# Output Peak Control of Nonhomogeneous Markov Jump System with Unit-Energy Disturbance

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**Abstract** This paper considers output peak controller design for discrete nonhomogeneous Markov jump systems under unit-energy disturbance. The mode-dependent output peak feedback controller is designed to ensure that the resulting closed-loop system is stochastically stable and the peak of the output is within a specified range. Furthermore, the optimal energy-to-peak gain indices of the mode-dependent and the mode-independent state feedback controllers are evaluated and compared. A numerical example is presented to illustrate the applicability of the results obtained.

**Keywords** Output peak control · Nonhomogeneous Markov jump system · Unit-energy disturbance

## 1 Introduction

The structures and parameters in many engineering systems tend to vary due to random variations caused by changes in subsystem interconnection, loss of communication, or sensor failures. These phenomena are known as "switching". It often occurs when a cluster of systems is involved, especially for Networked control systems (NCSs) [16–18]. Switched systems have been extensively studied in the past 20 years. See

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[25,27] for model reduction, and [26] for asynchronous control. Among switched systems, linear Markov jump systems (MJSs) belong to a special class of stochastic switched systems where the switching is driven by a Markov process. The transition probability (TP), a crucial factor in a Markov process (or chain), determines the behavior and performance of the linear MJS. This type of system has been studied under the assumption that the TP is fixed or partially unknown for over two decades and some systematic results are now available (see [13] for control, [20] for gain scheduling, [14] for sliding mode control, [15] for 2-D Markov system, [2,12] for filter, [6] for finite-time control, and [22] for partly unknown transition probability). However, these works are under the assumption that the TP is homogeneous, i.e., the underlying Markov process (or chain) is time invariant. In reality, this assumption may not be satisfied. A typical example can be found in networked control systems, where the delay as well as the packet loss is distinct at different time intervals [7]. Thus, the TP matrix keeps varying throughout the whole time horizon of the control system. In this case, the TP is time varying, and hence the underlying Markov process (or chain) is nonhomogeneous [8]. Similar phenomena are also observed in many engineering systems, such as electronic circuits [5]. Thus, there is an urgent need to develop new theories and methodologies for dynamical systems with nonhomogeneous TP. It has attracted an increased interest among the control community, and consequently, some important results have been obtained. See, for example, [1,21]. More precisely, a two-level nonhomogeneous Markov chain, which involves arbitrary variation as well as stochastic variation, is considered in [21]. A feasible and effective scheme to estimate a time-varying uncertainty by using a bounded convex polyhedron is proposed in [1]. The model considered in [21] is a special case of that considered in [1], where sufficient conditions for stochastic stability of the nonhomogeneous Markov system are derived.

On the other hand, external disturbances always exist in real life [3]. In fact, it is a detrimental factor causing the reduction of the performance of a controller. Thus, controller design with disturbance has always been an active research topic in the control community. Many control problems can be equivalently transformed into the controller design problems using the index of "energy-to-peak gain", "$l_2 - l_\infty$ gain", or "generalized $H_2$ performance" [9–11]. The goal is to ensure that the feedback system is internally stable, while the closed-loop mapping from the disturbance to the controlled output is small. In this way, disturbance rejection as well as specifications of robustness will both be achieved. During the past decade, many results focusing on external disturbances for MJS have been reported (see [4,23,24] for $H_\infty$ control and filtering, [19] for model reduction), where the $H_\infty$ index is widely used to evaluate the input-output performance of the system under $l_2$ disturbance.

In this paper, we consider an important class of problems, where the objective is to design an output peak controller for nonhomogeneous Markov jump systems in the form of [1] under unit-energy disturbance such that the output amplitude stays within a certain range. Furthermore, a sufficient condition is derived for the design of a output peak controller which will ensure that the output peak performance is satisfied.

The rest of the paper is organized as follows: In Sect. 2, the dynamical model of the system is defined and the purpose of the paper is stated. Section 3 gives the definitions of stochastic stability and output peak performance. In Sect. 4, a sufficient condition expressed in terms of LMIs is derived for the design of the output peak controller for the nonhomogeneous Markov system. In Sect. 5, a numerical example is provided to illustrate the applicability of the results obtained, and finally, Sect. 6 concludes the paper.

Notations. In the sequel, the notation $R^n$ stands for an $n$-dimensional Euclidean space, the transpose of the matrix $A$ is denoted by $A^\top$, $E\{\cdot\}$ denotes the mathematical statistical expectation of a stochastic process or vector, $l_2^n[0, \infty)$ stands for the space of $n$-dimensional square summable vector-valued functions over $[0, \infty)$, a positive-definite matrix is denoted by $P > 0$, $I$ is the unit matrix with appropriate dimension, and $*$ means the symmetric term in a symmetric matrix.

## 2 Problem Statement and Preliminaries

Let $(M, F, P)$ be a probability space, where $M$, $F$, and $P$ represent, respectively, the sample space, the $\sigma$-algebra of events, and the probability measure defined on $F$. We consider the following discrete-time MJS:

$$\begin{cases} x_{k+1} = A(r_k)x_k + B_1(r_k)w_k + B_2(r_k)u_k \\ z_k = C(r_k)x_k + D(r_k)u_k \end{cases} \tag{2.1}$$

where $x_k \in R^n$ is the state vector of the system, $u_k \in R^m$ is the input vector of the system, $z_k \in R^p$ is the controlled output vector of the system, and $w_k$ is the external disturbance vector of the system. $\{r_k, k \geq 0\}$ is a discrete-time Markov stochastic process which takes values in a finite state set $\Gamma = \{1, 2, 3, \ldots, \sigma\}$, and $r_0$ represents the initial mode, the transition probability matrix is defined as: $\Pi(k) = \{\pi_{ij}(k)\}$, $i$, $j \in \Gamma$, $\pi_{ij}(k) = P(r_{k+1} = j|r_k = i)$ is the transition probability from mode $i$ at time $k$ to mode $j$ at time $k+1$, which satisfies $\pi_{ij}(k) \geq 0$ and $\sum_{j=1}^{\sigma} \pi_{ij}(k) = 1$. For given vertices $\Pi^s(k)$, $s = 1, \ldots, N$, the time-varying transition matrix $\Pi(k)$ of the nonhomogeneous Markov jump systems is constructed as:

$$\Pi(k) = \sum_{s=1}^{N} \alpha_s(k)\Pi^s(k) \tag{2.2}$$

where

$$0 \leq \alpha_s(k) \leq 1, \quad \sum_{s=1}^{N} \alpha_s(k) = 1$$

Hence, the nonhomogeneous transition probability matrix of system (2.1) belongs to a polytope described by its vertices. It is noted that if $\Pi(k)$ is a constant matrix, the system becomes a homogeneous MJS. For brevity, when $r_k = i$, $i \in \Gamma$, the matrices $A(r_k)$, $B_1(r_k)$, $B_2(r_k)$, $C(r_k)$, $D(r_k)$, $H(r_k)$, and $K(r_k)$ are denoted as $A_i$, $B_{i1}$, $B_{i2}$, $C_i$, $D_i$, and $K_i$.

To proceed further, we need some preliminaries.

**Definition 2.1** For any initial mode $r_0$, and a given initial state $x_0$, the discrete-time Markov jump system (2.1) (with $w_k = 0$) is said to be stochastically stable if

$$\lim_{\upsilon\to\infty} E\{\sum_{k=0}^{\upsilon} x_k^\top x_k | x_0, r_0\} < \infty \tag{2.3}$$

**Lemma 2.1** *Let $\overline{M}$ and $\overline{N}$ be positive definite symmetric matrices. Then,*

$$\overline{M} + \overline{M}^\top - \overline{N} \le \overline{MN}^{-1}\overline{M}^\top$$

*Proof* Since $\overline{N}$ is a positive definite symmetric matrix, we have

$$(\overline{M} - \overline{N})\overline{M}^{-1}(\overline{M} - \overline{N})^\top \ge 0$$

Subsequently, the following inequality is derived

$$\overline{MN}^{-1}\overline{M}^\top - \overline{M} - \overline{M}^\top + \overline{N} \ge 0$$

This completes the proof.

**Definition 2.2** The disturbance $w_k$ is said to be a unit-energy disturbance if the following condition is satisfied:

$$\sum_{k=0}^{T} w_k^\top w_k \le 1 \tag{2.4}$$

## 3 Stochastic Stability and Output Peak Performance

In this section, the focuses are to discuss stochastic stability and output peak performance of system (2.1).

**Lemma 3.1** *[1] Consider system (2.1) ($u_k = 0$ and $w_k = 0$). Suppose that there exists a set of symmetric positive definite matrices $P_i^s > 0$, such that*

$$\Phi = A_i^\top \left( \sum_{j=1}^{\sigma} \sum_{s=1}^{N} \sum_{s=1}^{N} \alpha_s(k+1)\alpha_s(k)\pi_{ij}^s P_j^s \right) A_i - \sum_{s=1}^{N} \alpha_s(k) P_i^s < 0 \tag{3.1}$$

*Then, system (2.1) with nonhomogeneous TP defined by (2.2) is stochastically stable.*

**Theorem 3.1** *Consider system (2.1) ($u_k = 0$ and $x_0 = 0$) with nonhomogeneous TP defined by 2.2. Suppose that for a given $\gamma > 0$, there exist symmetric positive definite matrices $P_i^s > 0$, $\forall i \in \Gamma$, such that*

$$\Xi_1 = \begin{bmatrix} A_i^\top \left( \sum_{j=1}^{\sigma} \sum_{s=1}^{N} \alpha_s(k+1)\pi_{ij}^s P_j^s \right) A_i - P_i^s & A_i^\top B_{i1} \\ * & -I + B_{i1}^\top \sum_{j=1}^{\sigma} \pi_{ij}^s P_j^s B_{i1} \end{bmatrix} < 0 \tag{3.2}$$

$$\Xi_2 = \begin{bmatrix} -P_i^s & C_i^\top \\ * & -\gamma^2 I \end{bmatrix} < 0 \quad \forall i \in \Gamma \tag{3.3}$$

*Then, system (2.1) with unit-energy disturbance defined by (2.4) is stochastically stable and a prescribed output peak performance index $\gamma$, i.e.,* $\sup E\{\|z(T)\|\} < \gamma$, *is satisfied.*

*Proof* Construct a potential Lyapunov function as

$$V(x_k, r_k = i, \alpha_k) = x_k^\top \sum_{s=1}^{N} \alpha_s(k) P_i^s x_k \quad (i \in \Gamma)$$

Then,

$$\begin{aligned} &\Delta V(x_k, i, \alpha_k) \\ &\quad = E\{V(x_{k+1}, r_{k+1}, \alpha_{k+1})\} - V(x_k, r_k, \alpha_k) \\ &\quad = x_{k+1}^\top \left( \sum_{j=1}^{\sigma} \sum_{s=1}^{N} \sum_{q=1}^{N} \alpha_s(k)\beta_q(k+1)\pi_{ij}^s P_j^q \right) x_{k+1} - x_k^\top \sum_{s=1}^{N} \alpha_s(k) P_i^s x_k \\ &\quad = (A_i x_k + B_{i1} w_k)^\top \left( \sum_{j=1}^{\sigma} \sum_{s=1}^{N} \sum_{q=1}^{N} \alpha_s(k)\beta_q(k+1)\pi_{ij}^s P_j^q \right) (A_i x_k + B_{i1} w_k) \\ &\qquad - x_k^\top \sum_{s=1}^{N} \alpha_s(k) P_i^s x_k \end{aligned}$$

Clearly,

$$\Xi = \Delta V(x_k, i, \alpha_k) - w_k^\top w_k = \varsigma_k^\top \Xi_1 \varsigma_k$$

where $\varsigma_k = [x_k^\top w_k^\top]^\top$, and $\Xi_1$ is defined in (4.12). Thus, under the assumption $w_k = 0$, if $\Xi < 0$ and $\Delta V(x_k, i, \alpha_k) < 0$, it follows from Lemma 3.1 that $\lim_{\upsilon \to \infty} E\{\sum_{k=0}^{\upsilon} x_k^\top x_k | x_0, r_0\} < \infty$. The system is stochastically stable.

To establish the output peak performance for system (2.1), let us consider the following performance index $J = E\{\|z^\top(T) z(T)\|\}$. If (4.12) holds, then $\Xi < 0$, i.e.,

$$\Delta V(x_k, i, \alpha_k) - w_k^\top w_k < 0 \tag{3.4}$$

under zero initial condition, (i.e., $V(x_k, r_k, \alpha_k)|_{k=0} = 0$). Taking sum on both sides from 0 to T, we have

$$E[\sum_{k=0}^{T} \Delta V(x_k, r_k, \alpha_k)] = E[V(x_T, r_T, \alpha_T)] - V(x_0, r_0, \alpha_0)$$
$$\leq E[V(x_T, r_T, \alpha_T)] < \sum_{k=0}^{T} w_k^\top w_k \leq 1 \tag{3.5}$$

If (3.3) holds, then it follows from (3.5) that

$$E\{\|z^\top(T)z(T)\|\} = x_k^\top C_i^\top C_i x_k < \gamma^2 x_k^\top \sum_{s=1}^{N} \alpha_s(k) P_i^s x_k = \gamma^2 V(x_T, r_T, \alpha_T) < \gamma^2 \tag{3.6}$$

which implies that

$$sup\ E\{\|z(T)\|\} < \gamma, \forall T > 0 \tag{3.7}$$

This means that system (2.1) is stochastically stable and a prescribed output peak performance index $\gamma$ is satisfied. This completes the proof.

## 4 Output Peak Controller Design

In order to suppress the effect of the disturbances, we will design a mode-dependent state feedback controller for system (2.1) under unit-energy disturbance, such that the peak amplitude of the output stays below a specified level.

Now, we are ready to present our main results as given below:

**Theorem 4.1** *Consider system (2.1) with initial condition $x_0 = 0$ and nonhomogeneous TP defined by 2.2. Suppose that for a prescribed $\gamma > 0$, there exist a suitable mode-dependent state feedback controller $u_k = K_i x_k$ and symmetric positive definite matrices $P_i^s > 0$, $\forall i \in \Gamma$, such that*

$$\Xi_3 = \begin{bmatrix} (A_i+B_{i2}K_i)^\top \left( \sum_{j=1}^{\sigma} \sum_{s=1}^{N} \alpha_s(k+1)\pi_{ij}^s P_j^s \right)(A_i+B_{i2}K_i) - P_i^s & (A_i+B_{i2}K_i)^\top B_{i1} \\ * & -I + B_{i1}^\top \sum_{j=1}^{\sigma} \pi_{ij}^s P_j^s B_{i1} \end{bmatrix} < 0 \tag{4.1}$$

$$\Xi_4 = \begin{bmatrix} -P_i^s & (C_i + D_i K_i)^\top \\ * & -\gamma^2 I \end{bmatrix} < 0 \qquad \forall i \in \Gamma \tag{4.2}$$

*Then, system (2.1) under the unit-energy disturbance defined by (2.4) is stochastically stable and the prescribed output peak performance index $\gamma$, i.e.,* $\sup E\{\|z(T)\|\} < \gamma$, *is satisfied.*

*Proof* Consider the mode-dependent state feedback controller $u_k = K_i x_k$. Then, system (2.1) can be rewritten as

$$\begin{cases} x_{k+1} = (A_i + B_{i2}K_i)x_k + B_{i1}w_k \\ z_k = (C_i + D_i K_i)x_k \end{cases} \tag{4.3}$$

Construct a potential Lyapunov–Krasovskii function for system (2.1), expressed in terms of symmetric positive definite matrices $P_i^s$, as given below:

$$V(x_i, r_k, \alpha_k) = x_k^\top \sum_{s=1}^{N} \alpha_s(k) P_i^s x_k \quad (i \in \Gamma)$$

Apply Theorem 3.1, where the feedback controller $u_k = K_i x_k$ is used. Substituting $A_i$ and $C_i$ by $(A_i + B_{i2}K_i)$ and $(C_i + D_i K_i)$, respectively, it follows that $\Xi_3 < 0$ and $\Xi_4 < 0$ will guarantee that system (4.8) is stochastically stable and a output peak performance index $\gamma$ is satisfied, where

$$\Xi_3 = \begin{bmatrix} (A_i+B_{i2}K_i)^\top \left( \sum_{j=1}^{\sigma} \sum_{s=1}^{N} \alpha_s(k+1)\pi_{ij}^s P_j^s \right)(A_i+B_{i2}K_i) - P_i^s & (A_i+B_{i2}K_i)^\top B_{i1} \\ * & -I + B_{i1}^\top \sum_{j=1}^{\sigma} \pi_{ij}^s P_j^s B_{i1} \end{bmatrix} < 0 \tag{4.4}$$

$$\Xi_4 = \begin{bmatrix} -P_i^s & (C_i + D_i K_i)^\top \\ * & -\gamma^2 I \end{bmatrix} < 0 \qquad \forall i \in \Gamma \tag{4.5}$$

This concludes the proof.

*Remark 4.1* Theorem 4.1 provides a conceptional method for the design of a mode-dependent state feedback controller under which system (4.8) is stochastically stable and the output peak performance index $\gamma$ is satisfied. Next, the results of Theorem 4.2 will be formulated in terms of LMIs.

**Theorem 4.2** *Controller design under prescribed output peak index*

*Consider the discrete-time MJS with nonhomogeneous TP defined by 2.2 with zero initial condition* $x_0 = 0$ *and unit-energy disturbance* $\sum_{k=0}^{T} w_k^\top w_k \leq 1$. *Suppose that there exist positive definite symmetric matrices* $G_i$, $Q_i^s$, *and* $R_i$, *such that*

$$\Xi_5 = \begin{bmatrix} Q_i^s - (G_i)^\top - G_i & 0 & \sqrt{\pi_{i1}^s}(A_iG_i + B_{i2}R_i)^\top & \dots & \sqrt{\pi_{i\sigma}^s}(A_iG_i + B_{i2}R_i)^\top \\ * & -I & \sqrt{\pi_{i1}^s}B_{i1}^\top & \dots & \sqrt{\pi_{i\sigma}^s}B_{i1}^\top \\ * & * & -Q_1^q & 0 & 0 \\ * & * & * & \ddots & 0 \\ * & * & * & * & -Q_\sigma^q \end{bmatrix} < 0 \tag{4.6}$$

$$\Xi_6 = \begin{bmatrix} Q_i^s - (G_i)^\top - G_i & (C_iG_i + D_iR_i)^\top \\ * & -\gamma^2 I \end{bmatrix} < 0 \tag{4.7}$$

*Then, system (4.8) under the mode-dependent state feedback controller* $u_k = K_i x_k$ *is stochastically stable and the output peak performance index* $\gamma$*, i.e.,* $\sup\{E\,\|z(T)\|\} \leq \gamma$*, is satisfied, where the controller gains are given by* $K_i = R_i(G_i)^{-1}$ *and* $Q_i^s = (P_i^s)^{-1}$.

*Proof* Consider a mode-dependent state feedback controller $u_k = K_i x_k$. Then, system (2.1) under such a feedback controller can be rewritten as

$$\begin{cases} x_{k+1} = (A_i + B_{i2}K_i)x_k + B_{i1}w_k \\ z_k = (C_i + D_iK_i)x_k \end{cases} \tag{4.8}$$

By Schur complement Lemma, the two matrix inequalities involved in (4.4) and (4.5) can be transformed equivalently to

$$\Xi_7 = \begin{bmatrix} -P_i^s & 0 & \sqrt{\pi_{i1}^s}(A_i + B_{i2}K_i)^\top & \dots & \sqrt{\pi_{i\sigma}^s}(A_i + B_{i2}K_i)^\top \\ * & -I & \sqrt{\pi_{i1}^s}B_{i1}^\top & \dots & \sqrt{\pi_{i\sigma}^s}B_{i1}^\top \\ * & * & -Q_1^q & 0 & 0 \\ * & * & * & \ddots & 0 \\ * & * & * & * & -Q_\sigma^q \end{bmatrix} < 0 \tag{4.9}$$

Note that $\Xi_7 < 0$ is equivalent to $(\hat{G}_{i1}^s)^\top \Xi_7 \hat{G}_{i1}^s < 0$, or equivalently

$$\Xi_8 = \begin{bmatrix} -(G_i)^\top P_i^s G_i & 0 & \sqrt{\pi_{i1}^s}(G_i)^\top(A_i + B_{i2}K_i)^\top & \dots & \sqrt{\pi_{i\sigma}^s}(G_i)^\top(A_i + B_{i2}K_i)^\top \\ * & -I & \sqrt{\pi_{i1}^s}B_{i1}^\top & \dots & \sqrt{\pi_{i\sigma}^s}B_{i1}^\top \\ * & * & -Q_1^q & 0 & 0 \\ * & * & * & \ddots & 0 \\ * & * & * & * & -Q_\sigma^q \end{bmatrix} < 0 \tag{4.10}$$

where

$$\hat{G}^s_{i1} = diag\left\{ G_i \ I \ I \ \ldots \ I \right\}$$

From Lemma 2.1, it follows that

$$(G_i)^\top P^s_i G_i \geq (G_i)^\top - Q^s_i + G_i$$

$$\Xi_5 = \begin{bmatrix} Q^s_i - (G_i)^\top - G_i & 0 & \sqrt{\pi^s_{i1}}(A_i G_i + B_{i2} R_i)^\top & \ldots & \sqrt{\pi^s_{i\sigma}}(A_i G_i + B_{i2} R_i)^\top \\ * & -I & \sqrt{\pi^s_{i1}} B^\top_{i1} & \ldots & \sqrt{\pi^s_{i\sigma}} B^\top_{i1} \\ * & * & -Q^q_1 & 0 & 0 \\ * & * & * & \ddots & 0 \\ * & * & * & * & -Q^q_\sigma \end{bmatrix} < 0 \tag{4.11}$$

Let $R_i = K_i G_i$. Then, it is easy to show that the condition $\Xi_5 < 0$ in (4.16) implies $\Xi_8 < 0$. Similarly, $\Xi_4 < 0$ is equivalent to

$$\Xi_4 = \begin{bmatrix} -P^s_i & (C_i + D_i K_i)^\top \\ * & -\gamma^2 I \end{bmatrix} < 0 \qquad \forall i \in \Gamma \tag{4.12}$$

or equivalently $(\hat{G}^s_{i2})^\top \Xi_4 \hat{G}^s_{i2} < 0$, i.e.,

$$\Xi_9 = \begin{bmatrix} -(G_i)^\top P^s_i G_i & (C_i G_i + D_i R_i)^\top \\ * & -\gamma^2 I \end{bmatrix} < 0 \tag{4.13}$$

By Lemma 2.1 and Schur complement Lemma, it follows that $\Xi_9 < 0$ is guaranteed by $\Xi_6 < 0$, where

$$\hat{G}^s_{i2} = diag\left\{ G_i \quad I \right\}$$

This completes the proof.

*Remark 4.2* By examining the set of the coupled matrix inequalities in Theorem 4.1, we see that it is not easy to be solved directly. To overcome the difficulty, we introduce slack variables $G_i$. In this way, it becomes a problem of finding a feasible solution to a LMI.

The following corollary follows readily from Theorem 4.2.

**Corollary 4.1** ***Controller design under optimal output peak index***
*Consider the discrete-time MJS (2.1) under unit-energy disturbance* $\sum_{k=0}^{T} w_k^\top w_k \leq 1$ *with nonhomogeneous TP defined by 2.2 and zero initial condition* $x_0 = 0$. *. Suppose*

*that there exists a set of positive definite symmetric matrices $G_i$, $R_i$, $Q_i^s$, such that the following problem has a solution*

$$min\,\gamma \tag{4.14}$$

*s.t.*

$$\Xi_{10} = \begin{bmatrix} Q_i^s - (G_i)^\top - G_i & 0 & \sqrt{\pi_{i1}^s}(A_i + B_{i2}R_i)^\top & \dots & \sqrt{\pi_{i\sigma}^s}(A_iG_i + B_{i2}R_i)^\top \\ * & -I & \sqrt{\pi_{i1}^s}B_{i1}^\top & \dots & \sqrt{\pi_{i\sigma}^s}B_{i1}^\top \\ * & * & -Q_1^q & 0 & 0 \\ * & * & * & \ddots & 0 \\ * & * & * & * & -Q_\sigma^q \end{bmatrix} < 0 \tag{4.15}$$

$$\Xi_{11} = \begin{bmatrix} Q_i^s - (G_i)^\top - G_i & (C_iG_i + D_iR_i)^\top \\ * & -\gamma^2 I \end{bmatrix} < 0 \tag{4.16}$$

*Then, system (2.1) under the mode-dependent state feedback controller $u_k = K_i x_k$ with $K_i = R_i G_i^{-1}$ and $Q_i^s = (P_i^s)^{-1}$ is stochastically stable and the optimal (minimum) output peak performance index $\gamma$, i.e.,* $\sup\{E\,\|z(T)\|\} < \gamma$*, is satisfied.*

*Remark 4.3* Theorem 4.2 and Corollary 4.1 present, respectively, the methodologies for mode-dependent controller design under the prescribed output peak index and that under the optimal output peak index. When the state feedback gain matrix is not dependent on the mode, it becomes a mode-independent feedback controller.

## 5 Illustrative Example

Consider the following nonhomogeneous MJS with two modes:

$$A_1 = \begin{bmatrix} 0.5 & -0.3 \\ 0 & 0.7 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -0.15 \\ 0.9 & 1.2 \end{bmatrix}$$

$$B_{11} = \begin{bmatrix} 0.5 \\ 0.9 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 0.2 \\ 0.6 \end{bmatrix}$$

$$B_{21} = \begin{bmatrix} 0.6 \\ 0.3 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} 0.5 \\ 0.3 \end{bmatrix}$$

$$C_1 = [1 \quad 0], \quad C_2 = [0.2 \quad 0.5], \quad D_1 = 0.5, \quad D_2 = 0.3$$

The nonhomogeneous transition probability matrices are defined as follows:

$$\Pi^1(k) = \begin{bmatrix} 0.2 & 0.8 \\ 0.35 & 0.65 \end{bmatrix}, \quad \Pi^2(k) = \begin{bmatrix} 0.65 & 0.35 \\ 0.4 & 0.6 \end{bmatrix}$$

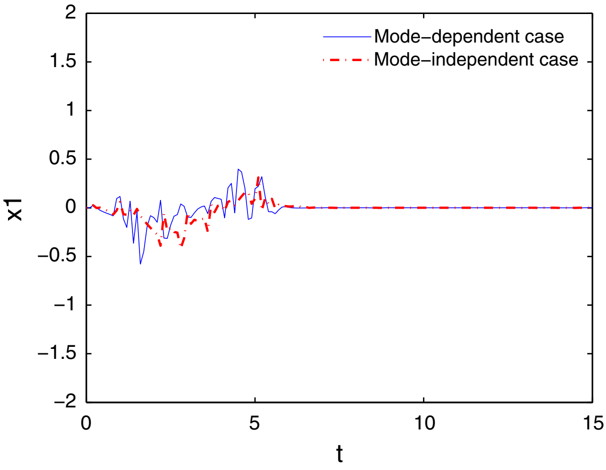


**Fig. 1** State trajectories x1

$$\Pi^3(k) = \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix}, \quad \Pi^4(k) = \begin{bmatrix} 0.5 & 0.5 \\ 0.25 & 0.75 \end{bmatrix}$$

The initial state is $x_0 = [0\ 0]^\top$ and the unit-energy disturbance is given by

$$w_k := \begin{cases} 0.3sin(k), & if \quad 0 \leq k \leq 5, \\ 0, & others, \end{cases}$$

By Theorem 4.2, we obtain the mode-dependent gain matrices $K_1 = [-1.6228\ -0.1978]$, $K_2 = [-0.4958 - 1.1693]$ and mode-independent gain matrices $K = [-1.0818 - 0.4531]$.

The state trajectories $x_1, x_2$ are shown in Figs. 1 and 2, and the output of the system is shown in Fig. 3. We can see that the amplitude of the system output $z$ under the designed mode-dependent state feedback controller is less than the prescribed output threshold value $\gamma = 0.5$ (the purple line), while the index $\gamma = 0.5$ under the mode-independent state feedback controller is not satisfied (the red dot dash line). This implies that the mode-dependent controller can better suppress the effect of disturbance when compared with the mode-independent controller. Actually, we can apply Corollary 4.1 to obtain the optimal (minimum) output peak performance index under mode-dependent as well as mode-independent state feedback controller. The details are presented in Table 1. The optimal (minimum) output peak performance index under mode-independent state feedback controller is $\gamma = 0.5757$. Clearly, the prescribed index $\gamma = 0.5$ under mode-independent state feedback is not satisfied. This is because mode-independent controller does not adequately utilize the mode transition information, and consequently the results obtained tend to be not as good as that under a mode-dependent state feedback controller.

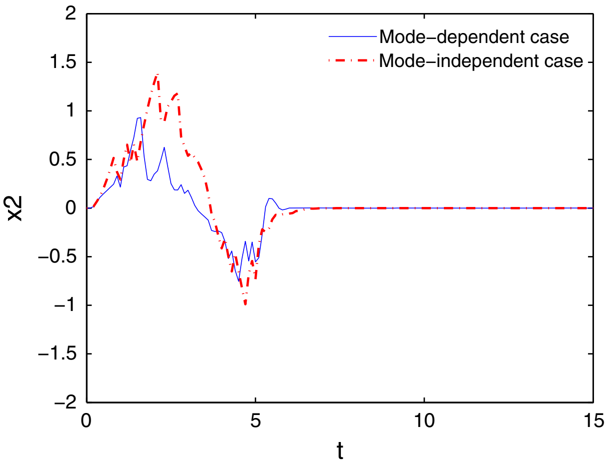


**Fig. 2** State trajectories x2

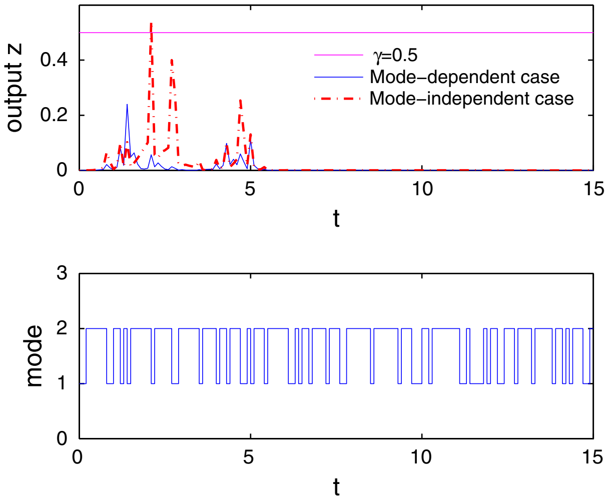


**Fig. 3** Output of the system

**Table 1** Optimal output peak performance index with different cases

| Case | Mode-dependent | Mode-independent |
|---|---|---|
| $\gamma_{min}$ | 0.1880 | 0.5757 |

## 6 Conclusions

In this paper, the issue on the output peak controller design for nonhomogeneous Markov jump system is addressed, where the nonhomogeneous transition probability matrix is described as a polytope. By introducing appropriate slack variables, an optimization problem is formulated where its constraints are expressed in terms of

LMIs. The simulation results show the potential of the proposed techniques. The results obtained appear to be extendable to nondeterministic switched systems. This is a future research topic.

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