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111 OPTIMAL STATE FEEDBACK FOR CONSTRAINED NONLINEAR SYSTEMS

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Abstract: In this paper, we consider a general nonlinear control system that is subject to both terminal state and continuous inequality constraints. The continuous inequality constraints must be satisfied at every point in the time horizon—an infinite number of points. Our aim is to design an optimal feedback controller that yields efficient system performance and satisfaction of all constraints. We first formulate this problem as a semi-infinite optimization problem. We then show that, by using a novel exact penalty approach, this semi-infinite optimization problem can be converted into a sequence of nonlinear programming problems, each of which can be solved using standard numerical techniques. We conclude the paper with some convergence results.

Key words: Optimal control; State feedback; Exact penalty function; Nonlinear programming.

1 PROBLEM FORMULATION

We consider nonlinear control systems in the following general form:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(t, \boldsymbol{x}(t), \boldsymbol{u}(t)), \quad t \in [0, T], \tag{1.1}$$

$$\boldsymbol{x}(0) = \boldsymbol{x}^0, \tag{1.2}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the *state*, $\boldsymbol{u}(t) \in \mathbb{R}^r$ is the *control*, $\boldsymbol{x}^0 \in \mathbb{R}^n$ is a given initial state, T is a given *terminal time*, and $\boldsymbol{f} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$ is a given continuously differentiable function.

System (1.1)-(1.2) is subject to the following *terminal state constraints*:

$$\Psi_i(\boldsymbol{x}(T)) = 0, \quad i = 1, \ldots, p, \tag{1.3}$$

where $\Psi_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, p$ are given continuously differentiable functions.

In addition, system (1.1)-(1.2) is subject to a set of *continuous inequality constraints* defined as follows:

$$h_j(t, \boldsymbol{x}(t), \boldsymbol{u}(t)) \leq 0, \quad t \in [0, T], \quad j = 1, \ldots, q, \tag{1.4}$$

where $h_j : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}$, $j = 1, \ldots, q$ are given continuously differentiable functions. Note that control bounds can be easily incorporated into (1.4).

Our aim is to design an optimal *state feedback control* for system (1.1)-(1.2). To this end, we assume that the control takes the following form:

$$\boldsymbol{u}(t) = \boldsymbol{\varphi}(\boldsymbol{x}(t), \boldsymbol{\zeta}), \quad t \in [0, T], \tag{1.5}$$

where $\boldsymbol{\zeta} \in \mathbb{R}^m$ is a vector of feedback control parameters and $\boldsymbol{\varphi} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^r$ is a given continuously differentiable function. Typical choices for the feedback controller (1.5) include *linear state feedback* *control* (in which $\boldsymbol{\varphi}$ is a linear function—see Khalil H. K. (2002)) and *PID control* (in which $\boldsymbol{\varphi}$ is the sum of linear, integral, and derivative terms—see Li B. (2011)).

The feedback control parameters ζ_k, $k = 1, \ldots, m$ are subject to the following bound constraints:

$$a_k \leq \zeta_k \leq b_k, \quad k = 1, \ldots, m, \tag{1.6}$$

where a_k and b_k, $k = 1, \ldots, m$ are given constants. Let Γ denote the set of all $\boldsymbol{\zeta} \in \mathbb{R}^m$ satisfying (1.6).

Substituting (1.5) into (1.1) gives

$$\dot{\boldsymbol{x}}(t) = \tilde{\boldsymbol{f}}(t, \boldsymbol{x}(t), \boldsymbol{\zeta}), \quad t \in [0, T], \tag{1.7}$$

where

$$\tilde{\boldsymbol{f}}(t, \boldsymbol{x}(t), \boldsymbol{\zeta}) = \boldsymbol{f}(t, \boldsymbol{x}(t), \boldsymbol{\varphi}(\boldsymbol{x}(t), \boldsymbol{\zeta})).$$

Let $\boldsymbol{x}(\cdot|\boldsymbol{\zeta})$ denote the solution of system (1.7) with the initial condition (1.2). Then the terminal constraints (1.3) become

$$\Psi_i(\boldsymbol{x}(T|\boldsymbol{\zeta})) = 0, \quad i = 1, \ldots, p. \tag{1.8}$$

Substituting the feedback control (1.5) into the continuous inequality constraints (1.4) gives

$$\tilde{h}_j(t, \boldsymbol{x}(t|\boldsymbol{\zeta}), \boldsymbol{\zeta}) \leq 0, \quad t \in [0, T], \quad j = 1, \ldots, q, \tag{1.9}$$

where

$$\tilde{h}_j(t, \boldsymbol{x}(t|\boldsymbol{\zeta}), \boldsymbol{\zeta}) = h_j(t, \boldsymbol{x}(t|\boldsymbol{\zeta}), \boldsymbol{\varphi}(\boldsymbol{x}(t|\boldsymbol{\zeta}), \boldsymbol{\zeta})).$$

Let Λ denote the set of all $\boldsymbol{\zeta} \in \Gamma$ satisfying (1.8) and (1.9).

We now consider the problem of choosing the feedback control parameters ζ_k, $k = 1, \ldots, m$ to minimize the total system cost subject to the constraints (1.8) and (1.9).

Problem P Choose $\boldsymbol{\zeta} \in \Lambda$ to minimize the cost function

$$J(\boldsymbol{\zeta}) = \Phi(\boldsymbol{x}(T|\boldsymbol{\zeta}), \boldsymbol{\zeta}) + \int_0^T \mathcal{L}(t, \boldsymbol{x}(t|\boldsymbol{\zeta}), \boldsymbol{\zeta}) dt,$$

where $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $\mathcal{L} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ are given continuously differentiable functions.

Note that (1.9) defines an infinite number of constraints—one for each point in $[0, T]$. Hence, Problem P can be viewed as a *semi-infinite optimization problem.* In the next section, we will use a novel exact penalty approach to approximate Problem P by a nonlinear programming problem.

2 AN EXACT PENALTY METHOD

Define a *constraint violation function* on Γ as follows:

$$\Delta(\boldsymbol{\zeta}) = \sum_{i=1}^{p} \Psi_i(\boldsymbol{x}(T|\boldsymbol{\zeta}))^2 + \sum_{j=1}^{q} \int_0^T \max\left\{\tilde{h}_j(t, \boldsymbol{x}(t|\boldsymbol{\zeta}), \boldsymbol{\zeta}), 0\right\}^2 dt.$$

Clearly, $\Delta(\boldsymbol{\zeta}) = 0$ if and only if $\boldsymbol{\zeta} \in \Lambda$.

Let $\bar{\epsilon} > 0$ be a given constant. We consider the following *penalty function* defined on $\Gamma \times [0, \bar{\epsilon}]$:

$$G_\sigma(\boldsymbol{\zeta}, \epsilon) = \begin{cases} J(\boldsymbol{\zeta}), & \text{if } \epsilon = 0,\ \Delta(\boldsymbol{\zeta}) = 0, \\ J(\boldsymbol{\zeta}) + \epsilon^{-\alpha}\Delta(\boldsymbol{\zeta}) + \sigma\epsilon^{\beta}, & \text{if } \epsilon \in (0, \bar{\epsilon}], \\ \infty, & \text{if } \epsilon = 0,\ \Delta(\boldsymbol{\zeta}) \neq 0, \end{cases} \tag{2.1}$$

where $\epsilon \in [0, \bar{\epsilon}]$ is a new decision variable, α and β are fixed constants such that $1 \leq \beta \leq \alpha$, and $\sigma > 0$ is a penalty parameter.

In the penalty function (2.1), the last term $\sigma\epsilon^{\beta}$ is designed to penalize large values of ϵ, while the middle term $\epsilon^{-\alpha}\Delta(\boldsymbol{\zeta})$ is designed to penalize constraint violations. When σ is large, minimizing (2.1) forces ϵ to be small, which in turn causes $\epsilon^{-\alpha}$ to become large, and thus constraint violations are penalized very severely. Hence, minimizing the penalty function for large σ will likely lead to feasible points satisfying constraints (1.8) and (1.9). On this basis, we can approximate Problem P by the following *penalty problem.*

Problem Q Choose $(\boldsymbol{\zeta}, \epsilon) \in \Gamma \times (0, \bar{\epsilon}]$ to minimize the penalty function

$$G_\sigma(\boldsymbol{\zeta}, \epsilon) = J(\boldsymbol{\zeta}) + \epsilon^{-\alpha}\Delta(\boldsymbol{\zeta}) + \sigma\epsilon^{\beta}.$$

Problem Q only involves bound constraints and is therefore much easier to solve than Problem P. In the next section, we will present some convergence results that formally link Problem Q with Problem P. First, however, we discuss how to solve Problem Q.

Problem Q can be viewed as a nonlinear programming problem in which the feedback control parameters ζ_k, $k = 1, \ldots, m$ and the new decision variable ϵ need to be chosen to minimize the penalty function G_σ. Numerical algorithms for solving such problems typically use the gradient of the cost function to compute *descent directions* that lead to more profitable areas of the feasible region (Luenberger D. G. (2008)). Notice, however, that $\boldsymbol{\zeta}$ influences G_σ *implicitly* through the dynamic system (1.7), and thus computing the gradient of G_σ is not straightforward. Nevertheless, the techniques developed by Vincent and Grantham (Vincent (1981)) and Loxton et al. (Loxton R. (2008)) can be used to derive formulae for the gradient of G_σ. This is described below.

First, for each $k = 1, \ldots, m$, consider the following *variational system*:

$$\dot{\boldsymbol{\phi}}^k(t) = \frac{\partial \tilde{\boldsymbol{f}}(t, \boldsymbol{x}(t|\boldsymbol{\zeta}), \boldsymbol{\zeta})}{\partial \boldsymbol{x}}\boldsymbol{\phi}^k(t) + \frac{\partial \tilde{\boldsymbol{f}}(t, \boldsymbol{x}(t|\boldsymbol{\zeta}), \boldsymbol{\zeta})}{\partial \zeta_k}, \quad t \in [0, T], \tag{2.2}$$

$$\boldsymbol{\phi}^k(0) = \mathbf{0}. \tag{2.3}$$

Let $\boldsymbol{\phi}^k(\cdot|\boldsymbol{\zeta})$ denote the solution of the variational system (2.2)-(2.3). We have the following result.

Theorem 2.1 *For each* $k = 1, \ldots, m$,

$$\frac{\partial \boldsymbol{x}(t|\boldsymbol{\zeta})}{\partial \zeta_k} = \boldsymbol{\phi}^k(t|\boldsymbol{\zeta}), \quad t \in [0, T].$$

Proof. First, note that

$$\frac{\partial}{\partial \zeta_k}\{\boldsymbol{x}(0|\boldsymbol{\zeta})\} = \frac{\partial}{\partial \zeta_k}\{\boldsymbol{x}^0\} = \mathbf{0}. \tag{2.4}$$

Thus, $\partial \boldsymbol{x}(\cdot|\boldsymbol{\zeta})/\partial \zeta_k$ satisfies the initial condition (2.3).

Now, by (1.7),

$$\boldsymbol{x}(t|\boldsymbol{\zeta}) = \boldsymbol{x}(0|\boldsymbol{\zeta}) + \int_0^t \tilde{\boldsymbol{f}}(s, \boldsymbol{x}(s|\boldsymbol{\zeta}), \boldsymbol{\zeta})ds = \boldsymbol{x}^0 + \int_0^t \tilde{\boldsymbol{f}}(s, \boldsymbol{x}(s|\boldsymbol{\zeta}), \boldsymbol{\zeta})ds, \quad t \in [0, T]. \tag{2.5}$$

It can be shown that $\boldsymbol{x}(t|\boldsymbol{\zeta})$ is a continuously differentiable function of ζ_k, $k = 1, \ldots, m$ (Loxton R. (2011)). Hence, by using Leibniz's rule to differentiate (2.5) with respect to ζ_k, we obtain

$$\frac{\partial \boldsymbol{x}(t|\boldsymbol{\zeta})}{\partial \zeta_k} = \int_0^t \left\{ \frac{\partial \tilde{\boldsymbol{f}}(s, \boldsymbol{x}(s|\boldsymbol{\zeta}), \boldsymbol{\zeta})}{\partial \boldsymbol{x}} \frac{\partial \boldsymbol{x}(s|\boldsymbol{\zeta})}{\partial \zeta_k} + \frac{\partial \tilde{\boldsymbol{f}}(s, \boldsymbol{x}(s|\boldsymbol{\zeta}), \boldsymbol{\zeta})}{\partial \zeta_k} \right\} ds, \quad t \in [0, T], \tag{2.6}$$

where

$$\frac{\partial \tilde{\boldsymbol{f}}(s, \boldsymbol{x}(s|\boldsymbol{\zeta}), \boldsymbol{\zeta})}{\partial \boldsymbol{x}} = \frac{\partial \boldsymbol{f}(s, \boldsymbol{x}(s|\boldsymbol{\zeta}), \boldsymbol{\varphi}(\boldsymbol{x}(s|\boldsymbol{\zeta}), \boldsymbol{\zeta}))}{\partial \boldsymbol{x}} + \frac{\partial \boldsymbol{f}(s, \boldsymbol{x}(s|\boldsymbol{\zeta}), \boldsymbol{\varphi}(\boldsymbol{x}(s|\boldsymbol{\zeta}), \boldsymbol{\zeta}))}{\partial \boldsymbol{u}} \frac{\partial \boldsymbol{\varphi}(\boldsymbol{x}(s|\boldsymbol{\zeta}), \boldsymbol{\zeta})}{\partial \boldsymbol{x}}$$

and

$$\frac{\partial \tilde{\boldsymbol{f}}(s, \boldsymbol{x}(s|\boldsymbol{\zeta}), \boldsymbol{\zeta})}{\partial \zeta_k} = \frac{\partial \boldsymbol{f}(s, \boldsymbol{x}(s|\boldsymbol{\zeta}), \boldsymbol{\varphi}(\boldsymbol{x}(s|\boldsymbol{\zeta}), \boldsymbol{\zeta}))}{\partial \boldsymbol{u}} \frac{\partial \boldsymbol{\varphi}(\boldsymbol{x}(s|\boldsymbol{\zeta}), \boldsymbol{\zeta})}{\partial \zeta_k}.$$

Differentiating (2.6) with respect to time yields

$$\frac{d}{dt}\left\{ \frac{\partial \boldsymbol{x}(t|\boldsymbol{\zeta})}{\partial \zeta_k} \right\} = \frac{\partial \tilde{\boldsymbol{f}}(t, \boldsymbol{x}(t|\boldsymbol{\zeta}), \boldsymbol{\zeta})}{\partial \boldsymbol{x}} \frac{\partial \boldsymbol{x}(t|\boldsymbol{\zeta})}{\partial \zeta_k} + \frac{\partial \tilde{\boldsymbol{f}}(t, \boldsymbol{x}(t|\boldsymbol{\zeta}), \boldsymbol{\zeta})}{\partial \zeta_k}, \quad t \in [0, T]. \tag{2.7}$$

Equations (2.4) and (2.7) show that $\partial \boldsymbol{x}(\cdot|\boldsymbol{\zeta})/\partial \zeta_k$ is the solution of the variational system (2.2)-(2.3). This completes the proof. □

We are now ready to derive formulae for the gradient of G_σ.

Theorem 2.2 *The partial derivatives of G_σ are given by*

$$\frac{\partial G_\sigma(\boldsymbol{\zeta},\epsilon)}{\partial \zeta_k} = \frac{\partial J(\boldsymbol{\zeta})}{\partial \zeta_k} + \epsilon^{-\alpha}\frac{\partial \Delta(\boldsymbol{\zeta})}{\partial \zeta_k}, \quad k=1,\dots,m, \tag{2.8}$$

and

$$\frac{\partial G_\sigma(\boldsymbol{\zeta},\epsilon)}{\partial \epsilon} = -\alpha\epsilon^{-\alpha-1}\Delta(\boldsymbol{\zeta}) + \beta\sigma\epsilon^{\beta-1}, \tag{2.9}$$

where

$$\frac{\partial J(\boldsymbol{\zeta})}{\partial \zeta_k} = \frac{\partial \Phi(\boldsymbol{x}(T|\boldsymbol{\zeta}),\boldsymbol{\zeta})}{\partial \boldsymbol{x}}\boldsymbol{\phi}^k(T|\boldsymbol{\zeta}) + \frac{\partial \Phi(\boldsymbol{x}(T|\boldsymbol{\zeta}),\boldsymbol{\zeta})}{\partial \zeta_k} + \int_0^T \left\{\frac{\partial \mathcal{L}(t,\boldsymbol{x}(t|\boldsymbol{\zeta}),\boldsymbol{\zeta})}{\partial \boldsymbol{x}}\boldsymbol{\phi}^k(t|\boldsymbol{\zeta}) + \frac{\partial \mathcal{L}(t,\boldsymbol{x}(t|\boldsymbol{\zeta}),\boldsymbol{\zeta})}{\partial \zeta_k}\right\}dt,$$

$$\begin{aligned}\frac{\partial \Delta(\boldsymbol{\zeta})}{\partial \zeta_k} &= 2\sum_{i=1}^{p}\Psi_i(\boldsymbol{x}(T|\boldsymbol{\zeta}))\frac{\partial \Psi_i(\boldsymbol{x}(T|\boldsymbol{\zeta}))}{\partial \boldsymbol{x}}\boldsymbol{\phi}^k(T|\boldsymbol{\zeta}) \\ &\qquad + 2\sum_{j=1}^{q}\int_0^T \max\left\{\tilde{h}_j(t,\boldsymbol{x}(t|\boldsymbol{\zeta}),\boldsymbol{\zeta}),0\right\}\left\{\frac{\partial \tilde{h}_j(t,\boldsymbol{x}(t|\boldsymbol{\zeta}),\boldsymbol{\zeta})}{\partial \boldsymbol{x}}\boldsymbol{\phi}^j(t|\boldsymbol{\zeta}) + \frac{\partial \tilde{h}_j(t,\boldsymbol{x}(t|\boldsymbol{\zeta}),\boldsymbol{\zeta})}{\partial \zeta_k}\right\}dt.\end{aligned}$$

Proof. From Theorem 2.1, we have

$$\begin{aligned}\frac{\partial J(\boldsymbol{\zeta})}{\partial \zeta_k} &= \frac{\partial \Phi(\boldsymbol{x}(T|\boldsymbol{\zeta}),\boldsymbol{\zeta})}{\partial \boldsymbol{x}}\frac{\partial \boldsymbol{x}(T|\boldsymbol{\zeta})}{\partial \zeta_k} + \frac{\partial \Phi(\boldsymbol{x}(t|\boldsymbol{\zeta}),\boldsymbol{\zeta})}{\partial \zeta_k} + \int_0^T \left\{\frac{\partial \mathcal{L}(t,\boldsymbol{x}(t|\boldsymbol{\zeta}),\boldsymbol{\zeta})}{\partial \boldsymbol{x}}\frac{\partial \boldsymbol{x}(t|\boldsymbol{\zeta})}{\partial \zeta_k} + \frac{\mathcal{L}(t,\boldsymbol{x}(t|\boldsymbol{\zeta}),\boldsymbol{\zeta})}{\partial \zeta_k}\right\}dt \\ &= \frac{\partial \Phi(\boldsymbol{x}(T|\boldsymbol{\zeta}),\boldsymbol{\zeta})}{\partial \boldsymbol{x}}\boldsymbol{\phi}^k(T|\boldsymbol{\zeta}) + \frac{\partial \Phi(\boldsymbol{x}(T|\boldsymbol{\zeta}),\boldsymbol{\zeta})}{\partial \zeta_k} + \int_0^T \left\{\frac{\partial \mathcal{L}(t,\boldsymbol{x}(t|\boldsymbol{\zeta}),\boldsymbol{\zeta})}{\partial \boldsymbol{x}}\boldsymbol{\phi}^k(t|\boldsymbol{\zeta}) + \frac{\mathcal{L}(t,\boldsymbol{x}(t|\boldsymbol{\zeta}),\boldsymbol{\zeta})}{\partial \zeta_k}\right\}dt.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\partial \Delta(\boldsymbol{\zeta})}{\partial \zeta_k} &= 2\sum_{i=1}^{p}\Psi_i(\boldsymbol{x}(T|\boldsymbol{\zeta}))\frac{\partial \Psi_i(\boldsymbol{x}(T|\boldsymbol{\zeta}))}{\partial \boldsymbol{x}}\boldsymbol{\phi}^k(T|\boldsymbol{\zeta}) \\ &\qquad + 2\sum_{j=1}^{q}\int_0^T \max\left\{\tilde{h}_j(t,\boldsymbol{x}(t|\boldsymbol{\zeta}),\boldsymbol{\zeta}),0\right\}\left\{\frac{\partial \tilde{h}_j(t,\boldsymbol{x}(t|\boldsymbol{\zeta}),\boldsymbol{\zeta})}{\partial \boldsymbol{x}}\frac{\partial \boldsymbol{x}(t|\boldsymbol{\zeta})}{\partial \zeta_k} + \frac{\partial \tilde{h}_j(t,\boldsymbol{x}(t|\boldsymbol{\zeta}),\boldsymbol{\zeta})}{\partial \zeta_k}\right\}dt \\ &= 2\sum_{i=1}^{p}\Psi_i(\boldsymbol{x}(T|\boldsymbol{\zeta}))\frac{\partial \Psi_i(\boldsymbol{x}(T|\boldsymbol{\zeta}))}{\partial \boldsymbol{x}}\boldsymbol{\phi}^k(T|\boldsymbol{\zeta}) \\ &\qquad + 2\sum_{j=1}^{q}\int_0^T \max\left\{\tilde{h}_j(t,\boldsymbol{x}(t|\boldsymbol{\zeta}),\boldsymbol{\zeta}),0\right\}\left\{\frac{\partial \tilde{h}_j(t,\boldsymbol{x}(t|\boldsymbol{\zeta}),\boldsymbol{\zeta})}{\partial \boldsymbol{x}}\boldsymbol{\phi}^k(t|\boldsymbol{\zeta}) + \frac{\partial \tilde{h}_j(t,\boldsymbol{x}(t|\boldsymbol{\zeta}),\boldsymbol{\zeta})}{\partial \zeta_k}\right\}dt.\end{aligned}$$

Equation (2.8) follows immediately from these equations. Equation (2.9) is obtained using standard differentiation rules. □

On the basis of Theorem 2.2, we can compute the gradient of G_σ using the following procedure: (i) Combine the original control system with the variational systems to form an expanded initial value problem; (ii) Solve this expanded initial value problem using a numerical integration method; (iii) Substitute the solution of the initial value problem into (2.8) and (2.9). This procedure can be integrated with a standard gradient-based optimization method—e.g. sequential quadratic programming (Nocedal J. (2006))—to solve Problem Q as a nonlinear programming problem.

3 CONVERGENCE RESULTS

In this section, we describe the mathematical theory relating Problem P with Problem Q. We begin with the following result proved by Lin et al. (Lin Q. (2012)).

Theorem 3.1 *Let $\{\sigma_l\}_{l=1}^{\infty}$ be an increasing sequence of penalty parameters such that $\sigma_l \to \infty$ as $l \to \infty$. Furthermore, let $(\boldsymbol{\zeta}^{l,*},\epsilon^{l,*})$ denote a global solution of Problem Q. Then the sequence $\{(\boldsymbol{\zeta}^{l,*},\epsilon^{l,*})\}_{l=1}^{\infty}$ has at least one limit point, and any limit point is a global solution of Problem P.*

Theorem 3.1 suggests that we can obtain a solution of Problem P by solving Problem Q sequentially for increasing values of the penalty parameter. As mentioned in the previous section, Problem Q is essentially a nonlinear programming problem that can be solved using standard numerical optimization techniques.

One disadvantage of Theorem 3.1 is that it requires the *global* solution of Problem Q. Problem Q is non-convex in general, and thus we will usually only be able to solve it locally. Nevertheless, by making some mild assumptions, one can show that a local solution of Problem Q converges to a local solution of Problem P as the penalty parameter increases.

We assume that for each feasible point $\boldsymbol{\zeta} \in \Lambda$ of Problem P, the following conditions are satisfied:

(A1) The vectors $\partial\Psi_i(\boldsymbol{x}(T|\boldsymbol{\zeta}))/\partial\boldsymbol{\zeta}$, $i = 1, \dots, p$ are linearly independent (when $p \neq 0$).

(A2) There exists a vector $[\eta_1, \dots, \eta_m]^\top \in \mathbb{R}^m$ and negative real numbers $\vartheta_1 < 0$ and $\vartheta_2 < 0$ such that

$$\sum_{k=1}^{m} \eta_k \frac{\partial\Psi_i(\boldsymbol{x}(T|\boldsymbol{\zeta}))}{\partial\zeta_k} = 0, \quad i = 1, \dots, p,$$

$$\sum_{k=1}^{m} \eta_k \left\{ \frac{\partial\tilde{h}_j(t, \boldsymbol{x}(t|\boldsymbol{\zeta}), \boldsymbol{\zeta})}{\partial\boldsymbol{x}} \boldsymbol{\phi}^k(t|\boldsymbol{\zeta}) + \frac{\partial\tilde{h}_j(t, \boldsymbol{x}(t|\boldsymbol{\zeta}), \boldsymbol{\zeta})}{\partial\zeta_k} \right\} < \vartheta_1,$$

$$t \in \{\, s \in [0, T] : \tilde{h}_j(s, \boldsymbol{x}(s|\boldsymbol{\zeta}), \boldsymbol{\zeta}) \geq \vartheta_2 \,\}, \quad j = 1, \dots, q,$$

$$\eta_k \begin{cases} > 0, & \text{if } \zeta_k = a_k, \\ < 0, & \text{if } \zeta_k = b_k. \end{cases}$$

(A3) There exists a constant $L > 0$ and a neighbourhood $\mathcal{N}$ of $\boldsymbol{\zeta}$ such that for each $j = 1, \dots, q$,

$$\max\left\{\tilde{h}_j(t, \boldsymbol{x}(t|\boldsymbol{\zeta}'), \boldsymbol{\zeta}'), 0\right\}^2 \leq L \int_0^T \max\left\{\tilde{h}_j(s, \boldsymbol{x}(s|\boldsymbol{\zeta}'), \boldsymbol{\zeta}'), 0\right\}^2 ds, \quad (\boldsymbol{\zeta}', t) \in \mathcal{N} \times [0, T].$$

Under Assumptions (A1)-(A3), we have the following result proved by Lin et al. (Lin Q. (2012)).

Theorem 3.2 *Let $\{\sigma_l\}_{l=1}^{\infty}$ be an increasing sequence of penalty parameters such that $\sigma_l \to \infty$ as $l \to \infty$. Furthermore, let $(\boldsymbol{\zeta}^{l,*}, \epsilon^{l,*})$ denote a local solution of Problem Q. Suppose that $\{G_{\sigma_l}(\boldsymbol{\zeta}^{l,*}, \epsilon^{l,*})\}_{l=1}^{\infty}$ is bounded. Then there exists a positive integer l' such that for each $l \geq l'$, $\boldsymbol{\zeta}^{l,*}$ is a local solution of Problem P.*

Theorem 3.2 implies that when the penalty parameter σ is sufficiently large, the values of the feedback control parameters in a locally optimal solution for Problem Q will also be locally optimal for Problem P. On this basis, we propose the following algorithm for solving Problem P:

(1) Choose $\boldsymbol{\zeta}^0 \in \Gamma$ (initial guess), $\sigma^0 > 0$ (initial penalty parameter), $\rho > 0$ (tolerance), and $\sigma_{\max} > \sigma^0$ (upper bound for the penalty parameter).

(2) Set $\bar{\epsilon} \to \epsilon^0$ and $\sigma^0 \to \sigma$.

(3) Starting with $(\boldsymbol{\zeta}^0, \epsilon^0)$ as the initial guess, use a nonlinear programming algorithm (e.g. sequential quadratic programming) to solve Problem Q. Let $(\boldsymbol{\zeta}^*, \epsilon^*)$ denote the local minimizer obtained.

(4) If $\epsilon^* < \rho$, then stop: take $\boldsymbol{\zeta}^*$ as a local solution of Problem P. Otherwise, set $10\sigma \to \sigma$ and go to Step 5.

(5) If $\sigma \leq \sigma_{\max}$, then set $(\boldsymbol{\zeta}^*, \epsilon^*) \to (\boldsymbol{\zeta}^0, \epsilon^0)$ and go to Step 3. Otherwise stop: the algorithm cannot find a solution of Problem P.

References

H. K. Khalil, *Nonlinear Systems*, 3rd edition, Prentice Hall, New Jersey, USA, 2002.

B. Li, K. L. Teo, C. C. Lim, and G. R. Duan, "An optimal PID controller design for nonlinear constrained optimal control problems," *Discrete and Continuous Dynamical Systems – Series B*, vol. 16, no. 4, pp. 1101-1117, 2011.

Q. Lin, R. Loxton, K. L. Teo, Y. H. Wu, and C. Yu, "A new exact penalty method for semi-infinite programming problems," submitted.

R. Loxton, K. L. Teo, and V. Rehbock, "Optimal control problems with multiple characteristic time points in the objective and constraints," *Automatica*, vol. 44, no. 11, pp. 2923-2929, 2008.

R. Loxton, K. L. Teo, and V. Rehbock, "Robust suboptimal control of nonlinear systems," *Applied Mathematics and Computation*, vol. 217, no. 14, pp. 6566-6576, 2011.

D. G. Luenberger and Y. Ye, *Linear and Nonlinear Programming*, 3rd edition, Springer, New York, USA, 2008.

J. Nocedal and S. J. Wright, *Numerical Optimization*, 2nd edition, Springer, New York, USA, 2006.

K. L. Teo, C. J. Goh, and K. H. Wong, *A Unified Computational Approach to Optimal Control Problems*, Longman Scientific and Technical, Essex, UK, 1991.

T. L. Vincent and W. J. Grantham, *Optimality in Parametric Systems*, John Wiley, New York, USA, 1981.