad \hat{\theta}_n^a$ (second line: $\underline{\hat{\theta}_n^b \quad \hat{\theta}_n^a}$)

Figure 5.67
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gamma $(3,\frac{1}{\sqrt{3}},-\sqrt{3})$, $\theta_o = 0.8$ and $n = 100$.

Estimator $\quad \hat{\theta}_n^b \quad \hat{\theta}_n^{ls} \quad \hat{\theta}_n^a \quad \hat{\theta}_n^c$

Figure 5.68
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gamma $(3,\frac{1}{\sqrt{3}},-\sqrt{3})$, $\sigma^2 = 1$, $\theta_o = 0.5$ and $n = 100$.

Estimator $\quad \hat{\theta}_n^b \quad \hat{\theta}_n^{ls} \quad \hat{\theta}_n^a \quad \hat{\theta}_n^c$

Figure 5.69
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gamma $(3,\frac{1}{\sqrt{3}},-\sqrt{3})$, $\sigma^2 = 1$, $\theta_o = 0.2$ and $n = 100$.

Estimator $\quad \hat{\theta}_n^b \quad \hat{\theta}_n^{ls} \quad \hat{\theta}_n^a \quad \hat{\theta}_n^c$

Figure 5.70
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gamma $(3,\frac{1}{\sqrt{3}},-\sqrt{3})$, $\sigma^2 = 1$, θ_o = -0.2 and n = 100.

Estimator	$\underline{\hat{\theta}_n^b}$	$\underline{\hat{\theta}_n^{ls}}$	$\hat{\theta}_n^a$	$\hat{\theta}_n^c$

Figure 5.71
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gamma $(3,\frac{1}{\sqrt{3}},-\sqrt{3})$, $\sigma^2 = 1$, θ_o = -0.5 and n = 100.

Estimator	$\underline{\hat{\theta}_n^b}$	$\underline{\hat{\theta}_n^c}$	$\underline{\hat{\theta}_n^{ls}}$	$\underline{\hat{\theta}_n^a}$

Figure 5.72
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gamma $(3,\frac{1}{\sqrt{3}},-\sqrt{3})$, $\sigma^2 = 1$, θ_o = -0.8 and n = 100.

Estimator	$\underline{\hat{\theta}_n^c}$	$\underline{\hat{\theta}_n^b}$	$\underline{\hat{\theta}_n^{ls}}$	$\underline{\hat{\theta}_n^a}$

5.4 Conclusions

For all the error distributions considered in this study, the results show that all the estimators, both currently available and new, perform comparably among themselves with respect to the bias criterion if $1-\theta_o^2$ is close to unity. For other values of $1-\theta_o^2$, $\hat{\theta}_n^b$ tends to be more biased than the rest of the estimators, which still behave similarly to one another. According to the same criterion, the performance of the estimators improves with increasing size of the time series sample. Turning to the behaviour of the estimators with respect to the asymptotic variance criterion, the results suggest that the estimated asymptotic variances are small for small values of $1-\theta_o^2$ and show an increase as $1-\theta_o^2$ tends to 1. We note that the asymptotic variance of the least squares-estimator for the uncensored case, $\hat{\theta}_n^{ls}$, is given by $1-\theta_o^2$ (see, e.g., Basawa and Rao, 1980, page 42). Hence, the similarity between the behaviour of the estimators and the behaviour of $1-\theta_o^2$ is consistent with this theoretical result. As in the case of the bias criterion, the results also show that the performance of the estimators improves with increasing sample size. All of these justify the use of the estimators in practical applications. For Gaussian errors, based on our simulations, about two-thirds of the time, the performance of the estimators deteriorates as the proportion of censored observations increases. If the errors are from the Laplace distribution, it is estimated that about one-half of the time, the performance deteriorates with increasing proportion of censored observations. In the other one-half of the time, the performance either first deteriorates and then improves or vice versa. For Gamma errors, about two-thirds of the time, the peformance of the estimators is not sensitive to changes in the censoring rate. In the remaining one-third of the time, the performance either first remains unchanged or deteriorates and then improves as the proportion of censored observations increases.

It is also important to assess the performance of the asymptotic variance estimators. The results suggest that for each estimator and for a fixed value of θ_o, when the error distribution is fixed and the censoring rate is held constant (Tables 1-18 vs corresponding tables from Tables 19-36), the asymptotic formula leads to a value which is asymptotically insensitive to the censoring pattern. Wu and Zubovic (1995) reached a similar conclusion from simulations of the asymptotic variance estimator of the Buckley-James estimator proposed by Ritov (1990) in the linear regression set-up. This is a favourable property of the proposed asymptotic variance estimators which justifies their use as variance estimators in practical applications.

We now turn to the comparison of the estimators among themselves on the basis of their estimated asymptotic variance. For Gaussian errors with unit variance and variance equal to 2, all the estimators behave similarly to each other for small values of $1 - \theta_o^2$. For values of $1 - \theta_o^2$ near unity, $\hat{\theta}_n^{ls}$ and $\hat{\theta}_n^b$ behave similarly to each other and better than $\hat{\theta}_n^a$, $\hat{\theta}_n^{zb}$ and $\hat{\theta}_n^c$ which are not significantly different among themselves. The MLE and $\hat{\theta}_n^{dag}$, on the other hand, behave similarly to estimators in both groups. When the variance of the errors is equal to 2, the superiority of $\hat{\theta}_n^{ls}$ and $\hat{\theta}_n^b$ over $\hat{\theta}_n^a$, $\hat{\theta}_n^{zb}$ and $\hat{\theta}_n^a$ is slightly enhanced.

Recall that for Laplace and gamma errors we compare only the least-squares estimator for the uncensored case with the new estimators, which are also compared among themselves. As mentioned in section 5.1, the reason is that in their present form, the currently available estimators are only applicable for Gaussian errors. For errors drawn from the Laplace distribution, the behaviour of the new estimators is similar to their behaviour in the Gaussian case. Once again, the estimators perform comparably for small values of $1 - \theta_o^2$. For values of $1 - \theta_o^2$ close to 1, $\hat{\theta}_n^b$ and $\hat{\theta}_n^{ls}$ have a similar behaviour and perform better than $\hat{\theta}_n^a$ and $\hat{\theta}_n^c$, which are also comparable to one another. For gamma errors, the estimators behave somewhat different to their behaviour for Gaussian and Laplace errors. About one-half of the time, they behave similarly among themselves. In the other one-half of the time, $\hat{\theta}_n^{ls}$, $\hat{\theta}_n^b$ and $\hat{\theta}_n^a$ tend to behave comparably. During this time, in some cases $\hat{\theta}_n^c$ compares favourably with $\hat{\theta}_n^a$ and $\hat{\theta}_n^{ls}$ and in other cases it performs worse than the rest of the estimators.

The similarity in the behaviour of $\hat{\theta}_n^a$, $\hat{\theta}_n^{zb}$ and $\hat{\theta}_n^c$ can be attributed to the way in which they are obtained. They are all obtained by replacing the time series rv's Y_i's in the least-squares estimating function for the uncensored case by their conditional expectations given the censoring. For $\hat{\theta}_n^{zb}$ and $\hat{\theta}_n^c$ the new rv's are computed sequentially, conditional on the censoring at the index time i and given the Y_{i-1} equals the value of the conditional expectation computed for the index time $i-1$. For $\hat{\theta}_n^a$ these expectations are conditional on the sigma-field generated by the censored observations corresponding to j for $j \leq i$. Note that the estimator $\hat{\theta}_n^{ls}$ is unrealistic for the censored time series data problem since it corresponds to the uncensored case and hence there is no loss of information due to censoring. Thus, it is expected to perform better than the other estimators. The simulation results are consistent with this intuition. They also show that $\hat{\theta}_n^b$ compares favourably with $\hat{\theta}_n^{ls}$. In a way, this compansates for the poor performance of the former estimator with respect to the estimated bias criterion. Since the estimating function obtained in the case of the MLE is optimal in the sense of Godambe (1960), the similarity in the behaviour of the new estimators with the MLE justifies the use of these estimators in practical applications.

Chapter 6

Summary, Conclusions and Future Prospects

The need for methods of analyzing censored time series data is largely self-evident. Firstly, for many studies in the physical and medical sciences and in business and economics in which data are collected sequentially in time, an exact value can be recorded only if it falls within a specified range due to upper or lower limits of detection. For example, in the physical sciences measurement of rainfall is limited by the size of the raingauge, and subject to evaporation. In medical studies, one may be recording daily bioassays of hormone levels in a patient. This gives rise to censored time series. Other examples of censored time series can also be given. One can fit an autoregressive model to account for the time dependence. Secondly, Likelihood procedures for autoregressive models have been well studied. Further, regression models for censored independent data have been investigated extensively. Yet, very little is known about the use of autoregressive models with censored data.

In this thesis, we are principally concerned with the estimation of parameters in autoregressive models with censored data. For convenience, attention is restricted to the first-order stationary autoregressive (AR(1)) model in which the sensitivity of the measurement has an upper limit of detection (right-censored). We propose that, extension to the AR(p) model, where $p > 1$, and to left-censored data can be easily accomplished by using ideas developed for the AR(1) model with right-censored data. However, this is a subject for possible future investigations. In their present form, the methods of estimation for censored autocorrelated data which were already available prior to the present investigation can only be applied to AR processes with Gaussian errors. Use of these methods in AR processes with non-Gaussian errors requires, essentially, rederivations of the estimators. Hence, we were prompted to develop new estimators which are robust in the sense that they can be applied with minor or no modifications to AR models with non-Gaussian errors. We proposed three estimators, the first two of which require knowledge of the form of the error distribution. Of these two, one, termed $\hat{\theta}_n^a$, is obtained by replacing each of the response rv's (the ith, say) in the least-squares estimating equation for the uncensored case by its conditional expectation given σ $\{(Z_j, \delta_j), j \leq i\}$ where $(Z_1, \delta_1),\ldots,(Z_n, \delta_n)$ are the censored observed data defined by $Z_i = \min(Y_i, T_i)$ and $\delta_i = I(Y_i \leq T_i)$, the Y_i's being the underlying time series rv's and T_i's, are the censoring rv's with distribution independent of the distribution of the Y_i's. The second estimator, $\hat{\theta}_n^b$ is also obtained by modifying the least squares estimating function for the uncensored case. In obtaining this estimator, differences between the successive sums of the estimating function are replaced by their corresponding conditional expectations given σ $\{(Z_j, \delta_j), j \leq i\}$. The third estimator, termed $\hat{\theta}_n^a$, is a distribution-free estimator based on the Kaplan-Meier estimator of the distribution function

Given in Chapter 2 of this thesis, was a summary of the various estimators in linear regression with censored i.i.d. data. Also given in Chapter 2, were detailed descriptions of the estimators for regression with censored autocorrelated data, which were already existing before the current investigation. These are the MLE, the PL estimator of Zeger and Brookmeyer (1986) and the PL estimator of Dagenais (1986). These were termed $\hat{\theta}_n^{mle}$, $\hat{\theta}_n^{zb}$ and $\hat{\theta}_n^{dag}$, respectively. Chapter 3 described the new estimators. We also proposed asymptotic variance estimators for each of these estimators. In Chapter 4, we established some asymptotic results for $\hat{\theta}_n^a$ and $\hat{\theta}_n^b$. We established that for each of these estimators, under suitable conditions on the moments of the distribution of the errors, if the estimator is consistent, then it is also asymptotically normally distributed. We also investigated the consistency of $\hat{\theta}_n^a$. We found that if the errors are Gaussian and alternate observations are censored, then $\hat{\theta}_n^a$ is consistent. Hence, for this special case, $\hat{\theta}_n^a$ is consistent and asymptotically normally distributed.

We have not investigated the consistency of $\hat{\theta}_n^b$ or the large sample properties of $\hat{\theta}_n^c$ and the currently available estimators. However, simulations in Chapter 5 of this thesis revealed that $\hat{\theta}_n^a$ is comparable with the currently available methods, which perform comparably among themselves. Thus, under possibly different conditions, it is perceivable that $\hat{\theta}_n^a$ and the currently available estimators have the same large sample properties. A further investigation is needed to establish the large sample properties of the currently available estimators. Also, extension of the above-mentioned consistency result of $\hat{\theta}_n^a$ to incorporate general censoring patterns involving more than one observation in a 'block' of consecutive censored observations. For the same estimator, i.e., $\hat{\theta}_n^a$, consistency could also be investigated for general error distributions. A theoretical study to investigate the consistency of $\hat{\theta}_n^b$ could also be conducted. As in the case of $\hat{\theta}_n^a$, the study could proceed by investigating conditions under which the estimating function for $\hat{\theta}_n^b$ crosses the θ-axis in a sufficiently small neighbourhood of the true value of the autoregression parameter θ. Also, the asymptotic behaviour of the distribution-free estimator proposed in the current research could be investigated. This could make use of the available asymptotic theory of Kaplan-Meier estimator of the error distribution function.

In Chapter 5, we used simulations to assess and compare the performances of the new estimators and the estimators which were available before the present research. This simulation study also includes the least-squares estimator for the uncensored case, $\hat{\theta}_n^{ls}$. To conduct the simulation study, we used three error distributions and three censoring distributions. The three error distributions were Gaussian, Lapalace and Gamma and the corresponding censoring distributions were Laplace, Gaussian and Gamma. For Gaussian errors, three criteria were considered in comparing the estimators, estimated bias, MSE, estimated asymptotic variance. According to the bias criterion, when $1-\theta_o^2$ is close to unity, all the estimators behave similarly among themselves. For other values of $1-\theta_o^2$, $\hat{\theta}_n^b$ performs worse than the other estimators, which are comparable among themselves. However, bias alone precludes the variance of the estimator. The MSE is a better criterion since it is given by the variance of the estimator plus squared bias. According to this criterion, when $\theta_o = \pm 0.2$, $\hat{\theta}_n^{ls}$ and $\hat{\theta}_n^b$ behave similarly and perform better than the rest of the estimators, which are comparable among themselves. When $\theta_o = \pm 0.5$, all the estimators tend to perform comparably among themselves. When $\theta_o = \pm 0.8$, $\hat{\theta}_n^b$ performs worse than the rest of the estimators, which are comparable among themselves. Clearly, according to the bias and the MSE, $\hat{\theta}_n^b$ tends to perform differently from the other estimators, which behave similarly among themselves. Further, the MSE is a better criterion than bias alone. Therefore, we base our conclusions about the behaviour of $\hat{\theta}_n^b$ in relation with the estimators on the MSE. For the same reason that the rest of the estimators perform equivalently with respect to bias and MSE, we base our conclusions about their behaviour on the asymptotic variance criterion. With respect to this criterion, $\hat{\theta}_n^{ls}$ and $\hat{\theta}_n^b$ behave similarly and perform better than $\hat{\theta}_n^a$, $\hat{\theta}_n^{zb}$ and $\hat{\theta}_n^c$, which are comparable among themselves. The MLE and $\hat{\theta}_n^{dag}$ tend to behave similarly to estimators in both groups.

For Laplace and Gamma errors, only the new estimators were compared among themselves and with $\hat{\theta}_n^{ls}$. This is because, as mentioned earlier in their present form, the currently available estimators are not applicable to non-Gaussian error distributions. Also, only the bias and the asymptotic variance criteria are used to compare the estimators. We found that, we respect to the bias and asymptotic variance criteria, the estimators exhibit a similar behaviour for Laplace errors to their behaviour in the Gaussian case. For Gamma errors, the estimators are comparable among themselves with respect to the bias criterion, apart from $\hat{\theta}_n^b$ which tends perform poorly. Their behaviour is slightly different to their behaviour for Gaussian and Gamma errors. In about one-half of the time, they behave similarly among themselves. In the remainder of the time, $\hat{\theta}_n^{ls}$ and $\hat{\theta}_n^b$ and $\hat{\theta}_n^a$ behave similarly among themselves. In the other half of the time, $\hat{\theta}_n^{ls}$, $\hat{\theta}_n^b$ and $\hat{\theta}_n^a$ are comparable while $\hat{\theta}_n^c$ has a similar behaviour to $\hat{\theta}_n^a$ $\hat{\theta}_n^{ls}$ in some cases and performs worse than the rest of the estimators in other cases.

The simulation results of the asymptotic variance estimators suggest that for each estimator and for a fixed value of θ_o, when the error distribution is fixed and the censoring rate is constant, the asymptotic variance formula leads to a value which is asymptotically insensitive to the censoring pattern. Also, the estimated asymptotic variances decrease with increasing sample size and their behaviour with respect to changes in θ_o is consistent with the behaviour of the asymptotic variance of the least-squares estimator for the uncensored case, which is given by $1 - \theta_o^2$ (see e.g., Basawa and Rao, 1980, p.43). These are favourable properties of the proposed asymptotic variance estimators which also justifies their use in practical applications.

The results suggest that, for Gaussian errors, choice of the preferred estimator can be made from any of the estimators $\hat{\theta}_n^{mle}$, $\hat{\theta}_n^{zb}$, $\hat{\theta}_n^{dag}$ and the new estimators, $\hat{\theta}_n^a$, $\hat{\theta}_n^b$ and $\hat{\theta}_n^c$. With respect to the computational time it takes to obtain the estimates, the estimators can, in general, be listed in order of increasing CPU time as: $\hat{\theta}_n^{zb}$, $\hat{\theta}_n^c$, $\hat{\theta}_n^a$, $\hat{\theta}_n^{dag}$, $\hat{\theta}_n^b$, $\hat{\theta}_n^{mle}$ for light censoring and as: $\hat{\theta}_n^{zb}$, $\hat{\theta}_n^c$, $\hat{\theta}_n^{dag}$, $\hat{\theta}_n^a$, $\hat{\theta}_n^b$, $\hat{\theta}_n^{mle}$ as the proportion of censored observations increases. The CPU time for $\hat{\theta}_n^{mle}$ is much higher than for the rest of the estimators. Hence, $\hat{\theta}_n^{mle}$ would be considered only if computer time is inexpensive. As noted earlier, with this estimator, numerical procedures that avoid nonconvergence problems need to be considered (see Zeger and Brookmeyer, 1986). On the other hand, not only the new estimators are computationally feasible but also they compare favourably with $\hat{\theta}_n^{mle}$. The estimators $\hat{\theta}_n^{zb}$ and $\hat{\theta}_n^{dag}$ also have this advantage. However, unlike the new estimators, in their present form, these estimators and $\hat{\theta}_n^{mle}$ cannot be applied for distributions other than the Gaussian. For errors from the Laplace and gamma distributions, choice of the preferred estimator can be made from $\hat{\theta}_n^a$, $\hat{\theta}_n^b$ and $\hat{\theta}_n^c$. The estimator $\hat{\theta}_n^c$ can be applied in practical applications where the form of the error distribution is unknown since the it replaces this distribution by its Kaplan-Meier estimator based on the observed data. Hence, it can be applied in a wide variety of situations.

Appendix

A.1 Contribution of censored observations

As mentioned in chapter 2, the expression for the contribution of censored observations to the MLE score function in Zeger and Brookmeyer (1986) has an error (possibly typographical). We correct this through the following lemma.

Proposition A.1.1 *Let $\{Y_i, i \in \mathcal{Z}\}$ be a zero-mean, possibly censored first-order stationary autoregressive process. Suppose the process is Gaussian. Then the contribution of the censored observations, $\underline{Y}_j^c$, to the likelihood score function is given by (2.3.22-c) in section 2.3.1.*

Proof

Let

$$\tilde{S}_{\underline{Y}_{j,1},\ldots,\underline{Y}_{j,m}|\nu_j^c,X_j,Y_j^u}(t_1,\ldots,t_m|m,X_j,Y_j^u;\theta) = \frac{\partial}{\partial\theta}[ln\ \tilde{\Phi}_{j,\nu_j^c}(t_1,\ldots,t_m,X_j,Y_j^u)],$$

and

$$\begin{aligned}&\tilde{T}_{\underline{Y}_{j,1},\ldots,\underline{Y}_{j,m}|\nu_j^c,X_j,Y_j^u}(t_1,\ldots,t_m|m,X_j,Y_j^u;\theta)\\ &= \frac{\int_{t_m}^{\infty}\cdots\int_{t_1}^{\infty} \underline{s}'\frac{\partial}{\partial\theta}[(\Sigma_j^c)^{-1}]\underline{s}\ \ \phi_{j,\nu_j^c}[(\Sigma_j^c)^{-1}(\underline{s}-\eta_j^c)]\Pi_{k=1}^m ds_k}{\tilde{\Phi}_{j,\nu_j^c}(t_1,\ldots,t_m,X_j,Y_j^u)}\\ &= \frac{\int_{t_m}^{\infty}\cdots\int_{t_1}^{\infty} \underline{s}'\frac{\partial}{\partial\theta}[(\Sigma_j^c)^{-1}]\underline{s}\ \ \phi_{j,\nu_j^c}[(\Sigma_j^c)^{-1}(\underline{s}-\eta_j^c)]\Pi_{k=1}^m ds_k}{\int_{t_m}^{\infty}\cdots\int_{t_1}^{\infty} \phi_{j,\nu_j^c}[(\Sigma_j^c)^{-1}(\underline{s}-\eta_j^c)]\Pi_{k=1}^m ds_k}.\end{aligned}$$

and note that the total contribution for all $\underline{Y}_j^c$ is

$$\sum_{j=1}^{K} \tilde{S}_{\underline{Y}_{j,1},\ldots,\underline{Y}_{j,m}|\nu_j^c,X_j,Y_j^u}(\underline{T}_{j,1},\ldots,\underline{T}_{j,m}|m,X_j,Y_j^u;\theta).$$

A simplification for $\tilde{S}_{\underline{Y}_{j,1},...,\underline{Y}_{j,m}|\nu_j^c,X_j,Y_j^u}(t_1,...,t_m|m,X_j,Y_j^u;\theta)$ is given by

$$\begin{aligned}
&\tilde{S}_{\underline{Y}_{j,1},...,\underline{Y}_{j,m}|\nu_j^c,X_j,Y_j^u}(t_1,...,t_m|m,X_j,Y_j^u;\theta)\\
&= \frac{\frac{\partial}{\partial\theta}\,\tilde{\Phi}_{j,\nu_j^c}(t_1,...,t_m,X_j,Y_j^u)}{\tilde{\Phi}_{j,\nu_j^c}(t_1,...,t_m,X_j,Y_j^u)}\\
&= \frac{\int_{t_m}^{\infty}\cdots\int_{t_1}^{\infty}\frac{\partial}{\partial\theta}\,\phi_{j,\nu_j^c}[(\Sigma_j^c)^{-1}(\underline{s}-\eta_j^c)]\Pi_{k=1}^m ds_k}{\int_{t_m}^{\infty}\cdots\int_{t_1}^{\infty}\phi_{j,\nu_j^c}[(\Sigma_j^c)^{-1}(\underline{s}-\eta_j^c)]\Pi_{k=1}^m ds_k}.
\end{aligned}$$

But

$$\begin{aligned}
&\frac{\partial}{\partial\theta}\,\phi_{j,\nu_j^c}[(\Sigma_j^c)^{-1}(\underline{s}-\eta_j^c)]\\
&= \{-\frac{1}{2|\Sigma_j^c|}\frac{\partial|\Sigma_j^c|}{\partial\theta} - \frac{1}{2}(\underline{s}-\eta_j^c)'\frac{\partial}{\partial\theta}[(\Sigma_j^c)^{-1}](\underline{s}-\eta_j^c)\\
&+(\frac{\partial\eta_j^c}{\partial\theta})'(\Sigma_j^c)^{-1}(\underline{s}-\eta_j^c)\}\{\phi_{j,\nu_j^c}[(\Sigma_j^c)^{-1}(\underline{s}-\eta_j^c)]\}
\end{aligned}$$

$$\begin{aligned}
&= \{-\frac{1}{2|\Sigma_j^c|}\frac{\partial|\Sigma_j^c|}{\partial\theta} - \frac{1}{2}\underline{s}'\frac{\partial}{\partial\theta}[(\Sigma_j^c)^{-1}]\underline{s}\}\{\phi_{j,\nu_j^c}[(\Sigma_j^c)^{-1}(\underline{s}-\eta_j^c)]\}\\
&-\frac{1}{2}\{-2(\eta_j^c)'\frac{\partial}{\partial\theta}[(\Sigma_j^c)^{-1}]\underline{s} + (\eta_j^c)'\frac{\partial}{\partial\theta}[(\Sigma_j^c)^{-1}]\eta_j^c\}\{\phi_{j,\nu_j^c}[(\Sigma_j^c)^{-1}(\underline{s}-\eta_j^c)]\}\\
&+\{(\frac{\partial\eta_j^c}{\partial\theta})'(\Sigma_j^c)^{-1}(\underline{s}-\eta_j^c)\}\{\phi_{j,\nu_j^c}[(\Sigma_j^c)^{-1}(\underline{s}-\eta_j^c)]\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\tilde{S}_{\underline{Y}_{j,1},...,\underline{Y}_{j,m}|\nu_j^c,X_j,Y_j^u}(\underline{T}_{j,1},...,\underline{T}_{j,m}|m,X_j,Y_j^u;\theta)\\
&= -\frac{1}{2|\Sigma_j^c|}\frac{\partial|\Sigma_j^c|}{\partial\theta} - \frac{1}{2}\tilde{T}_{\underline{Y}_{j,1},...,\underline{Y}_{j,m}|\nu_j^c,X_j,Y_j^u}(\underline{T}_{j,1},...,\underline{T}_{j,m}|m,X_j,Y_j^u;\theta)\\
&-\frac{1}{2}\{-2(\eta_j^c)'\frac{\partial}{\partial\theta}[(\Sigma_j^c)^{-1}]\hat{\underline{Y}}_j^c + (\eta_j^c)'\frac{\partial}{\partial\theta}[(\Sigma_j^c)^{-1}]\eta_j^c\} + (\frac{\partial\eta_j^c}{\partial\theta})'(\Sigma_j^c)^{-1}(\hat{\underline{Y}}_j^c-\eta_j^c)
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2|\Sigma_j^c|}\frac{\partial|\Sigma_j^c|}{\partial\theta} - \frac{1}{2}\tilde{T}_{\underline{Y}_{j,1},...,\underline{Y}_{j,m}|\nu_j^c,X_j,Y_j^u}(\underline{T}_{j,1},...,\underline{T}_{j,m}|m,X_j,Y_j^u;\theta)\\
&-\frac{1}{2}\{(\hat{\underline{Y}}_j^c-\eta_j^c)'\frac{\partial}{\partial\theta}[(\Sigma_j^c)^{-1}](\hat{\underline{Y}}_j^c-\eta_j^c) - (\hat{\underline{Y}}_j^c)'\frac{\partial}{\partial\theta}[(\Sigma_j^c)^{-1}]\hat{\underline{Y}}_j^c\}\\
&+(\frac{\partial\eta_j^c}{\partial\theta})'(\Sigma_j^c)^{-1}(\hat{\underline{Y}}_j^c-\eta_j^c)
\end{aligned}$$

$$= -\frac{1}{2|\Sigma_j^c|}\frac{\partial|\Sigma_j^c|}{\partial\theta} - \frac{1}{2}(\hat{\underline{Y}}_j^c-\eta_j^c)'\frac{\partial}{\partial\theta}[(\Sigma_j^c)^{-1}](\hat{\underline{Y}}_j^c-\eta_j^c) + (\frac{\partial\eta_j^c}{\partial\theta})'(\Sigma_j^c)^{-1}(\hat{\underline{Y}}_j^c-\eta_j^c)$$

$$-\frac{1}{2}\{\tilde{T}_{\underline{Y}_{j,1},\ldots,\underline{Y}_{j,m}|\nu_j^c,X_j,Y_j^u}(\underline{T}_{j,1},\ldots,\underline{T}_{j,m}|m,X_j,Y_j^u;\theta)-(\hat{\underline{Y}}_j^c)'\frac{\partial}{\partial\theta}[(\Sigma_j^c)^{-1}]\hat{\underline{Y}}_j^c\}$$

$$=-\frac{1}{2|\Sigma_j^c|}\frac{\partial|\Sigma_j^c|}{\partial\theta}-\frac{1}{2}(\hat{\underline{Y}}_j^c-\eta_j^c)'\frac{\partial}{\partial\theta}[(\Sigma_j^c)^{-1}](\hat{\underline{Y}}_j^c-\eta_j^c)$$
$$+(\frac{\partial\eta_j^c}{\partial\theta})'(\Sigma_j^c)^{-1}(\hat{\underline{Y}}_j^c-\eta_j^c)-\frac{1}{2}tr[\frac{\partial}{\partial\theta}[(\Sigma_j^c)^{-1}]V_j^c],\ \Box$$

A.2 Results in martingale limit theorems, stationarity, ergodicity and other results

In this section, we present results that were used in the proofs of the results in Chapter 4 of this thesis.

Theorem A.2.1 *(Theorem 2.18 of Hall and Heyde, 1980). Let $\{S_n=\sum_{i=1}^n X_i,\ \mathcal{F}_n, n\geq 1\}$ be a martingale and $\{U_n, n\geq 1\}$ a nondecreasing sequence of positive rv's such that U_n is $\mathcal{F}_{n-1}$-measurable for such n. If $1\leq p\leq 2$ then*

$$\sum_{i=1}^{\infty} U_i^{-1}X_i \quad \textit{converges a.s.}$$

on the set $\{\sum_{i=1}^{\infty} U_i^{-p}E(|X_i|^p|\mathcal{F}_{i-1})<\infty\}$, and

$$\lim_{n\to\infty} U_n^{-1}S_n=0 \ \textit{a.s.}$$

on the set $\{\lim_{n\to\infty}U_n=\infty,\ \sum_{i=1}^{\infty}U_i^{-p}E(|X_i|^p|\mathcal{F}_{i-1})<\infty\}$. If $2<p<\infty$, then (A.2.1) and (A.2.2) both hold on the set

$$\left\{\sum_{i=1}^{\infty}U_i^{-1}<\infty,\ \sum_{i=1}^{\infty}U_i^{-1-p/2}E(|X_i|^p|\mathcal{F}_{i-1})<\infty\right\}.$$

Theorem A.2.2 *(Theorem 3.2 of Hall and Heyde, 1980). Let $\{S_{ni},\mathcal{F}_{ni}, 1\leq i\leq k_n, n\geq 1\}$ be a zero-mean, square-integrable martingale array with differences X_{ni}, and let η^2 be an a.s. finite rv. Suppose that*

$$\max_i |X_{ni}| \xrightarrow{P} 0,$$

$$\sum_i X_{ni}^2 \xrightarrow{P} \eta^2,$$

$$E\left(\max_i X_{ni}^2\right) \quad \textit{is bounded in } n,$$

and

$$\textit{the } \sigma\textit{-fields are nested}: \ \mathcal{F}_{n,i} \subseteq \mathcal{F}_{n+1,i} \quad \textit{for } 1 \le i \le k_n, \ n \ge 1.$$

Then $S_{nk_n} = \sum_i X_{ni} \xrightarrow{\mathcal{D}} Z$ *(stably), where the rv* Z *has characteristic function* $E exp(-\frac{1}{2}\eta^2 t^2)$.

Corollary A.2.1 *(Corollary 3.1 of Hall and Heyde, 1980). If (A.2.3) and (A.2.5) in Theorem A.2.2 are replaced by the conditional Lindeberg condition*

$$\forall \ \varepsilon > 0, \ \sum_i E[X_{ni}^2 I(|X_{ni}| > \varepsilon)|\mathcal{F}_{n,i-1} \xrightarrow{P} 0,$$

If (A.2.4) is replaced by an analogous condition of the conditional variance

$$V_{nk_n}^2 = \sum_i E(X_{ni}^2|\mathcal{F}_{n,i-1}) \xrightarrow{P} \eta^2,$$

and if (A.2.6) holds, then the conclusion of Theorem A.2.2 remains true.

Corollary A.2.2 *(Serfling, 1980). For any random variable* X *and real number* $r > 0$,

(a) $E|X|^r = r\int_0^\infty t^{r-1} P(|X| \ge t)\, dt$

and

(b) if $E|X|^r < \infty$, *then* $P(|X| \ge t) = o(t^{-r})$, $t \to \infty$.

Proposition A.2.1 *(Proposition 6.6 of Breiman, 1968). Let* $X_1, X_2, \ldots$ *be stationary,* $\varphi(x)$ *measurable* $\mathcal{B}^\infty$, *then the process* $Y_1, Y_2, \ldots$ *define by*

$$Y_k = \varphi(X_k, X_{k+1}, \ldots)$$

is stationary.

Proposition A.2.2 *(Proposition 6.31 of Breiman, 1968). Let $X_1, X_2, \ldots$ be a stationary and ergodic process, $\varphi(x)$ measurable $\mathcal{B}^\infty$, then the process $Y_1, Y_2, \ldots$ define by*

$$Y_k = \varphi(X_k, X_{k+1}, \ldots)$$

is ergodic.

Theorem A.2.3 *(The ergodic theorem, e.g., Theorem 6.21 of Breiman, 1968). Let T be measure preserving on $(\Omega, \mathcal{F}, \mathcal{P})$. Then for X any rv such that $E|X| < \infty$,*

$$n^{-1} \sum_{k=0}^{n-1} X(T^k \omega) \xrightarrow{a.s.} E(X|\mathcal{J}),$$

where, $\mathcal{J}$ is a class of invariant sets (a σ-field).

Corollary A.2.3 *(corollary to the ergodic theorem, e.g., corollary 6.23 of Breiman, 1968). Let T be measure preserving and ergodic on $(\Omega, \mathcal{F}, \mathcal{P})$. Then for X any rv such that $E|X| < \infty$,*

$$n^{-1} \sum_{k=0}^{n-1} X(T^k \omega) \xrightarrow{a.s.} E(X),$$

A.3 Simulation results

In Chapter 5, analyses of variance and Fisher's LSD analyses were carried out on the asymptotic variance estimates for each of the tables (Tables 1-72) to compare the estimators. As mentioned therein, for Tables 1-18, the analyses were carried out on the MSE's as well. The results of the LSD analyses are summarized in Figures A1-A18 below. Figure A1 summarizes the results of the LSD analysis for Table 5.1, Figure A2 summarizes the results of the LSD analysis for Table 5.2, etc. In each figure, the estimators are arranged in order of increasing MSE.

Figure A1

Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,1), $\theta_o = 0.8$ and $n = 25$.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{c}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{b}$

Figure A2

Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,1), $\theta_o = 0.5$ and $n = 25$.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{c}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$

Figure A3

Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,1), $\theta_o = 0.2$ and $n = 25$.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{zb}$

Figure A4
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,1), θ_o = -0.2 and $n = 25$.

Estimator	$\hat{\theta}_n^b$	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^a$	$\hat{\theta}_n^c$
	———	———					
		———	———				
			———	———	———	———	———

Figure A5
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,1), $\theta_o = -0.5$ and $n = 25$.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^a$	$\hat{\theta}_n^b$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^c$
		———	———	———	———	———	
				———	———	———	———

Figure A6
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,1), θ_o = -0.8 and $n = 25$.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^a$	$\hat{\theta}_n^c$	$\hat{\theta}_n^b$
	———	———	———	———	———	———	

Figure A7

Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,1), $\theta_o = 0.8$ and $n = 50$.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{c}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{b}$

Figure A8

Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,1), $\theta_o = 0.5$ and $n = 50$.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{c}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{dag}$

Figure A9

Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,1), $\theta_o = 0.2$ and $n = 50$.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{dag}$

Figure A10
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,1), θ_o = -0.2 and n = 50.

Estimator	$\hat{\theta}_n^b$	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^a$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^a$
	———	———					
			———	———	———	———	———

Figure A11
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,1), θ_o = -0.5 and n = 50.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^a$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^c$	$\hat{\theta}_n^b$
		———	———	———	———	———	———

Figure A12
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,1), θ_o = -0.8 and n = 50.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^a$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^c$	$\hat{\theta}_n^b$
	———	———	———	———	———	———	

Figure A13

Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,1), $\theta_o = 0.8$ and $n = 100$.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{c}$	$\hat{\theta}_n^{b}$

Figure A14

Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,1), $\theta_o = 0.5$ and $n = 100$.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{c}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{b}$

Figure A15

Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,1), $\theta_o = 0.2$ and $n = 100$.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{b}$	$\hat{\theta}_n^{c}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{a}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{dag}$

Figure A16
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,1), θ_o = -0.2 and n = 100.

Estimator	$\hat{\theta}_n^b$	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^a$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^c$
	—	—	—	—	—	—	—

Figure A17
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,1), θ_o = -0.5 and n = 100.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^a$	$\hat{\theta}_n^c$	$\hat{\theta}_n^b$
	—	—	—	—	—	—	—

Figure A18
Fisher's LSD comparison of the estimators. Estimators underlined by a common line are not significantly different at the 0.05 level. Errors are from Gaussian (0,1), θ_o = -0.8 and n = 100.

Estimator	$\hat{\theta}_n^{ls}$	$\hat{\theta}_n^{dag}$	$\hat{\theta}_n^{mle}$	$\hat{\theta}_n^{zb}$	$\hat{\theta}_n^a$	$\hat{\theta}_n^c$	$\hat{\theta}_n^b$
	—	—	—	—	—	—	

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