Science and Mathematics Education Centre

THE INFLUENCE OF MULTIPLE REPRESENTATIONS ON THE LEARNING OF CALCULUS BY ESL STUDENTS

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Abstract

The goals of this study were to research the learning difficulties among a group of four pre-university introductory calculus students who were mainly international students studying *English as a Second Language* (ESL). The intention was to create a constructivist-style classroom environment in order to determine if it could improve students' knowledge about the use and management of multiple representations (that is, graphical, numerical, symbolic, pictorial, linguistic or diagrammatic approaches for problem representation), increase their classroom communication as a means to improving ability in the modelling of calculus word problems, and to develop, implement and evaluate a teaching package that encouraged the use of multiple representations as a means of improving conceptual understanding.

The achievement of these goals was sought by means of the development, implementation and evaluation of a number of calculus extended tasks that encouraged the use of multiple representations. These activities facilitated the compilation of a menu of approaches to the solution of mathematical problems, while the longitudinal nature of the study allowed for the monitoring of student changes in their preferred approach. A traditional calculus curriculum was used for the study, but the instructional emphasis was based more on students' understanding of concepts in a classroom environment utilising a constructivist approach rather than on their memorising computational techniques. Reading, writing, and discussion were emphasised in small group settings to develop language skills and to foster an appreciation of the alternative solution strategies of individual students. The study was conducted at an International College north of Perth in Western Australia, and the majority of students in the sample were from *Non-English-Speaking-Backgrounds* (NESB).

A range of methods was used to collect qualitative and quantitative data in order to increase the credibility of the research. These methods included audio recordings of structured task-based interviews with each of the four students in the sample; teacher analysis of student worksheets; my classroom observations; the analysis of alternative student conceptions on assessment tasks obtained through

post-test interviews, and my personal reflections. Quality controls were employed to ensure the credibility of the data collected. As classroom teacher and principal researcher, it was possible for me to treat each of the four students involved as an individual case study. Descriptive questionnaires were used in order to gain information regarding the course and the use of graphics calculators.

The results are applicable to ESL introductory calculus students only, and the nature of the sample implies a number of study limitations detailed in Chapter Five. There was extensive evidence of the benefits of the use of a multi-representational mode and evidence also of the benefits of encouraging the use of a diversity of modes of classroom instruction. Outcomes of the study were qualified by the difficulties ESL students face in coordinating conflicting information and interpreting the language demands of problem presentation. It is expected that this study will assist in extending the knowledge and understanding of the learning difficulties faced by ESL students in the area of pre-university calculus.

Results of this study suggest that instructional material has an important influence on ESL students' use and management of multiple representations. However, there are often limitations to the influence of the material due to student preferences, mathematical ability and firmly held beliefs as well as on the amount of detail presented in a problem. Secondly, small group learning environments based on a constructivist approach were found to influence student ability to model calculus word problems in a positive manner, provided there is teacher support to overcome cognitive obstacles. Finally, it was established that an effective teaching package could be developed to assist ESL students in calculus learning. The teaching package's evaluation highlighted the need for matching language use in problem presentation with the current mathematical language register of each student.

Table of Contents

Appendices

List of Tables

List of Figures

Chapter One: Introduction

Background To The Study

Aim of the Study

This study arose out of a need to ensure that international students, particularly those with a second language background, were receiving adequate calculus instruction at the pre-university level. Its aim involved the investigation of the influence and preference for multiple representations (including the use of graphic calculator technology) when real world problems were presented to ESL students studying introductory calculus.

The need to highlight the teaching and learning of mathematical concepts in a multi-representational way has been recommended in a summary of recent research involving the upper secondary years (White & Pegg, 1996). This is particularly necessary in calculus teaching where modern ideas suggest that the usual formal approach should be changed to one involving the more exploratory use of graphics calculators to assist in conceptual learning (Kissane, 1995a). The use of *multiple representations* as a possible approach for improving the level of student understanding of calculus, has created researcher interest (in preference to a solely computational approach). This term refers to a teaching repertoire that extends beyond the use of a symbolic, algorithmic approach and typically uses a combination of plots, tables and notation – commonly referred in the literature as graphical, numerical and symbolic representations, respectively (Ferrini-Mundy & Graham, 1994). Along with other modes of representation, including pictorial and linguistic, the aim is to extend the presentation and solution approaches possible for mathematical tasks. It is my belief that this multi-representational approach to calculus problems may be of particular benefit to ESL students. The possibility existed to redesign calculus course material to emphasise the learning and teaching of calculus. This could be achieved through the use and promotion of multiple representations in problem presentation and the creation of a learning environment that encourages student-focused instruction.

The Problem

Poor results in calculus in many university courses highlighted the need for research at preuniversity level to ensure students are being adequately prepared. Over the years researchers have been aware that students attending university have been achieving at unsatisfactory levels and that there was a need to improve the way calculus was taught in order to improve students' preparation (LaTorre, Fetta, Kenelly, Nicholson, Proctor & Reneke, 1990; Porzio, 1994). These ideas led to the development of a curriculum resource known as the *Resources for Calculus* project. Director, Roberts explains its origins and the problems they encountered in the following words:

> Beginning with a conference at Tulane University in January, 1986, there developed in the mathematics community a sense that calculus was not being taught in a way befitting a subject that was at once the culmination of the secondary mathematics curriculum and the gateway to collegiate science and mathematics. Far too many of the students who started the course were failing to complete it with a grade of C or better, and perhaps worse, an embarrassing number who did complete it professed either not to understand it or not to like it, or both. For most students it was not a satisfying culmination of their secondary preparation, and it was not a gateway to future work. It was an exit.(cited in Straffin, 1993, p.vii)

The *Calculus Reform Movement* came into prominence, referring to research into improvements (in calculus teaching) such as student-centred classroom environments. The movement was widely publicised by the Mathematical Association of America (Dudley, 1993) and many of its essential features have been described by Joiner (1998, p. 1):

> Calculus reform reflects wider educational reform, with the same central tendencies, such as constructivism, and diverse implementation strategies, such as collaborative or computer-assisted learning. The calculus reform movement has been trialed or implemented in over one-fifth of all United States universities and is active in at least three Australian universities.

A variety of reforms has been attempted in calculus instruction under the banner of calculus reform projects, each with the underlying goal of balancing the use of procedural algorithms (or syntax) with an emphasis on conceptual understanding (or semantics). The aims of many of these projects were overlapped, although they all encompassed one or more of the following ideals (Joiner, 1998, p. 5):

- to increase the relevance of the calculus taught, both through more interesting, real-world examples and by adjusting the order and coverage of material taught,
- to create a curriculum that involves students more interactively,
- to give students multiple representations and importantly more visualisation of calculus to improve their understanding,
- to create a more learner-centred pedagogy in the classroom,
- to develop higher order understanding of the calculus theories rather than emphasising lower-order computational skills,
- to develop a broader awareness of the changing interpretation and use (hermeneutics) of calculus rather than an 'absolute' or 'right and wrong' perception of calculus that teaching only computational skills emphasises,
- to use more student mentoring and collaboration to improve students' assimilation of new knowledge and to improve their ability to communicate mathematically with that knowledge,
- to increase the independence and confidence of students in learning calculus,
- and to use new knowledge to help the above aims.

Constructivism, a learning theory currently popular in research into mathematics education (Aspinwall, Shaw & Presmeg, 1997; Forster, 1997; Magidson, 1992; von Glasersfeld, 1995), forms an integral part of many calculus reform projects. Its development was advanced with the work of Piaget and his efforts to understand student learning through clinical interviews. Constructivism argues that students must actively learn from experiencing, rather than being passive receptacles, and that learning involves organising and extending conceptual structures based on past and present events, rather than seeking to memorise absolute truths (Truran & Truran, 1998). The application of this paradigm to the clinical interview seems natural with the creation of a supportive, nonauthoritarian approach where the researcher is keen to listen to student ideas. Teaching that is inspired by a constructivist epistemology has the potential to develop a problem solving ontology that can be transferred:

> The solving of problems that are not precisely those presented in the preceding course of instruction requires conceptual understanding, not only of certain abstract building blocks but also of a variety of relationships that can be posited between them. Only the student who has built up such a conceptual repertoire has a chance of success when faced with novel problems. Concepts cannot simply be transferred from teachers to students - they have to be conceived.(von Glasersfeld, 1995, p. 5)

The use of graphic calculator technology also goes hand-in-hand with a constructivist approach. This is because the technology facilitates a classroom wherein students actively produce and validate their own ideas and solutions and where the teacher becomes more of a facilitator. Understanding of the influence of graphics calculators on student learning can be quantified using a qualitative approach such as clinical interviews:

> By analysing the specific content of assessment items and students' responses, and by probing students' conceptual understanding through interviews, researchers can paint a more detailed picture of the effects of graphing calculator-based instruction on students' learning.(Dunham & Dick, 1994, p. 441)

Graphics calculators have a number of potential advantages for calculus instruction. These advantages have been highlighted in a range of research articles over the past seven years (Agostinelli, 1999; Croft, 1998; Dunham & Dick, 1994; Penglase & Arnold, 1996). Used as an introductory tool for calculus exploration, the graphics calculator constitutes a starting point for a subsequent rigorous, formalised treatment of calculus concepts and skills (Kissane, 1995a). Students can more easily compare numerical, symbolic, and graphical displays and investigate a range of problems that were previously too complex using pencil and paper. Tedious and time-consuming procedures such as evaluating functions, solving excessively complex equations and inequalities and plotting graphs, are avoided, and there is more time available to investigate relevant and interesting topics. The graphics calculator can also be used to introduce more advanced topics to students as well as for checking answers.

Many calculus students studying *English as a Second Language* (ESL) require additional assistance with the reading and comprehension of mathematical language. Typically, ESL students need tuition in vocabulary from the mathematics register and an increased level of reinforcement or repetition of ideas during instruction. Evidence exists to suggest that classroom tasks that involve writing, reading and discussion of calculus concepts in small-groups are more beneficial to ESL students than those tasks involving passive listening to teachers and the answering of their questions (Kessler, 1992). Furthermore, technical and everyday language integration in ESL students' writing and lesson presentation has been recommended to improve their understanding of calculus concepts (Frid, 1993a).

Unfortunately, most teachers have neither the resources nor the training needed to provide adequate support for ESL students in mathematics classes (Bishop & Larkin, 1994). Additionally, such students typically come from different cultural backgrounds; consequently they arrive in the classroom with mathematical concepts developed through a complex interaction of linguistic and cultural factors. It is recommended (Australian Education Council, 1991; Curriculum Council, 2001) that any classroom learning materials should view mathematics in a context related to students' prior cultural experiences (a constructivist notion). This would give ESL students the opportunity to take advantage of their prior strengths and experiences, applying them to the instructional material, as well as extending the range of contexts in which they can solve mathematical problems (Australian Education Council, 1991). The goal of this study was to design a teaching package to address this goal.

Study Participants

Calculus students from the Western Australian International College (WAIC) in Perth were involved in this action research study over a four-semester period. This college is part of the Australian Institute for University Studies, one of the first private tertiary educational institutions in Australia, and provides students with the opportunity to progress from pre-university studies to graduate level on a single campus. The campus has an international atmosphere, the student body consisting of persons from Australia as well as Indonesia, Malaysia, Singapore, Brunei, Hong Kong, India, Sri Lanka, Pakistan, Bangladesh, Taiwan, Thailand and Japan. International students not meeting the English language proficiency requirements of a course may take an intensive English (ELICOS) course prior to commencing formal study, and they later study ESL or English concurrently with other subjects such as calculus.

Students eligible to study ESL are usually those who have been resident in Australia for seven years or less and for whom English is neither their native language nor the language spoken at home. Singapore students, for example, are not required to study ESL, as their level of reading and comprehension ability in English is usually of a high level. Exemptions to the above rules are available by applying to the Curriculum Council in Western Australia.

It will become clear from later comments that ESL students may not be homogeneous in their first language ability. Australian-born children from non English-speaking backgrounds can also be eligible to study ESL, but in the present study this group was not represented. Typically, English is the language of instruction at WAIC, however the influence of bilingualism in classroom discourse – that is, the use of English and perhaps the native language jointly in classroom learning – was supported in the present study but not analysed directly as part of the research.

Typically, ESL students face a number of difficulties not experienced by students taught in their native language. One of the major difficulties is problem comprehension. Students may have developed an understanding of concepts and procedures in their native language but they can experience translation difficulties when similar tasks are presented to them in English (Galligan, 1993). For example, being asked to rearrange a formula can result in a range of responses, with many students feeling the question has not been expressed in enough detail. One approach for enhancing comprehension of mathematical problems is to encourage small-group discussions where students test their understanding of vocabulary, word-order and the role of prepositions (Kessler, 1992). These discussions could be supported with graphics calculators where a range of representations can often be used simultaneously to assist with understanding. Thus the students' skill with the use of multiple representations can help to increase the comprehension of calculus tasks. The diversity of information provided by such an approach can help overcome the cultural constraints faced by ESL students even though the influence of culture is very prominent:

> All experience is cultural through and through; we experience our "*world*" in such a way that our culture is already present in the very experience itself.(Lakoff & Johnson, 1980, p. 57)

The proportion of errors of different types made by ESL students when completing calculus problems is likely to vary considerably from the proportion produced by first language users. Apart from comprehension, errors have been traditionally classified into reading, transformation, careless and process-skill types. Earlier studies have indicated that the major source of errors for students result from the reading, comprehension and transformation stages of problem solving (Ellerton & Clements, 1996). It is likely that the limited vocabulary of ESL students may influence their reading ability. Their ability to comprehend the problem, however, will depend on its semantic structure and this may cause the greatest difficulty.

In any mathematics class, however, ESL students may be amongst the highest achievers. In particular, ESL students born overseas with good ability in their first language as well as English, may have an advantage in mathematical tasks – an observation that is referred to by Cummins as the *threshold hypothesis* (as cited in Ellerton & Clements, 1996) and as observed in the Victorian (Australia) public examination extended task trial for secondary students. The hypothesis refers to the phenomenon where overall facility in two or more languages can possibly provide an overall cognitive structure that assists students in some mathematical tasks, including language-based tasks (Cummins, 1979, 1986). Further examination of these ideas is beyond the scope of the present study, however, its relevance may have had an influence on classroom learning as second language discussions were encouraged:

> Cummins took it for granted that, with learners who speak two or more languages, there is clearly an interplay in the learning process between the language codes. This may assist or may detract from learning. If the use of more than one language by individual learners is an accepted part of classroom culture then this can produce a cognitive advantage once a threshold of competence in the two (or more than two) languages is reached.(Ellerton & Clements, 1996, p. 211)

Rationale for the Study

In general, students may use a range of methods to translate an unfamiliar mathematical situation into a form from which they can find a solution. Typically they rely on the use of language, a spatial option or a symbolic one (Buxton, 1997). Of particular importance is that different students may prefer other options than the one indicated as a favourite one by the teacher. To accommodate all students, instruction should use a range of methods or representations. Students who may not be very clever, or who are from ESL backgrounds may respond better than expected if a visual approach is used. According to Krutetskii (Aspinwall et al., 1997) very bright students may feel that a visual approach is unnecessary. The preferences for representations and advantages for the use of different representations for different types of students needs further research. An attempt has been made to address these issues in this study.

Ferrini-Mundy and Lauten (1993, p.159) have described multiple representation questions in mathematics as the possible translation between the following six modes of representation: real phenomenon, verbal rule, diagram, table, graph and formula as well as the recognition of two representations of the same function given as distinct formulae. School calculus texts which students are regularly expected to read typically contain any of a variety of representations: symbolic, graphical, tabular or diagrammatic. Knowledge of a diversity of modes is considered essential for mathematical understanding and effective planning in problem solving.

The cognitive objectives for the Western Australian Tertiary Entrance Examination (T.E.E.) *Calculus* course includes the following references to the use of technology and multiple representations (Secondary Education Authority, 1997b, p. 29):

- comprehend information in oral and written forms including graphical, diagrammatic and tabular presentations
- select and use appropriate forms for representing mathematical data and relationships
- select and use appropriate technologies

In most schools in Western Australia, computing or graphic calculator tools are integrated into the mathematics program and may be used in external examinations such as the Tertiary Entrance Examination (TEE). In order to achieve the cognitive course objectives, an attempt was made in the present study to integrate graphics calculators into weekly classroom tasks. The HP38G calculator which all students used has been designed with plotting, symbolic and numerical features to effectively apply and link multiple representations. The role of calculators in mathematics education is now changing and there are many potential benefits for students in their use:

> Not only do they allow students who would ordinarily be turned off by traditional mathematics' tedious computations and algorithms to experience true mathematics, but they also help students to more quickly and readily develop number sense, gain mathematical insight and reasoning skills, value mathematics, and cultivate mathematical

understanding, while they enjoy what they are learning.(Pomerantz, 1997, p. 4)

The benefits of the use of technology, combined with the use of multiple representations in mathematics classrooms, has been identified and extended with the *Calculus Reform Movement* in the USA (Joiner, 1998; Kaput, 1994; Lauten, Graham & Ferrini-Mundy, 1994). This movement introduced a number of innovations into the traditional approach to teaching calculus. Smaller class sizes with more staff-student interaction, and heavy use of technology, especially graphics calculators, are two distinct features of the approach. Tasks are taught from three standpoints: algebraically, numerically and graphically. The purpose of the reform program is to enable students to learn calculus conceptually. The technology is there solely as a means to an end. However, the availability of 2nd generation graphical calculators (with more advanced features such as matrix algebra, multiple function memory and parametric equations), their extensive adoption for use in class and their use for assessment has prompted a continual flood of research into their potential benefits (Berger, 1998; Dunham & Dick, 1994; Kissane, 1995c; Penglase, & Arnold, 1996; Pomerantz, 1997; Smith, 1996; Swincicky, 1994).

Data Collection

Action research (an interpretative approach) using a case study approach was the chosen research method. The method allows feedback from classroom activities to be used in the design of other classroom tasks. Graphics calculators allow a greater focus on interpreting and analysing information as well as understanding concepts (Agostinelli, 1999). Interviews meant that the understanding of assessment tasks were linked to the level of student understanding rather than solely to assessment results. The study allowed the use of many interventions, an approach that can be more effective than using a single static intervention at one point in time. The use of interviews with ESL students was considered to give research advantages. These included the indication of student conceptual problems (allowing students to redo work once they understood where they needed help), to identify where comprehension problems occurred, and in developing a closer bond with students. The interview approach demonstrated a direct interest in the student. This could aid learning, improve results, and create a focus on reasoning – asking for explanations rather than a yes/no or right/wrong approach. Finally, interviews could be used to assist in the assessment of the effectiveness of tasks already completed in order to design better tasks (see also Chapter 3).

Clinical interviews with each student were extensively used in the study to assist in better understanding the student use of multi-representations. Interviews were conducted at the conclusion of each calculus topic, being task-based, and focussing on student assessment scripts using Confrey's ideas (as cited in Lauten, Graham & Ferrini-Mundy, 1994, p. 226):

> Task-oriented, flexible interviews involve a student and interviewer, wherein the interviewer is expected to follow and pursue the student's thinking, asking questions until the student's reasons for response are understandable to the interviewer.

There is renewed interest in the use of clinical interviews as part of mathematics research for the probing of students' conceptual understanding (Dunham & Dick, 1994; Truran & Truran, 1998; von Glasersfeld, 1995). This often involves questioning about the content and responses to previous assessment items, or about responses to a particular issue or written problem presented to students during the interview. In the present study an attempt was made to measure the level of conceptual understanding by a detailed and well-organised approach such as that used by Ferrini-Mundy and Graham (1994), Kaput (1994) and Forster (1997).

AIM OF THE STUDY

The general aim of this study was to examine the learning difficulties of a class of pre-university introductory calculus students by creating a constructivist-style classroom environment. The specific aims were to:

1. Enhance knowledge about the use and management of multiple representations in a calculus course,
2. Increase ESL student classroom communication as a means to improving ability in the modelling of calculus word problems, and
3. Develop, implement and evaluate a mathematics " teaching package " for ESL students that encourages the use of multiple representations in order to improve conceptual understanding.

RESEARCH QUESTIONS

In order to meet these aims it is necessary to find answers to the following research questions:

1. Can appropriate instructional material introduced over an extended time in an introductory calculus course enhance ESL students' use and management of multiple representations?

2. Can ESL students' ability to model calculus word problems be enhanced by using a study environment with small group student-centred learning focussing on skills of comprehension and translation?

3. Can a teaching package utilising a representation mode of study be developed to assist ESL students in calculus learning?

SIGNIFICANCE OF THE STUDY

The presence of an increasing number of students in Australia for whom English is not the first language of instruction means that teachers may need to use new instructional methods and materials to better facilitate the acquisition of mathematical concepts and knowledge by these students. This study should assist teachers to do this by providing an extensive summary of observed conceptual problems about which they should be aware. In addition, guidelines were developed in this study to facilitate the best use of spoken and written language to develop procedural and conceptual skills in pre-university ESL calculus students. Other Western Australian research has given some insight into the needs of ESL students with differing mathematical ability, and has also suggested individual instruction on some aspects of graphics calculator use (Swincicky, 1994). The present study attempted to add to the literature on ESL student learning when solving calculus word problems by providing a range of representation systems including spoken and written language.

The findings of the study will be used to assist teachers in the use of graphic calculator technology at secondary level where rapid change is likely to occur in the near future. The Tertiary Entrance Examination (TEE) in Western Australia is taken by up to ten thousand mathematics students each year who will now mostly all need at least one graphics calculator for use in the TEE as well as in classroom instruction. Hence continuing research into graphics calculator efficacy is topical and highly relevant. A significant outcome of the present study was the development and evaluation of a series of extended tasks suitable for use with graphics calculators and which also emphasise the use of multiple representations, including reading, writing and communication skills. The teaching package developed in the study should contribute to the current emphasis in encouraging a diversity of approaches to calculus instruction and the provision of tasks of a multi-representational nature.

Only a limited number of studies have looked at the impact of graphics calculators on pre-university assessment (Forster & Mueller, 1999, 2000; Haimes & Webster, 2000; Malone, Haimes, Forster & Mueller, 2002; Senk, Beckmann & Thompson, 1997). New kinds of assessment items need to be developed to encompass the complexity of functional thinking in multi-representational environments such as those generated by the HP38G (Schwarz & Dreyfus, 1995).

OVERVIEW OF THE METHODOLOGY

Qualitative methods are often used to interpret the spoken and written language used by students in a mathematics classroom, and they are particularly useful in interpretative research (Frid, 1993b; Goldin, 1993; Schaller & Tobin, 1998; Shepard, 1993; Thompson, 1994; Truran & Truran, 1998). These methods have developed alongside quantitative methods, but there are extra benefits in their use when assessing educational parameters such as language comprehension or relational understanding (Truran & Truran, 1998, p. 61). In the present longitudinal study a qualitative approach was used in examining the progress of a small group of students. There were two parts to the study: a pilot study where data was collected on students studying an identical course to the main study and wherein instruments could be developed; then the main study that followed over a two-semester period. The students in the main study were assessed and interviewed in cycles in an interactive process that led to changes in instrumentation as further information was collected about student preferences, ideas and mathematical understanding. In using such an approach, detailed contextual information was recorded so that factors such as the extent of transferability of the research results could be determined (Lincoln & Guba, 1985). Further details on the study's methodology appear in Chapter three.

Sample Selection

The present study was modelled on a number of research studies such as those that analysed a small group of calculus students intensively over a period of time (Lauten et al., 1994; Thompson & Thompson, 1994), and those that analysed students from one or more classrooms in the same college (Forster, 1997; Shield & Swinson, 1997; Swincicky, 1994; Thompson, 1994). The samples used for the pilot and main studies consisted of all *Introductory Calculus* students (8 students) from the intake over a four-semester period at the Western Australian International College in Joondalup, Western Australia. These students were those belonging to my class. A case-study approach was adopted with the main sample of students during the last two semesters of the study. This sample consisted of three ESL students and one Australian student.

Validity, Reliability and Trustworthiness

Four important considerations must be addressed by any researcher: how can one establish confidence in the "truth value" of research findings, to what extent can the findings be applied elsewhere, are the findings dependable if the research process is repeated on similar students and is there confidence that no bias was introduced by the researcher because of his or her own background and approach (Lincoln & Guba, 1985). To provide reliable and valid information through the creation of an effective triangulation process (a process where data is repeated more than once and often using a range of methods), a range of instruments and approaches was used. This diverse set of 'probes' was necessary to ensure student understanding was interpreted in a consistent and effective manner (White & Gunstone, 1992). As observations could be viewed differently by other researchers, an attempt has been made to provide extensively detailed transcripts so that readers can form their own conclusions. Combined together with extensive information on the conduct of the study and its context, the degree of trustworthiness (the ability to repeat the data collection and obtain similar results with the same or different samples) of the qualitative aspects of this study, including transferability aspects, has been addressed (Guba & Lincoln, 1989).

In the present study, no researcher triangulation took place (only one researcher was used for the entire study) and observations relied exclusively on the data provided by myself. This introduced a high degree of subjectivity into the results. This was a decision justified by the nature of the study, which required detailed records to be collected over an extended period of time on individual students, as is customary in a case-study approach. The data analysis needs to be viewed in the light of this approach. The use of only one teacher and therefore only one interviewer, eliminated inconsistencies between investigators, but meant that I may not have interpreted the study's outcomes objectively. To overcome this problem, the data collected from the present study was informed by an extensive collection of data from all sample members over an extended period of time. Student scripts have been included, in detail, in the appendices so that other researchers can make their own observations and decide if any of my conclusions were reasonable or valid. The inclusion of several levels of evidence has been suggested by research; following are examples of two such levels:

> The first level of evidence is the data on hand. Second, there is, also, a holistic understanding of the project itself, its history, the intensions of the developers, the ongoing relationships within the project, and so on. This understanding about a project or program is frequently unarticulated, a part of that vast body of tacit knowledge that we all have. And lastly, the researcher / evaluator has a store of knowledge and understanding about the social world which allows such projects and evaluations of them to exist. A store of information about how schools operate, what school goals are, how classrooms operate, what teachers do, and so on. It takes all of these levels to provide good explanations around the data collected through triangulation strategies.(Mathison, 1988, p. 16)

Due to the qualitative nature of interpretations from interview discussions used in the present study it was expected that any convergent, inconsistent or contradictory evidence would be available for other researchers to scrutinise and then draw their own conclusions. Attempts have been made to adequately describe the environment in which the current study took place, the nature of the instructional approach along with data collection methods, results and scripts. Credibility was improved by ensuring findings and interpretations were reviewed with contributors as the interview process continued, and that all aspects of the study were thoroughly documented.

Instrumentation

Data was drawn from students in a single college over a four-semester time period at varying times within a school day. This required the construction of the following data collection instruments:

- A pre-test of concept understanding, computational skill, heuristic knowledge, representational preference, mathematical language knowledge, attitudes to mathematics classes, and the use of graphics calculator technology. The pre-test was essential in order to identify initial student representational preferences, as well as to provide a basis from which to measure changes in these attributes during the study. Tasks obtained through a literature review were used where appropriate for the pre-testing. The pre-test contained an extended worksheet on traffic tasks involving distance, speed and acceleration problems. The Maths-CLES (Taylor, Fraser & Fisher, 1993) was adapted for use as a graphics calculator environment questionnaire.
- Task-based interview protocols for students were formulated. The aim was to measure students' preferences for multi-representational types of tasks and to develop extended tasks for later use in the study. Interviews were audiotaped for later analysis.
- Extended tasks were prepared by myself. These were designed to complement and extend the students' knowledge of graphics calculators, as well as improve reading, writing and communication skills. At times, this involved the development of new graphics calculator programs called *Aplets.*
- Regular tests and examinations were used to measure achievement as part of the external requirements of the Introductory Calculus course in place at the college where the study was conducted.

Extensive transcripts and scripts of student work and interviews were retained.

Data Collection

To make the results of the present study potentially replicable, valid, comparable and generalizable, the use of a range of methods suggested by researchers using both qualitative and quantitative methods was adopted (Goldin, 1993; White & Gunstone, 1992; Zevenbergen, 1998). As the project took place in a classroom, it was important to minimise the affect of intervention points (where data was collected for the study) with the achievement of the course objectives. My presence as a researcher/teacher in the classroom meant I carried dual responsibilities, with the potential for conflicts of interest as noted by other researchers (Swincicky, 1994; White, 1998; Wong, 1995). The following describes the collection process and illustrates the diversity of methods used:

- The use of student assessments available for teacher analysis; this process focussed on quantitative data related to mathematical achievement and provided information on the perceived level of difficulty, language skills and representational preference. The assessments included tests, extended worksheets and examinations.
- Introduction of graphic calculator extended tasks where the researcher could measure changes in representational preference.
- The use of structured task-based interviews with each student to explore student cognitive frameworks, language misinterpretations, and alternative heuristic strategies. These interviews were conducted after important interventions in the early stages of the study.
- My classroom observations collected during the twice-weekly classes as students were attending to extended tasks usually with the assistance of graphic calculators. These included anecdotal notes on technology use issues.
- Descriptive data on student interest in the topics presented, their perceived relevance to the student's course of study, the effectiveness and usefulness of graphics calculators and students' feelings about their use.

Data Analysis

The analysis involved a range of measures including synthesis of descriptive data collected from interviews, as well as the preparation of summary data from tests and examinations. Student scripts were reviewed and results extensively summarised with changes over time in the student's facility with multiple representations compared after the intervention of graphics calculators. Summaries of the audiotaped interviews, classroom observations and assessment task analysis were included. Thus, there was an *audit-trail* of information which included components of the six classes of raw records suggested by Halpern (Lincoln & Guba, 1985, pp. 319-320):

- Raw data including test and interview scripts and audio transcripts
- Data reduction and analysis products including quantitative summary tables
- Data reconstruction and synthesis products including the Chapter 5 summary with connections to the existing literature
- Process notes such as to track the credibility of conclusions
- Materials relating to my intentions and dispositions including the research aims
- Instrument development information

Use of Case-Studies

In studying the students' changing facility with the concepts of velocity, distance, time and acceleration and any changing preferences in problem solving approach, a case-study methodology seemed appropriate. The aim was to understand the processes used individually to comprehend the real world problems presented. Typically, a problematic situation was isolated, and then the focus was narrowed to learn more about it, and data was collected on how the student's ideas changed over a period of time using a range of methods as recommended by Merriam (1988. p.10):

> Case study (research) does not claim any particular method of data analysis. Any and all methods of data gathering from testing to interviewing can be used in a case study.

Much of the sample data has been presented to the reader as student scripts to assist other researchers to better understand how conclusions were drawn and to enable the transferability of outcomes where appropriate. Thus indications of time and context become more relevant rather than the interpretation of statistical confidence intervals as might occur in a statistical report from a scientific study. Changes in student outcomes were thus analysed from interview transcripts and other data collection items rather than from a comparison of means and variances. Tasks in the present study were developed for students in a sequential process influenced by the results of earlier assessment results, interviews, or other data collection details.

Uniqueness of the Research Questions and Methodology

An extensive review of the literature revealed that very few *calculus reform* projects have addressed the effects of their unique approach on ESL students. This study attempted to add to the effective research on this second-language group, and at the same time develop practical exercises for effective learning which were based on a suitable balance of language demands and the encouragement of a rich representational structure. The use of case studies makes this piece of research methodologically significant as a contribution to research into the effectiveness of constructivist learning environments.

Summary

Part of this research study involved the design, implementation and evaluation of a teaching package suitable for pre-university ESL calculus students. The need for teachers to change the learning environment to one where conceptual understanding and small-group tasks were emphasised, as well as the need to prepare students in technical graphics calculator skills, was addressed through the development of appropriate resources. More generally, this study emphasised the advantages that can occur when classroom teachers listen to students through an interview approach and so gain a better understanding of students' conceptual knowledge. As a result of this study, teachers have been provided with a range of resources necessary for them to become researchers in their own classrooms.

Delays in the preparation of this thesis were caused by family concerns requiring a leave of absence totalling twelve months. Thus there is a four year time delay between the final data collection and the submission of this work. Influences that this may have on the data analysis are discussed in Chapter five and must be considered by other researchers when reviewing this document.

Thesis Layout

The thesis follows the traditional approach of reviewing past research in the field, outlining the research method, collecting and analysing the data, interpreting the results, discussing them and making recommendations. Chapter Two is based on a research. Limitations of the study follow. Appendices are extensive due to the extended nature of the study. They include important resource documents – for example, scripts and teaching package material.

Finally, a word on style. The essence of the presentation has been to include original scripts, assessments, interviews and personal notes wherever possible to allow the reader to make his or her own conclusions. Fellow researchers have been quoted extensively throughout the thesis to provide a sense of grounding on commonly accepted research ideas and results.

Chapter Two: Review of the Literature

REVIEW PROCEDURES

In this chapter, recent research literature perspectives on the teaching and learning of calculus are reviewed. Research studies involving the use of a constructivist viewpoint in teaching have been emphasised, but this was only one of the criteria used. Other criteria covered research into calculus learning involving ESL students, the efficacy of multiple representations in pre-university calculus, graphics calculator efficacy in pre-university calculus, as well as research studies into calculus learning using qualitative studies. The information from the literature review was used to guide the direction of this study and to form a framework for decisions about the study design.

The research review begins with a section containing discussion of the significance of constructivism and its use as a referent when researching conceptual knowledge changes among mathematics students. The application of a constructivist ideology established the foundation for the present study, and this epistemological approach is very effective for situations where new technology is applied (Aspinwall, Shaw, & Presmeg, 1997; Joiner, 1998). Research areas covered in this first section include learning calculus concepts, the use of a constructivist perspective, co-operative learning, multiple representations, instrumental vs. relational thinking, the role of language and the approach to assessment. These particular topics were thought most relevant to the nature of the present study. This section is then followed by a review of relevant research into calculus learning for ESL students. It covers the research areas of the language demands of calculus instruction, learning styles of ESL students and language factors.

Research on the efficacy of multiple representations in pre-university calculus is considered next, including the benefits of using many perspectives to solve a problem, and the necessity to do so for thorough mathematical understanding and knowledge. Sections on *Calculus Reform* are included, along with a description of the need for translation models, the efficacy of using multiple representations, the role of visual modes of instruction, and a consideration of research into representation systems other than the visual. Also included are sections on the need for specific teacher preparation, the efficacy of a function concept, as well as research into the topic of rates of change involving velocity, distance and acceleration concepts. This is followed by a review of recent research on graphics calculator efficacy in pre-university calculus.

First, a description is given of the role and implications of the *Calculus Reform Movement* (Atkins, 1994; Dudley, 1993), then the effects of improvements in graphics calculator technology as well as learning and assessment issues are detailed. Research into calculus learning using qualitative studies is the final topic discussed. A number of important issues relevant to qualitative research, covering the topics of trustworthiness criteria, clinical interviews, action research and case-study approaches is then described, and details about the advantages in the use of classroom environment surveys are also provided. A chapter summary highlighting the main ideas introduced concludes the chapter.

The research literature reviewed was selected on the basis of a number of criteria relevant to the present study. First, the review focused on research articles concerned with the use of a variety of mathematical representations in the teaching of introductory calculus, particularly where the emphasis was on problem solving in a constructivist environment. Of particular interest was research into classrooms where students were introduced to a range of representational formats or problem solving approaches, and where personal preference or teacher advice determined the solution approach. This would imply the research articles were based on a classroom situation where a student-centred instructional approach consistent with a constructivist perspective was used. The metaphor of *teacher as tour guide* (Tobin, 1993, p. 223) has been used to identify this learning and instructional approach where the classroom environment allows student discussion of calculus problems and the teacher acts as a support person.

The second criterion for the choice of research articles was relevance to ESL students. Most of the sample for the present study consisted of ESL students, since the thesis research was carried out at an international college in Western Australia.
A third criterion was that the studies selected for review were to involve students in the pre-university sector, since the study itself was on students in the 16-19 age

range. Relevant studies on students in the 12-15 age group, and the results from research studies on university students were also included where appropriate.

A fourth criterion was the selection of studies based largely on the relevance of the research method combined with its application to mathematics students. In particular, emphasis was given to studies involving interview analysis, or a longitudinal approach in preference to those using simply a short-term mathematical intervention with control group comparisons. Care was taken to include recent studies where ESL students were involved so that the research ideas were topical and included language aspects.

With these criteria in place, a large number of studies using interview techniques and analysing fundamental mathematical ideas such as functions, rate of change and the interpretation of graphs were found by sourcing prominent journals and library databases – for instance, the Dissertation Abstracts International (Curtin University, Perth, Western Australia). During this process, it was noted if any studies had been criticised for their lack of design considerations and consequent confounding anomalies, the aim of this check being to ensure that no such design problems would occur in the present study. Problems in design were noted, in particular in a recent review of intervention studies involving the use of graphics calculators (Penglase & Arnold, 1996). The need to carefully design qualitative studies has been noted by other researchers (Truran & Truran, 1998).

The sequence of presentation of these research findings commences with a section dealing with constructivism and its relevance to this study. Then follow sections on the special needs of ESL students in calculus classrooms, the advantages of a multi-representational approach to classroom instruction, the influence of graphics calculators on classroom instruction and learning, and the role of qualitative research methods in recent research studies. Research concentrating on case-studies or clinical interviews is the main focus of the last section on qualitative research. The use of qualitative methods in the present study was consistent with the constructivist theme of this thesis and was also in keeping with the uniqueness of the study. These methods were combined with quantitative classroom assessments measured over a four-semester period.

Studies conducted at the university level were only included if they contained unique features considered relevant to this study. Approaches using interview techniques and graphics calculators tended to be amongst the research recently conducted into mathematics education, indicating the relevance and importance of the present study which involved these aspects. The combination of calculus learning research and research into the learning needs of ESL students is a broad and increasing area of interest (Chapman, 1997; Ellerton, Clements & Clarkson, 2000).

Learning From A Constructivist Perspective

Learning Calculus Concepts

Calculus is the field of mathematics that uses techniques for predicting, monitoring and explaining change and it has extensive applications in science and technology. As such it forms a pivotal role in preparing students for university mathematics and science classes where confidence in its use and understanding of its concepts are essential. However, before the calculus class even commences, students already have an epistemological perspective structured by prior experiences in and out of the classroom, and differing attitudes and anxieties about mathematics. Unfortunately, traditional calculus instruction can reinforce any negative mathematical expectations students may have. According to Cooley (as cited in Joiner, 1998, p. 12):

> Calculus presents a different kind of mathematics from what most students have previously encountered. Precalculus mathematics is static in that it solves fixed problems that can be represented by simple equations or diagrams. Calculus is dynamic and deals with change. It examines infinitesimally small as well as infinitely large domains. It also requires the blending of visual and algebraic representations of problem situations and solution strategies, something students are not used to doing.

To change the calculus classroom with its established historical techniques, ways of thinking and language, radical constructivism has been suggested (Confrey, 1995; Tobin, 1993; von Glasersfeld, 1995). This approach allows the student to worry less about memorising knowledge solely for assessment tasks in order to achieve a good grade and to focus more on integrating new information into his or her current cognitive framework. The present study, in introducing classroom instructional materials presented in a multi-representational format, encouraged students to investigate, discuss and learn new ways of presenting information in a classroom environment rich in problem solving activity. Students were more involved and responsible for the paths their learning took as they struggled to integrate new knowledge with old and produce workable approaches for problem solutions:

> The radical constructivist program assumes that the individual makes sense of experience in order to satisfy an essential need to gain predictability and control.(Confrey, 1995, p. 194)

The choice of a constructivist perspective for the present study was motivated by two factors. First was the desire to approach learning from an individual student perspective as well as incorporating influences from the social setting of the classroom. Advantages were created by the use of a small class of four students, providing the opportunity to interview them, observe their progress and assist them on a one-to-one basis as the traditional role of the classroom teacher changed to one of researcher and facilitator, or *tour guide* (Tobin & Tippins, 1993). The nature of the role of teachers as they interact with students has been discussed by a number of researchers, particularly Vygotsky and Piaget, and more recently by Bauersfeld, Wertsch and Toma (Confrey, 1995, p. 225). The importance of social interaction of some form has been emphasised by all of these researchers.

Second, the present study's use of graphics calculator technology with its implied emphasis on student directed experimentation and group discourse meant a less structured student-centred learning environment had to be created. The research approach was, however, constrained by the school curriculum and the need to move quickly between topics in order to cover all required material. Under such time constraints it is still appropriate for the teacher to learn more about student learning and give them the opportunity to be challenged:

> My experience has been with a compelling task that as the teacher strives to understand the students' view of the problem, the students clarify their own ideas and often follow with rapid progress. To describe this as primarily the appropriation of the adult's goals underestimates the importance of students' autonomy. It neglects the role of reflection and self-regulation in the individuals' constructive processes.(Confrey, 1995, p. 205)

As part of this process of students gaining understanding, there is a need to create an environment where students are given the opportunity to view a given mathematical situation from a number of perspectives or representations. This is not just to improve understanding, but in many cases to provide a perspective that makes understanding feasible, particularly for ESL students. Hence, the present study used the graphics calculator as a technology aid to assist in this regard, and the student was presented with a teaching package rich in a multi-representational format. Additionally, learning by using a constructivist approach, and with curriculum materials that encourage the use of multiple representations, allowed the students to seek out the type of representation with which they were familiar, or which brought them understanding:

> Regardless of whether the pedagogy is verification or induction, from a constructivist perspective a multitude of representations maximises the chances that one or more of those representations will be the key to constructing personal meaning.(Joiner, 1998, p. 21)

Use of a Constructivist Perspective

Concepts are cognitive structures that enable us to recognise items that we have never seen before as examples of a familiar kind. Students enter the calculus classroom with cognitive structures built up from prior knowledge and experiences from school and elsewhere, and they have preferences for certain mathematical representations. In the classroom, the teacher can interpret what the student does and says, and try to build up a *model* of the student's conceptual structures (von Glasersfeld, 1995, p. 14) by listening to the student. In using our past knowledge of concepts and their representation, students can recognise patterns in a new situation that relate to some past experience. This process of recognition will be made easier if the concept has been learnt through an instructional process that develops a more robust understanding of a topic. In the words of von Glasersfeld:

> So we can recognise several particular experiential items, in spite of differences they may manifest, as belonging to the same kind, we must have a concept that is flexible enough to allow for a certain variability.(Steffe, 1991, p. 49)

The present study charts new waters by focussing on ESL students and aims to develop a robust basis for conceptual understanding while at the same time working from the student's past experience. The research of von Glasersfeld (1990) has provided a list of implications for instruction based on a constructivist model. Inherent is the suggestion of a flexible teaching approach where correct answers are only part of the goal. Listening to student reasoning becomes an important consideration too:

> 1. Whatever a student says in answer to a question (or "problem") is what makes sense to the student at that moment. It has to be taken seriously as such, regardless of how odd or "wrong" it might seem to the teacher.
> 2. The knowledge they have is the only basis on which they can build more. Hence it is crucial for the teacher to get some idea of where they are (what concepts they seem to have and how they relate them).
> 3. If teachers want to modify a student's concepts and conceptual structures, they have to try and build up a model of the particular student's own thinking.
> 4. Asking students how they got to the answer they gave, is a good way of discovering something about their thinking and opens the way to explaining why a particular answer may not be useful under different circumstances.
> 5. If you want to foster students' motivation to delve further into questions which, at first are of no particular interest from the students' point of view, you will have to create situations where the students have an opportunity to experience the pleasure inherent in solving a problem.(von Glasersfeld, 1990, p. 15)

The present study adopted the above ideas in following the learning habits of a class of four students, and the procedures followed aimed to influence student thinking by recording and then making use of their interactions with multi-representational examples. Along with classroom assessments, students participated in interviews with myself. To develop models of their thinking processes, the study followed the methods of von Glasersfeld (1995) suggesting a process where student responses were analysed to determine how they were derived.

Mathematics learning is built ideally from a variety of sensory experiences or diverse representational systems, but in the constructivist classroom there are no prescriptive guidelines on how best to use these to improve the learning of calculus concepts, especially for ESL students. A constructivist approach does, however, suggest that students will learn, or change their conceptions when actively involved in a discovery process which encourages them to assimilate differing points of view in the light of prior knowledge of a topic (Confrey & Smith, 1994; Joiner, 1998; Tobin

& Tippins, 1993). Typically, this would involve the student in some form of socially negotiated control of the learning environment; regular use of peer group discussions allowing ideas to be compared and adjusted, and the role of the teacher changing to one of facilitator. Research has an important and often neglected role to investigate appropriate learning methods, particularly when technology such as graphics calculators is involved:

> Little research is available which addresses the ways in which students construct their mathematical knowledge or the misconceptions that students have at various levels within the course when tools such as graphics calculators and computers are available.(Swincicky, 1994, p. 23)

As knowledge of the advantages of the use of multiple representations became more widespread, the nature of the function concept has matured hand-in-hand and been extended to include a *proceptual* view. Representations can be viewed as providing a dynamic view of mathematics in agreement with the viewpoint of Piaget:

> Knowing an object does not mean copying it - it means acting on it. It means constructing systems of transformations that can be carried out on or with this object. Knowing reality means constructing systems of transformations that correspond, more or less adequately, to reality. Knowledge, then, is a system of transformations that become progressively adequate.(Confrey, 1995, p. 196)

Learning involves a movement from the introduction of difference, or perturbation, through action to *reflective abstraction.* These actions on objects that Piaget observed (*reflective abstractions)* are based not on actions or representations, themselves, but more importantly on the co-ordination of actions. This movement towards a stable concept by introducing change involves a diversity of representation systems including semiotic functions such as language and mathematical notation through to drawing and modelling. To Piaget, however, language was not necessarily the most important of the systems or the predecessor of others but simply one of many ways to interact with objects. Another interpretation by Vygotsky, suggests that language development occurs through the interaction with humans in a social environment where a child plays with a knowledgeable adult leading to internalisation, decontextualisation, and new concepts (Vygotsky, 1962; Wertsch. & Toma, 1995). These two competing theories on student interaction cannot be discussed in more detail in a study such as this, but they do indicate the variety of views on appropriate student interaction for robust concept formation.

One important feature of the constructivist classroom environment is the use of interviews to improve the information available about each student (Cobb & Steffe, 1983; Tobin & Tippins 1993; von Glasersfeld, 1995). In traditional mathematics teaching, where teacher-focused instruction often encouraged a rote manipulation of symbols, the amount of knowledge the teacher has of student levels of understanding has often been limited to that of procedural skills shown during regular tests. With a greater diversity of representation systems now being encouraged in both lessons and assessment through the availability of graphics calculators, the opportunity is there for the level of student understanding to be investigated across a broader range of methods. This was relevant in the current study, where technology was applied in a constructivist classroom environment with pre-university level calculus students. Case-studies of an Australian male student and three male ESL students were compared as the students confronted introductory calculus concepts with the aid of a teaching package. Students were interviewed many times, a procedure that provided further insight into the steps used to form concepts as well as giving insight into the reasons for student responses. Caution was necessary when making interpretations from interview data because of the influence of *Voice* and *Perspective*:

> When the interviewer analyses the data to build a model of what the student's actions, words, schemes and concepts are like, the interviewer's own understanding of the task influences that model. Voice is used to describe the model of the child - and interviewing is an attempt to give voice to a child's ways of approaching a problem. Perspective surrounds voice, as the context from which the interviewer hears, probes and develops and sets new tasks.(Confrey & Smith, 1994, p. 32)

It would seem to be the case that the use of graphics calculators in the teaching and learning of mathematics will encourage the increasing use of student-centred learning practices (Penglase & Arnold, 1996). Certain teaching styles may be more compatible with using the extended capabilities of the graphics calculator as a means to enhance the use of multiple representations. A teacher who perceived the graphics calculator as a computational tool may simply stress content-oriented goals and view learning as teacher-talk, restricting the potential of students to test their cognitive structures and to understand the relevance and importance of translation between representational modes. Those believing in a constructivist approach, however, would see the graphics calculator as an instructional tool applicable to student-centred learning where extended tasks replace routine instruction and the teacher acts as a facilitator. There would seem to be more scope to use an inductive approach using *example, rule, practice* rather than the traditional deductive approach using *rule, example and practice.* Such an inductive approach could encourage students to do more conjecturing, discussing and generalising. The inductive teaching approach was most relevant when investigating ESL students in the present study where instructional tasks and teaching methods were designed to increase both relational and translational understanding among the students.

Co-operative Learning Research

There is an argument, particularly for ESL students, that a more open classroom with small group interaction may improve learning and the term *Co-operative Learning* has been used to describe this process (Kessler, 1992). This approach and others considered to create new learning environments in the classroom were an appropriate response to earlier research such as that of (Gardner, 1974, p. 75) showing student difficulties with scientific language leading to recommendations to investigate more appropriate instructional approaches. One suggestion was for teachers to talk less, and allow students to talk more amongst themselves. In order to change the traditional classroom teaching norms it is necessary to reflect on the conditions existing there and think of ways of doing things better. However, the type of learning environment the ESL student is culturally familiar with, possibly one where the student listens and the teacher imparts knowledge, means ESL students in particular may feel unnatural and be disadvantaged in such an open learning environment. The attitude to instruction can also be influenced by the mathematical ability of the students, leading to the following recommendation for the instruction of low-achieving students:

> The results of this study suggest that, for international students attending pre-university courses at senior colleges in Australia, it may be necessary to identify low-achieving students whose attitude to cooperative learning is unfavourable and to help them develop more favourable attitudes prior to any group work being required of them. The manner in which this

> could be achieved requires further research.(Swincicky, 1994, pp. 115-116)

ESL students are often familiar with more traditional classroom norms where the teacher takes a dominant role and students listen and repeat with little social emancipation. Rogers (1995, p. 176) describes the conditions prevailing when she was at school, and which provide an argument for a more hermeneutic approach:

> We were rarely given the opportunity to play with mathematical ideas or to construct our own meanings (except on our own at home). Instead, through the medium of the polished lecture (or textbook), mathematics came to me finished, absolute and pre-digested. A pedagogy that emphasises "product" deprives students of the experience of the "process" by which ideas in mathematics come to be. It perpetuates a view of mathematics in which right answers are the exclusive and sole property of experts. Such a pedagogy strips mathematics of the context in which it was created and reinforces misconceptions about its very nature.

It is important to not underestimate the influence of the social environment of the classroom for mathematics development. The recognition of the importance of the classroom social setting, its many aspects, influences and constraints on the individual has been stressed by Confrey (1995, p. 215) where she summarised the important features of social interactions, and their *personal, public, sociocultural* and *universal* aspects:

1. social vs. private require actual face-to-face social interactions among people at some time in the constructive process
2. public vs. personal signal the *public* or societal meaning of an idea, as contrasted to one's own personal meaning
3. sociocultural vs. universal distinguish the ever-present influence that society exerts on all acts of construction, including one's choice and conceptualisation of problems, one's selection of tools and appropriate course of action, one's assessment of completion, and one's choice of language or representation for reflection or communication.

The present study concentrated on facilitating classroom interactions with an understanding that there were aspects beyond the personal influencing the level of student understanding. The classroom instructional focus rejected the traditional teacher directed instrumental learning approach and introduced student task-based interviews where discussions encouraged relational understanding of real-world calculus problems, even though there was expected resistance from the ESL students with such a change. A student-focus places more responsibility on the student and researchers have highlighted the preference of many calculus students for a more traditional manipulative, instrumental focus (Cohen, 1994; Confrey, 1995; White & Mitchelmore, 1996). Special assistance must be given to students to assist them to work at more abstract levels and work effectively in groups. The use of the term *abstract-apart* understanding referred to by White and Mitchelmore (1996) is the type of learning we wish students to overcome, since it involves a decontextualised symbolic approach. The aim is to achieve a more generalised understanding which can be achieved through using a more *social* classroom environment, and the introduction of instructional material rich in multi-representations of concepts can assist this process:

> It is easy to see why many students prefer to learn in an abstract-apart fashion and become more comfortable with decontextualised problems than with contextual problems. Abstract-apart ideas are easier to learn because they are limited to a purely symbolic context (sometimes only the two variables x and y). All the decontextualised problems students can solve look very similar, and the appropriate procedures are therefore easy to formulate. Success in such a narrow context can even lead to a sense of satisfaction. By contrast, learning abstract-general concepts requires the formation of links among a wide variety of superficially different contexts. This must take longer and must be intellectually more demanding. But the resulting abstract-general concepts can be "seen" in contexts where there are no visible cues, and the learned relationships can then be used to solve the most diverse problems.(White & Mitchelmore, 1996, p. 92)

To encourage the development of *abstract-general* concepts there is the need for an approach by the teacher that encourages student thinking beyond the instrumental level with its typical symbolic focus. Douglas Barnes (Shell Centre for Mathematical Education, 1986, p. 227) suggested this in referring to the teaching style of '*Reply*':

> When a teacher *replies* to his pupils he is by implication taking their view of the subject seriously, even though he may wish to extend and modify it. This strengthens the learner's confidence in actively interpreting the subject-matter; teacher and learner are in a collaborative relationship.

Since the interaction with other students allows them to compare concepts and can create perturbations, and since the change process requires perturbations to occur in the students' cognitive structures, it is the very co-operative or collaborative classroom environments that should involve greatest learning. Collaborative learning becomes an important element of any situation where students are developing their own concepts, by building on and changing concepts learnt previously. von Glasersfeld (Joiner, 1998, p. 46) has extensively used the term *collaboration* in such a context.

Thus it was important at the beginning and during the present study to analyse the sociocultural nature of classroom interactions and reinforce those patterns of interaction that were compatible with constructivist goals in the light of research by Vygotsky:

> Students and teachers learn appropriate and expected ways of acting and reacting, many of which have become habitual and were learned early in adult-child activities to be nearly invisible in classroom exchange. Thus, the Vygotskian perspective can make the actors in teaching and learning processes more aware of the patterns of interaction that are unreflective and routine; for some of these patterns, alternative forms that are more compatible with constructivist goals for instruction can be integrated into the classroom. Seeing how these can be rewoven into the fabric of classroom interaction can only be accomplished by viewing the sociocultural character of classrooms and considering their influence on the epistemological enterprise.(Confrey, 1995, p. 214)

A change in classroom culture can bring with it teacher resistance. The teacher's authority can be challenged in an open-ended classroom and this has led to resistance to change from the traditional teacher-focused methods where the teacher was more familiar with the expected classroom management rules and their implementation. This increases the resistance to implement a constructivist approach since in many cases teachers felt satisfied with their results-driven classroom culture:

> One symptom of this is an unhealthy preoccupation with students' efficiency in arriving at solutions to problems; students are rewarded for short efficient non-original routines and sometimes even penalised for longer yet original routines. It is often more time efficient for teachers to state a well-established method than to guide students to perform their own personal constructions. Add to this the prevailing emphasis on skills tests and one begins to get a clearer picture of the difficulties faced by those who advocate a constructivist approach to the teaching and learning of mathematics.(Green, 1994, p. 68)

Multiple Representations

Research suggests that student discussion of alternative points of view, usually involving the use of everyday language, can lead to a more mature expression of ideas in mathematical language (Wheatley, 1993). Situations where conflicts occur in a student's current concept image assist the change process, and this can be facilitated through the use of multiple representations. Research question one of this study asks whether instructional material can enhance ESL students' use and management of multiple representations. The application of instructional material using different representations over the four-semester period of the present study ideally gave the student the ability to transfer between modes and so give them the ability to solve a greater range of problems, as well as possibly leading to an increase in conceptual understanding. A section on the efficacy of the use of multiple representations in calculus follows (see page 52), as their use has been a focus of recent research in mathematics education (Croft, 1998; Forster, 1997; White & Pegg, 1996).

Ideally, instruction should be aimed at helping students learn concepts, which involves making appropriate mental constructions. These constructions can be reinforced and altered when instruction combines not only verbal and symbolic, but also visual representations. In practice, the linkage of these representations to form a more general concept may not be so easy to achieve, but it is an important skill as it can lead students from abstract-apart to abstract-general concepts and provide a more general view of function, as will be discussed later in this chapter. If the conceptual knowledge of students is such that they treated variables as symbols to be manipulated, they had an *abstract-apart* view of function. An *abstract-general* view of function occurs where variables are treated as quantities to be related involving a greater level of understanding (White & Mitchelmore, 1996). Traditionally the symbolic form, such as a mathematical equation where the terms can be easily manipulated, has been emphasised in modern mathematics. The term *action* notation has been used to describe symbol use (Janvier, 1987b), as distinct from the use of a *static* representation system such as a graph. Kaput (1994) suggests an image of one possible *translation* between representations in Figure 2.1 following.

The transfer of information from a graph to an equation is fundamental to understanding calculus, but is a process in which many students struggle. Historically, it was an important step in the history of calculus (Kaput, 1994). There are important pedagogical implications for the use of only a single representation since the use of equations or symbolism in instruction creates a concept of function that may limit a student's ability to then successfully translate to other representations such as a graph (Dubinsky, 1994a, 1994b). By using only equations in instruction, a limited understanding of concepts can occur, and there may be no understanding of the linkage to other representations, resulting in a limited function concept on the part of the student.

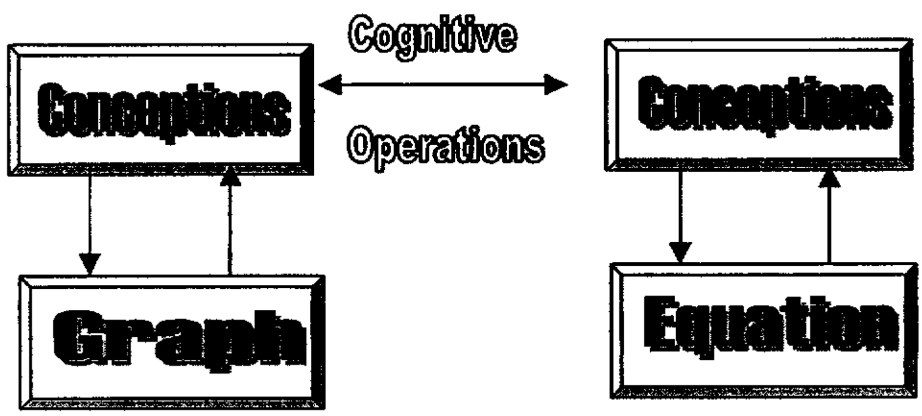


Figure 2.1: Translations (Source: Kaput (1994))

The present study compared the preferences for representations systems among a sample of mainly ESL students, and researched their ability to learn new translations and change their preferences over time. Students were presented with problem-solving situations where more than one type of representation was involved. The nature of the tasks given has an important influence on the methods students use. In using calculus real-world problems, the goal was to motivate students to tackle realistic and challenging tasks and research their conceptual understanding through the use of a number of instruments. The term *translate* has been used above to represent a conceptual change from using a graph to an equation, but the term is commonly used to represent the step in problem solving following the reading and comprehension of a mathematical problem. The present study examined ESL student preferences for representations as used in problem solving, but also considered the *translation* problems faced by ESL students as they commenced calculus word problems and were required to first read, comprehend and then create a mathematical model. Two steps are involved in any translation process as outlined by White and Mitchelmore (1996, p. 82):

> The first step is the definition of new variables and the symbolic expression of relations between variables known as (algebraic) *modelling*. The selection of a calculus concept and its expression in symbolic form we shall call *symbolisation*. Modelling and symbolisation together constitute *translation*.

A number of factors including classroom instructional material, familiarity of task and stage of the problem solving process affect student choice of representations (Ferrini-Mundy & Graham, 1994). Students can show preferences for either a visual or a verbal approach to problem solving, and this may depend on the stage they are at in working through the problem, as well as the type of task, sociocultural influences and the amount of detail in the problem presentation. A visual approach is typically preferred when there is uncertainty during the early stages of a concrete problem. These factors were important in the present study in order to correctly identify representational preferences.

Students develop concepts through a combination of the use of procedural knowledge and intuitive thought processes. Intuition involves parallel processing quite distinct from the step-by-step sequential processing required in formal deductive mathematics (Tall, 1991a, p. 107). Visual information has been connected with intuitive thought since information is processed simultaneously, or as a Gestalt, where unifying patterns are important. During the present study, collaboration in small groups in the classroom provided an opportunity for student discussion on instructional material rich in visual data, providing the opportunity to balance the use of these two processes. Traditionally, the two thought processes were linked with the use of visual or verbal representations and connected by *dual coding theory*:

> According to dual coding theory, problem-solving performance is mediated by the joint activity of the individual's verbal and imagery systems. The relative contribution of each system will depend on the characteristics of the task and the cognitive abilities and habits of the problem solver. The theory predicts that the more concrete and nonverbal the task, the greater the contribution of the imagery system.(Dawe and Anderson, 1993, p. 224)

To encourage the use of multi-representations such as visual modes, the idea of a *link-sheet* has been suggested with the aim of facilitating writing about mathematics (Shield & Swinson, 1997, p. 7). Lower school mathematics students were asked to divide an A4 page into four rectangular sections, one each for a mathematical example, an everyday example, a diagram/picture, and an explanation. This technique encourages students to extend their thinking away from a procedural or instrumental approach to one involving higher-level processes. This idea has been incorporated into some of the graphics calculator worksheets used in the present study, in a modified form with both a graph and a table shown together (see Appendix I).

Instrumental vs Relational Thinking

The attempt to promote the students use of both intuitive and procedural thought processes, which has advantages for a students mathematical understanding, has not been successfully achieved to any extent in the teaching of pre-university calculus. In an attempt to co-ordinate these processes, one possible theoretical framework has been suggested for mathematical problem solving – namely, incorporating the idea that the development of conceptual understanding should involve four levels of *language*, as well as a number of affective aspects (Goldin, 1987, 1993). Figure 2.2 following depicts Goldin's model that incorporates the following components:

- syntax features of written and verbal language
- imagery (visual/spatial, auditory, kinaesthetic encoding) related to task content and context
- notational language accounting for the effects of task structure
- planning, monitoring, and executive control (heuristic aspects).
- affective moods and emotions during problem solving

Goldin (1987, p. 60) used the word *language* when referring to one type of representation but he also applied it to identify the four key levels of the problem solving process. Student facility at using all *language* levels is thought to be essential for competency in mathematical problem solving and should be incorporated into any instructional process. Ideally, students will use several of these *language* features in problem solving. This important expression of a structure to aid classroom instruction guided the development of instructional material in the present study so that skills such as moving from images to notation (symbols), instruction in heuristic planning and image processing were encouraged.

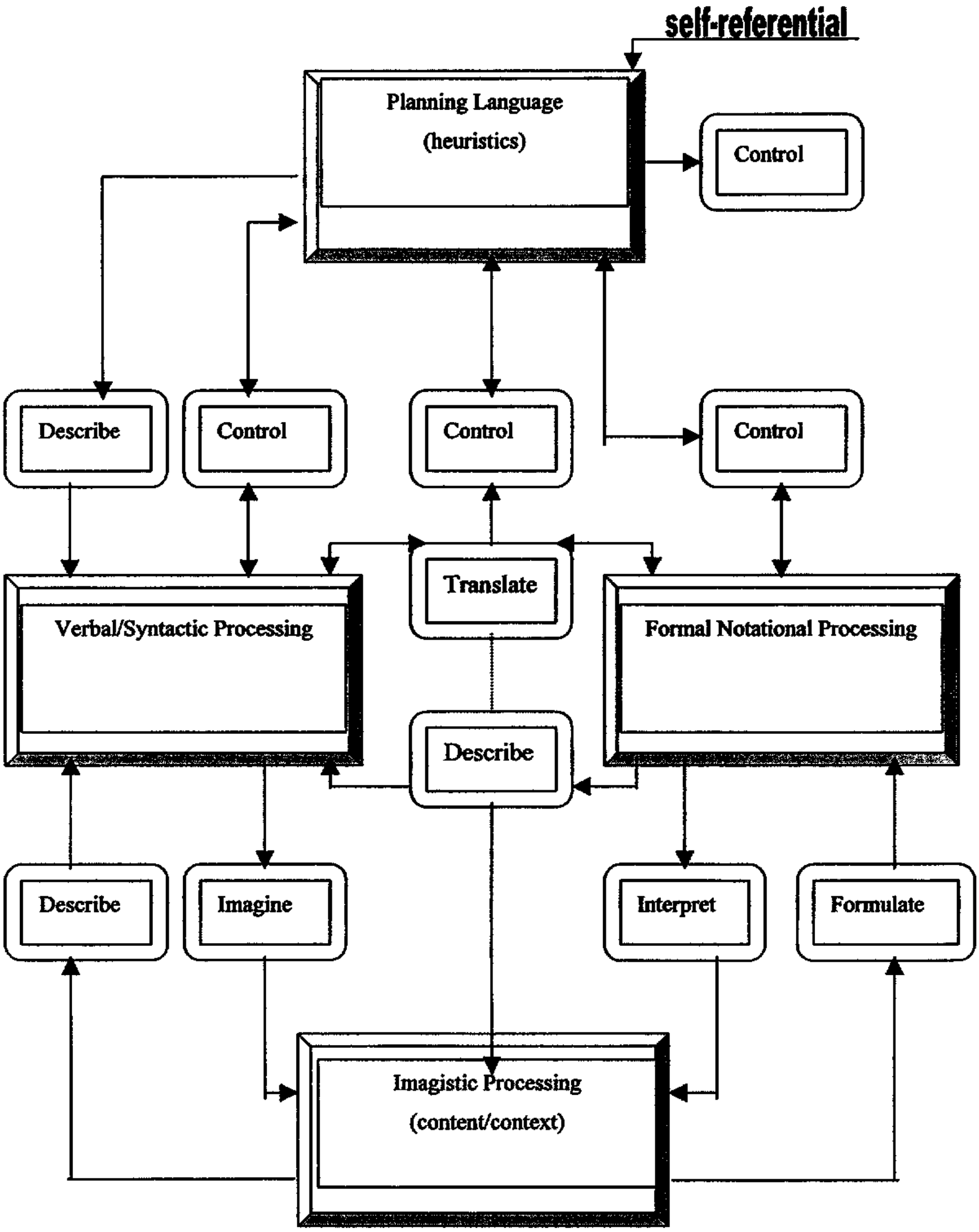


Figure 2.2: A Model for Competency in Mathematical Problem Solving (Source: Goldin (1987))

Knowledge of these language levels is thought to be important for ESL students and an attempt was made to design a teaching package for the present study that incorporated all *language* levels into the instructional material.

Figure 2.2 illustrates the many pathways that are necessary, or appropriate, to successfully solve a mathematical problem. Initially *Verbal/Syntactic Processing* refers to the reading and comprehension of the problem. *Imagistic Processing* is important to clarify and assist in problem analysis, whereas *Planning Language* is essential for the design and selection of a solution approach. The initial problem is *Translate*d into mathematical language and solved in a process indicated as *Formal Notational Processing*. This whole process can cycle in a loop when initial calculations lead to a reclarification of the problem. There are many links between the steps in the model allowing, for example, feedback on the solution in a visual form, which can lead to further reformulation of the approach. This type of model was consistent with the use of multi-representational modes during problem presentation and processing used in the present study. The idea is to assist the student to identify an approach that is appropriate for them, and so enable them to commence and solve mathematical problems.

The above ideas on a multi-step approach to problem solving have implications for the teaching style required. An instructional approach focused on only procedural or *instrumental* understanding has been criticised by many researchers as giving a limited view of mathematics (Anderson, 1996, p. 813; Durkin & Shire, 1991; Lepik, 1990). The instrumental approach is linked to students jumping in with well-tried and trusted methods and applying them in new problem situations in order to quickly finish and hopefully get a correct answer. Often students will apply techniques in an incorrect way when a problem situation requiring a different technique is involved, with their knowledge being restricted to particular situations. In the words of Skemp:

> We are able to get some of the students to write some of the notes without them ever hearing the larger melody, let alone the interplay of melodies. So their achievements are in small pieces, based on remembering rules for the manipulation of character strings.(Kaput, 1994, p. 132)

Traditional teaching where a presentation moves directly from verbal to notational language modes and skips an emphasis on images and planning gives students only a limited comprehension of the problem situation. This limited instructional approach is consistent with the student only achieving instrumental understanding and a better approach would encourage greater use of visual representations and instruction in heuristic planning rather than simply proceeding directly to the notational processing step. By incorporating all four of Goldin's (1993) language levels into classroom practice, students can be assisted to develop a range of solution strategies so that they can better solve problems in particular contexts as well as confront the issue of personal and cultural preferences. The greater the number of solution strategies and representational modes available to students potentially results in greater mathematical competence. This notion was encouraged in the present study.

As students move to greater mathematical competence they achieve *relational understanding* (Skemp, 1987). To obtain this level of understanding, students can typically transfer their knowledge to a wider context as well as being able to complete the mechanistic calculations of an instrumental approach. Student knowledge of *function* will be discussed later, as an example of student understanding achieving a relational level as they move towards a more abstract concept of function. The present study used multi-representational learning contexts in an attempt to promote and measure relational understanding. The terms instrumental and relational understanding have been used in a similar way by Piaget and Garcia but in terms of a stage theory of conceptual development (Kaput, 1994, p. 84). The first stage, *intraoperational*, leads to the interoperational and finally to the transoperational stage. The first stage actions are performed with attention only to the properties of the mode involved. This can be related to Skemp's *instrumental level* of understanding. Next the interoperational stage shifts attention to relationships and transformations between objects within a mode to another mode, and also to invariant properties. This stage of development has similarities to a relational level of understanding. The last stage is a metacognitive one, or the process of reflection about the structures being used. The term reification has also been used to refer to this last stage:

> Reification involves the leap from an operational mode to a structural mode where a process becomes an object in its own right.(White & Mitchelmore, 1996, p. 80)

The stages from a procedural or process conception to a more intuitive and abstract approach involve learning and understanding the methods of translation, reversibility and generalisation. This change process has been supported with technology in the present study using graphics calculators. Dubinsky and Schwingendorf (Kaput, 1994, p. 118) suggest the use of technology to encourage the development of abstract thinking:

> The passage from process conception to conceptual entity is difficult, but it is traversable with appropriate forms of deliberately designed experience such as defining and manipulating a wide variety of functions in a computer environment.

The Role of Language

In overcoming the language barriers ESL students may face in the mathematics classroom, research has shown that special structures used in our language (such as metaphors) have an important influence. Lakoff and Johnson (1980, p. 3) have highlighted the role of metaphorical language in our lives:

> Metaphor is pervasive in everyday life, not just in language but also in thought and action. Our ordinary conceptual system, in terms of which we both think and act, is fundamentally metaphorical in nature...The essence of metaphor is understanding and experiencing one kind of thing in terms of another.

There is a strong link between the idea of metaphor and multi-representation systems where any particular mode of representation of a problem such as a table, graph or equation is one way of conceptualising a calculus problem and so represents a metaphor. The metaphor is often used to help explain a process before a more structured approach is used. The appearance of metaphors in classroom practice needs to be highlighted and elaborated and a link between the use of metaphor and that of real world examples has been researched by Brown (1990) with 10-year-old students based around the ideas of Valerie Walkerdine (1989) who encouraged group involvement in mathematical activities that were rich in discussion:

> One aspect of metaphor is seeing the connection between pieces of mathematics and situations in the real world they seem to model. For

> example, within mathematics we can talk of rectangles, squares and triangles and develop theories about them and these seem to work when such shapes are found in the real world. But it is also helpful to see metaphoric association more generally as recognising similar mathematical phenomena in different contexts.(Brown, 1990, p. 15)

Brown (1990) discussed the movement to mathematical understanding through the initial use of metaphors when students designed a pathway around a lawn of a given shape and size by working in units of square metres of paving. Students moved between sketches, talk and tables as possible metaphoric representations of the lawn details. For example, a table was prepared to show the change in path length as the lawn size increased. The table was one mode of representation, or one metaphor for the information relevant to this problem. If students worked independently in one of these modes and extended their understandings to other applications at a more general and abstract level without referring back to previous representations, then they were said to be working at a more abstract level (known as a metonymy) than that of metaphor. An example is where students were able to discuss the connections between lawn shapes and areas in general without referring to tables or sketches. In the present study, a similar process of abstraction was observed to occur among ESL students when asked to reflect on the properties of distance-time graphs of car movement, understand the metonymic rules, and create individual pictures of new car situations, which could then be communicated in writing. Any lack of knowledge of the rules required for a given situation meant students were still thinking at a metaphoric level of processing and could not create new situations of their own (see Chapter Four results pages 156-159 for examples).

The movement from metaphors to metonymy was considered an important aspect of student learning for the present study. Students were introduced to examples on the distance, speed and acceleration of individual cars, and to a later concentration on the more abstract calculus idea of rate of change. In studying distance-time graphs as one metaphoric representation, the students attempted to understand the rules in their formation by referring to a selection of examples. If they understood the essential rules without referring back to other metaphors such as pictures, tables, or time-lapse diagrams, and created their own graphs, then it was possible to assume they were exploring the metonymic field of all distance-time graphs. They had reached an abstract-general level for this concept. As they wrote about their newly created distance-time graphs, they were exploring another metaphoric representation and this use of a written language metaphor can be an advantage for concept understanding:

> In translating work represented in Polyhedron models, scale drawings, tables and spoken description into a written description, the child will reflect on the rules and relations inherent in each of these metonymic fields. The child is actively moving within a metonymic field in attending to one representation but in the act of writing the child is actively placing different metaphoric representations alongside each other. I feel this active movement between metonymic fields (ie making metaphoric leaps between metonymic fields) underlies children developing an active relationship to the mathematics they are doing.(Brown, 1990, p. 17)

Metaphors also refer to expressions that come into common use in language but which also appear in mathematics lessons and form part of the mathematics register. Thus confusion can arise between the use of technical and everyday language. This aspect of metaphor usage was observed during the present study but a detailed analysis was beyond the constraints of the research (see Lakoff and Johnson (1980) for further examples).

In the present study, language terms that caused students difficulty were recorded during classroom observations but there was not sufficient time to analyse teacher use of metaphors. The application of metaphors in particular cases has been referred to in research by Kissane (1995c) and Kissane, Kemp and Bradley (1995) who used the idea of metaphor to describe the ways students respond to new technologies such as graphics calculators. One such metaphor was *tool*, which implied the use of the calculator in preference to a conventional analytic approach in a situation where the students' aim was to simply complete a troublesome mathematical task rather than to understand the process. The other metaphors used were *laboratory device, teaching aid, curriculum device, cheating device, and status symbol.* A possible seventh was *nuisance*, this term implying that the time to learn the calculator, the errors that can be made in its use, and the reduction in time for traditional instruction meant that less effective learning was taking place. Kissane (1995c, p. 3), described the metaphor of a *laboratory device* which was particularly relevant for the present study:

> In the laboratory approach we associate calculators with exploration, experimentation, purposeful play, and the discovery of new things. It is suggestive of an environment for learning, a playground, and

collaboration within a small group. In a laboratory, we expect to find out for ourselves things we didn't know before, perhaps even things that nobody has known before. An overriding aspect is that we are engaged in doing something: we are busy thinking, noticing, responding, discovering and acting, in contrast to being busy watching, listening and attending to others.

The Approach to Assessment

In discussing constructivist attitudes to learning, Davis, Maher and Noddings (1990) have suggested that traditional marking approaches in many cases give students a negative self-image and that a marking scheme should, instead, give rewards for original thinking and heuristic planning, and also give encouragement and suggestions for alternative solution methods. Students should be encouraged to move away from an approach where they answer a problem with no working, nor details of solution planning. The latter approach frustrates enthusiastic teachers, as they cannot see where in the solution process the students have taken a different path, and it is very difficult to allocate any marks based on student planning or multi-representational use. These sentiments are applicable to this study with its approach to the marking of assessment instruments during the longitudinal evaluation. Students were given feedback on their results through interviews as well as being given the opportunity to repeat problems that were inadequately dealt with.

The constructivist attitude to learning includes a changed perspective on misconceptions (errors) in line with researchers such as Ferrini-Mundy and Graham (1994) whose research treated student misconceptions in the light of the student's own prior concept images. Thus there was an explanation allowed for any student answer, and hence interviews took on a renewed importance as a method to find out what students were thinking. For the present study, with its sample from a high school, the time demands of the curriculum meant a reduction in the number and scope of the interviews possible. It was also necessary to use an interview approach that was task-based as this created a more structured approach to allow interviews in short class periods. Traditional assessment approaches that were time effective replaced qualitative methods, thus emphasising right and wrong answers rather than student misconceptions.

Calculus Learning For ESL Students

Language Demands

The move to increase the application of greater levels of language use in mathematics assessment recognises the broad range of social interactions in the mathematics classroom, the nature of the linguistic demands made on students and the importance of the mastery of the written representation mode (Chapman, 1997; Galligan, 1993; Shield & Swinson, 1997). There have been moves in Victoria, Australia, to increase the level of language use by mathematics students at the senior high school level by the implementation of a greater degree of written assessment in mathematics courses (Leder, Rowley, & Brew, 1995). Students completing the Victorian Certificate of Education (VCE) were required to prepare a fifteen hundred word written report in response to a challenging mathematics problem:

> Students produced polished versions of mathematical writing in which mathematical concepts, principles and applications were explained, interpreted and developed. In other words, language factors were accorded a more direct and different role in mathematics learning than ever before.(Leder et al., 1995, p. 204)

For the sample of mainly ESL students in the present study, prior learning in their first language has occurred using possibly a different linguistic structure, especially for those from Asia now studying in a non-Asian society such as Western Australia. It has long been recognised that the ability of ESL students to learn mathematical concepts depends on their linguistic mastery, not only in their first language but also in their second language (Austin & Howson, 1979). Encouraging student collaborative learning to increase the language demands, in both written and spoken forms, can improve linguistic mastery in mathematics and overcome two of the difficulties ESL students face (Zepp, Monin, & Lei, 1987, p. 1):

1. Student's ability to comprehend textbooks and teachers in the second language is not adequate for the complexity of the subject matter. High school textbooks have, in the past, been written using sophisticated vocabulary and complex sentence structure not suited to second language learners. Teachers imported from developed countries have not appreciated that they should simplify their classroom language for the students.
2. A person's thinking and logical processes are dependent on the first language. To attempt logical reasoning in a second language is

> difficult, if not impossible, because the logic in the mother tongue is different and pre-emptive.(Whorf, 1956)

In the written form, the *Whorf hypothesis* suggests we are obliged to create the situation and represent it to ourselves, and this demands detachment from the actual situation. The process has been described as a move from oral speech to inner speech to written speech where written language is a separate linguistic function differing from oral speech in both structure and mode of functioning (Vygotsky, 1962, p.99-103). It is unsatisfactory to teach using traditional methods with their focus on oral speech. Rather the teacher should adapt to the needs of ESL students by using appropriate methods that incorporate the use of explanation in a group setting supplemented with suitable written tasks (Kessler, 1992). Kessler encourages the application of second language teaching methods across the curriculum, including mathematics. Small groups of students work together on tasks with the teacher being a facilitator. Discussion of ideas is an important feature, with students endeavouring to reach consensus.

Mathematics has its own specialised language that must be mastered by ESL students. The present study recorded the occurrence of both words and phrases that were causing difficulty for students attempting assessment tasks. In particular, three Chinese students from the sample were interviewed and their language difficulties recorded. However the present study did not examine the more complex issues related to the derivation of word meanings. Many technical terms used in mathematics, such as the word *parabola*, were derived from Greek morphemes or roots and their meanings would not be widely known by international students studying in Australia. The translation of the meaning of *parabola* as *throw something up* may convey more meaning to second language students based on earlier practical experiences with moving objects. The Mandarin language used as a first language by three of the sample students in the present study generally has fewer morphemes than English (Galligan, 1993).

An examination of the structure of a student's first language was beyond the scope of the present study, but structure influences their level of understanding. Learning difficulties of ESL mathematics students have been researched by Galligan (1993, p.

277) and she highlights not only the extensive use of language in mathematics, but also the difficulties faced by Chinese students:

> In the contextually reduced realm of much of mathematics, language can be merely an unimportant embellishment to the mathematics problem, but often single words or phrases can cause confusion or misinterpretation. This could be due to some of the typology of many Asian languages. Sometimes slight rewording can improve the interpretation of the problem for the ESL student.

Learning Styles of ESL Students

Since the present study involved mainly Asian student partcipants, it was important to consider their learning styles. The learning styles of Asian University students were researched and summarised in a study by Niesten (1993). Typically, Asian students were found to be reproductive in their approach to their studies, simply attempting to absorb information presented to them, as compared to an analytic style which would involve more original thought. However in group work, these same students were found to typically have high commitment and expectations, which made this a productive style for them. The use of small-group work with less teacher talk can give students the confidence to change their traditional reproductive learning styles.

Learning styles can be influenced by performance on mathematical tasks as students who gain success in, for example, an algebraic mode will continue to use this approach. A number of linguistic and symbolic misconceptions can affect mathematics students too. Norman and Prichard (1994) have summarised these misconceptions: First, reading errors from the interpretation of natural language, syntax, synonyms and homonyms. Second, comprehension errors in relating details of sentence semantics and contextual information. Third, transformation errors such as the name-process dilemma and the incorrect transfer of operations to a new situation. Finally, process skill errors such as those resulting from concatenation, unitising and general distributivity. The extension of skills to unfamiliar problems causes many of these misconceptions. The present study will concentrate on the reading, comprehension and transformational aspects of students' errors and measure changes in *cognitive competence*, a term due to Cummins:

> By surface linguistic competence is meant those 'visible' features of language which are relatively easy to measure, eg. pronunciation, vocabulary, grammar, fluency, etc. By cognitive competence is meant the ability to make effective use of the cognitive functions of the language, ie, to use language effectively as an instrument of thought and represent cognitive operations by means of language.(Dawe, 1983, p. 332)

As students gain a better understanding of a mathematical concept, their use of mathematical language and their confidence in discussing that concept will increase. By confronting ESL students with multiple representations of a situation, there is the potential to extend their mathematics register, giving them greater confidence in classroom discussions which can lead to concept development. The move from less mathematical language to more mathematical language has been encouraged in the present study, and is an important aspect of classroom learning (Frid, 1993b). Mathematical meanings are constructed as the students take on this transformational shift in language use. When a high degree of mathematical language is used the combination of terms is mostly metonymic: it has little or no metaphoric content. In the present study, ESL students were confronted with increasing levels of mathematical language via a teaching package that gave a rich metaphoric experience with which to discuss ideas, and lead to a greater level of conceptual abstraction and use of mathematical language (Chapman, 1997, p. 166).

In the context of student discussions, the use of less mathematical language may be appropriate to assist in communication with other students, and so can indicate a greater level of understanding of a problem. Under teacher direction, an able student should also be able to express ideas mathematically. In peer group discussions the power structure shifts from the teacher, and as students gain confidence, a greater degree of certainty or modality is achieved in the expression of ideas. This greater certainty often involves the use of precise mathematical language.

The use of language in calculus lessons, for both instruction and student-student interaction, should move from metaphor to metonymy as mathematical understanding becomes more abstract (Brown, 1990; Chapman, 1997). In discussing a concept, a high degree of metaphoric language may benefit ESL students, but there is an eventual selection of spoken language, used by the teacher and later by the student, which is mathematical. This implies a shift from less mathematical language to more mathematical language, where colourful, illustrative real world examples are translated to specific details in the symbolic language of mathematics. During this process, the steps involve the selection of examples (metaphor stage), the combination of ideas into a single, often more abstract theme (metonymy), and a greater degree of certainty in the students thinking often linked to the language they use (modality). Interview language should also be carefully chosen as research has recorded the effect of dysfunctional instructional interactions when the language used in interviews causes uncertainty in student thinking (Thompson & Thompson, 1994).

Language Factors

A National Statement on Mathematics for Australian Schools (Australian Education Council, 1991) suggests that a command of mathematical terminology is essential in learning and is an important aspect of numeracy. The ability of students to explain and use language in mathematics classrooms has been widely studied and lists of words causing difficulty have been compiled (Durkin & Shire, 1991; Gardner, 1974; Malone & Miller, 1993; Miller, 1993). These word lists are a starting point for the teacher and indicate the need for carefully selected classroom language as well as the pre-testing of student knowledge of mathematical terminology. The way sentences are used in the language of mathematics forms a unique system known as the language register. An essential aspect of effective student communication in mathematics is the acquisition of a comprehensive mathematics register that involves not only words, but also how these are combined in a particular context (Chapman, 1997, p. 159).

A detailed analysis of word use and language forms, such as logical connectives, was carried out by Gardner (1974) in studying changes in proficiency in word understanding as students move through high school. Words commonplace in a calculus course were amongst a list of words causing difficulty. These include *rate* (25% correct use in first year high school to 58% correct use after 3 more years), and *relative* (41% and 74% respectively). Better understood were *instantaneous* (77% and 96% respectively) and *scale* (76% and 94% respectively). For situations involving the meeting or collisions of cars, such as examples used for the present study (Appendix C), the term *simultaneous* was less successfully understood (54% and 77% respectively).

Australasian research summaries into language factors in mathematics learning directly affecting the instruction of ESL students have been prepared over a number of years (Ellerton & Clarkson, 1992; Ellerton & Clements, 1996; Ellerton, Clements & Clarkson, 2000). Five types of errors commonly found in problem solving have been classified. The first three involve reading, comprehension, and transformation into mathematical symbols, as referred to in research question two. The last two involved carelessness and process skills such as formula use. The present study used a series of student interviews to assist in the identification of error types.

Further research into linguistic understanding was carried out at high school and for a sample of students entering university (Malone & Miller, 1993; Miller, 1993). Terms which students had trouble defining include *function*, *variable*, *polynomial* and *proportion*. Acceptable responses to around 90% of the questions were expected in the research, but for the above words the results were 45%, 64%, 67% and 66% respectively. In the Miller study, students were allowed to use diagrams, symbols or examples as definitions, and not just words alone and there was confusion amongst academic colleagues about acceptable definitions for these words. The words *function* and *variable* will be extensively used in the present study.

It is clear that there is an identified need for the teacher to analyse his or her own use of vocabulary and grammatical patterns in order to improve instruction. Patterns of language use by teachers can influence learning since research has identified a range of technical and non-technical words that were misunderstood. There is a need to check student understanding of words and one approach is by extending the language demands on students to create opportunities to increase word familiarity. Only some of these aspects could be tackled in this research study. Instruction tasks with a major language component were used in the present study.

The mathematics register includes homonyms – words with different everyday meanings to their meaning in the mathematics classroom. An example is *differentiation*, which has the same form but different meanings in a mathematical context. These are distinct from homophones, which sound the same but have different meanings. An example of a homophone is the use of *sum* and *some*. Presentation of tasks within a multi-representational framework, where students are exposed to more than just written information is necessary, to remove the influence of homonyms. The use of interviews in the present study helped locate the presence of ambiguous words as a means to improve problem comprehension. Durkin and Shire (1991, pp. 73-74) have prepared a list of eighty common ambiguous words that can serve as a guideline:

> Most of the ambiguous words listed are either homonymous or polysemous (words with two or more meanings in different contexts). These types of ambiguity are of interest because of the possibility that we can identify the basis for particular misinterpretations by pupils, and hence develop teaching strategies that circumvent or exploit such tendencies.

The need to understand the semantics of a language may cause special problems for ESL students. According to Lemke (Chapman & Lee, 1990, p.280) words by themselves are not enough to clarify the meaning of a situation. To learn mathematics the student must construct meaning from the particular patterns of language used in mathematical writing. This further emphasises the need for multi-representational modes to provide the ESL student with additional information to assist with underlying comprehension problems. Zepp et al. (1987), in his studies of Chinese second language students, considers the cultural background of the student as an influence on the understanding of mathematical language, an idea known as the *Whorf Hypothesis* (Whorf, 1956) that was defined on page 46 of this thesis. Studies of the ideas of researchers such as Whorf are beyond the scope of the present study.

The Efficacy Of Multiple Representations In Pre-University Calculus

Background on Calculus Reform

This study continues the traditions of *calculus reform* projects where the use of *The Rule of Three* using symbolic, graphical and numerical aspects in calculus instruction was incorporated into an infrastructure designed to enhance students' conceptual learning and enjoyment of mathematics (Joiner, 1998; Schwarz & Dreyfus, 1995). Past research in the area of calculus reform has encompassed a broad range of issues:

- constructivist approaches to teaching and learning (Green, 1994; Joiner, 1998; Malone, & Ireland, 1996).
- broad cultural reform in mathematics (Anderson, 1997; Ellerton, Clements, & Clarkson, 2000; Rogers, 1995).
- qualitative understanding through multiple representations (Bell, Brekke, & Swann, 1987; White & Mitchelmore, 1996).
- learning in greater depth and less breadth (Drijvers & Doorman, 1995; Lauten, Graham, & Ferrini-Mundy, 1994).
- greater real-world contextualisation (Barnes, 1991; Borlaug, 1993; Solow, 1993).
- more student social interaction and cognitive reinforcement (Brown, 1990; Geiger, 1998).
- the use of technology with possibly graphics calculators (Beckmann & Sundstrom, 1992; Forster, 1997).

These issues relevant to *calculus reform* make a number of assumptions about the learning environment. Firstly, a student-focused instructional approach is recommended for ESL students so they can discuss their currently held ideas with other students. This encompasses the ideals of cultural reform and constructivism by supporting the attitudes, beliefs and mathematical experiences of each student. Secondly, the teacher must be interested in the psychology of learning and be willing to encourage the use of multiple representations as a means to improve learning. The term *qualitative pedagogy* has been used to suggest an approach where a combination of graphical, numerical and tabular representations, as well as the traditional algebraic ones are used along with encouragement of both verification and induction (Duit, Treagust & Mansfield as cited in Joiner, 1998):

> both strategies play a certain role in that there is a complimentarity between verification and induction, with induction helping one avoid being blinded totally by a theory that actually is not well suited to understanding students' understanding.(Joiner, 1998, p. 21)

To initiate a *calculus reform* project a classroom teacher must reorganise the classroom and change the goals of instruction. Teaching in greater depth on a smaller range of real world topics must be combined with the encouragement of the use of a problem-solving framework using a range of representations in order to develop relational understanding:

> At one end of the continuum of beliefs teachers believe that their role is to transmit information and that students learn by memorising procedures and practising many questions. This view is often associated with a traditional approach to teaching mathematics. At the other end of the continuum is a belief that students learn by being challenged to think about their current understandings and by having to reorganise existing knowledge. This mode of teaching involves new approaches and innovative practices including the extensive use of problem solving situations.(Anderson, 1997, p. 7)

The need for *calculus reform* is evident as the types of student studying mathematics changes. Today, as a greater percentage of students wish to move to university, there is a need to teach an increasing number of students at high school who, in general, will have poor mathematical skills. The type of students reaching university has changed since lower entrance levels have followed as a greater percentage of students continue on at school beyond the compulsory age and those attempting calculus courses can often only operate at an abstract-apart level (White & Mitchelmore, 1996). This has created a need for a more contextualised calculus approach as well as leading to recommendations to include technology tools (such as graphics calculators), and to add extra steps in the instructional sequence in order to ensure all students attain a suitable level of understanding of calculus concepts. Joiner's description is as follows:

> Steps in instruction:
> 1. Reach a student's current level of knowledge, including if necessary appropriate contexts
> 2. Help them to construct new knowledge
> 3. Help them abstract or generalise that knowledge
> 4. Have them practice transferring that knowledge to new problems that are at differing complexity and in new contexts.(Joiner, 1998, p. 29)

Need for Translation Models

In an attempt to help students to better generalise their knowledge and create an improvement in comprehension and translation of calculus problems, the present study introduced instructional material requiring translations between representations. This allowed for the extensive use of multiple representations in problem solving in both classroom exercises and assessments. Research Question

Three (Chapter One, p. 11) related to measuring the efficacy of a teaching package incorporating a multi-representational format on ESL student understanding of calculus. Since the study involved three ESL students as well as one Australian student, it was important to create an instructional approach that encouraged the *language* components of problem solving as generalised by Goldin (1987). Earlier in Chapter One (p. 7), possible translations between mathematical representations included language contexts, pictures, symbols, graphs and tables were described. According to Bruner (Chapman & Lee, 1990, p.285) students require a very thorough knowledge of the mathematics behind the different representation modes to effectively use translation methods. Their mathematical knowledge of a representation is intensified during the translation process, and a degree of student reflection is usually involved.

A number of researchers have identified the types of translations suitable for instruction (Ferrini-Mundy & Lauten, 1993; Janvier, 1987b; Lesh, Post, & Behr, 1987). Translations are a transfer from one type of representation to a different one. One example is the translation from a calculus word problem to a graph, called sketching. Intermediate steps of comprehension, translation into mathematical symbols or selection of a solution approach can be involved. It is important to identify the preferred translations students use in problem solving in order to develop a model of their conceptual thinking. Representational preference was recorded longitudinally as part of the present study.

Table 2.1 following illustrates a summary of the simplest – form translations along with the mathematical terminology (in each cell of the table) used to describe them (Janvier, 1987b, p. 28). A translation involves a *source* representation such as an equation, and a *target* representation such as a graph. The source representations were listed in the first column of Table 2.1 and were linked to a target representation along the top row of the table. Typically, a calculus problem involves a translation from one type of representation to another. Multiple translations may be involved in problem solving. The central diagonal of the table is shaded since translations to the same representation have been excluded (such as the manipulation of an equation that uses only a rearrangement of formulae).

Table 2.1: Translation Processes
(Source: Janvier, 1987b)

Source \ Target	Situation, Verbal Description	Tables	Graphs	Formulae
Situation, Verbal Description		Measuring	Sketching	Modelling
Tables	Reading		Plotting	Fitting
Graphs	Interpretation	Reading off		Curve fitting
Formulae	Parameter Recognition	Computing	Sketching	

In solving calculus problems we may move from one representation to another in a series of steps involving a number of translations. Table 2.1 shows the four common types of representation in the left hand column:

- *Situation, Verbal Description* represents the use of text, diagrams or a verbal description of a problem
- *Tables* represent the use of numerical data for the variables
- *Graphs* represent the use of mathematical plots
- *Formulae* represent the use of mathematical symbols to represent the variables of a problem along with suitable equations

Formulae representations allow direct manipulation of mathematical symbols and are known as *Action* modes. All of the other representations shown in the first column of the table are known as *Display* modes. The *Action* mode has traditionally been used extensively in pre-university instruction but has two major drawbacks: one is the emphasis on mechanical operations, without much understanding of what is behind them; the second is errors made transferring the process to new situations, usually when concepts are more complicated. As an example of incorrect use from Krutetskii (Norman & Prichard, 1994) the exponential function $f(x) = a^x$ is often incorrectly differentiated using the power rule $\frac{df}{dx} = x\,a^{x-1}$ rather than being identified as a special case that involves logarithmic differentiation. The power rule should in fact only be used for polynomial functions such as $f(x) = x^a$ with

$\frac{df}{dx} = a\,x^{a-1}$ $(a \neq 0)$.

Lesh et al. (1987) further illustrated the representations that can occur in mathematics problem solving (Figure 2.3 following). The use of *Manipulative Models* as a representation mode has been suggested in addition to those used in Table 2.1. It was important for the present study to identify representations preferred by ESL students in problem solving contexts involving multiple representations. The sample students investigated appropriate translations for effective problem solving (research questions one and three).

When one moves from one mode of representation to another, such as from pictures to the use of symbols, a psychological process is involved (Janvier, 1987a). Representations have been modelled by using an iceberg with five points that are all connected but only one of which may be showing *above the water*. The points of the iceberg are the representations of table, graph, formulae, verbal description, and real object. Students are encouraged to move between representations when instructional material is prepared where the use of a number of representations is required.

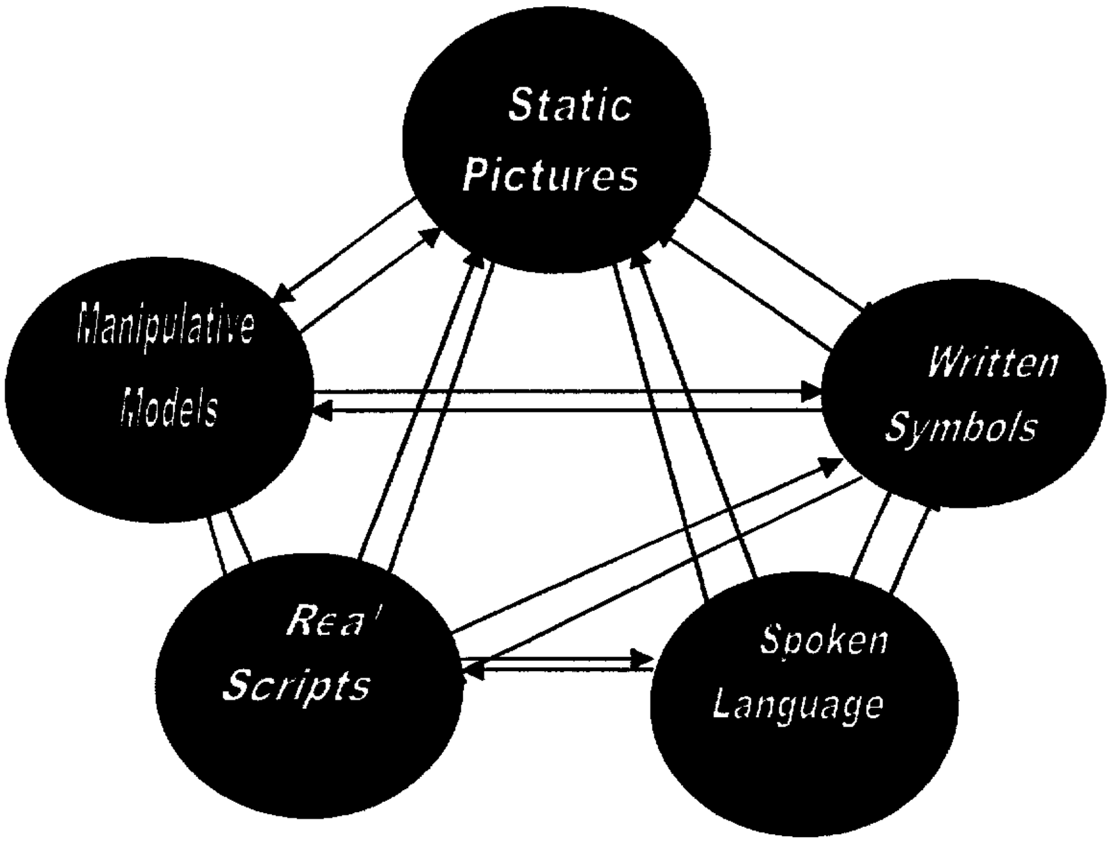


Figure 2.3: Mathematical Representations (Source: Lesh et al. (1987))

Dufour-Janvier, Bednarz and Belanger (1987, p. 113) found that representations such as graphs or equations were seen as tools, or objects but not as a means towards a solution to a problem (ie. knowledge of them was not enough to provide the planning components for problem solving - the heuristics). They concluded that consistency of meaning and context were not always important for students, that different representations were not seen as embodying the same situation and that students did not expect the same answer using different methods. The concerns identified above have been reinforced by research on representations by Ferrini-Mundy and Graham (1994, p. 42):

- The ability to co-ordinate formulae and graphical contexts may differ substantially across concepts.
- Competing, conflicting conceptions and conclusions are held quite comfortably and routinely by students in the development of calculus concepts.

- Students actively develop meanings and connections that may differ from those of adults, in particular from those of the teacher. The influence of previous experience and knowledge counts.
- Startling inconsistencies exist between performance, particularly on procedural items, and conceptual understanding and this may be hidden by traditional assessment items.

Extensive research appears to be necessary in the area of student use of representations. The instructional approach can help to eliminate many of the difficulties and levels of belief students have with certain representations. In order to establish any connections between representations, the present study used a longitudinal approach to identify changes in student knowledge. This enabled me to gain a deeper understanding of individual differences and attempt to overcome problems in the linkage of representations identified by Ferrini-Mundy and Graham (1994, p. 44):

> Concepts, methods of reasoning and the repertoire of heuristics are radically different maybe in each representation and if students are comfortable with the inconsistencies, contradictions and competing meanings that emerge then connections are hard to establish.

Schwarz and Dreyfus (1995) used the term *settings,* instead of representations, to refer to graphical, tabular and algebraic approaches. They recommend these be introduced in the problem solving activities used in the classroom to effectively instruct students in translation processes. Within each setting there are many representations, such as the many graphs of the same data using zoom features on a graphics calculator, but it is the interplay between settings that is the primary interest of the research.

Efficacy of Multiple Representations

The ability of students to use and connect representations has been linked to the instructional approach by a number of researchers (Forster, 1997; Porzio, 1994). In a study of university students and their use and understanding of multiple representations, Porzio (1994) compared three instructional approaches: the use of a computer package (Mathematica), use of graphics calculators, and the use of a traditional approach with no use of these technologies. Results showed that for students to use and connect different representations, the instructional approach itself must incorporate at least the use of symbolic, graphical and numerical approaches.

The present study aimed to extend the above results by using appropriate instructional material with pre-university ESL students, and using a multi-representational instructional approach. Thus the present study fits into the field of earlier studies but breaks new ground with its emphasis on ESL students in a high school setting.

A number of other studies have investigated the use of technology tools with university calculus students (Hart, 1992; Porzio, 1994; Thompson, 1994). These studies used technology in the form of either a computer algebra system (CAS) or graphics calculators, and showed that the successful use of multiple representations depends on factors such as student knowledge, and on their ability to manage and connect different representations. These studies show it is important to determine which representations help concept development and how these representations should be used. The instructional approach itself, as well as student prior knowledge of the technology involved, were important control variables applicable to the present study.

The present study aimed to create instructional tasks that were rich in language content but also rich in language demands. Participants had the opportunity of adapting to the linguistic structure used in the instructional material through a longitudinal approach. The integration of the use of multiple representations and extra language demands aimed to give ESL students better long term concept understanding. Instruction in the transfer of mathematical information between the types of representations mentioned earlier (Table 2.1) can be tackled with the use of real-world examples similar to those of Borlaug (1993). Her research paper on the introductory calculus topics of differential velocity, distance and acceleration was very relevant to the present study. It described how a Tonka truck toy was used to develop representations of two-dimensional motion. Fundamental to her study were the following assumptions concerning the graphical representation:

- The curve will rise with forward motion and fall with backward motion but will continue to move to the right as time passes
- The faster the truck moves, the steeper the graph
- Constant velocity is the easiest kind of motion with a straight-line graph

- As the truck slows down in the same direction the curve becomes flatter

The use of multiple representations is often linked to technology tools in the classroom. The results from research into computer technology can have beneficial results applicable to the present study, where the desktop computer has been replaced with a graphics calculator:

> The computer gives us an unrivalled opportunity of building new curricula which redress the balance towards a more versatile form of thinking appropriate for the new technological age. With the computer the teacher can introduce activities that encourage the development of holistic thinking patterns, linking them to sequential, deductive thinking. Our new versatile learners, introduced to topics by the teacher and supported with the computers are more likely to be successful in mathematics at the higher levels where the ability to switch one's viewpoint of a problem from a local analytical one to a global one, in order to be able to place the details as part of a structured whole, is of vital importance.(Cadby, 1993, p. 6)

The Role of Visual Modes

One mode of classroom communication is the spoken language. Student perceptions of the verbal modes of communication between teacher and student are a fundamental aspect of knowledge development in mathematics classrooms (Chapman, 1997; Chapman & Lee, 1990; Ellerton, Clements, & Clarkson, 2000; von Glasersfeld, 1991). When listening to a teacher, the student interprets the language, but may represent the information both internally and externally in a different way to what was expected. The verbal representation used by the teacher thus may produce an internal representation in the student, which differs from the teacher's original perception. To overcome this learning problem the present study used triangulation methods in an attempt to better understand both the process of student conceptual thinking and interpretations of teacher-talk. The approach included exposing students to a range of representation systems including visual modes to supplement the information collected.

Visual modes are a common element in the representation systems of Ferrini-Mundy and Lauten (1993), Janvier (1987b), Kaput (1994), Lesh et al. (1987) and Schwarz and Dreyfus (1995). There is a growing awareness that visual representations play a central role in concept acquisition, particularly during the discovery phase of problem solving, and when the novelty of the task is present (Dawe, 1983, p. 350; Dawe & Anderson, 1993; Presmeg, 1992). The visual mode is linked to intuitive aspects of learning and can help guide students towards the more formal solution of a problem, or just get them started.

Historically the symbolic aspects of mathematics have always been emphasised. These can have the effect of inducing success in mathematics in a limited way and this is often linked to a traditional teacher-focused instructional style. For ESL students where the need for better language instruction is required, the linguistic aspects of problem solving are rarely given suitable emphasis in instruction (Chapman, 1997; Durkin & Shire, 1991; Lepik, 1990). The present study attempted to increase the use of visual modes by combining a student-focused classroom environment and a focus on appropriate language use in a multi-representational framework. A teaching package was designed to assist in the implementation of visual modes and a student-centred instructional approach will give students more flexibility in choosing representations, including the visual mode.

Universal agreement on the potential advantages of using visual representations in all situations does not exist. The research of Aspinwall et al. (1997) discusses situations where visual images can be a hindrance to problem solving in the case of a participant called Tim. In a case study of that student (Tim), who was struggling in connecting his visual image of the derivative of a parabola with a symbolic linear equation, it appeared that graphic understanding was a more powerful referent than symbolic understanding. A conflict in conclusions led Tim to use the former. The ability of students to prefer certain representations and overlook inconsistencies from the use of more than one representation is an important aspect of the student use of multiple representations. It demands further investigation.

Earlier research into mathematics education has shown that students generally have very weak visualisation skills in calculus (Goldenberg, 1988; Tall, 1991a), and that the role of visual thinking in successful mathematical problem solving has not been seriously considered in classroom instruction (Zimmerman, 1991). This may be compounded in a traditional classroom environment where procedural work often emphasises the use of algebraic expressions using written symbols (symbolic representations). Instructional practice in visual modes was incorporated into the present study, and an overhead projector calculator pad was used on a regular basis to benefit students in learning how to use a graphics calculator screen for plotting and for other translation activities. Tall (1991a, p. 118) noted the advantages of an approach using visual representations:

> By introducing suitably complicated visualisations of mathematical ideas it is possible to give a much broader picture of the ways in which concepts may be realised, thus giving much more powerful intuitions than in a traditional approach.

A number of difficulties can occur in the comprehension of the visual components of a mathematics problem. One is the misconception that a graph of motion against time such as one of velocity versus time follows a similar shape to that of a picture of the situation (Bell, Brekke & Swann, 1987). A golf ball, for example, has distance greatest above the ground when vertical speed is a minimum, and in such a situation the student may link the time of highest distance above the ground with the time of greatest velocity. The vertical speed will be greatest at impact with the club and also at impact with the ground. At both these times the distance above the ground is zero. The graph of the speed versus distance of an object such as a golf ball, or the movement of an accelerating car shown on a distance versus time graph, were interpreted quite incorrectly by ESL students in the present study (Chapter four). Prior knowledge transfer is often being used incorrectly. Such counter-intuitive results, where the shape of the path of the object shown in the picture affects student decision-making, illustrate that there are extra variables influencing any translation process involving visual information. The present study used interview discussions as an important tool to discover more about the presence of graphical misinterpretations when translating from pictures.

Other Representation Systems

Another representation system used in calculus teaching is that of approximation, often used within the algebraic setting, where a formula is used to estimate a graph, a method made easier with the plotting features of the graphics calculator (Ruthven, 1990). The use of approximation in calculus is common, such as where the differential is used as a linear approximation to the derivative at a point (Frid, 1993b; Mochon, 1996). In a naturalistic study of undergraduate calculus students where approximation methods were a feature, Frid (1993a) compared the three instructional approaches of *technique-oriented, concepts-first* and an *infinitesimal approach*, and recorded any apparent effects on calculus misconceptions. Extensive examination of oral and written responses from task-based interviews enabled her to analyse the use of symbols, technical and everyday language. Instruction using everyday and technical language influenced the students' use of language forms with the *infinitesimal* instructional approach leading to fewer mathematical misconceptions. Her approach used advanced computer software but modern graphics calculators can give similar exposure to approximations. The aspect of the research important to the present study was the language use. Students in the present study were introduced to the concept of differentiation using magnification (zooming) of plots, a technique that can be easily applied with a graphic calculator. This dynamic method for the interpretation of graphs made the concept of limit accessible:

> That is, the visual mechanism of blowing up or infinitely magnifying a curve serves as a visual, physically accessible means by which to examine related limiting notions.(Frid, 1993a, p. 262)

Teacher Preparation

Teachers in Western Australia have been motivated to change their instructional approach with the requirement for the use of graphic calculators in examinations. The teacher – skill issue includes the serious question of whether the capabilities of the calculator are known well enough by teachers to qualify them to advise students on their effective use in problem solving. Teacher expertise levels are an important aspect of graphic calculator implementation in the classroom and possibly the most urgent issue according to Croft (1998, p. 30):

> Development of expertise can be effectively broken into two aspects. The first level is simply the development of the necessary skills by the teacher to use the calculator effectively as a tool. The second level, for which the first is a pre-requisite, is the effective incorporation of the calculator into lessons as a teaching and learning tool.

Along with the more frequent use of graphics calculators comes the need for teachers to be more aware of reliability issues, since as operating systems become similar to those of a computer, it is possible for the graphic calculators to freeze. This occurred in the present study where there were three cases of freezes during examinations, with one student experiencing two such *crashes*.

A number of schools offer access to pre-packaged graphics calculator applications, which can be used directly in lessons to assist teachers in their preparation. The St Hilda's Anglican School for Girls Home page (www.sthildas.wa.edu.au) gives access to the work of Colin Croft, a senior mathematics teacher at the college, who has prepared links to other relevant Internet pages, including calculator software called Aplets. The Aplets can be transferred directly to the Hewlett Packard HP 38G calculator. Aplets can not only assist students in exploration but also have the potential to improve examination performance. The application of these tools in a classroom is assisted if the learning environment is to be constructivist in nature, with student-centred learning a feature, as used for the present study. This requires extra teacher training:

> It is clear then that there is no shortage of resources that might be used to facilitate a constructivist approach to the teaching and learning of mathematics. Even so, the commitment to encourage constructivist teaching approaches will not be fully realised without an equal commitment to provide teachers with greater access to and training in using such resources.(Green, 1994, p. 67)

Efficacy of the Function Concept

Fundamental to any mathematics research dealing with mathematical representations is the notion of a functional relationship. The range of different function representations presented in instruction has a large influence on the knowledge and ability of students to apply, translate between, and develop their individual preferences for representations. Recent research highlighted a range of student misconceptions in understanding many of the fundamental function representations (Confrey & Smith, 1994; Joiner, 1998; Ryan, 1994) as well as showing that students in many cases were unable to successfully apply the function concept to different contexts (Slavit, 1997; White & Mitchelmore, 1996). There is increasing international interest in research on student conceptions of function in the mathematics education research community and their influence on the use of different representational forms makes the topic of function an integral part of the present study.

One common approach to define the function concept is through the idea of action/process/object levels. The simplest and most common understanding of function for students is an action view, a non-permanent construct that is developed with a disregard for pattern and dependency relationships:

> An action view pertains to the computational aspects associated with functions, such as an arithmetic process or a 'function machine'. For example, one can consider the function $f(x)=3x^2-7$ to be an algorithm used to compute numeric values for a given input. This conception does not require an awareness of patterns and regularities that may exist between the numeric values of successive inputs and outputs, nor attention to causal and dependency relationships between inputs and outputs. An action conception is concerned with the computation of a single quantity for a single numeric value via a given algorithm or rule of association.(Slavit, 1997, p. 261)

When seen in symbols as an *action*, a function takes an input, manipulates it, and produces an output. This is the usual approach when a formula has been given, and an ordered pair of values results. The *process* view of function, however, goes beyond a simple calculation of a pair of values in isolation, and involves student understanding of the transformation of pairs as a group, a transformation of *mental objects* (Breidenback, Dubinsky, Hawks & Nichols, 1992). Looking at the properties of a function through a process view can assist abstraction to a more general, or "objective" view, however this process is not automatic (Sfard, 1991a, 1991b). The process approach can create an understanding of the equivalence of procedures in different notational systems. This is an important aspect of the present study. An example is when a student realises the equivalence of intercepts on a graph and the solution of an equation. Extensive research has been based around a process view of function (Quesada & Maxwell, 1994; Ruthven, 1990; Schwarz & Dreyfus, 1995), and much of it suggests that an unhealthy student focus on variables rather than processes can lead to a context-dependent understanding of function (White & Mitchelmore, 1996, p. 91):

- students develop an inadequate concept of variable, especially independent variable
- students are preoccupied with visible symbols (the 'x, y syndrome')
- a manipulation focus on rules based on visible symbols was observed
- no covariance view (explained later) of variables – they are seen as symbols to be manipulated rather than as quantities to be related.

By exposing first year university students to rate of change problems with a range of presentation formats, including the use of word problems involving applications of calculus, White and Mitchelmore (1996) studied the conceptual knowledge of students including their level of relational understanding. Students treated variables as symbols to be manipulated, or had an *abstract-apart* view of function, rather than an *abstract-general* view of function, where variables are treated as quantities to be related involving a greater level of understanding. The research questions of the present study relate to the solution of calculus word problems (Chapter One, p. 10) and the longitudinal investigation of students' thinking as they move towards the achievement of greater levels of relational understanding and abstraction:

> Abstraction can be described as a process of identifying certain invariant properties in a set of varying inputs (Skemp, 1986). The act of abstracting is based on generalising these properties to other inputs, but is seen as qualitatively different from simply identifying patterns in a set of examples. It is a many-to-one function where generalisations are synthesised from many inputs to form a new abstraction. Dreyfus (1991) summarises the process as a sequence of generalising → synthesising → abstracting.(White & Mitchelmore, 1996, p. 80)

The instructional method and instructional material can influence the mental images of a function students' use and their preferences for graphic, symbolic or other approaches to function representation. A change in the teaching emphasis when developing function concepts can ensure symbolic, decontextualised representations as well as graphs, and tables are included. This teaching change can influence the current limited student view of function according to research by Cooley (as cited in Joiner, 1998):

> Most students understand a function to be a formula. A graphical or tabular representation with no formula has no meaning for most calculus students. They have a strong subservience to symbolism. There is a notion that a relation is a function only when it can be represented by a single formula. Also, students tend to view algebraic data and graphical data as being independent of one another.(Joiner, 1998, p. 25)

Researchers have noted comprehension problems from table to graph and the preference to rely on only one of these representations (Bell et al., 1987; Ryan, 1994). These student problems can involve a misunderstanding of the correspondence and covariance aspects of a table of values, the *discrete* and *Gestalt* views respectively. Traditional mathematics courses, where symbols have been emphasised, have commonly used the idea of a correspondence between values in which one initially builds a rule that allows one to determine a unique y-value from any given x-value. This approach is emphasised in conventional algebraic notation in which we write: y = f (x) (Confrey & Smith, 1994, p. 137). A covariation approach is linked more to changes in one variable in a table of values relating to a pattern of changes in the other variable, typically as you move from x_m to x_{m+1}. The present study introduced this approach for the concepts of velocity and acceleration. It is a sign of greater mathematical ability if both the correspondence and covariation concepts can be grasped:

> The subtleties of the tabular form are not easily seen. In grasping the underlying functional structure of a table of values, a student needs to see both correspondence and co-variance. This involves a double perspective of individual input-output match and overall matching of the two sets of data (x and y). I suspect that this may be very difficult for most students.(Ryan, 1994, p. 6)

A new emphasis on a property-oriented curriculum has been advocated in a move away from an action/process emphasis (Slavit, 1997, p. 266). A property-orientated view of function deals with the gradual awareness of specific functional growth properties of a local and global nature, followed by the ability to recognise and analyse functions by identifying the presence or absence of these growth properties. Properties such as the recognition of rate of change, in the form of acceleration or velocity shown on distance-time graphs, have been investigated during the present study. Through experience with function exemplars, students may develop a concept image of function as a related set of procedures as well as functional properties, within a variety of notational systems. This suggests a change in the type of functions used in instruction and illustrates the importance of a broad range of instructional formats:

> This situation suggests that the question of the sufficiency of the kinds of functions that are currently most often studied in today's schools be re-examined in regard to allowing students access to a broader array of functional properties. These could include discontinuities, finite domains, multivariable functions, or non-functional relations. One dimension on which this question should be addressed is the type of

> functions that give rise to contexts and situations that support investigation of algebraic and functional ideas.(Slavit, 1997, p. 278)

The use of the object-oriented views of the property, correspondence or covariation perspectives have been said to lead to a "proceptual" understanding of functional notations, a term due to the research of Gray and Tall recorded by Slavit (1997). A proceptual level of understanding has been recommended for all calculus students and is an important goal for any calculus class, including the sample from the present study:

> Students who see functional notations proceptually have the flexibility to think about function as an action, an object, or both. Further flexibility arises when the student incorporates different perspectives into his or her concept image, such as correspondence and covariance views. These views also allow the student to better understand actions performed on a function, such as a 'shift' translation (eg, changing f (x) to f (x+3)) or taking a derivative. It would be quite difficult for a student to completely understand an action he or she performed on a function if an object-oriented view of function was not yet achieved.(Slavit, 1997; p. 270)

Rate of Change

Student understanding of rates of change in kinematics examples involving distance travelled under acceleration can indicate potential problems later in the Fundamental Theorem of Calculus and indicate a poorly developed image of functional covariation (Thompson, 1994). The present study introduced instructional material to specifically identify student misconceptions in rate of change problems. One aspect of change that needs further research is student use of proportional correspondence for quantities such as distance and time under constant velocity. Students use speed as a distance travelled, and think of a time interval as a sum of distances without forming any correspondence relation (Thompson & Thompson, 1994). In the present study the main exercises in the early part related to distance, velocity and acceleration concepts involving rates. A polynomial model used in instruction was supplemented with the inclusion of approaches using both correspondence and covariance ideas, allowing students to develop a well-rounded concept of the change process.

In any mathematical modelling of situations involving motor vehicles there is typically a segmentation of the process of car movement into, say, hourly intervals, with invariance across segments for constant velocity situations (Confrey & Smith, 1994, p.42). Thus a suitable correspondence will be best written in terms of a starting position and a unit consisting of the movement of the car in one hour, usually just the size of the constant velocity in the simplest case. An example where the start is at 40 metres and velocity is 200 metres per second would be d= 40+ 200t, where *t* is the time in seconds. The next step was to extend the real-world scenarios to situations where the velocity was not fixed over a time interval, hence introducing the idea of acceleration. This was an important step in students' development of mathematical ideas before the introduction of derivatives in calculus:

> We believe that the ability to recognise variation in a rate of change is essential for the transition to calculus, and is the point where ratio and rate concepts depart ways. In calculus, all non-linear functions exhibit varying rates of change. We are concerned that placing the emphasis in defining rate as a description of the invariance across equal proportions may create an obstacle to understanding varying rates of change for it tends to neglect the importance of one's experiential understanding that a rate can vary over time (for example, an understanding of the changing speed for a freely falling body). We seek to establish a rate construct that allows one to explain both the uniformity of unit to unit comparisons (homogeneity) and the variation in rates over time (non-homogeneity).(Confrey & Smith, 1994, pp. 154-155)

Mori, Kojima and Deno (1976) identified early childhood conceptions of speed amongst ESL students formed from a combination of practical experience outside the classroom as well as from earlier classroom experiences, in a different cultural environment. Mori etal. used the idea of differential velocity to help overcome misconceptions based solely on which car arrived first, or which car moved the greatest distance in order to overcome some speed and distance illusions. In this case the first arriving car was not necessarily the fastest moving, as the starting positions as well as accelerations of the two cars involved varied. These misconceptions occurring at an early age (the sample was a group of primary school students) illustrates the need for any research study to identify currently held beliefs.

Graphics Calculator Efficacy In Pre-University Calculus

The Movement to Calculus Reform

A movement to reform the teaching of calculus in US high schools commenced in the 1980s. Central to the ideas being suggested was that of an extension and change in the traditional focus on computational or procedural skills with an increasing emphasis on the learning and application of concepts in a diverse range of situations. One consequence of this *calculus reform movement* was that many teachers turned to the use of computer technology with its use of graphics components, tracing of points, tables of values and colour 3-dimensional plotting (Dudley, 1993; Solow, 1993; Thomson, 1994). Traditionally calculus has involved the use of symbolic as well as graphic representation systems with the 14^{th} century contributions of Oresme, who introduced vertical heights above an axis, and the 17^{th} century contributions of Descartes and Fermat with their contributions linking symbolic and graphical translations (Kaput, 1994). As an alternative to the use of computers, graphics calculators can directly link symbolic, numeric and graphic representations, and it is relevant to research areas of calculus teaching that will benefit from their use (Forster, 1997; Joiner, 1998; Kissane, 1995a). The decision to use the graphics calculator for the present study was linked to a number of factors present at the time of the study:

- The availability of relatively cheap, good quality, graphics calculators
- The existence of learning programs (Aplets) which could be adapted for use as part of a new curriculum package
- The educational demand that graphics calculators were mandatory for use in external examinations in Western Australia
- The convenience, portability and availability of graphics calculators compared to organising access to computer laboratories and selecting software.

With the availability of graphics calculators, further questioning of traditional mathematics teaching methods emphasising the use of algebraic methods has occurred (Forster, 1997; Joiner, 1998; Shield & Swinson, 1997). There are two major drawbacks with using solely an algebraic approach: first, the very power of this notation means students in schools will learn to perform mechanical operations without much understanding of the ideas involved; and second, the skills in using symbols are complex, and in memorising these, students in many cases cannot easily transfer to unfamiliar situations. Hence there is a need to extend the classroom teaching methods to include other notation systems in a complementary and supplementary way.

Personal computers with appropriate graphing software have allowed a change in the way calculus concepts were presented. Tall (1991a) demonstrated an example related to the introduction of the derivative. The usual sequence of teaching involves a number of sub-tasks involving the notion of limit, the limit of the gradient at a fixed point, and the value of this limit at other points. A dilemma occurs when the restricted contexts of each sub-task can lead to confusion. A more holistic approach suggested by Tall (1991b), can assist in overcoming students' misconceptions:

> There are two ways to attempt to solve this dilemma: one is to research the cognitive obstacles so that they may be addressed appropriately at a later stage, the other is to use a "deep-end" approach (Dienes, 1960) in which the whole concept is met early on in a rich, but more informal, context designed to offer a cognitive foundation for a more coherent concept image. In the case of the derivative the "deep-end" approach can be done by first considering the informal idea of the gradient of a curved graph through magnification. It is based on the idea that a differentiable function is precisely one that looks "locally straight" when a tiny portion of the graph is highly magnified.(Tall, 1991b, p. 16)

The same holistic approach to problem solving can be used when the personal computer is replaced with graphics calculators. Technology in the form of graphic calculators can allow a range of problem solving activities and increased mathematical understanding through the use of an important learning theory, described by Pomerantz (1997) as *multiple linked representations.* This theory suggests that mathematical concepts can be explored by including numerical, graphical and symbolic representations. The calculator allows students to move freely between the three notations so that each student can be using a different approach. Being able to use a method that best suits the student or the problem at hand causes students to gain improved results (Quesada & Maxwell, 1994). More complex real life problems can be included in classroom tasks because of the technical capabilities of graphics calculators. Examples include the ability to graph an equation and solve numerically for a situation where the solution is algebraically too difficult for students at an introductory level. The algebraic solution of a problem can also be checked graphically and numerically to verify results and bring about a better understanding of a problem, a broader knowledge of heuristics, and greater retention of learning:

> When students are able to choose between several methods of solving a problem, it is more likely that they will remember how to solve it and be able to solve a similar type of problem the next time they see one.(Pomerantz, 1997, p. 18)

Preferably, students will find a consistency of outcomes using different notations, but in a given context there should be individual choice of a representation or notation to use. Flexibility in the use of notations or the problem solving approach is consistent with a constructivist ideology only if there is viability of the mathematical concepts understood, therefore that they are adequate in a particular context (von Glasersfeld, 1995). Student freedom to choose notations must be linked to the responsibility to use an appropriate method:

> There will always be more than one way of solving a problem or achieving a goal. This does not mean that different solutions must be considered equally desirable. However, if they achieve the desired goal, the preference for a particular way of doing this cannot be justified by its rightness, but only with reference to some other scale of values such as speed, economy, convention, or elegance.(von Glasersfeld, 1995, p. 8)

The use of graphics calculators has become increasingly widespread in pre-university mathematics classes over the past fifteen or so years since they're first use in mathematics education. During 1998 in Western Australia the use of graphics calculators in senior high school classes increased exponentially, due to their classification as a required aid in the external tertiary entrance examinations (TEE), approved by the Curriculum Council of Western Australia. There has been widespread success in the application of graphics calculators in mathematics education in recent years, including a report on a study conducted over a three-year period by Drijvers and Doorman (1996, p. 425):

> Observation of the students' behaviour during the experimental lessons supports the premise that the graphics calculator can stimulate the use of

> realistic contexts, the exploratory and dynamic approach to mathematics, a more integrated view of mathematics, and a more flexible behaviour in problem solving.

The sample group for the present study was selected from year 11 *Introductory Calculus* students. Students in Western Australia study this subject to provide an introduction to differential and integral calculus and its applications. This traditional curriculum is limited to functions involving powers of a variable, exponentials or logarithms. The subject *Calculus* usually follows this course in year 12, the year prior to university entrance. This advanced calculus course extends the theory and techniques of differential and integral calculus, first studied in *Introductory Calculus,* and combines them with trigonometric and vector methods, and complex numbers.

HP38G Calculator Efficacy

The present study used the Hewlett Packard HP38G graphics calculator which has captured around thirty percent of the market share in Western Australian senior high school mathematics classes. The tool has the same capabilities as a scientific calculator as well as a library of additional features; the most extensively used being the graphing of functions of one-variable and their derivatives when entered in explicit form. The graphs of these functions can be linked to a range of applications including the calculation of zeros, areas, maxima and minima, and intersection points as well as numerical data in tables. The introduction of specially designed calculator models created primarily for high school use (such as the Hewlett Packard HP38G) has led to an increase in the popularity of graphics calculators for instruction in High School. As part of an innovation over the period 1989-1990 at Clemson University, USA, the HP48 calculator was introduced into most of the mathematics courses, and extensive research was encouraged into its use (LaTorre, Fetta, Kenelly, Nicholson, Proctor & Reneke, 1990). A simpler and cheaper version, the HP38G is now one of the many calculators available for pre-university use, and extensive research continues in to the efficiency and effectiveness of this calculator, including the present study with a sample of ESL introductory calculus students in a pre-university course.

The HP38G graphics calculator has been developed for high school use and enables translation simultaneously between graph, formula and table modes. This can be particularly beneficial with ESL students where translations from verbal instructions and diagrams may be an aspect requiring extra emphasis to assist these students to comprehend a problem. Research has suggested that the instruction in modes of representation is best developed in (symmetric) pairs as illustrated earlier in Table 2.1 (Janvier, 1987b, p. 29). An example is the interpretation of a graph along with the different skill of either sketching or plotting from an equation. This *reversibility* of modes is an important component of understanding of mathematical concepts along with the processes of *flexibility* and *generalisation* according to Krutetskii (Norman & Prichard, 1994). The use of *flexibility* is encouraged by an approach using multiple representations where many ways are used to approach a problem. *Generalisation* is encouraged with an approach where students search for problem solutions rather than being given answers straight away, and require the ability to identify problem details that will lead to selection of a recognised solution approach. *Reversibility* can be used to initiate understanding of a problem and is important to facilitate the sketching of graphs from equations, a skill where students have difficulty.

The permanent memory of a graphics calculator, and its ability to store functions and images is a new feature which earlier scientific calculators used in pre-university did not possess. This feature is very useful for students in developing a repertoire of images to draw on as a guide to problem solving or as an aid in understanding concepts and principles. According to Zimmerman (1991, p. 133):

> The content of calculus provides a repertoire of simple visual images which are as natural and useful to retain as the musician's repertoire of familiar melodies, and which are an effective means of retaining and integrating one's knowledge and understanding.

Hence, to prepare for learning or assessment the storage capabilities are an essential aspect to improve learning or achievement. Each time the calculator is turned off the current work is not lost but returns later. The traditional mathematics tables book can be replaced with specialised aplets, notes and sketches to create an effective learning environment. Research has suggested that with technology tools like computer algebra systems or graphics calculators, more complex problems can be presented to students, so increasing conceptual obstacles and facilitating learning

(Tall, 1991a). Instead of studying the traditional curve: *f (x) = sin (x)* you can introduce *f (x) = sin (x) + abs (sin 100x) / 100*. This second curve has corners when magnified 100 times and can challenge students' conceptual image of a differentiable function, and provide valuable learning experiences about the Zoom features of the calculator. Such an approach to learning can be seen in the instructional materials of Appendix I.

At a cost of around $200, students can own a calculator and use it in school and at home. The specialised Aplet applications which can contain instructional text, sketches and typically follow a sequence of steps that students use at their own pace, can be developed by the teacher, purchased or downloaded from the Internet. Aplets are typically transferred from the teacher's to the students' calculators before a lesson commences via an infrared port. This takes only a few minutes and then an illustration can be given using the calculator interactive screen enlarged using an overhead projector. The importance of Aplets for instruction has been emphasised in the present study where a series of kinematics Aplets were created as part of the teaching package (Appendix J & K). Particle-motion problems have been identified as an area of mathematics where the visual representations of graphics calculators can be an advantage and where the features of the graphics calculator enable a focus on the use of real-world problems (Demana & Waits, 1993). Such a focus on real-world problems is a feature of the present study.

The graphics calculator allows for a new instructional approach as well as a change in the sequence of topics, making traditional textbooks less effective. This is because textbooks usually emphasise the computational at the expense of the conceptual, so the order of chapters may not be suitable for an approach where a student-focus is commonly used, and the use of prepared instructional material and Aplets make the textbook less effective. Textbooks usually lack real applications and give ready access to the *correct* answer, which does not encourage student thinking (Green, 1994, p. 60). Electronic lessons in the form of HP38G graphics calculator Aplets were developed for the present study, the use of which can imply an involvement with more realistic contexts, more exploration, integration across representations, dynamics of use, flexibility in the choice of representations and a broader coverage of multiple representations including approximations and limits. *Realistic*

Mathematics Education (Drijvers & Doorman, 1995, p. 428) capitalises the advantages of the calculator and claims the following advantageous characteristics:

- A variety of solution strategies
- A high degree of student input
- Use of informal strategies and informal knowledge
- Footholds for reflection
- Stimulus for raising the level, for generalisation and for formalisation.

Technology Issues in the Mathematics Classroom

The mathematics education community in Western Australia has readily embraced the use of graphing calculators with their capability of symbolic, numeric, and graphic representations in an attempt to improve student learning in mathematics. Replacing the traditional mathematics classroom teaching approach, where typically only written symbols, text and verbal descriptions were used to represent a problem situation or concept, with an approach that includes the mathematical representations of pictures, graphs and tables, may bring about desirable changes in conceptual understanding. By creating problems with relevance to real phenomenon as a motivator, the present study used alternative representation modes as a further source of mathematical learning.

A full understanding of a mathematics problem from many viewpoints can be an advantage (Chapman, 1997, p.164). Students instructed in the linkage of representations gained better understanding since different representations of the same situation produce similar meaning constructions, but they use different internal structures. The knowledge and ability to apply a range of representation systems is considered essential for the development of mathematical understanding at both the instrumental and relational levels. There is evidence that graphical approaches are particularly beneficial:

> The growing interest in graphical approaches to calculus instruction as a way of improving conceptual understanding accompanies and reinforces the movement away from manipulative skill-based approaches previously associated with calculus learning. When coupled with the increasing functionality of handheld mathematics technology, the role of the graphics calculator in the teaching and learning of calculus assumes critical significance, and the demands for clear and informative research become more pressing.(Penglase & Arnold, 1996, p. 66)

The graphics calculator enables regular use and practice with graphs and this should increase knowledge of function families and transformation properties. With the use of an overhead device for instruction, more lessons may include graphs, and students can be encouraged to use them to commence problem solving rather than relying only on symbolic methods (Ruthven, 1990; Forster, 1997). Students have access to a better quality of information because they can check within *analytic-construction* approaches, where equations are formed and checked graphically before constants or coefficients are changed. This should improve the chances of success and is likely to improve student confidence, provided those students develop an interest in using the graphics calculator tool.

To use the tool effectively in problem solving it is important for students to not only learn the motor skills of the calculator, but also to make sound decisions about when to use them, and combine these decisions with the use of suitable representations for a particular calculus problem. Thus the intelligent use of the tool should become an outcome of education (Kissane, 1995c). Its influence can determine the sequence of topics in the curriculum and the timing of their introduction. Certainly, the ability to find approximate solutions and the maximum and minimum of functions can allow more flexibility in the sequence of instruction in introductory calculus. The present study incorporated these ideas into the instructional material.

There is now more emphasis being placed on exploratory exercises for graphics calculators in the design of calculus texts. Previously, Australian calculus books varied greatly in their presentation style with many using examples followed by extensive exercises (Sadler, 1992, 1998). Barnes (1991) is an exception, where an extensive number of real world problems were provided for investigation. Now the computational ease of use of the graphics calculator (for trained users) allows a much greater degree of contextualisation in problem presentation. This provides the teacher with the opportunity to initiate learning with a greater emphasis on intuition through the use of real applications (Mochon, 1996). Atkins (1994, p. 14) discusses this change:

> Open ended, real problems could be posed with an emphasis on problem formulation and interpretation of results. The distraction of the

computational aspects that accompany these problems could be significantly reduced.

Students need a thorough preparation before they can use the graphics calculator effectively. This was achieved in the present study by using worksheets with real-world problems that encouraged the use of multiple representations in order to investigate any benefits of the use of the calculator. The problems were given in the context of space (i.e. kinematics of motion) with the aim of broadening the appeal of the subject. With the use of graphics calculators the nature of the classroom changes with an increasing use of a student-focused instructional approach. Research by Drijvers and Doorman (1996) suggest five hypotheses applicable when graphics calculators are used in the classroom. Their use:

- Enables the attention to shift from purely algorithmic operations to translating realistic problems into a mathematical model and interpreting the results.
- Stimulates the posing of new questions and the generalisation of problems with "active investigation".
- Encourages the integration of geometric and algebraic activities.
- Visually illustrates the changing behaviour of magnitudes in their mutual relation so encouraging a dynamic manner of observing analytical models.
- Shifts emphasis away from "rigid techniques" towards a more flexible solution procedure, in which a critical attitude is developed with respect to numerical results.(Drijvers & Doorman, 1996; pp. 437-439)

During the present study, student familiarity with instructional material and research methodology occurred during a trial period of nine weeks. During this time a pre-assessment period of calculator awareness was conducted including the use of training worksheets similar to that suggested by Dion (1990), who collected a set of examples from student observations as well as from her own personal exploration.

This trial period was arranged to overcome problems experienced in earlier research where calculator errors may have been due to a lack of training or unfamiliarity with worksheet requirements (Tuska, 1993; Penglase & Arnold, 1996). The role of interviews were also discussed with students along with their format, as well as skills in the transfer of electronic lesson aplets between teacher or student calculators. Students were encouraged to be involved in classroom discussion groups while using calculator worksheets and they were encouraged to express their results verbally, in English or their first language, and in writing.

The integration of the calculator into daily classroom life should occur preferably before the commencement of any research study. Lack of student preparation for the use of the graphics calculator has been sighted as a poor design feature of some past research studies (Penglase & Arnold, 1996). This meant that the effects of students' familiarity with the facilities of the graphics calculator could not be addressed. Whether the results of such studies would remain true if students were conversant with a range of representations as well as the use of any technology used, prior to the study commencement, is uncertain in such cases. This technology-change argument incorporates a number of other variables, the most important being the instructional approach and materials used, and their influence affects the results of research studies. Changes in the type of worksheets completed by the students which differed in form and content from those used previously, as well as any substantial change in instructional style or philosophy, are details that were extensively reported during the present study.

As students learn the use of the graphics calculator they must be made aware of any inadequacies that occur due to the small screen and its low resolution. There can be visual distortions associated with the graphic calculator screen (Kissane, 1995b): First, the graph may not appear continuous, as the smooth curve shape may become a series of ugly lines, each being short and sometimes not connecting. The distortion effect can be improved by using a suitable scale on the axes. Second, a curve may appear to be constant when it is increasing or decreasing in value. This is as much a function of the axis scales as the graphing pixel, and is due to the relatively poor resolution quality compared to a personal computer. Finally, for discontinuous functions, the result may give completely inaccurate values since the poor resolution near asymptotes gives awkward looking graphs. To overcome these problems most calculators make use of dots rather than a continuous curve, along with the option of using a table of values, as well as extensive zoom facilities to fine tune the graph. These problems do not only occur on graphics calculators:

> While the same problems are just as serious on hand-drawn graphs, graphs drawn in a textbook and graphs on blackboards, the scales of

> graphic calculators make them more evident earlier. In any event, students need to eventually form mental images of smooth curves or points of discontinuity, despite the visually defective information displayed on a calculator screen, a blackboard or a piece of paper.(Kissane, 1995a, p. 13)

Confusing the difference between, say, stretching and shrinking when using zoom features involves two kinds of action (Schwarz & Dreyfus, 1995, p. 265): the first kind changes the function itself. Examples are squaring, shift and stretch transformations. The second kind is used to obtain a different view of the same function. Examples are scaling (or zooming) or simply decreasing the x-values in a table. Invariance with respect to properties such as smoothness, concavity and slope can possibly be better understood when these two features can be distinguished.

In one of the major studies directly looking at student errors in graphics calculator use, Tuska (1993) analysed multiple choice examination question results and identified eight misconceptions. Many of these misconceptions were caused by errors in using the calculator window, and in making approximations. These types of errors must be addressed before the commencement of a research study on these calculators. The types of errors observed by Tuska have been summarised by Penglase and Arnold (1996, p. 73):

> These errors were seen to fall into four distinct categories, three of which demonstrated incomplete understanding concerning the domain of a function, end behaviour and asymptotic behaviour of functions, and the solution of inequalities. The fourth was the apparent belief that every number is rational.

Learning and the Calculator

The efficacy of graphics calculator use remains inconclusive, one reason being that many studies have failed to distinguish the use of the tool from the context of its use, so it is not clear whether any positive changes in students knowledge are due to the calculator alone (Penglase & Arnold, 1996). Studies on the role of the graphics calculator in the learning of calculus now take on an urgent priority. Extensive research summaries have been carried out into the use of graphics calculators in pre-university calculus (Andrews & Kissane, 1994; Dunham, 1992; Dunham & Dick, 1994; Penglase & Arnold, 1996; Smith, 1996) and the effects on examination use

(Forster & Mueller, 2000). The need for further research has been identified and the present study aims to contribute to such research:

> To ensure that the use of this tool benefits students' learning it remains important to identify factors associated with graphics calculator use that can inhibit or enhance students' development of conceptual understanding.(Penglase & Arnold, 1996, p. 74)

There are benefits for student learning from the application of graphics calculators. In a study by Ruthven (1990), the method used by students in translating functions from graphic to symbolic form was investigated. There appeared to be improved performance by calculator users on symbolism items amongst the three popular methods of solution but superior performance by the comparison group on interpretation items. It was also found that calculator users tended to use the calculators' graphing facility to repeatedly modify a symbolic expression as a method of solution (*graphic-trial approach*) in preference to an *analytic-construction* approach (constructing a precise symbolisation from information available in a given graph and then altering the variable coefficients in its function) or a *numeric-trial* approach (formulating a symbolic conjecture, trialing this using a small number of coordinates and modifying, repeatedly if necessary):

> The findings illustrate that, under appropriate conditions, access to information technology can have an important influence both on the mathematical approaches employed by students and on their mathematical attainment. The finding of superior performance by the project group over the comparison group on the symbolisation items, but not on the interpretation items, suggests that the treatment effect is not an artefact of the design of the study, but genuinely attributed to the use of graphics calculators in the project classes.(Ruthven, 1990; pp. 431-447)

It is not clear which approach demonstrates a more complete understanding of graphing concepts, but the study above informs their use. Suggestions can be made for an instructional approach that will capitalise on the strengths of a graphical approach. A number of studies by Giamati and Asp, Dowsey and Stacey (as reported by Penglase & Arnold, 1996) have identified the usefulness of a table of function values as part of the process of understanding the function concept as distinct from simply graphing a formula using a graphics calculator. Thus the simultaneous use of numeric, symbolic and graphic forms gave greater understanding.

The present study continues the focus of research on the action/process/object views of function presented through a teaching package to allow ESL students more flexibility in the way they approach problem solving. The focus on ESL students, in particular, gives the present study a new focus compared to much of the earlier research. A large number of students in the present study were found to rely on a property-oriented view of function, which influenced their problem solutions and created a number of misconceptions. Ruthven (reported in Slavit, 1997) describes a study using the property-oriented view of function where the addition of the use of graphics calculators was found to be a positive influence:

> Ruthven (1990) found that students who used graphing calculators were better able to describe a given graph in symbolic terms than those who did not use the calculators, and that their ability to do this relied heavily on their knowledge of functional properties. These students were also better able to identify and distinguish between classes of functions.(Slavit, 1997, p. 270)

There were alternatives to the selection of the graphics calculator for use in the present study, and particularly in light of the limitations that have been mentioned above, there were reasons to consider other technology tools. The clarity of results shown on the graphics calculator screen may be a problem, and a computer spreadsheet was recommended by Forster (1997) as having advantages to simultaneously show numeric, symbolic and graphic data. Her research identified further problems in designing worksheets suitable for all levels of students when supplying a graphics calculator Aplet on exponential equations. Only half the sample group used relational thinking, and the weaker students were frustrated by the task and found the instructional questions too general:

> Displaying all three data representations on a single spreadsheet allows students to observe related changes as they occur and may help students to link mentally the changes. On the other hand, having the representations on separate calculator screens can encourage a "guess and check" approach which is preferable to merely observing the results.(Forster, 1997, p. 55)

Assessment and the Calculator

The use of graphics calculators in examinations has been the focus of considerable research recently (Forster & Mueller, 1999, 2000; Kemp, Kissane & Bradley, 1996;

Malone, Haimes, Forster & Mueller, 2002). Studies on the effectiveness of graphics calculator technology have often been based on traditional achievement tests which fail to incorporate suitable items on conceptual understanding where the calculator may be an advantage. Where a constructivist philosophy underlies the instructional process and may be a contributing factor for changes in student achievement apart from the influence of the calculator, this factor cannot be isolated from the results of the study. The graphics calculator was used in the present study to examine knowledge in at least three subcategories – procedural, relational and translational across representations as recommended by Tolias (1993), and these levels all need assessment. A range of methods were used to supplement traditional test and examinations.

Barling (1994) suggested strategies for the introduction of graphics calculators in mathematics courses after three years experience at Swinburne University of Technology where students had continuous access to a graphics calculator. The suggestion is again made to fully integrate and teach graphic calculator use before research can be effective:

- use the calculator in class as much as possible and demonstrate its usefulness
- integrate the calculator into the course from the first lesson through to the examination
- develop connections between multiple representations
- keep staff up-to-date on what you are doing

In a recent study of assessment items used in year 12 mathematics in Western Australia (Malone, Haimes, Forster & Mueller, 2002) in the course *Applicable Mathematics* (Curriculum Council, 2001) it was apparent that there was considerable variation in the way the graphics calculator was expected to be used in assessment items (which indicated considerable room for the calculator to have a wider impact). The purpose of their study was twofold:

> .. to ascertain the extent to which the availability of the graphics calculator was reflected in teachers' assessment practices; and to develop a means of categorising assessment items that teachers might find of use when developing tasks for their students.(Malone et al., 2002, p. 3)

In related research (Forster, Mueller, Malone, & Haimes, In press) assessment items were collected on the topic of *random variables and their distributions*, an area identified as one where graphics calculators had many applications. Eight high schools were selected and examples of assessment items were analysed. The research noted that:

> For the assessment items that we collected, advantageous and essential use of graphics calculators on test and examination questions on random variables extended to passing integration and equation solving to the technology and calculating probabilities. The form of the questions has not changed but the presence of the calculators has led to the use of a greater range of functional relationships.(Forster et al., In press, p. 12)

New Learning Problems

The introduction of graphics calculators into classrooms on a daily basis can create new learning problems. Swincicky (1994) has identified a number of problems related to the plotting set-up and the autoscale function when students were asked to translate from an equation to a graph. Low achieving students in particular associated the increase of the size of the graph in a graphics calculator window (the zooming in on a particular feature) to be directly linked to making the x and y minimum and maximum ranges bigger, rather than smaller. Students need training in the use of the zoom features of the calculator in this case. The process of producing a suitable graph where features such as turning points and intercepts are present includes, at times, the creation of many different screens. This can be done by a reselection of x and y minimum and maximum values for plotting, or by the use of Zoom facilities. The latest graphics calculators now have the capacity to label axes, which may reduce the difficulties of students in learning this skill.

The advent of the HP38G calculator, which can plot axis scales with the graph, may overcome much of the earlier confusion where students could not see the scales and the graph simultaneously. This process where students are searching for a better graph or even trying to place a graph on the centre of the screen, has been likened to a big spotlight searching for a plane in the sky (Swincicky, 1994, p.95). To make a graph bigger to show clearer details of turning points, or points of intersection needs a range that is smaller (known as zooming in), but preferably closer to the position required. The autoscale feature of the HP38G can help in this task by selecting suitable y values once the x values have been chosen. If the student has limited knowledge of the x range then this function will be of little value, however. These new skills require student training before research commences.

Good examples of learning difficulties with graphics calculator use have been recorded by Dion (1990) and illustrated in Appendix G. A stress test for calculators has been designed to highlight some common errors arising from the misuse or misguided use of the graphics calculator (Dion, 1990). Some tasks on this test either cannot be completed, or will give misleading results. Computationally, the accuracy of the calculator to twelve decimal places can make some calculations involving the combination of small and large numbers inappropriate. An example is $10^{50} + 812 - 10^{50}$ which returns a value of zero when the answer is clearly 812. In graphing applications, students can have trouble finding a suitable viewing screen, especially where a polynomial has many turning points or a curve contains an asymptote. An example from Swincicky (1994) involves the curves: $Y_1 = -0.5\ (x - 200)^2 + 50$ *and* $Y_2 = 20\ x - 3910$. Students are required to find the points of intersection. This requires a series of zooms or range selections. Since the HP38G calculator can only identify one intersection point at a time (the one closest to the cursor) a number of steps are needed to find the solutions. An alternative approach is to redefine a function as a difference between the two curves and search for zeroes in the number table. The 'Stress Test' designed by Dion (1990), was to be used by precalculus students and was designed to make students aware of difficulties solely caused by technical calculator issues, as well as by misuse. The stress test:

- Illustrates some common errors that arise from the misuse or misguided use of calculators.
- Furnishes some examples of tasks that cannot be completed or will give misleading results on a calculator when they are approached in a straightforward manner.
- Identifies problems in finding an appropriate viewing screen. Frequently, the default screen on a calculator generates an incomplete or confusing graph. Even an apparently reasonable viewing screen sometimes produces a graph that defies interpretation.(Dion, 1990, pp. 567-572)

The next section deals with the approach taken for the collection of data in the classroom and considers research where the use of qualitative as well as quantitative methods has been shown to have an advantage.

Qualitative Studies In Calculus Research

Trustworthiness Criteria

As this study develops models of student thinking rather than just the recording of assessment results, methods for the effective collection of data that are reliable and credible, and also which will allow the transfer of results, must be used. The present study introduced a number of criteria recommended for effective research evaluation. The term *Trustworthiness* has been used in the literature to refer to a range of criteria that are appropriate to use in such situations (Guba & Lincoln, 1989; Lincoln & Guba, 1985). Firstly, internal validity or credibility measures can improve the accuracy of specific data items and should include:

- prolonged engagement, persistent observation and triangulation at specific intervals. The triangulation process should involve both multiple copies of one type of source, such as interview data, as well as different sources of the same information, such as interview recollections and transcripts where any differences can be moderated. An independent audit is kept to ensure products are consistent with the data and methods are carried out effectively
- member checking through discussion and interview to check misinformation, preconceptions, relevance, ambiguity, and verification (the *single most crucial* approach to establish credibility)
- peer debriefing - which may involve an external supervisor where discussion occurs systematically on research steps, findings and decisions in order to challenge, design future steps, or legitimise methods
- negative case-analysis to refine the working hypothesis until it covers all known cases. When a reasonable number of cases are satisfied then there is greater confidence in any conclusions
- referential adequacy, such as literature review feedback

Secondly, a measure of external validity or transferability was created in the present study through the provision of "thick descriptions" of the data collection environment to provide extensive research details to other researchers. This measure provides details on the following aspects (LeCompte & Goetz, 1982):

- Selection of the research participants

- Setting - provide sufficient descriptive data about the context
- History - unique features specific to that period
- Construct effects specific to the study

Finally, measures of dependability and confirmability were introduced which involve to a large extent the credibility and audit-trail requirements already mentioned above. Lincoln and Guba (1985) have detailed the importance of the above measures:

> In summary, we believe it to be the case that the probability that findings (and interpretations based upon them) will be found to be more credible if the inquirer is able to demonstrate a prolonged period of engagement (to learn the context, to minimise distortions, and to build trust), to provide evidence of persistent observation (for the sake of identifying and assessing salient factors and crucial atypical happenings), and to triangulate by using different sources, different methods, and sometimes multiple investigators, for the data that are collected.(Lincoln & Guba, 1985, p. 307)

Clinical Interviews

An important aspect of the present study was the use of a range of data collection tools including task-based clinical interviews. There is renewed confidence in the use of qualitative research methods such as student interviews to provide feedback on student understanding (Aspinwall etal., 1997; Denzin & Lincoln, 2000; Ferrini-Mundy & Graham, 1994; Kerlinger, 1986; Ruthven, 1990). In many studies, interviews are repeated on the same students through a course topic over a period of time (Forster, 1997; Swincicky, 1994), and they are often used to collect extensive data in a case study approach with only a few students (Breidenbach etal., 1992; Ferrini-Mundy & Graham, 1994; Thompson & Thompson, 1994). In mathematics research, task-based interviews are commonly used as a means to understand students' conceptual thinking. Although time consuming, interviews can be very informative compared to the usual scientific research analysis into pre-university assessment and instruction which relies on the yes/no correction process commonly used for the analysis of interventions. However, in the case of ESL students, the interview process does not guarantee that the researcher learns about the true conceptual understanding of the students involved as research suggests that there might remain a culture-specific reasoning process that inhibits the ESL student's ability to abstract and generalise:

> The underlying problem is that different types of thinking may lead to one and the same cognitive product, which could be mistakenly perceived as a sign of similar reasoning. If someone enters an unfamiliar cultural or linguistic system the contextual usage of the words may easily form pseudoconcepts. This functional achievement does not prevent the newcomer from continuing to reason in categories that may differ substantially.(Vygotsky, 1962, pp. 164-165)

The extra difficulty of research involving ESL students has prompted the implementation of a constructivist classroom environment for the present study. Radical constructivism maintains that through the use of triangulation methods including clinical interviews an awareness of student thinking can be more successfully explored, and the knowledge the students gain from this process can lead them to do things differently or better (Confrey, 1991). Students are assumed to make sense within their own conceptual framework and the researcher investigates this through the use of flexible probes, such as task-based interviews. An interview approach is suitable if any particular representation of a situation has a conceptual basis and gives the opportunity to investigate both skills-based rote, manipulative learning often prevalent in pre-university students, and relational and translational understanding across representations (Kaput, 1994). The latter skills need to be emphasised as they give the student the ability to generalise and apply techniques in new contexts (White & Mitchelmore, 1996).

Research studies have reported extensively on the use of clinical interviews in mathematics education. Often only a small number of participants are used in a longitudinal approach of relevance to that used in the present study. Separate case reports of ten subjects along with a multicase analysis of data, collected from three one-hour task-based interviews, were recorded by Ellison (Penglase & Arnold, 1996, p.66). Similarly, Confrey (1991) analysed a series of nine one-hour interviews on the topic of exponential functions with one university student enrolled in a remedial algebra course. She later extended the case study to include five other students. Ferrini-Mundy and Graham (1994) used task-based interviews with six students studying calculus. They were guided by the work of Piaget in presenting students with problematic situations and observing their progress.

The research community has encouraged the use of an approach using clinical interviews (Dunham & Dick, 1994) and the present study extends the research by using such an approach with ESL students. A common feature of previous interview studies as with the present study was the desire to understand the processes of student thinking, including the influence of language use. Frid (1993a) used task-based interviews with seventeen students to inquire into the nature and role of students' language use in a mathematical calculus course. Students were presented with calculus problems and gave oral and written responses based around a range of different instructional approaches presented to them.

In descriptions of some earlier research (Goldin, 1993), the interviewer takes an approach where rather than telling students correct answers, all responses can be accepted during the interview. Goldin (1993) suggested this approach with the use of a technique involving forty-five minute task-based interviews with students. In the present study the curriculum limitations meant that this approach was not directly appropriate. Studies by Lauten, Graham and Ferrini-Mundy (1994) used a series of clinical interviews with a small group of students doing an *Advanced Placement Calculus* course incorporating graphic calculator use. Such an approach was used with introductory calculus students in this study. Audiotaping of interviews was used for the present study, an approach recommended by a number of researchers (Aspinwall et al., 1997; Forster, 1997; Geiger, 1998), including Breidenback et al. (1992), who used this technique during interviews in order to study the process of student's thinking:

> To study the process of students' thinking rather than simply the products of their thinking activities, four clinical interviews were conducted with each of six students enrolled in a Calculus I class. During the interviews students were videotaped or audiotaped as they solved a series of calculus problems.(Breidenback et al., 1992, p. 266)

Action Research

The present study applies the theory and practice of *action research* as the teacher/researcher collected and analysed a range of classroom data (McNiff, 1992, pp. 22-33). In terms of method, action research uses a self-reflective spiral of cycles from observation and analysis, planning for change, action, observation and reflection (Carr & Kemmis, 1988). It is a feature of the method that students are involved in a process of reflection on what they have done, as well as in self-reflection or meta-analysis. The reflection process can benefit from discussions with other students involving the challenging of ideas. Students can probe their own understanding by being presented with a greater variety of classroom activities (Hook, 1981; White & Gunstone, 1992) and by being challenged on their current alternative conceptions (Bodner, 1986).

In the present study, data was first collected mainly from ESL students in a pre-university introductory calculus course over a two-semester period. For the main study, data was collected on a similar sample of students involved in collaborative learning in a small class of four students. As students completed assessment tasks, a cycle of steps followed involving the analysis and design of new tasks in order to develop appropriate aspects of the students' calculus problem solving skills and to create models of student learning.

In action research the teacher/researcher assumes primary responsibility for identifying the research issues, collecting data, and analysing and interpreting the data. Those outside the school community adopt the role of facilitator, observer or equal participant (Wong, 1995). In the case of the arrival of new technology, it is necessary to examine the best ways, if any, that this new technology can be an effective addition to the repertoire of possible instructional approaches. In applying new technology, the instructional approach may be involved in structural changes in order to achieve perceived objectives. Structural changes in a number of study variables simultaneously make it difficult to approach analysis from a standard cause-effect method. For the present study, a student-focused classroom environment was implemented in a longitudinal study allowing for the analysis of student case studies. It was necessary to extensively document the study to improve the credibility of any interpretations made from the data.

A Case-Study Approach

The case-study approach has been used by calculus researchers such as Slavit (1997, p. 271) who observed changes in students' view of the function concept as they discussed and interpreted exercises on a property-oriented view of function. Over a two-semester period when case-study data was collected, students used different notational systems as they developed function concepts, and it was possible to trace student development over time. Thompson and Thompson (1994) studied and interviewed a student, Ann, as she worked on and interpreted tasks on rates of change for a period of four days. The teacher's use of language and the student's interpretation of the task were issues central to the study. Instructional actions that adapt to students' thinking were major themes and the interviews were allowed to develop so that teaching inadequacies were exposed. Aspects of this approach pertinent to the present study have been detailed by Lincoln and Guba (1985). The approach:

> 1. Is a primary vehicle for emic inquiry – the naturalistic inquirer tends toward a reconstruction of the respondent's constructions (emic)
> 2. Demonstrates the interplay between inquirer and respondents. Both data collection and analysis, on the one hand, and interpreting and reporting, on the other, are heavily influenced by this interplay
> 3. Allows the reader an opportunity to probe for internal consistency since each new item of information provides another point of leverage from which to test interpretations
> 4. Provides a detailed description so necessary for judgements of transferability
> 5. Provides a grounded assessment of context – phenomena take meaning from and depend for their existence on their contexts, so it is essential the reader receives an adequate grasp of the study context.(Lincoln & Guba, 1985, p. 359)

The present study takes advantage of the benefits of the case-study approach in the main study by the use of a two-semester observation period. The researcher becomes familiar with the sample members and can measure any preference change in representational use and possible influences of language components on achievement. The case-study method allows greater detail to be collected on a few students, and has many benefits for the transferability of results, as recommended in the research of Lincoln and Guba (as cited in Aspinwall et al., 1997):

> Our purpose in conducting a single case study was to provide the *thick descriptions* necessary to enable readers to reach conclusions about whether transfer could be expected in other contexts. This establishment of transferability in a case study is different from the external validity sought in statistical studies. While statistical reports are expected to contain precise statements about external validity, for example in the form of a confidence interval, reports of a case study set out to splice hypotheses together with a description of time and context in which they were found to hold.(Aspinwall et al., 1997, p. 305)

Classroom Environment Surveys

Important research has been conducted in Western Australia to enable the collection of information on student perceptions of the classroom learning environment, not only on its *actual* state, but also on its *preferred* nature (Fraser, Treagust, Williamson & Tobin, 1987). Their research enabled the later development of the CLES-Maths or *Constructivist Learning Environment Survey for Mathematics* by Taylor, Fraser and Fisher (1993). This is an instrument to gain understanding of student feelings about the instructional process in order to measure the benefits of a constructivist approach and act as a basis for guiding improvements:

> Teachers can conveniently use these instruments to measure the success or otherwise of pedagogical practices aimed at enhancing student learning and understanding of mathematics. The instruments usually contain several scales, which may be easily hand scored to arrive at a measurement or score for various aspects of the classroom learning environment. The CLES-Maths allows one to apply the potentially useful tradition of quantitative assessment to this important aspect of teaching and learning. The easily administrable, hand-scorable nature of the CLES-Maths should encourage teachers to assess, reflect upon, and more easily make the transition towards a constructivist learning environment.(Green, 1994, p. 57)

For the present study a survey form related to the CLES-Maths was developed to anticipate student ideas on the efficacy of graphics calculator use in the classroom. Since the instructional approach was changing as the calculators were implemented it was important to measure student ideas on teaching effectiveness. This survey would give a measure of the success of any classroom interventions, and allow rapid changes to be made, in line with the action research nature of the study. The versatility of this instrument made it a convenient tool for obtaining qualitative data on student perceptions. The collection of data from this survey before the commencement of the main study was considered to be an effective method to measure the effectiveness and reasonableness of using a constructivist approach for classroom instruction.

Conclusion And Summary

All too often ESL students are treated no differently from other Australian students in the context of the mathematics classroom. For a number of reasons, not enough is known nor investigated about their intellectual, academic and personal needs. In the area of pre-university calculus it is difficult to locate research directly concerning the needs of ESL students. This study attempted to overcome these limitations by providing a number of case studies on ESL students, along with an appropriate teaching package providing guidelines to teachers. Academically, to develop a better understanding of ESL students' conceptual thinking, constructivism has been recommended to provide the lessons with an appropriate structure.

By introducing real-world problem solving situations using textual and pictorial representations, the present study aimed to both increase the exposure of ESL students to mathematical problem formats where they had some experience, and to encourage them to apply a range of representation modes. At the same time ESL students were challenged to improve their language skills in order to efficiently and effectively solve calculus word problems. To implement this approach in the calculus classroom it was appropriate to apply the methods of the *Calculus Reform Movement* where the use of real-world problem situations is encouraged along with collaborative learning in small groups:

> Some forms of calculus reform seek to challenge the students by providing research projects in calculus that they can solve collectively, using their metacognitive skills and computer-based resources. The students are given control of their learning through exploratory or discovery learning in either computer-based microworlds, collaboratively on projects in small groups, or both. The predominant feature of calculus reform is that of active, constructivist learning. Last, students' curiosities are aroused by contextualising the problems in real-world examples and using computer technology to solve problems; consonant with the way industry uses computers and mathematics to solve problems.(Joiner, 1998, p. 46)

This chapter reviewed the extensive research literature on calculus instruction and considered a number of relevant issues focussing on: the role of constructivism, calculus learning for ESL students, multiple representations, graphics calculator technology, and qualitative research methods. The following gives a brief summary of the information within each section:

Conceptual Knowledge from a Constructivist Perspective

It was important to create an appropriate learning environment with a suitable format for a student-focused instructional process and a choice of real-world problems where the delivery of content is secondary to the attempt to understand learning from the student's perspective (Malone & Ireland, 1996, pp. 125-126). The style of presentation at the start of a section of work, and during classroom instruction, was completely different for the participants in the present study (who worked in a constructivist-style learning environment) than for students in a traditional classroom. This change began with the very first lesson:

> Often the most crucial stage of a unit of work is the initial lesson used to introduce the concept. The constructivist teacher capitalises on this by using this lesson to create a classroom environment that will guide students into making their own connections. A carefully sequenced selection of activities, explorations, investigations and discussions will need to be included in the lesson. These will be in the form of printed resources and manipulative materials which allow students to construct and develop their own understanding of concepts.(Green, 1994, p. 62)

Calculus Learning for ESL Students

More research needs to be done relating conceptual understanding, in various topics in mathematics, with ESL students' linguistic skills and prior learning from different cultural environments. The interview approach used in the present study has major benefits for understanding the way ESL students learn, and in determining their current concept development. The definition and measurement of initial language understanding associated with an appropriate mathematics register is an important part of the ongoing research process suggested by Cuevas (1984) who compiled a list of suitable research questions relevant to the present study:

> 1. How does the verbalisation of a mathematical concept affect the way it is learned?
> 2. How does instruction in language skills that are related to content affect concept formation in mathematics?
> 3. What reading and general language difficulties do selected aspects of school mathematics present?(Cuevas, 1984, p. 142)

Language-based instructional experiences were designed for the present study to provide students with the capacity for reducing their misconceptions as measured in written or oral communication on calculus concepts. The use of approximations when combined with the zoom features of computer algebra systems has benefits for the use of mathematical language by students (Frid, 1993b).

The Efficacy of Multiple Representations in Pre-University Calculus

The earlier research of Tall (1991a) emphasised the need to explore examples which work and examples that fail, using a graphical context amongst others, as a means for students to gain the visual intuitions necessary to provide a pathway to later formal problem solution. The role of multiple representations is pivotal to the present study and is linked to the important idea of *flexibility* of approach by students, which is encouraged with the use of graphics calculators in a student-focused environment:

> *Flexibility* refers to the ease with which a student switches from one method of solving a problem to another method of solving the same problem. From the students' perspective flexibility is exemplified by ease of students' use of guess-and-test, computers, traditional algorithms, manipulatives, and so on. From the teacher's perspective flexibility is developed by modelling it in mathematics instruction and by providing students with a rich repertoire of problem solving strategies and heuristics. From a constructivist perspective, flexibility can be viewed as the ability to create connections between problems and strategies, and to see in new problem solving situations structures previously encountered.(Green, 1994, p. 45)

Graphics Calculator Efficacy in Pre-University Calculus

Results that should encourage further research into the use of technology were contained in a meta-analysis of graphics calculator research by Smith (1996), and summarised below:

- there is an improvement in the ability to learn concepts using graphic calculators
- there is an improved ability to solve word problems using graphic calculators
- there is an improving attitude using graphic calculators
- computational skills are maintained at levels similar to non-graphic calculator users
- no observed improvement in the ability to learn the process of graphing.

Other research has observed the need for students to identify the points of graphs as a different skill to observing the overall trend or Gestalt of a graph (Ferrini-Mundy & Lauten, 1993). Penglase and Arnold (1996) have noted the scarcity, but importance of studies that investigate the use of graphics calculators in the acquisition of mathematical modelling skills. In the area of mathematical modelling, where students have major difficulties, the following recommendations were made (and have been incorporated into the design of the present study) based on benefits arising from students' ready access to graphics calculators:

> Studies of longer duration with students of different ages, engaged in different levels of study, need to be carried out before more definite conclusions concerning the potential of the graphics calculator as a tool for mathematical modelling can be drawn. Such studies must delineate clearly between the effects of tool use and those that result from the learning environment, including the instructional program and the effects of the teacher. It is evident, however, that the graphics calculator may serve as a catalyst, making possible the implementation of programs of study in mathematics that enable exploration and discovery. From the evidence cited in this review it is clear that such a partnership is frequently associated with significant gains in conceptual understanding and achievement in mathematics.(Penglase & Arnold, 1996, pp. 72, 82)

Qualitative Studies in Calculus Research

There is a need to change the classroom ideas on mathematics teaching and encourage teacher/researcher investigations. The diagnosis of student errors can improve conceptual understanding by changing the traditional approach of studying only right and wrong answers. By reviewing errors there is the possibility of relating the student's response to cognitive obstacles of which they were not aware:

> Recognising that errors are acceptable and part of the learning process may require a change in the beliefs held by students. If students hold the belief that "you cannot learn from your mistakes" then they probably perceive that there is no connection between right and wrong ways of doing mathematics. A student holding this belief would ignore any errors and go back to the beginning of a question rather than try to find a flaw in the solution or argument.(Anderson, 1997, p. 9)

In a process known as *diagnostic teaching* (Bell et al., 1987) wherein student misconceptions are identified and tasks involving cognitive conflict developed, the present study collected feedback on students' conceptual understanding. The *roller* *coaster* exercise (Appendix C) is one example of the picture-graph misconception given to students where their responses were discussed at later interviews.

Final Note

It is my view that a learner-focused instructional style was appropriate for the present study and that the instructional style had a direct influence regarding the development, implementation and evaluation of the mathematics teaching package. The constructivist approach of the study influenced the methodology used in the evaluation component within the study. The approach also facilitated the advantages of small group work to be emphasised, wherein students could internalise group talk through their questioning and challenging of others, leading to the checking and possibly changing of thoughts and ideas (Davis et al., 1990). Using a teaching package that recognised the significance of student-student social interaction in the teaching and learning process, real-world examples such as the Pre-test Traffic worksheet (Appendix C) encouraged student discussion and provided opportunities for the development of discourse skills (Cohen, 1994).

Worksheets chosen to emphasise a multiple representational format encourage a focus on language interpretations, including the comprehension and translation of calculus word problems and the writing of conclusions (Ellerton & Clements, 1996). Finally, with the ESL student focus of the present study, a move away from a traditional mathematics classroom to one involving a student-focus has been recommended for second language learners, not only in English classes, but also across the curriculum (Kessler, 1992).

Chapter Three details the research methodology used in the present study. Data collection involved both quantitative as well as qualitative methods including task-based interviews to build a model of student conceptual understanding.

Chapter Three: Research Methodology

INTRODUCTION

This chapter commences with a description of the research methods adopted for the present study and first considers the study's theoretical basis. Justification is provided for first, the establishment of a constructivist learning environment at the research site where both quantitative and qualitative data was collected, and then for the use of a case-study approach as a means of addressing the research questions stated in Chapter One. Following this, information on the data sources, data collection methods, quality controls, and instrumentation details is presented. The chapter concludes with a discussion of the teaching package development process.

The need for reforms in calculus education is highlighted in Chapter One (p. 1-4) and this study examines the effects of a multi-representational instructional approach to the teaching of a sample of mainly ESL introductory calculus students. According to Penglase and Arnold (1996), there is a need for well-designed and well-focused studies in this area, since many researchers appear to intertwine instructional change, teacher variation and graphics calculator use together, making it difficult to extract the real effect of any single intervention. The graphics calculator is now included during assessment in class and external examinations at pre-university level in Western Australia where the present study was conducted. It has therefore become important to understand how best to use this technology.

The *Western Australian International College,* where the research was conducted, has been supportive of all aspects of the study. A large proportion of the students at the college come from Asian countries and these students use English as a second language. These students must not only become proficient with the English language, but they must also adapt to teaching methods used in Australian schools, that are frequently unfamiliar to them. The present study was designed to accommodate the existing curriculum framework of the College, while the instructional approach used in the study was developed specifically for the purpose of this research.

The methods adopted to investigate the research questions posed in Chapter One (p. 11) will be discussed below. The research questions were:

1.Can appropriate instructional material introduced over an extended time in an introductory calculus course enhance ESL students' use and management of multiple representations?

2.Can ESL students' ability to model calculus word problems be enhanced by using a study environment with small group student-centred learning focussing on skills of comprehension and translation?

3.Can a teaching package utilising a representation mode of study be developed to assist ESL students in calculus learning?

To answer these questions it was considered appropriate to combine both qualitative and quantitative research methodologies in an ethnographic approach. The use of a case study approach for research questions one and two will be justified in this chapter. A longitudinal approach (where data was collected and analysed over a two-semester period) was adopted to develop the teaching package as it was believed that this would allow for greater diversity in the data collection, and at the same time strengthen the findings by providing extensive research data.

Background and Literature on Method

There are three parts to this section. The first deals with the special case of ESL students and the use of an appropriate epistemology to facilitate the measurement of changes in their learning. The remaining parts deal with the general research approach, and the data collection arrangements.

The study lends itself to an ethnographic research design because of the nature of the quantitative measurements of achievement represented by interventions over a longitudinal time frame. An ethnography embeds the researcher within a cultural perspective over an extended period. However, as many changes occur within the classroom environment, it was difficult to approach the study from a simple cause-effect stance – as an ethnographic study would imply – because the role of the teacher/ researcher introduces potential bias (Chapter One, p 13). One special feature of the present study included the approach from an action research perspective, where I collected classroom data and used it to develop new instruments. Another special feature was the trialing of graphics calculator worksheets and calculator Aplet programs where the learning outcomes were unpredictable.

Special Case of ESL Students

A student-centred classroom environment (defined below) was considered appropriate with the ESL students involved in this study, since the development of linguistic skills is facilitated when discussion becomes an important feature of student learning. Such an environment for ESL students has been recommended by Morris:

> For the child who has to begin to learn in a second language, whether on entering school or later, the linguistic concepts and structures have to be taught...Unless the linguistic concepts are presented in concrete and dynamic form, the language used by the teacher will only be a mystery to the hearer. This means that mathematics must not be taught by the teacher writing symbols on the blackboard, rearranging them, getting "answers", asking the class to copy the process and to learn it by heart. Instead the teacher must be trained to involve the children in carefully structured activities, investigations and discussions that will ensure understanding. In short, the teaching of mathematics in a second language must, in effect, adopt the principles that govern the methods of teaching a second language as a language.(Cuevas, 1984, p. 138)

This quotation suggests that the use of a small group learning approach where there can be some sense of shared goals or rewards and differing levels of interdependence. The present study was based around a small group of students working interactively (often in two languages) on clearly defined mathematical tasks, but without any clearly defined small group structure other than as a discussion forum. Cohen (1994) has reviewed the use of small groups and the means to make them productive. This often involves the careful preparation of tasks if one wishes the group to share goals, either as achievements or rewards. Typically, groups are involved in extensive discourse without continuous staff involvement. The term cooperative learning has been used extensively in the literature to describe this activity:

> For the purposes of this review, *cooperative learning* will be defined as students working together in a group small enough that everyone can participate on a collective task that has been clearly assigned. Moreover, students are expected to carry out their task without direct and immediate supervision of the teacher.(Cohen, 1994, p. 3)

General Research Approach

Typically, action research in schools involves the investigation of an intervention used in the classroom. The present study was typical in that an instructional innovation such as a worksheet was tried, the outcome was uncertain, changes were made immediately to the lessons based on any feedback, and the results analysed again in a continuing cycle (Carr & Kemmis, 1988; Kessler, 1992; McNiff, 1992). The type of interventions used in the study involved the use of classroom material with a multi-representational focus where graphics calculators were often an advantage. A teacher/researcher approach was used as this introduces a number of distinct advantages. With their ability to react and adapt to student needs, and with their knowledge of the classroom, teachers are in a pivotal position to create and implement research. An entire area of professional development activity has come into use in the research community:

> The movement for teachers to study themselves and take professional responsibility for decisions about educational practice is termed *action-research.*(Hook, 1981, p. 285)

The literature on research into graphics calculator use in classrooms indicates a number of research studies where the methods used have been inadequate. The mix of new instructional approaches, materials, and teacher variability, along with intensive graphics calculator use, leads to the possibility of methodological flaws, making it difficult to ascertain the cause of any changes in student understanding or achievement (Penglase & Arnold, 1996). Additionally, the use of traditional achievement instruments with their emphasis on algebraic aspects of problem solving which do not require the use of the features of a graphics calculator, can hinder the measurement of outcomes which become more appropriate with graphics calculator use. This leads to assessment results biased towards traditional instructional approaches. Examples of topics not represented in traditional assessment instruments include the co-ordination of skills across representations and the interpretation of calculator output. The evaluation of a study becomes difficult if outcomes relevant to the use of multiple representations had little chance of being tested during the assessment phase.

Aspects of the present study made it difficult to isolate cause and effect since the introduction of graphics calculators goes hand in hand with the use of new materials, instructional methods and assessment, thus making it difficult to evaluate the influence of many of the interventions used in the study. This does not necessarily detract from the usefulness of the current research but indicates that results from this study should be qualified with those constraints in mind. The study's credibility has been partly addressed through an extensive documentation process. The term *systemic* has been used to label a classroom environment where the research data collection uses a holistic approach to describe classroom events:

> A systemic paradigm uses whole systems as its unit of analysis, assumes that variables and events are inseparable from each other, and that they are in constant flux.(Salomon, 1992, pp. 249-263)

One disadvantage in a systemic approach is the restrictions placed on forming causal links between study outcomes and interventions. The present study overcame that disadvantage by the use of a variety of quantitative and qualitative measures along with their comprehensive documentation. This assisted the formation of conclusions and provided directions for future research. In any *calculus reform* project such as the present study, it is essential to report on both affective and conceptual information, as well as results from the traditional student achievement instruments. The use of interviews was one approach of the present study to collect information at a broader level, including researcher-developed student learning models.

The present study attempted to provide a formative approach to evaluation in line with details described in a calculus reform study by Joiner (1998). Here it was suggested that a number of questions should be used to frame the evaluation of any calculus reform project. The use of a different technology tool (the graphics calculator) for the present study, rather than the use of computers (as in the Joiner research) does not change the requirement that a broad range of questions are evaluated from the study. These same questions are relevant for any calculus-reform project and the use of graphics calculators was simply an additional classroom environment variable introduced into the study. Appropriate instruments, resources and approaches, mentioned in more detail later in this Chapter, were designed to measure the following list of evaluation questions suggested by Atkins (1994, p. 39):

- Is conceptual understanding a primary focus of instruction?
- Does the approach instil positive attitudes towards mathematics?
- Do students feel the activities and materials are beneficial to their learning?
- Do the course materials reflect a multifaceted approach?
- Are paper and pencil manipulative skills of calculus compromised?
- How accessible is the technology?
- Do the graphics calculator assignments take too long to complete?
- Does the assessment reflect the desired learning?
- Do students demonstrate a relatively deep understanding of calculus concepts?

The above questions need to be asked in any effective calculus reform study and were regarded as an important set for the present study, embedded as they are within the Aims detailed earlier in Chapter One (see page 10).

Data Collection Summary

Qualitative research calls for a number of quality controls, including the triangulation process as discussed in Chapter One (pp. 13-14). Triangulation involves a diverse range of data collection instruments to provide a better sense of verification of the results (see also p.111). As recommended by Fraser (1994) and Atkins (1994), the data collected must include background information on the students, including language, mathematics and graphics calculator ability, as well as information on the researcher and program (*antecedent*). Also important is data influenced by the purpose of the college program (*rationale*); implementation of the program (*transaction*); and effects of the program, including assessment (*outcomes*). The terms *antecedent*, *rationale*, *transaction* and *outcomes* defined above are due to Stake (as cited in Atkins, 1994, pp. 44-45). The data collection thus involves a structured sequence of tasks in line with course and research requirements.

The advantages of using qualitative research methods were noted in Chapter Two (pp. 83-84) where a number of studies reflecting the increasing use of interview techniques in mathematical research were reviewed. The present study adopts a combination of qualitative and quantitative techniques with a sample of pre-university ESL students. One data collection method was the use of individual interviews to give a clearer understanding of student learning, and in particular to encourage the development of relational understanding on calculus topics. Interview data led to changes in research instruments used with the main sample. The use of technology was recommended in Chapter One (pp. 8-9) to increase student exposure to a range of problem strategies and techniques and enable a more effective learning environment. Students in the present study were trained in the use of graphic calculators before this tool was used in the research process, as recommended in a review of recent graphics calculator research studies (Penglase & Arnold, 1996).

THEORETICAL BASIS FOR THE STUDY

This section deals in detail with the research methods and begins with a discussion of the use of action research as one aspect of the classroom environment being created. This is followed with details about the applicability of the research approach to *calculus reform*, a discussion about the effects of triangulation on the robustness of the results of the study, and concludes with a section on the researcher/teacher paradox (relevant when the researcher and teacher are the same person). The data source and data collection methods follow, and include a description of the research site, data collection procedures, student profiles, and interviews. Instrumentation selected and developed for the present study includes information about worksheets, the design of tasks, and the teaching package along with details on learning to use the graphics calculator and details of student scripts. Next, the evaluation criteria for the research study are discussed in terms of both qualitative and quantitative components. Of importance was the overlap between these two approaches, with the qualitative aspects essential for the development of student learning models. Finally, the chapter ends with a summary. The results of the study are presented in Chapter Four.

Action Research and Constructivism

Action research usually involves a structured sequence of steps involving the collection and analysis of a range of classroom data (McNiff, 1992, pp. 22-33). The classroom teacher takes a direct role in the research process using a variety of classroom activities that allow students to discuss and change their understanding of concepts (Carr & Kemmis, 1988; White & Gunstone, 1992). In action research the teacher assumes primary responsibility for identifying the research issues, collecting data, and analysing and interpreting the data (Wong, 1995). This approach is especially applicable when new technology such as graphics calculators makes it necessary to examine the best ways to produce effective learning, including the selection of instructional approach and material.

Since the subjects in the present study were from a small class, I could afford the time to treat each of the four students as a case-study. With this approach it was appropriate to collect data from each student over an extended period, in this case, four semesters. It was also important to create a productive learning environment where students would want to make sense of the material, and where they would feel comfortable in the negotiation of ideas with other students. The use of small groups can facilitate the development of higher order thinking:

> Some researchers advocate small groups because they believe that small-group processes contribute to the development of higher order thinking skills (Noddings, 1989). Noddings sees this school of thought as originating in the work of Dewey and the social constructivism of Vygotsky (1978). Because these researchers assume that such outcomes cannot be achieved without the creation of suitable discourse or conversation within the small groups or without a process of discovery, they define productive small groups as those that are engaged in high-level discourse.(Cohen, 1994, p. 3)

There is a link between the epistemological perspective informed by constructivism and the use of instructional material designed with a student-focus (Chapter Two pp.25-26 and 27 inform us on this point). A learning environment is created where small groups often occur, students discuss results between themselves and the amount of teacher-centred learning is reduced. According to research by Barnes and Todd (Cohen, 1994, p. 5), creating an effective small group learning environment requires student social skills, including the ability to modify and use different viewpoints. Along with the willingness to give mutual support, cognitive skills required in small groups include the ability to construct meaning for a given question, and recreate prior experiences. The pedagogical approach was clearly defined and the effect of the graphics calculator technology could be more easily separated from the classroom context. In other words, interventions involved teaching methods and technology tools at the same time and it was important to at least consider measuring their individual effects on learning. This allowed cause-effect interpretations of the research data. Jost (Penglase & Arnold, 1996, p. 80) linked interpretation of research data to the teaching style when graphics calculator technology was used:

> Jost found that teachers who tended to employ interactive or inquiry-oriented methodologies used the calculators during instruction more often than teachers who used other teaching approaches. She also reported that there was a correlation between specific perceptions concerning the use of graphics calculators and specific teaching approaches. Teachers who perceived the graphics calculator as a computational tool tended to stress content-orientated goals and viewed learning as listening. Teachers who saw it as an instructional tool had student-centred goals and discipline goals, interactive inquiry-driven teaching styles and student-centred views on learning.

Applicability to Calculus Reform

A number of researchers have studied small groups of students over an extended period of time. The extensive review of graphics calculator research by Penglase and Arnold (1996) reports on two studies using a longitudinal approach similar to that adopted in the present study. The 1993 study by Martinez-Cruz (reported in Penglase & Arnold, 1996) followed the progress of eight high school students over a period of nine months as they studied function concepts integrated with the use of graphics calculators:

> An attempt was made to learn more about the effect of graphing technology on precalculus students' concept images and concept definition of functions, the development of procedural and conceptual knowledge of functions, students' application of functions, and on the stages of development involved in gaining an understanding of the function concept.(Penglase & Arnold, 1996, p. 68)

Also reported in Penglase and Arnold (1996, p. 66) was a study by Ellison. Separate case reports were made of ten subjects as they learnt concept images of the derivative. Over a period of time, Ellison recorded three one-hour task-based interviews with each subject, and collected data from tests and class observations. The type of questions being researched in the approaches of Martinez-Cruz and Ellison varied but were in-line with the requirements of a genuine *calculus reform* project. These two studies have in common the integration of graphics calculators with the research program, but due to the close interaction of technology and the instructional program, the individual effect of both aspects on the study outcomes is not clear. The present study complements their approach, but in addition analysed changes in student preferences and knowledge of representations rather than providing a review from a computational skill perspective, or assessing a single intervention.

In this study, after suitable intervals, assessment tasks, and interviews, followed as an extra means of analysing task outcomes. Complex interactions occur in a classroom and the clinical interview approach of Piaget or the facilitating approach of Vygotsky (one aspect of which is the idea of a *Zone of Proximal Development*) involve participation in activities only with the teacher. In a classroom there is a far more complex interaction, with social interactions occurring directly between student-teacher as well as student-student interactions impacting on the learning taking place. The social interaction of the interview was an integral part of the assessment activities, but because a task-based interview with a focus on psychological processes was used, this data collection approach fell within the realm of a neo-Piagetian clinical method perspective:

> From the neo-Piagetian perspective, social interaction is typically treated as a catalyst for otherwise autonomous mathematical development. Thus, although social interaction is considered to stimulate individual development, it is not viewed as integral to either this constructive process or to its products, increasingly sophisticated mathematical conceptions. Vygotskian perspectives, on the other hand, tend to elevate interpersonal or social relations above psychological processes.(Cobb & Whitenack, 1996, p. 215)

According to von Glasersfeld, the elements of the Piagetian clinical method are indispensable for finding out anything about the reflective thought of calculus students (Janvier, 1987a, p.11). The approach consists of following the mathematical development of the students by the use of carefully selected task-based interviews, conducted at regular intervals. In the present study the interest was in both psychological variables (such as interest in mathematics) as well as conceptual learning encouraged by the social interaction between teacher-student and student-student in a classroom environment. The interviews were contextualised by the task, as well as the classroom environment, and the students were asked causative (because) and adversative (but, although) questions. Students were also challenged on tasks that were not completed and the teacher assisted them in a Piagetian role:

> For Piaget, progress in conceptual development lies in the child's co-ordination of actions, movement from physical action to more abstract mental operations and structures, use of a variety of representational systems in accomplishing the internalisation, and reflective abstraction of the ideas. The child is seen to enter with his or her own set of insights, commitments, experiences, and relationships, and these may not mirror those of an adult, but differ substantially. The task of the teacher is to provide the child with the opportunity to build stronger and more fertile conceptions.(Confrey, 1995, p. 211)

The present study involved research into the effectiveness of multi-representational tasks as outlined in the aims of the study. As with other *calculus reform* studies, it was important that students increased their conceptual understanding through the use of a multifaceted approach via the instructional materials. In order to gauge the effectiveness of any new instructional material, however, it is impossible to isolate the combined effects of other influences such as instructional style and the nature of interventions used in the study. To estimate the effectiveness of graphics calculators as part of a study, guidelines have been provided by Penglase and Arnold (1996, p. 62):

> In particular, studies which make claims regarding the effects of graphics calculator use must carefully distinguish between the tool and the context in which it is used, while those that purport to judge effectiveness must make explicit their assumptions concerning both the method and focus of assessment procedures.

To provide guidance to students on the effective use of graphics calculators, appropriate tasks must be prepared and graphics calculators must be used in the instructional approach. A number of studies have used a graphics calculator overhead screen (Aspinwall et al., 1997; Quesada & Maxwell, 1994) and its use has been recommended as a teaching aid by other researchers (Kissane, 1995c). A traditional overhead projector and a new electronic pad allow the HP38G calculator's display to be shown on a screen. This enables the teacher to introduce new features on the calculator, since the student can see the display and replicate what has been shown instantaneously. Additionally, the pad can provide a link between the students and the teacher to facilitate a more interactive lesson style. For instance, after introducing the tangent to a curve close to a point of interest, and illustrating the variable $(f(x+h) - f(x))/h$, the student is asked to try this with values of h reducing in powers of 10, and comment on calculator accuracy as unusual results start to appear. The teacher can complete the lesson by summarising.

The present study attempted to provide course material with real world examples and encouraged the integration of representations. Resources are documented in the Appendices. The evaluation of the present study, discussed later in this chapter, includes a caution to other researchers to be careful when making conclusions based on the interplay of many different factors that may have been introduced. The provision of student scripts in Chapter Four is one aspect of the reporting process that should assist in alleviating the concerns of other researchers about the credibility of the findings.

Reproducibility, Validity and the Use of Triangulation

Underlying the evaluation of traditional quantitative research studies are the measures of validity and reliability. However, the qualitative nature of the present study, with its case-studies of four students, would not normally allow any degree of reproducibility of observations. The very nature of much of the subjective measurements makes repeatability difficult. The approach to case-study evaluation must rely on a different set of criteria (Lincoln & Guba, 1985). With only a single researcher as participant-observer, reliability was not related to factors such as observer agreement, but to a measure of repeatability or reproducibility. To overcome the interpretation bias of a single researcher, extensive details of transcripts were supplied to allow for the checking of conclusions. This would apply to certain aspects of the data analysis more so than to others. This was because the test and examination scripts were a more objective measure that would allow a more effective transfer of results.

Task-based interviews with a student can suggest a range of conceptual patterns which are observable within other students, even though there is a different context, and the interventions may vary in detail. There is a need to compare interview outcomes between different students, and in different contexts if any degree of generalisation of results is to occur. To enable the results of the study to be repeatable and to add a degree of generalisability, detailed student transcripts including text and diagrams as well as summary tables are given in Chapter 4 of the present study. Other researchers then know the exact conditions and contexts that occurred and are aware of criteria used when making conclusions from the data collected. As with experimental research, however, the replication of studies can help improve their validity so the results from this study will possibly lead to similar research, which can lead to a broader range of results from which to generalise.

The consistency of research judgments, known as the reliability, is subject to parameters at times not under control in interventions with individual students (Kerlinger, 1986; Vinner, 1997). Triangulation of the research data is designed to improve the reliability by assisting in the process of making fair judgements using a range of qualitative or quantitative measures. During the interview process it is necessary to identify theU targets ofunderstanding, whether they be concepts, single elements of knowledge, situations or communication skills (White & Gunstone, 1992). The types of data collected in this study were of the form of audiotaped discussions, images, procedural skills such as the use of formulae, motor skills in calculator dexterity, and cognitive skills such as sorting out relevant and irrelevant information. However, the effectiveness of a triangulation process is in doubt in a naturalistic study (one where a multitude of variables may change simultaneously) over a longitudinal time frame where the variables being measured are changing constantly (Guba & Lincoln, 1989). There is often a link between the observed verbal/intellectual ability of the student and their country of origin, but the instructional setting may also influence this behaviour. The interview approach must vary depending on the cultural background of the student:

> In developing an ethnography of a mathematics classroom, it is possible that the questions asked of the various participants would be different. In interviewing students, it may be necessary to access information through a variety of questions, because students will not have well-formed or

> articulate concepts related to the focus of the research. By contrast, other participants may have more articulate responses and hence more direct questions can be posed.(Zevenbergen, 1998, p. 23)

To assist in the design of a study with dependable data and to improve overall credibility of any study with a small sample size, a number of recommendations have been made (Lincoln & Guba, 1985). These include a peer debriefing process whereby ideas are challenged by a senior researcher as the study progresses, and it is clear then why the study emerged as it did. Second, a process of referential adequacy is created where raw data from preliminary assessments and interviews is used to assess dependability and credibility of later results during the study. Third, an extended time period for the study to allow adequate data collection. Finally, a process of member checking is essential where assessment responses are clarified through interviews, opportunities are created for the researcher to correct information and inferences, and a summary of any conclusions can be agreed on.

The present study has been informed by the use of four types of triangulation, where the term *triangulation* refers to the use of multiple methods, data sources and researchers to improve the validity of research findings. The four types of triangulation suggested by Denzin (1978, 1989) are listed below:

> (1) *Data triangulation* has three subtypes: (a) time, (b) space, and (c) person. Person analysis, in turn, has three levels: (a) aggregate, (b) interactive, and (c) collectivity. (2) *Investigator triangulation* consists of using multiple rather than single observers of the same object. (3) *Theory triangulation* consists of using multiple rather than single perspectives in relation to the same set of objects. (4) *Methodological triangulation* can entail within-method triangulation and between-method triangulation (Denzin, 1989, p. 237).

The present study has concentrated on the use of data and methodological triangulation involving one international college and a sample classroom, but using a longitudinal data collection timeframe. A single theoretical approach with a student focus formed the backbone for the study, but a number of data collection methods were used based around a single technology tool, the graphics calculator. While creating its own limitations, this approach to triangulation has a number of advantages that allow for measurement in detail of a whole range of variables not possible in a more general study. An example is the attempt to model student learning approaches over a two-semester period, rather than simply to summarise assessment results.

Apart from the crosschecking of specific data items of a factual nature, the triangulation process clarifies all details about the conduct of this study. Data collection methods used will be explained in this chapter and my dual role of teacher in a classroom, as well as researcher having responsibility for all data collection strategies, will be discussed. The school authority, the Western Australian International College, enrolled the sample students and I was chosen as the most suitable teacher for the *Introductory Calculus* class. All data was collected from this single classroom, which operated under the educational paradigms detailed in Chapter One. Interviews provided multiple examples of student responses and transcripts are detailed in Chapter Four and Appendices D and E. The availability of these scripts enhances a study's credibility and internal reliability and has been recommended by LeCompte and Goetz (1982). Independent researchers would produce similar scripts and hopefully come to the same conclusions provided they approached the research from the same point of view as myself.

Validity has been defined as making appropriate interpretations from the observation and measurement of behaviour, and it is the extent to which we are measuring what we say we are measuring (Kerlinger, 1986, p.488). Validity may suffer when interpretation has created inappropriate conclusions. This was overcome in the present study (where the changing mental constructions of participants were of interest) by incorporating checks on my interpretations of the data. In detailing the ethnography, it was important for me to let the student's point of view be heard when describing events observed in the classroom by giving direct student transcripts of selected field observations. This presents a more valid body of work for external interpretation. Content validity can best be emphasised by allowing readers to make their own judgements from very detailed interview data (Hook, 1981; Kerlinger, 1986). The data collection process in the present study had to conform to usual classroom assessment guidelines. It was possible to conduct student interviews as part of the usual formative diagnostic process where the goal was to assist student learning of calculus concepts, as well as to encourage the learning of mathematics.

Interviews can be used to supplement, clarify, or validate the data gained from other sources. Accordingly, they can be employed to gain access to the researcher and students' impressions, beliefs, assumptions, and justifications of observed events. It is then possible to represent participants in a way that will be seen as fair and true. Thus, the use of interviews can be one aspect of the triangulation process where extra evidence can be collected in an attempt to better understand student learning. In a sense, the verification of the results for the present study have been qualified by a more comprehensive data collection process, involving interviews as one aspect, and this makes the evaluation of the study results more viable. The purposes of interview use have been further summarised by Swincicky (1994, p. 37):

1. As a principal means of gathering information, having a direct bearing on the research objectives. Questions posed include: What does the student know (knowledge and information)? What does the student think (attitudes and belief)? What does the student like/dislike (value and preference)?
2. To test emergent hypotheses and to suggest new ones, that is, as a means of developing assertions.
3. To go deeper into the motivation of respondents and their reasons for responding the way they did.

Two major forms of interview are normally used in research, and these have been discussed extensively by Zevenbergen (1998). The researcher can firstly ask informal and unstructured questions to clarify and understand classroom events. This creates a degree of credibility to observations and enables students to express a point of view as a backup to your interpretations. Secondly, more formal and structured interviews can be conducted at certain points in the research process. In the present study, interviews were conducted during lessons within the classroom, and their function was mainly to clarify students' written responses on mathematical tasks as well as to encourage students to consider other approaches or ideas. Conjectures were made based on interviews, and it was necessary to be explicit about the reasons for these conjectures, including relevant contextual factors. The task-based approach used gave an immediate starting point. An appropriate method for analysis of interviews would be *Content Analysis,* where communications are studied and analysed in a systematic, objective and quantitative manner in order to measure the variables of interest (Kerlinger, 1986, p.477). There are distinct advantages in taking such a thorough approach to interview analysis:

1. The chance of discovering totally unexpected phenomena increased and new topics can be incorporated
2. Develop relevant concept indicators with manipulable variables.
3. The influence of situational variables is incorporated to maximise transferability and generalizability.(Hook, 1981, p. 108)

As well as the student interviews, the present study conducted summative tests to determine the quality of learning. It was vital to give students relevant tasks before the interviews in order to match the assessment goals of the Introductory Calculus course being studied. The evaluation of the student scripts must be conducted with care since the situation is one of nonexperimental research with major weaknesses:

> (1) The inability to manipulate independent variables, (2) the lack of power to randomise, and (3) the risk of improper interpretation. In other words, compared to experimental research, other things being equal, nonexperimental research lacks control; this lack is the basis of the third weakness: the risk of improper interpretation.(Kerlinger, 1986, p. 358)

I have attempted to involve the reader in the present study by the extensive inclusion of summaries of classroom conversations and student comments along with sketches of student work. The data was accurate and truthful, but an attempt has still been made to make the reader feel a part of the study in an interpretative research approach similar to a narrative format:

> Narrative is frequently used in the telling of stories because of its potential to contribute to the creation of understanding and knowledge by seducing the reader into an interactive relationship with the text. The researcher in essence becomes a filter through which personal experience is shaped and given meaning.(Schaller & Tobin, 1998, p. 57)

Credibility of the research findings was enhanced through the combination of the range of methods of data collection mentioned above and the longevity of the study over four semesters. Transferability of the findings would depend on how similar another sample of students would be to the group of three ESL students and one Australian student used for the present study. There would appear to be many common elements between colleges where the same course is taught to mixed classes involving ESL students, so that in many cases the level of transferability would be acceptable. Additionally, many of the interventions were based around tasks that can be used in any *introductory calculus* course, making them repeatable. Further, contextual details (e.g. second language ability) would need to be evaluated by a new researcher before any transferability of results would be recommended.

In Chapter One (page 14) a number of levels of information were suggested in the reporting of a research study by Mathison (1988). The present study focused on methodological triangulation, which employs a combination of methods including teacher observation, structured interviews and written assessments. Using more than one method reduces the chance of misleading results, and gives greater credibility to the research process. Classroom observation is a method open to widespread problems such as incorrect inferences:

> The major problem of behavioural observation is the observer himself. One of the difficulties with the interview, recall, is the interviewer, because he is part of the measuring instrument. This problem is almost non-existent in objective tests and scales. In behavioural observation the observer is both a crucial strength and a crucial weakness. Why? The observer must digest the information derived from observations and then make inferences about constructs.(Kerlinger, 1986, p. 487)

The credibility process involved a range of methods recommended to improve the chance of gaining useful results on student conceptual understanding (White & Gunstone, 1992), including projective methods involving a more unstructured approach (Kerlinger, 1986, p.471). Checks on the accuracy of observations and first hand responses from students required interventions such as interviews. Zevenbergen (1998) has recommended a degree of triangulation with audiotaped interviews that provide a number of advantages. A review of the tapes can be made later in more detail, and events missed in a manual interview are recorded permanently. The following triangulation methods were used in the present study:

- Multiple structured task-based interviews with each student after important interventions. Each was audiotaped and transcripts produced in the appendices. This will help identify nuances within the responses of each student.
- Quantitative assessments analysed, including tests, extended worksheets and examinations.
- Researcher's classroom observations collected during the twice-weekly classes.
- Pre and post-tests of concept understanding, computational skill, heuristic knowledge, mathematical language knowledge and attitudes to mathematics classes and the use of technology.

- Extended tasks on each calculus topic over a four-semester period designed to collect data on representational use, reading, writing and communication skills.

As a teacher-directed research project in an actual classroom, this study will be of value when it presents information that other classroom teachers can readily apply and generalise to their own classroom situations. The script extracts shown in Chapter Four and the Appendix worksheets will provide directly accessible research and teaching tools. The descriptive nature of much of the data required the use of a comprehensive audit trail that makes the research directly comparable and highly transferable (Guba & Lincoln, 1989). Triangulation has been used not just for improved reliability and credibility, but also to ensure correct conclusions and adequate information for making judgements:

> There seems little reason to pursue a triangulation strategy based on the assumption that bias will be cancelled out. Triangulation provides evidence. Triangulation does not make sense of some social phenomenon. The value of triangulation lies in providing evidence such that the researcher can construct explanations of the social phenomena from which they arise.(Mathison, 1988, pp. 14-15)

Researcher/Teacher Paradox

It was necessary to renegotiate current classroom norms, roles, values, and expectations for the teacher and students to integrate the research agenda of the present study into a traditional course. Since the quality of teaching takes priority over the quality of research, the research activities must integrate with the contexts and goals of the fixed curriculum for the *Introductory Calculus* course. A number of researchers have highlighted concerns about conducting research in the teacher's own classroom (White, 1998; Wong, 1995). White (1998, p. 169) discussed his role as a teacher/researcher with a Year 8 mathematics class and expressed concern about combining the multiple roles:

> In making sense of classroom incidents, I was often confronted with *possible* multiple descriptions reflecting the different roles: that of teacher (where the purposes of the lesson, the students' reactions and a description from the teacher's eyes were recorded); that of researcher (reflecting on the teacher-student circumstance and the interaction taking place, putting myself as the subject in the analytic framework of a social constructivist); and that of a collaborator (having discussions of classroom incidents with the observer-academic researcher). Although

> all these possible descriptions made by me overlap, no one description seems to cover the others. In order to accommodate all the aforementioned possible descriptions into a coherent whole, I found that no single existing theoretical framework such as constructivism or social constructivism was adequate.

The concept of a teacher as researcher is still popular (Ellerton et al., 2000; Forster, 1997; Joiner, 1998; Magidson, 1992; Swincicky, 1994; Thompson, 1994) and has been recommended for future research (White & Pegg, 1996). The present study incorporates this concept into the pre-university area to follow its extensive use at university level, such as the research by Frid (1993b) and Porzio (1994). A review of shortcomings in the design methods used in the teacher/researcher approach can be found in Penglase and Arnold (1996) with a discussion of more general issues by Wong (1995). In a study by Cobb and Steffe (1983) three reasons were suggested for the teacher adopting the role of researcher, the most important being the implementation of a classroom environment that is the most suitable context for learning mathematics:

> Firstly, it gives the researcher as teacher the opportunity to understand the significance of students' mathematical behaviour. Secondly, the experiences that students gain through interaction with the researcher as teacher influences their construction of mathematical knowledge and, thirdly, by acting as the teacher the researcher is able to form close personal relationships with the students and help these students to reconstruct the context within which students learn mathematics.(Cobb and Steffe, 1983, pp. 83-84)

Data Source

Research Site

I was completing the research as part of an external qualification but was lawfully bound by the demands of a State Government body to complete a fixed curriculum with the sample students. The college involved in the study encouraged academic excellence in a mature age student environment with few school rules, and where students at pre-university mingled with older students at university level. A degree of responsibility was encouraged, backed up by highly qualified staff, who were given the freedom to implement innovative instructional approaches as demanded by the unique student population of mainly second-language international students studying on an Australian visa.

It was appropriate to supply information about the present study to the school community as changes to classroom lessons, such as the introduction of new material, and the need for the supply of graphics calculator technology equipment, required appropriate approval. An ongoing development in all colleges in Western Australia at that time involved innovations related to graphics calculator use. It was important that the researcher's professional development was seen to be supportive of the school needs, in order for the continued approval of the study interventions.

Sample Details

The present research involved a naturalistic study involving a pre-university introductory calculus class. The human subjects were not chosen randomly. All students in *Introductory Calculus* over a 2-year period were involved, and a pilot study involved four ESL students for two semesters. This assisted in the development of interventions for the main study. The main research data was generated from a sample of three ESL and one Australian student enrolled over a two-semester period (immediately following the pilot study). The students attended the Western Australian International College in Perth, Western Australia and were enrolled in the *Introductory Calculus* course, which involved learning techniques and concepts for functions and their rate of change, along with related graphing and problem solving relevant to function concepts.

Colleagues were aware of the usage of audio recording equipment in some classrooms but there was little impact of this use on other classes. All of these issues were addressed in a systematic and ethical manner to maintain a positive influence on the school environment (McNiff, 1992, pp. 67-72). A number of other important administrative research issues used in the present study have been identified by Fitzpatrick and McTaggart (Hook, 1981, p. 290) who suggested that the researcher should:

- Apply observation instruments with a medium degree of interference (Kerlinger, 1986, p. 50).

- Destroy questionnaires and maintain the anonymity of data by summarising.
- Discuss the research with the students and involve them.

Data Collection

Introduction

This sub-section outlines all data collection procedures used in the present study. Aspects of the data collection process were mentioned earlier in this chapter and are summarised in Table 3.1 followed by details of the data collection steps. These were listed by using the column label categories *Data, Source, When, Data Type* and *Perspective*. As an example, the *Course Content* consists of the introductory calculus syllabus from the Curriculum Council (a State Government education body), which contains the background principles and details of the course. This material was prepared in advance of this study *(Prior)* and contains details such as course sequencing, which is considered to be of a qualitative nature.

The *Perspective* column in the table identifies the nature of the data collection, which includes both qualitative and quantitative aspects as is typical of *calculus reform* projects. Every aspect of the data collection has been included in Table 3.1 from *Course Content* and *Pedagogical Strategies,* to the preliminary data collection of *Current Math Level in Rates, Graphics Calculator Aptitude,* and the *Classroom Environment Survey. Student Achievement* and *Teacher Perceptions* complete the row headings. These latter items were mainly quantitative in nature. For each component in the left-hand column of the table, a number of classifications are listed as one moves across the rows.

The terms *antecedent, rationale, transaction* and *outcomes* are attributed to Stake (reported in Atkins, 1994). The term *antecedent* as used in Table 3.1 refers to background information on the students, including language, mathematics and graphics calculator ability, as well as information on the researcher and program (Fraser, 1994; Atkins, 1994). Also important is data influenced by the purpose of the college program (*rationale*); implementation of the program (*transaction*); and effects of the program, including assessment (*outcomes*).

The terms *prior, during and through* as used in Table 3.1 refer to the timing of a particular data collection episode. The term *prior* refers to any data collection completed before the main study commenced. Of importance also is data collected during the main study (*during*) and data collection on a continuous basis up until thesis submission (*through*). *The Red Box "Traffic" trial* refers to a particular data collection instrument (Shell Centre for Mathematical Education, 1986) used in the early part of the main study.

The use of *Teacher Perceptions* as a data collection category included the collection of information of a descriptive nature. With a sample of only four students, it was possible to use a case-study approach, and to observe changes over time in the student's conceptual thinking and facility with multiple representations. Summaries of interviews and classroom observations were included as an important feature and contributed to the measurement of change in student preferences for representations. Instructional style, resources and assessments were being developed as the study progressed, with the research questions being viewed as exploratory and adaptable to the research progress.

Field notes were summarised in Chapter Four (ie. participant observations and personal reflections of the researcher) consisting of both descriptive and reflective materials as defined by Swincicky (1994, p. 36):

> Descriptive fieldnotes attempted to capture a *word picture* of the events in the setting. These events included conversations and actions. Although the setting in this case may never have been completely captured, the aim was to transmit as much as possible to paper presenting a detailed account of the observations.
> Reflective fieldnotes tried to capture a personal account of the course of the inquiry. Reflective fieldnotes formed the basis of further research. They were used to document problems that were encountered throughout the research and to suggest ways that these problems may be overcome.

Table 3.1: Data Collected

Data	Source	When	Data Type	Perspective
Course Content	Curriculum Council	Prior	Rationale	Qualitative
Pedagogical Strategies	Past Research Observation Handouts	Prior During During	Rationale Transaction Transaction	Qualitative
Current Math Level in Rates	Red Box "Traffic" trial	During	Transaction; Outcomes	Quantitative
Graphics Calculator Aptitude	Worksheets	During	Transaction; Outcomes	Quantitative
Classroom Environment Survey	Instrument	During	Antecedent	Qualitative
Student Achievement	Worksheets Tests Examinations	During During During	Outcomes	Quantitative
Teacher Perceptions	Interview & Observation	Through	Transaction; Outcomes	Qualitative

Data Collection Procedures

The research data collection timetable is shown in Table 3.2 following. Students worked on a teaching package of algebraic, numerical and graphical items modelled around the use of a range of technology and instructional approaches.

While the research study operated in the classroom, assessment of the curriculum continued, forming an integral part of the study with assessment strategies including:

- paper-and-pencil tests and examinations
- worksheets on technology use
- structured interviews with students
- collected samples of students' independent work

Table 3.2 Chronology of the Data Collection Procedure

Month	Procedure
February 1996	Trial period of calculator instruction and instructional experimentation
May 1996	Start graphics calculator trial and instrument selection, design and testing
December 1996	Graphics calculator questionnaire survey
February 1997	Four case studies commence. Use of Red Box "Traffic" trial
March 1997	Start interviews A graphics calculator aptitude test based on work by Dion (1990).
April 1997	First Pencil and Calculator Test. Second Interviews and Post Interview Discussion
May 1997	Worksheet: acceleration of a car with Aplets
June 1997	First examination
August 1997	Worksheet: motion of a ball under gravity + aplets
November 1997	Second examination

The study commenced in February 1996 with the introduction of small group discussions followed by graphics calculator instruction and the development of instruments. The concluding procedure was the final examination in *Introductory Calculus* held with the main sample group in November 1997. The table illustrates the longitudinal nature of the study and the variety of data collection procedures from interviews and questionnaires to tests and examinations.

Longitudinal Student Profiles

A longitudinal study such as the present one enables the researcher to observe student behaviour and determine whether changes were due to patterns of student thinking or rather one-shot phenomena influenced by extraneous factors (Farrell, 1996). The analysis of audio recordings of classroom interviews continued over time and involved individual students and myself. Interviews were conducted as part of the normal social life of the classroom and in a manner designed to minimise disruption to other classroom activities. Combined with the assessment tasks and other data collection instruments, the recordings attempted to help achieve the first two aims of the research study: to investigate ESL student knowledge about the use and management of multiple representations in a calculus course, and to observe the effects of the study on student classroom communication about, and ability in, the modelling of calculus word problems. As part of these aims, students were encouraged to describe their experiences orally and in writing – a recommended approach for ESL students suggested by the Australian Education Research Council (1991).

The ethnographic nature of aspects of this study implies the systematic use of data on individual students in a cyclical process where provisional conjectures were continually tested (Cobb & Whitenack, 1996). Trustworthiness of the findings then depends on whether each step has been reasonable and justifiable given the researchers' interests and concerns. It is possible to backtrack through the interviews and assessments to check this. This record of the process of developing an analysis contributes to a degree of empirical grounding. By creating a framework that enables the data analysis to be verified over a period of time creates this sense of grounding. Additionally, the first-hand experience of dealing with a small group over an extended period constitutes a crucial source of insight with which to account for their activities.

Interviews

The decision to use interviews is drawn from the constructivist theme of the study. Underlying the constructivist paradigm is the view that students bring different experiences and perspectives to the classroom and naturally the interview can give the researcher access to student thinking approaches. It was important that student perspectives be identified in order to achieve a consensus of opinion on learning priorities. von Glasersfeld (Wheatley, 1993) has noted the prominent role of negotiation as a means to sense making in a classroom environment. Interviews assist this process with the teacher interacting and creating new models of the student's mathematical knowledge. As part of this process of developing student learning models, participants need encouragement to seek conceptual understanding:

> Conducting clinical interviews with students of all ages has led me to believe that rule-following characterises most students' stances and it is a rare student who attempts to make sense of school mathematics.(Wheatley, 1993, p. 129)

Individual interviews were used in the present study and helped isolate the perspectives of ESL and non-ESL students. Interviews about past events were the focus of this research, typically about recent calculus tasks. Audio recording was used, as it is inexpensive and gave a permanent record of words spoken, pauses and the clarity of the language used. However, audio recordings cannot record teacher non-verbal behaviour and student activities. In the transcription process, a classification system was needed to describe the situation presented to the students, the type of question you have put to them, and the type of response given by them. The task-based nature of the interview process helped give the audio recordings structure. It was necessary to record actual student statements, my discussions with the student, as well as provide a data resource to support my interpretations of what was said along with appropriate diagrams to help explain the situation (refer Chapter 4 and appendix D & E).

Inherent in the form of the interviews used were the theoretical underpinnings of the *Calculus Reform* approach. The aim was for students to understand calculus concepts rather than show mastery of computations. This produced a need for students to understand not only each representation type, but also their interaction through translations, and the advantages of each representational type in a particular problem situation. Aspects of the interview questions were based around an appropriate task, which was usually rich in written, graphical, pictorial or symbolic information. My focus was to learn more about the students' understanding of the representations presented to them. The interview format may sacrifice some of the speed with which the subject discusses their understanding of a problem, but this was outweighed by the need to understand student thinking in preference to speedily presented solutions. As the interviews were conducted in a school classroom, certain contextual constraints were evident and it is important to note these in the transcripts:

> Interviews do not take place outside of a social and psychological context. We observe that the child's expectations of an interview are influenced by the fact that it is conducted by a relative stranger (the

> clinician); it takes place in school (and thus might involve some kind of test that "counts" towards an evaluation, and the tasks are likely to have "right" and "wrong" answers); it involves tasks unrelated to a goal or purpose generated by the child; it may be taking place at a moment when the child is alert, tired, hungry, distracted, or excited; and so forth. Seemingly small, contextual aspects of the tasks themselves may have important effects.(Goldin, 1993, p. 307)

With any interview approach it is necessary to remember that attempts by a researcher to interpret student responses is simply a personal interpretation, within a given framework of student perceptions of their own and other learnt experiences. This should be kept in mind with any analysis of scripts. Apart from this bias, sample scripts are one of the most valuable sources of information for other researchers, whether they contain actual interview details or represent a summarised transcript. An example is a script from Lauten et al. (1994) who conducted a task-based interview with a first year university student, Amy, about her concept view of function and have identified inconsistencies in her use of concepts:

> When asked if f (1)=2, f (3)=6, and f (5)=9 represented a function, Amy made a table of values. In functional notation and in a table of values, Amy seeks out and sees clear distinction between x and y values. However, in the limit interview, it appears that once a diagram is drawn, Amy is content to allow x and y values to lose their distinction. Rarely did she refer to either axis to identify a value, whether it is the x value or y. The symmetry inherent in the Cartesian plane may be strong enough to carry over to Amy's treatment of both variables.... As a consequence of this equal-handed treatment of x and y, she did not seem to see a y value representing a vertical height on a graph.(Lauten et al., 1994, p. 231)

Interviews can be time consuming and open to control and interpretation by the researcher. Their effectiveness is influenced by the limited vocabulary of some students to express mathematical concepts in words. This was particularly relevant in the present study with ESL students. Regardless of these problems, the interview method is a much appreciated data collection instrument and is the most direct method of assessing a student's understanding. Its purpose is to find out as much as possible of what the person knows about a concept, so that this recorded or transcribed knowledge can be analysed to give measures or impressions of the person's understanding (White & Gunstone, 1992). Geiger (1998) used interviews to determine the views and attitudes of students. There appeared to be a number of immediate benefits that arose from the use of the classroom interviews in the study, for example:

- The freedom to teach during the interview, which could involve the introduction of new representation ideas directly to the student. This allowed me to focus on the individual student's area of need.
- Flaws in my explanation style were evident during the audiotape analysis. Changes were made which would affect the later data analysis (a changing explanation style would affect student understanding).
- Enthusiasm generated in the students as they were receiving very close attention and feedback. There was a chance to improve their marks, since a more accurate assessment of their skills could be made and the students were given the opportunity to redo work in cases where they misunderstood the questions.
- Student's learning style and approach to checking for inconsistencies were identified (White & Gunstone, 1992).

Instrumentation

Introduction

Our measuring instruments had to cater to the students' second language background as well as the relative novelty of the graphing calculator technology. Published tests and worksheets were used if they were deemed suitable for a multicultural context (Dion, 1990; Shell Centre for Mathematical Education, 1986). A description of the instruments and how they were developed follows. The action-research nature of the study necessitated the development of instruments as the study progressed. Trials of these data collection instruments were conducted with a small group of ESL students in a two-semester trial before the main study (a different group to the case-study one). The resources from textbook writers were used where appropriate (Andrews & Kissane, 1994; Barnes, 1991; Beckmann & Sundstrom, 1992; McCarter, 1992; Sadler, 1992). Exercises from other researchers were adapted for use with the latest graphics calculators (Borlaug, 1993; Ferrini-Mundy & Graham, 1994; Forster, 1997; Ruthven, 1990; Swincicky, 1994; Thompson, 1994).

Worksheets

There are a number of criteria for designing worksheet questions where a multi-representational format is used. For one, it is important to be aware of the demands of graphics calculator use, such as the need to view several screens to comprehend details. Unfortunately, research has shown that students readily accept whatever is displayed on the window of the graphics calculator (Aspinwall et al., 1997; Buecher, 1997; Goldenberg, 1988, 1991; Lauten et al., 1994; Steele, 1995; Zimmerman, 1991). There are also special problems with the representation of singularities (or asymptotes in situations where a function is undefined) on a graphics calculator due to low calculator resolution. Apart from interpretation misconceptions caused by factors such as symmetric or non-symmetric scales or geometric transformations, questions for worksheets must be related to the research objectives, involve an appropriate mix of representations and be clear and unambiguous (Kerlinger, 1986, p.444). Tuska (1993) summarised the results of research into calculator-associated misconceptions in one of the few studies in this area, but his use of a multiple-choice response format made the results unconvincing:

> Errors were seen to fall into four distinct categories, three of which demonstrated incomplete understanding concerning the domain of a function, end and asymptotic behaviour of functions, and the solution of inequalities. The fourth was the apparent belief that every number is rational.(Penglase & Arnold, 1996, p. 73)

Research suggests that students of calculus need preparatory studies on analysing increase and decrease patterns in numerical data, especially sequential data found in real-world applications such as plant growth or the number of people in the school canteen (Kaput, 1994). Exploration of the representation of quantity and change in quantity using symbolic, numerical and graphical notations was recommended to take students from an intraoperational level where they conceive and compute differences to a situation where differences are treated as entities themselves. This can best be emphasised in worksheets by a range of real-world applications leading to a context independent understanding. The initial exercises in the present study focused on real-world traffic scenarios and involved the concepts of change and difference (Appendix C).

Design of Tasks

The initial material used in a Pre-Test Traffic worksheet was selected from the Shell Centre *Red Box* material (Shell Centre for Mathematical Education, 1986). These materials were designed to create a method of teaching that encouraged the discussion of concepts between students:

> The materials in the *Language of Functions and Graphs* are written with the objective of provoking reformulation of children's ideas of graphs. The material focuses on the known errors or misconceptions by beginning with a rich exploratory situation that contains a conceptual obstacle. The questions are deliberately posed in such a way as to allow the errors to come to the surface. This is done by exploring the task in pairs or in small groups, consulting and discussing with each other, and working towards group consensus.(Bell, Brekke & Swann, 1987, p. 50)

Students were given a range of extended tasks to be completed in class and at home using the *Traffic* component of *The Language of Functions and Graphs* (Bell, Brekke & Swann, 1987). The idea was to prepare the students for later work on rates of change by introducing them to a range of representations, involving a lot of sketching and real life situations, including a written descriptive component. Of particular relevance to the study was the style of presentation which included the point form of a function. This has been recognised as being commonly missing from many instructional approaches where graphic calculators are used (Ferrini-Mundy & Graham, 1994). The difference technique of W.W.Sawyer was used, explained in his book *What is Calculus About?* (as cited in Kaput, 1994), by providing students with snapshots of motion from a helicopter camera:

> We are going to investigate speed, the speed of a moving object. How can we see clearly what a moving object is doing? We might make a "movie" of an object moving along a straight line. Suppose we have a camera that makes a picture every tenth of a second.(Kaput, 1994, p. 109)

Heights can be used on a graph when interpreting the camera snapshots of cars, as quantities are assumed to be more easily represented by line segments than by points, and by lengths rather than by locations. This original idea dates from Oresme and was developed further by Galileo, Barrow, and Newton (Kaput, 1994, p. 101). The data can very easily be organised on a graphics calculator as a table of values. Students were asked to translate information a variety of ways into a number of representations including the creation of written summaries. Speed and acceleration calculations were included with the introduction of the derivative at a much later stage of the study.

Each task had a discussion component in which questions were asked about the reasons for a particular approach. It was felt this was an important aspect of learning for the ESL students involved in the sample, since it would develop skills in expressing mathematical concepts. Students were later interviewed about the results of their tasks, and these interviews were audiotaped for later analysis. Comprehension of tasks was an important aspect of the research and the preparation of tables and graphs was common to many of the tasks, rather than the use of equations. This latter representation was to be discussed in class later using gravity examples involving the distance and velocity of a falling body. There was concern, at this stage of the study, whether graphics calculators would increase or decrease discussion in the classroom. When students did not understand what was required for a given task, and did not resolve this during lessons, they had the opportunity in interviews to discuss this, and were given the chance to repeat their work and change the assessment results. In this case it was felt the ESL students should be given extra opportunities in reading, comprehension and translation by introducing such an individual approach. This also gave me the opportunity to understand their reasoning where misconceptions had occurred.

I observed and provided assistance, during instruction, with students who discussed problems and received help from other students. The task situations were presented as a worksheet involving a real world scenario. The first task involved a picture of three cars, two moving and one stationary. Police photographs were taken at regular intervals (snapshot blanks) from a helicopter as a number of cars sped along a road at constant speed. A *ticker-tape* graph sheet was used with time on the x-axis and distance on the y-axis (Appendix C). Each snapshot corresponded to one point on the graph. Photographs were taken of the scene every second from the moving police helicopter as the cars changed position. Students were asked to plot points that were distance-time co-ordinates. The dots formed lines and the intersection of lines represented cars crossing. No use of an action view of function using equations was suggested at this stage (see Chapter Two pp.64-67).

During the first task students were not given equations directly and were not required to use graphic calculators. The aim was more to encourage students to use a range of representations while they familiarised themselves with fundamental calculus ideas on changing quantities. The six tasks each covered a new calculus idea, or considered ideas from the viewpoint of different representations. Tasks needed to be well designed to be effective and empirical research continues into the analysis of task transcripts (Cobb & Whitenack, 1996). In a review of task design for small groups, Barnes and Todd (as cited in Cohen, 1994) report on a study indicating some important task design considerations that were taken into account in the present study:

> Differences in the transcripts between groups carrying out different tasks led to the observation that the degree of unfamiliarity of the task to the students should be considered so as to keep the amount of uncertainty manageable. Other task dimensions that the investigators saw as important were how loosely or tightly structured was the task and whether there were one or multiple solutions to the problem. They (Barnes & Todd) also mentioned that having some concrete object for students to manipulate could make a difference in the effectiveness of the group.(Cohen, 1994, p. 6)

The Teaching Package

The present study examined the effect of multiple representations on the acquisition of mathematical modelling skills by ESL students. The study included the development, implementation and evaluation of a teaching package that emphasised conceptual understanding and small-group tasks by way of a multiple representation format. Tasks were designed for ESL students and prepared with the understanding that graphics calculators would be available. There is an increasing belief that this technology will assist the learning of conceptual understanding in mathematics by allowing the introduction of more diverse problems involving real-world phenomena (Swincicky, 1994). Using real-world situations involving the position, speed and acceleration of cars and falling bodies, students went through a problem solving framework similar to that suggested by Schoenfeld (1994), who suggested an eight step process of *read, analyse, explore, plan, implement, verify, introduce new information, and local assessment*. The development of mathematical modelling skills with real-world problems is very difficult and an area worthy of research when the graphics calculator is available:

> Studies of longer duration with students of different ages, engaged in different levels of study, need to be carried out before more definite conclusions concerning the potential of the graphics calculator as a tool for mathematical modelling can be drawn. Such studies must delineate clearly between the effects of tool use and those that result from the learning environment, including the instructional program and the effects of the teacher.(Penglase & Arnold, 1996, p. 72)

Research questions two and three focused on whether ESL students' ability at calculus modelling skills of comprehension and translation could be enhanced by a package utilising a representations mode of study. The teaching package contains a number of sections:

- introduction to the capabilities of the graphics calculator (Appendix H)
- detailed overview of graphing using the calculator (Appendix G)
- representations on the calculator (Appendix I)
- a set of graphics calculator Aplets (Appendix J, K)
- a pre test of representation skills (Appendix C)
- worksheets on velocity problems (Appendix C, F, I)
- a set of solved and unsolved velocity problems (Appendix F)
- a post test of representation skills (Appendix C)
- tests and examinations (Appendix C)

Learning to Use the Graphics Calculators

In any research study where graphics calculators are involved it was an important step to adequately prepare students in graphics calculator skills. Penglase and Arnold (1996, p.64) discuss the results of earlier research studies, and they comment that prior learning of graphics calculator skills before a study commenced is to be recommended. It was expected that results found during the research would depend on the technical skill level at using graphics calculators that the students possessed prior to the study. An integral part of the teaching package was detailed instruction on graphics calculator use. This was included in the present study in order to ensure effective research findings, since students need an extended period of graphics calculator preparation preferably before a study commences. Penglase and Arnold (1996) report on a study by Giamati, where she researched the effect of graphics calculator use when students studied families of functions and their transformations.

Students with partial or poor understanding of the relationship between graphs and equations were distracted by having to learn how to use the graphics calculator graphing facility. In her study, students with access to the calculator over a five and a half week period appeared to have worse results on an assessment. Unfamiliarity with certain calculator features was thought to be one reason for the graphics calculators' lack of effectiveness in that study, and this outcome suggests that effective calculator training lessons should be integrated into any teaching package in order to overcome the shortcomings of such earlier studies.

Students in the present study were presented with a range of calculus tasks in order to improve their technical skill with the graphics calculators. One example is shown in Appendix I where students were presented with a question from Sadler (1992, p. 108), which required them to find the rate of change of a mixed function with a quadratic and rational part. On a worksheet they were guided through a series of graphics calculator screens. These had been produced using software and a connecting cable designed by Hewlett-Packard. This allows the graphics calculator screen to be transferred to a personal computer as a bitmap, and later transferred to a Word document. One student, Ricky, entered the function equation as $p=5r^2-6/r$ instead of $p=5r^2+6/r$, which produced a mirror image of the function. Why the mirror image occurred was not instantly clear and showed how the teacher must be required to think on his/her feet in these situations. Each student was using a variety of zoom approaches and the numerical results at the minimum point showed the rate of change was not exactly equal to zero, although a zero result was expected. This was a point for both confusion but also discussion.

It was necessary to ensure students clearly understood the developmental nature of the instructional material they were to receive and the type of classroom interactions that would be acceptable. To ensure adequate details were collected on student learning models, students needed confidence in expressing their ideas openly – either in the classroom or to the researcher during discussions or interviews. Tests and examinations allowed the use of the graphics calculator, which can then have an influence on examinations results (Barling, 1994; Berger, 1998; Boers & Jones, 1994; Forster, & Mueller, 1999, 2000; Jones & McCrae, 1996; Senk, Beckmann & Thompson, 1997). Boers and Jones (1994) have analysed the influence of graphics calculators in University examinations by choosing a sample of student examination scripts from a one-semester *Introductory Calculus* course and found that the integration of algebraic and graphical information was difficult for students, especially when errors were made in calculator input. The effect of input errors was noticed in the first semester trial classes for the present study, evidenced in the work of the student called Ricky (Appendix I).

Trials with a year 11 *Introductory Calculus* class (using a Western Australian syllabus) commenced in February 1996 and graphics calculators were first introduced into the calculus lessons in May. At this time students were loaned a calculator only during class time and there was no teacher access to a calculator overhead projector panel. There were three boys and one girl in the group, with two students from Indonesia, one from Singapore and another from Hong Kong. Some tests developed at this time were used again in the main study. Qualitative data was collected with the trial students (Appendix A, B) and showed a positive attitude to the lesson format. The different instructional approach used, which involved new and varied material with a real-world emphasis, may have been one factor to cause this. Da Ponte used a similar approach in determining changes in student attitudes:

> Data on students' views and attitudes were collected by means of classroom observations, interviews with teacher and students, and a questionnaire designed by the class teacher.(Penglase & Arnold, 1996, p. 77)

Student Scripts

The study was conducted over a four-semester period with the first two semesters mainly being a preparation and instrument-testing phase. During the main study student test, worksheet and examination scripts were extensively analysed. A post-assessment interview was conducted after some of these assessment items. In line with the descriptive nature of aspects of the present study and the goal of producing reliable and credible results, many of the script details have been provided in Chapter Four and the Appendices so that researchers can make their own interpretations and be aware of transferability issues.

The Evaluation Process

The general aim of the present study was to develop the use and management of multiple representations with pre-university ESL calculus students in order to improve knowledge and communication when modelling calculus word problems. This called for an answer to the research questions based on the implementation of several instruments discussed earlier in the chapter. The following sub-sections describe the processes undertaken to obtain the answer to each research question. Specific details of the evaluations are discussed in Chapter Four. The naturalistic nature of the present study emphasises the importance of the development of the research questions over time as new data is collected (Hook, 1981, p. 107). During the pilot stage of the study this was possible with the research questions developed prior to the main study taking place.

Research Question One

> Can appropriate instructional material introduced over an extended time in an introductory calculus course enhance ESL students' use and management of multiple representations?

Research Question Two

> Can ESL students' ability to model calculus word problems be enhanced by using a study environment with small group student-centred learning focussing on skills of comprehension and translation?

Research Question Three

> Can a teaching package utilising a representation mode of study be developed that assists ESL students in calculus learning?

Quantitative Measurement Instruments

Students in the present study were administered a range of quantitative instruments as part of the school's approach to evaluate their mathematical knowledge. These instruments were given to the students at different stages of the study, as indicated earlier in Table 3.2. These assessments were necessary in order to meet the official requirements of the course of study, rather than being specifically designed as research instruments.

Pre-test traffic worksheet.

After the commencement of the third semester of the study a worksheet on pre-calculus concepts and preferential learning styles involving six tasks was given to the sample group. The worksheet was selected from *The Language of Functions and Graphs* (Shell Centre for Mathematical Education, 1986) and covered the topic of *Traffic*. This worksheet was chosen because it contained important features relevant to the study, including the extensive use of multiple representations. Students were guided through tasks of increasing difficulty but each task required knowledge of a range of representations. Students were presented with a range of representations to encourage reversibility (e.g. movement from graph to equation or the reverse), and shown the properties of a representation in different contexts leading to generalisation. These three features of flexibility, where students choose a solution format, reversibility and generalisation were recommended by Krutetskii as essential for mathematical learning (Norman & Prichard, 1994).

These extended worksheet tasks seemed ideal for small group discussions, and more importantly, they used word problems which were considered important for examining the ability of the sample of mainly ESL students to comprehend and translate problems. Student tasks included the interpretation of diagrams of car movements based on velocity and acceleration scenarios, creating distance-time graphs from word problems, and generalising properties of distance-time graphs. Also included was the sketching of distance-time graphs from pictures of cars involved in traffic situations, interpreting and creating *Snapshot Blanks* or plots based on photographs taken of a scene every second, and the creation of individual scenarios, where a student creates, for example, their own traffic pictures and analyses them. During this process the students' preferred learning styles were noted in terms of their preferred representational setting. The equivalent of a verbal and visual learning style inventory was therefore performed as a prerequisite to the research project, as suggested by Kirby, Moore and Schofield (Dawe & Anderson, 1993, p. 225).

Exponential function worksheet.

A worksheet on the development of the exponential function (McCarter, 1992) used the traditional idea of gradients of a decreasing interval – an area where students often have conceptual problems in calculus courses. It was possible to use the zoom features of the graphics calculator to visualise this concept in a more appropriate way (Frid, 1993b; Tall, 1991a, 1991b). This was followed by an in-class investigation under test conditions. The graphics calculator provided the background for analysing the effects of coefficients in the exponential equation.

Achievement test.

Many researchers do not redesign test items to take into account the new skills being tested when new representations become important or during the graphics calculator technology introduction. Penglase and Arnold (1996, p.67) quote studies by Alexander and Rich where function concepts were studied using algebraic and graphical means. Their testing instruments were not designed specifically to identify this type of learning. This can have the effect of either strengthening the positive results of graphics calculator use, or instead leading to better results for a control group who were adequately prepared for the traditional style of test. Test instruments should be designed primarily for the type of presentation format their lessons covered. Test results for these researchers become difficult to interpret:

> ...One wonders whether the results of the post-test would have been substantially different if a portion of the questions within the test had been devised specifically for the study of algebra using graphical rather than conventional, or algebraic, means, and for the testing of conceptual understanding rather than procedural skill.(Penglase & Arnold, 1996, p. 67)

Inconsistencies in research results can be caused by a number of factors, including instructional approach, assessment procedures and the extent of prior use and training with technical tools such as graphics calculators. The graphics calculator introduces a number of specialised features such as scale selection on the x and y-axes, emphasis on connections between symbol/graph representations, zooming capabilities to find graphical solutions, and the ability to solve more complex problems. These features may not be integrated into traditional assessment instruments.

The types of questions being asked in assessment tasks for the present study were selected to ensure a range of representations were being used. Research by Ruthven (1990) provides an example of suitably classified problems where half the exercises were designed to test whether the graphic calculators influenced performance on graphing tasks where the translation to a suitable equation was required. These tasks related to earlier historical work by Descartes where the graph is given and a suitable equation must be found. Ruthven (1990) found the graphics calculator influenced the solution method in the case when the graph is given, and improved the results, whereas on different tasks where a verbally contextualised graph was given, the graphics calculator was less influential.

Semester one and two examination.

The Semester One and Two examinations both allowed student use of graphics calculators. The questions were designed to test for flexibility in the use of representations and to measure both instrumental and relational understanding. These examinations were trialed in the first two semesters of the study and described in greater detail in Chapter Four. Copies of the examination papers are available in Appendix C.

Graphics calculator applets.

Specially designed graphics calculator lessons called Aplets were based on a design for a *Gravity Aplet* supplied by Hewlett Packard on their Internet site, and now available on the Internet (www.sthildas.wa.edu.au). Aplets can be electronically downloaded from the Internet and then transferred to a student's graphics calculator via an infrared link requiring no physical leads. The *Gravity Aplet* contained a menu of items for students to follow, as well as sketches showing the position vs. time and velocity vs. time of a moving body. Students can change the value of some variables, such as the starting velocity and position, but the fundamental equations of motion were fixed in the Aplet programs. The Aplets can be edited, but this is best done using a separate Aplet Development Kit supplied by Hewlett Packard. A number of Aplets were prepared for the present study (Appendix K).

Teaching package worksheets.

The earlier research of Tuska (1993) suggested the following guidelines for the creation of instructional materials and these ideas were used in the selection of instructional material for the present study:

- Use problems in an easy-to-difficult sequence with increasing divergence
- Match examples/non-examples
- Use a large variety of examples
- Emphasise verbalisation
- Take advantage of the power of multiple representations

A number of worksheets on investigating change were included from the Mary Barnes Australian series (Barnes, 1991). These worksheets required students to interpret graphs of physical situations. Also included were a number of assessment support materials provided for the course in *Introductory Calculus* by the Secondary Education Authority (1992) in Western Australia. These materials consisted of tasks involving exponential functions and curve sketching. Important for the present study were exercises on the motion of a ball moving under gravity and involving distance and instantaneous speed calculations. The latter task was used to introduce students to differentiation concepts.

Graphics calculator aptitude worksheets.

Graphics calculator aptitude worksheets based on work by Dion (1990) provided students with appropriate technical expertise and prepared them with graphics calculator skills (Appendix I).

Qualitative Measurement Instruments

A number of qualitative instruments were used in the study in order to better understand student needs, their feelings about the study and to discover more about their understanding of calculus concepts through such interventions as classroom interviews.

Graphic calculator environment inventory.

A graphics calculator survey was developed, based on the classroom environment questionnaires such as the CLES questionnaire (Taylor, Fraser & Fisher, 1993) and the earlier work of Fraser, Treagust, Williamson and Tobin (1987). The authors of this instrument advised me of its robustness which allows significant rewording of items as long as original 'sense' is not lost (personal communication, February, 2002). Despite this, I established the face validity of the revised instrument by having it examined and verified by Fraser and Fisher. The aim was to obtain details on students' attitudes to the learning environment when a student-focused approach was used in conjunction with graphics calculators. Classroom instructional material involved the use of specially developed worksheets where the learning of multiple representations was required. Assignments, investigations, tests (ongoing) and examinations gave detailed insight into progress at later stages of the study. Sections of the questionnaire measured the following four areas (Green, 1994, p. 118-120):

- *Intuition* - determining if students viewed mathematics as uncertain and intuitive rather than as procedural
- *Critical Voice* - determining if students felt involved in the quality of the learning experience
- *Shared Control* - determining if students were involved in classroom management
- *Student Negotiation* - determining if students felt free to discuss ideas with other students or the teacher

The three students who answered this pilot survey were international students from the first two-semesters of the study: Jatos speaking Javanese and Indonesian, Filda speaking Mandarin and Indonesian, and Ricky speaking Cantonese. The input from these students informed me about the instructional approach and the materials, and facilitated the effective design details of the main study.

Observations.

I was involved with the sample participants twice a week for two hours in a standard high school setting where the group worked in their own classroom. During this time it was possible to observe the students interacting when discussing worksheets or using the text, graphics calculators or class computer. Evidence on the type of student collaboration was gained from the observations to assist in the action-research process. The effects of classroom practices were used to make changes to the teaching package.

Summary

The research methodology used in the present study aimed to collect and analyse the data. It was then possible to check for improvement in ESL students' knowledge, use and management of multiple representations, their ability to model and solve calculus word problems, and their skill at comprehension and translation of calculus problems. This involved designing a number of extended tasks and assessment items (to become part of the teaching package), and observing, interviewing and testing students. During the classroom lessons students in the main research study worked together in a group of four, with graphics calculators being available twenty-four hours a day (the students could take them home but did not own them). Qualitative research in the classroom allowed anecdotal records to be kept over an extended period. To reduce research demands that would conflict with normal classroom activities, the interviews were limited and the content of instruments was in-line with curriculum demands. It was important to record instances where students had problems in using graphics calculators as any lack of skill could affect their problem solving method selection and results. An extensive sample of student scripts was collected for interpretation. Transcriptions of interviews are made available to the reader to reinforce trustworthiness criteria. Results of the study follow in detail in Chapter Four.

Chapter Four: Data Analysis And Interpretation

Introduction

This chapter first describes the connection between the research questions and the data collection instruments. Analysis of the data collected in the manner detailed in Chapter Three (Table 3.2) then follows. The bulk of the data collected has been presented in detail in order to enhance credibility, as well as to allow other researchers to make their own interpretations. In presenting the data in this way the reader is made aware of contextual details and can better make judgements about the accuracy of specific data items and inferences. According to Lincoln and Guba (1985, pp. 296-298) this can improve the credibility and transferability of the results. The chapter begins by reviewing the research questions for the present study and linking them to the data collection procedures. Answers to the research questions are presented in Chapter 5.

Research Question One

> Can appropriate instructional material introduced over an extended time in an introductory calculus course enhance ESL students' use and management of multiple representations?

The evidence will investigate whether students become more familiar with the potential of diverse heuristic approaches using a range of representation modes, and that they would coordinate these approaches to solve real-world problem situations, especially those involving the motion of cars. Changes in students' use of multiple representations were measured by following the progress of each student in a longitudinal study.

Observing, assessing and interviewing students (who were mainly using English as a second language) as they solved real-world problems was a simple procedure because of the case study approach used in the study. It involved following four students individually and recording their experiences gauged from sample scripts completed during classroom assessments, and further analysed with the use of audiotaped interviews. During the classroom instruction a variety of solution approaches were demonstrated by myself through the use of a graphics calculator overhead projector pad, thus influencing and encouraging a multi-representational approach with the use of graphics calculators. Students were exposed to a range of problem types, some of which required the use of graphics calculators while others did not. Students solved problems using their preferred combination of representational modes, and changes in preference were recorded over time.

Research Question Two

> Can ESL students' ability to model calculus word problems be enhanced by using a study environment with small group student-centred learning focussing on skills of comprehension and translation?

To create suitable text a translation process from graph to first language, and finally to second language is required. An idea expressed in a first language then requires similar expression in the second language. Evidence will suggest whether communications skills developed within a multi-representational approach would challenge and assist students to discover the appropriate relationships, and allow concepts to be integrated across the range of translations students needed to grasp. Transactional writing, concerned with informing and accurately explaining ideas, can greatly influence concept formation and was encouraged in the present study:

> With respect to conceptual development, it is the transactional (concerned with informing and accurately explaining ideas) type of writing which may best facilitate these cognitive changes. When students are required to explain, define, and describe concepts and procedures in their own words they are actively involved in creating personally meaningful conceptions of the subject matter. Engaging students in progressively more abstract and integrative forms of transactional writing should enhance the structural development of conceptual bodies of knowledge.(Shepard, 1993, p. 289)

Research Question Three

> Can a teaching package utilising a representation mode of study be developed that assists ESL students in calculus learning?

The evidence will suggest whether the teaching package of tasks, which required skills in translation from text and pictures to a variety of mathematical representations, and incorporated appropriate instruction in technology tools such as graphics calculators, enabled students to improve their skills at comprehending and translating calculus word problems.

Analysis Of The Results

A range of results are presented in the following pages along with summary tables and a synthesis of student script details including quotations where appropriate. The analysis begins by examining details from a traffic worksheet given to students in order to identify their understanding of fundamental calculus ideas, and to identify their strengths when translating between representations.

Pre-Test Traffic Task Worksheet (refer Appendix C)

A summary of the results of the pre-test are given in Table 4.1 following. The variation in results is considerable for the sample of three ESL international students (Tonton, Oliver, Welly) and the local Australian student John. In general, the results in Table 4.1 show that John had a good understanding of the tasks compared to the ESL students. The reasons for this were far more complex than a simple literacy score would indicate. The values in the table are the raw scores on the six tasks contained in the traffic worksheet.

Table 4.1: <u>Results of the Pre-Test</u>

Student	Task 1 (20)	Task 2 (16)	Task 3 (26)	Task 4 (33)	Task 5 (30)	Task 6 (11)	Total (136)
John	19	14	19	22	17	7½	98½ (72%)
Tonton	11	6½	11	13½	21	5	68 (50%)
Oliver	8½	6	6	24½	15½	0	60½ (44%)
Welly	16	4	9	13	18	8	68 (50%)

Tonton, for example, scored 11 out of 20 for Task 1 and achieved a total score of 68 out of 136, or 50%. It is clear from the table that there were many areas where the misunderstandings of all sample students were substantial , especially for Tasks 2 and 3. The tasks were presented to the students together as an extended classroom worksheet to be handed in on a set date.

Sample Student Scripts

Task details appear in italics on the following pages, along with relevant diagrams, selected sample student scripts and notes I collected. Full task details are shown in Appendix C, while extracts of original student scripts are included in Appendix D and audiotaped interview transcripts are reproduced in Appendix E. All tasks were analysed to determine a measure of their ease of reading, and results are shown in Table 4.2 following. For example a reading-ease or Readability score of 69% for the total of all samples is at the easy end of the *Standard* level (Flesch, 1974). The Readability and Human Interest scores are calculated using the tables in Appendix M and a detailed account of the process is given by Flesch (1974, pp. 247-251). The general idea of readability is that shorter sentences (smaller number of words per sentence) and shorter words (less syllables) will produce text that is easier to read and understand. The measure, however, is only a guideline:

> In a sense, our modern short sentences are an illusion; as far as ideas are concerned, our sentences are usually much longer and fuller than those that people wrote two or three centuries ago.(Flesch, 1974, p. 146)

The general idea of a *Human Interest* score is a measure of the involvement of the reader in the written passage. Personal sentences refer to those where quotes, questions, exclamations or commands are included that indicate a direct link with the reader. Personal words refer to the use of names, personal pronouns (he, she, I, you), masculine or feminine words (uncle, spinster) or the two words *people* or *folks.* These two measures are combined to produce a Human Interest score, which is a measure of the effect a passage has on motivating students to take an interest in a mathematical or other task. Table 4.2 shows that for the pre-test the tasks were relatively easy to read (overall 69%) with tasks two and three having the highest scores for ease of readability of 74% and 77% respectively. These two tasks in particular had both a lower number of syllables per 100 words as well as a higher percentage of personal sentences than the average. Note that task six asked a number of questions of the reader giving a 91% score for personal sentences but contained

0% personal words. The *Human Interest* score average of 28% indicates that scientific writing, rather than that used in a magazine article, has been used.

Table 4.2: Task Readability

Task	Words	Sentences	Syllables per 100 Words	Readability	Personal Sentences (%)	Personal Words (%)	Human Interest
1	155	12	151	66	33	0	10
2	173	15	144	74	60	1	23
3	219	19	140	77	58	7	44
4	320	21	152	63	33	4	25
5	113	11	154	66	64	3	31
6	109	11	156	65	91	0	29
Total	1089	89	149.5	69	54	3	28

Student scripts were analysed and sample extracts follow. The choice of extracts was based on important details that influenced the study results. The type of detail and solution approaches used by students is indicated for a range of problems and the introduction to each question is presented to set the context starting with Figure 4.1 and the problem labelled (c) in Figure 4.2 following.

Task One – Taking Photographs from a Helicopter

Suppose that a police officer in the helicopter takes a photograph of the scene every second. On a sheet of "snapshot blanks", complete a series of photographs taken at one-second intervals. Write down what happens. Now suppose that the black car (the parked car) had been travelling at 10 m/s...As the helicopter flies Northwards, it spots various other traffic situations on different roads...Investigate each of these situations, in turn, on a separate "snapshot blank". Be careful to give each car its correct speed and starting position. (The black car always starts at 0 metres along the road.) Write about what happens. Each of the other cars is labelled with a letter (A, B, C, etc.).

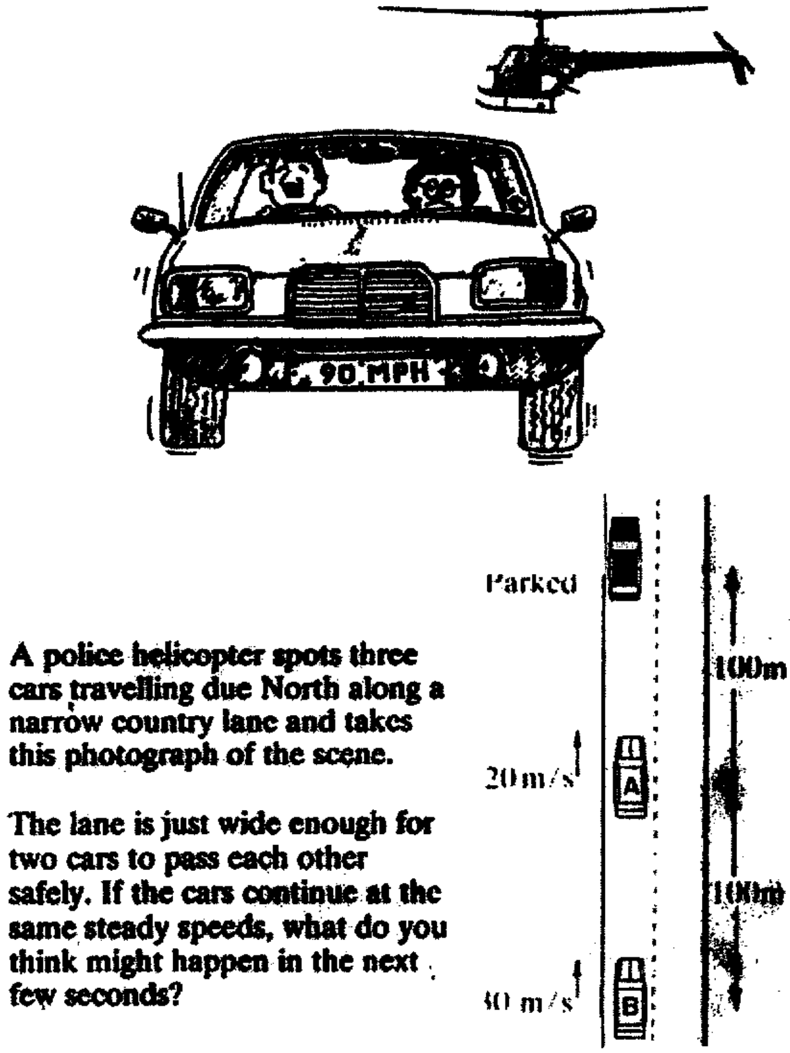


Figure 4.1: Taking Photographs from a Helicopter

(Source: Shell Centre for Mathematical Education.
Note: The black car is parked while *car A* and *car B* move at constant speeds of 20m/s and 30m/s respectively.)

The first extract is related to Figure 4.2 where four traffic scenarios are presented. Part (c) involves the greatest number of cars and the movement of traffic in two directions. Details of one solution are shown below for John the non-ESL student (see Appendix D, Extract 1.A for a full script). He created the graph in Figure 4.3 following. His approach involved discrete time calculations as distinct from the continuous-time calculations of the ESL students to be shown later.

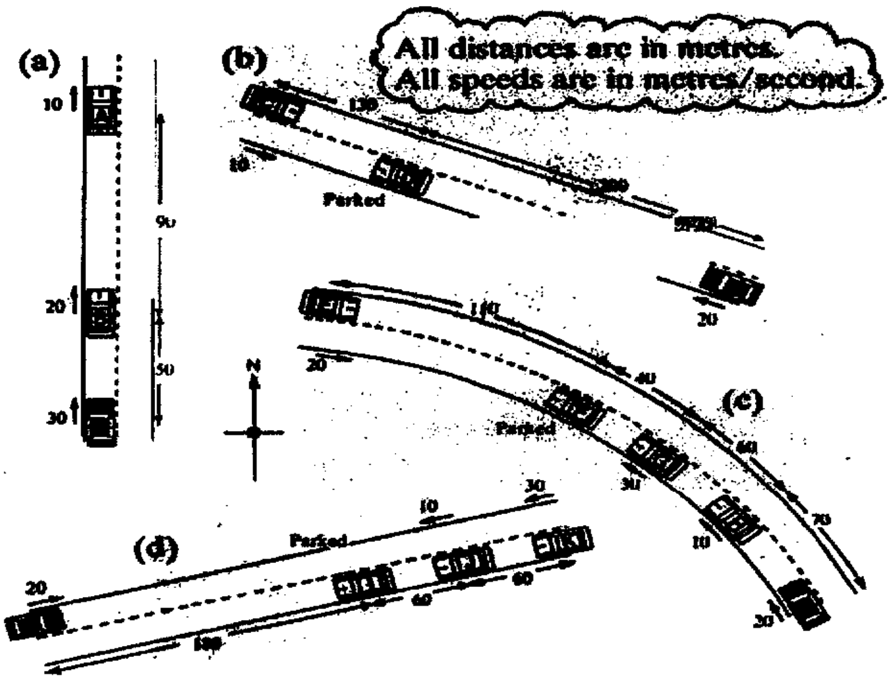


Figure 4.2 Task One Traffic Situations

(Source: Shell Centre for Mathematical Education)

The graph in Figure 4.3 shows five dotted lines (which can only be *seen* if you imagine a straight line of dots moving across the page, up or down, each representing a car at constant velocity), each one representing the movement over time of one of the cars in Figure 4.2 (c). The cars are labelled E, F, G, and H from left to right with the last car being black in colour. As dotted lines cross, the cars are overtaking or passing on opposite sides of a road. Only one of the dotted lines moves in a downwards direction, starting at a distance of 280 metres, and represents the single car moving in the opposite direction to the other four cars. Time calculations were evident from the script below:

> After $1^1/_3$ seconds car G overtakes car F, after $4^1/_6$ seconds more car E passes car F. The black car and car E crash while the black car overtakes car H.(Appendix D, p. 299)

John appeared to understand the question as the scenario above gives a correct solution with time calculations included. It was not necessary for the students to enter continuous lines at this stage. John's graph, shown in Figure 4.3, contained only plotted points without lines drawn, and was not directly used in any solution method. John used the distance travelled within each second, at a fixed speed, to find correct solutions. John preferred solving questions numerically. A symbolic approach using equations to find where the cars meet was not suggested by the format of the question. John was confident in generating a written discussion to explain the position of the cars and was the only student able to create a picture of a new traffic situation and discuss it.

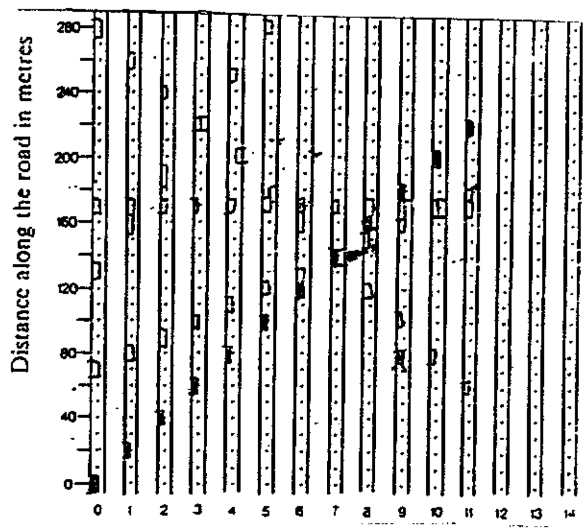


Figure 4.3: Task One (c) - John

Welly, an international student from Indonesia who uses English as a second language, produced the graph shown in Figure 4.4. The student script indicates the language level of the student, a different detail level in the solution (compared to John) and the preferred learning approach (for a full script see Appendix D, Extract 2.A):

> ..Cause car F parked on 170m is block the way for car G, H and black car to go. And the car B will passed each other at the distance 220m and 140m. (Appendix D, p. 326)

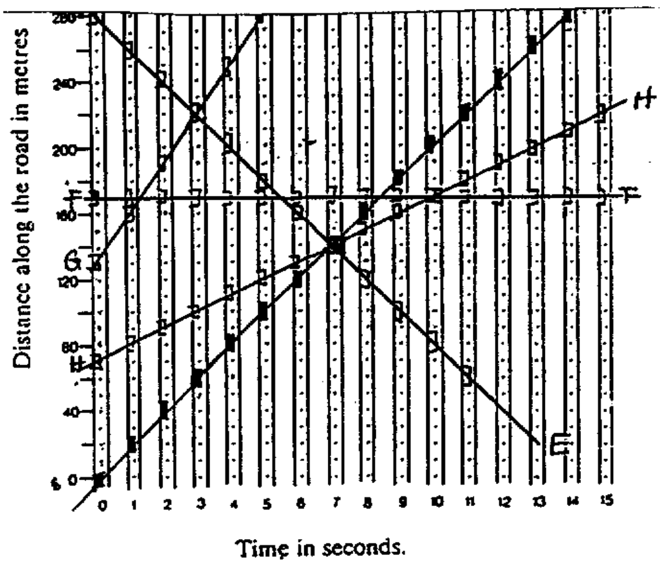


Figure 4.4: Task One (c) - Welly

The student obtained a correct interpretation of the problem, but preferred to concentrate on the position of the cars along the distance (vertical) axis. The preferred approach for this student involved using a line drawn on a graph sheet (see Figure 4.4) as the focus for all solutions. Even when the dots, representing time-distance pairs on the graph, were joined incorrectly, often due to the choice of an incorrect start point (t = 0), Welly trusted the line to supply all answers. No times were given for events such as collisions, so it is clear that a symbolic or numeric approach has not been used, or possibly the student assumed that this information was not necessary. This student used the meaning of the word *Write* to express a personal view of the traffic situation in everyday language, rather than to discuss his results in mathematical language. The intersection of lines was used to find car collisions and then the distance at which these occurred on a vertical axis scale was used. An awareness of the continuous time nature of car travel can be inferred from the lines drawn on the graph.

Tonton, another international student from Indonesia, uses English as a second language and produced the following script solution and graph, as shown in Figure 4.5 following (see Appendix D, Extract 3.A).

> The black car will hit the H car and the E car in 7 seconds and overtakes the F car. After the H car crashed on the black car, it will crash at the F car in another 3 seconds from the spot where the H car hit by the black car and the E car. (Appendix D, p. 352)

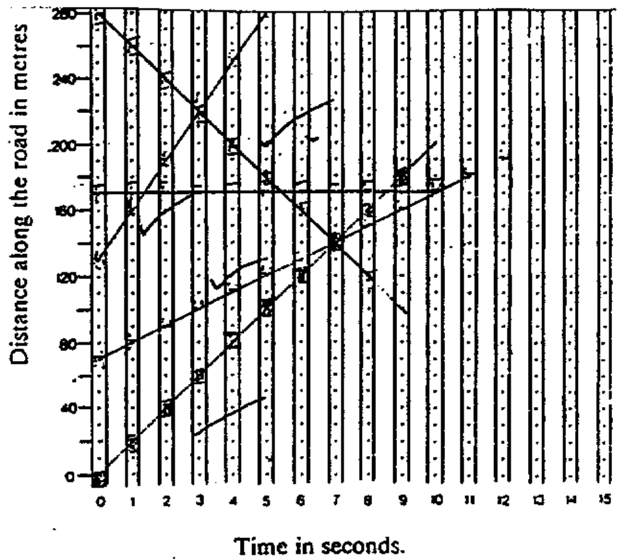


Figure 4.5: Task One (c) - Tonton

Once again we see a completely logical conclusion but based on different calculations from the other students. Time has been used as a variable in the solution, and only major collisions were summarised. Lines were constructed on the graph (see Figure 4.5) suggesting a continuous time approach, however, this student preferred to interpret events from a time point of view. It is clear from the graph that results were taken directly from it without any use of equations. In interpreting another picture of three cars travelling in the same direction (picture (a) in Figure 4.2), Tonton assumed the cars catch up on the graph, shown by lines intersecting, but collide during the next second. This may be an inconsistency in Tonton's thinking, though, as this idea is not repeated in later interpretations from this student.

Oliver, the other international Indonesian student uses English as his third language, and gave no written response to this question (see Appendix D, Extract 4.A). He drew solid lines on the graph (see Figure 4.6) in order to interpret collisions. No writing on time or distance calculations was given.

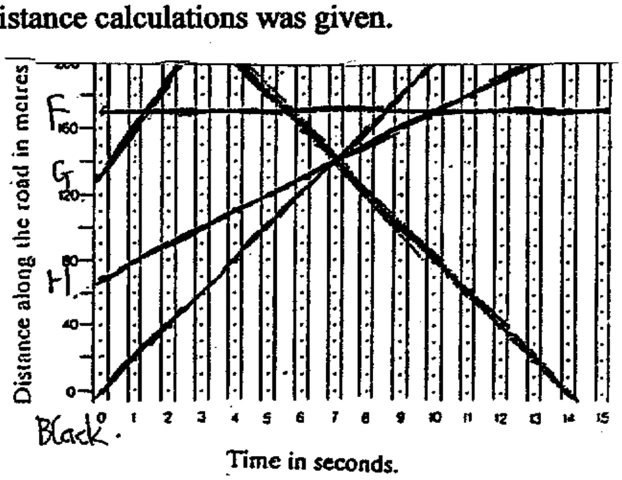


Figure 4.6: Task One (c) - Oliver

Task Two – From Photographs to Cine Film.

Attempt the problem below, before reading on: When will these two cars meet each other? How far will they be from the tree at that time? How accurate are your answers?

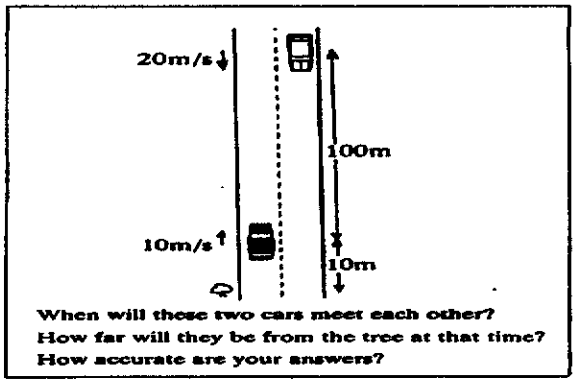


Figure 4.7 Task Two Traffic Situations

(Source: Shell Centre for Mathematical Education)

The police took photographs of this situation with three different cameras. Camera A takes one photograph per second. Camera B takes ?(calculate from the details shown) photographs per second. Camera C takes?(calculate from the details shown) photographs per second. What will the "snapshot" diagram look like if even faster cameras are used? Draw 'photographs' of the black car taken at 1 second intervals, but this time use a sheet of graph paper, as shown on the next page. Notice that the road is now represented by a single vertical line and the position of the 'car' is shown by a small black dot on it. Add extra dots showing the position of the black car at ½ sec intervals, then at 1/10 sec intervals, then at 1/100 sec intervals...What happens? Add the 'cinefilm' pictures for the white car. Find out as accurately as you can where they pass each other.

A distance-time graph was given to students showing time on the horizontal-axis and distance on the vertical-axis. Students were forced to move from a discrete set of time-distance points to a continuous model as they filled in the dots on the graph using smaller and smaller time intervals, thus encouraging a move from a correspondence view of function to a covariation view. Additionally, students needed to understand that cars moving in opposite directions on the same road at constant speed can be represented on a graph by a series of lines in different directions, with a downward sloping line representing the movement towards a reference point. The intersection of lines represents cars crossing. This task had a

Readability score of 74% and 60% *Personal Sentences* (refer p.141) so was expected to be one of the easier tasks to comprehend. No use of an action view of function using equations was suggested by the problem format.

John used the concept of velocity as giving equal distance in equal units of time in developing his solutions (see Appendix D, Extract 1.B):

> The gap is 100m and closing in at 30m/s means they will pass after $3^1/_3$ seconds. (Appendix D, p. 308)

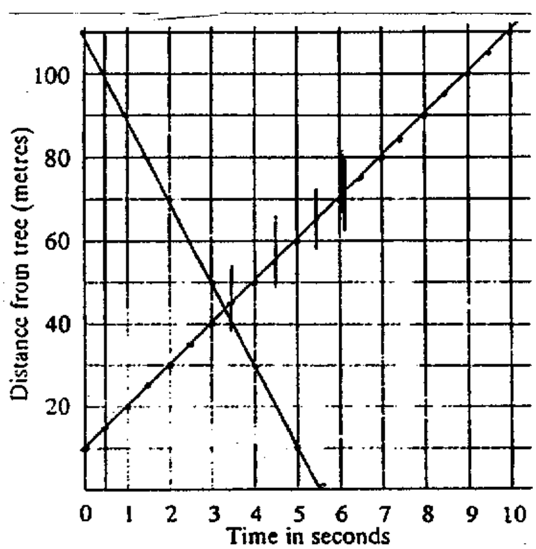


Figure 4.8: Task Two - John

John included time calculations automatically and correctly calculated values. His graph in Figure 4.8 was identical to that of the other students but was not directly used in any solution method. He used the relative speed of the two cars to find correct solutions. John enjoyed and preferred solving questions in this way rather than using a graphical or a symbolic approach. He confidently generated a written script to explain the position of the cars and confidently created his own original traffic situation along with a description. For him the instructions were clear and provided there was freedom to use any solution approach he could adequately solve this problem.

Welly (see Appendix D, Extract 2.B) gave no written response to the first part of the question. He appeared to understand the question correctly as he estimated the meeting point but gave no hint of the calculation method. It appeared the solution might have been estimated from the graph sheet that was identical to that in Figure

4.8. Welly was not confident in giving a written explanation of the position of the cars or in creating an imaginary script of his own and did not answer many of the questions for this task.

Tonton (see Appendix D, Extract 3.B) responded:

> 3 seconds for the white car, 4 seconds for the black car.(Appendix D, p. 359)

Tonton appeared to understand the question as he sketched two lines on the graph sheet that were identical to those shown in Figure 4.8 drawn by John, but values from the graph were not used for his conclusions. Tonton compared the positions of the two cars and noticed when the positions were equal regardless of the value for the time. The different times corresponded with the same position on the distance axis for the two cars since the White car travelled at 20m/s and moved from 110m towards the tree (110-3x20=50) while the Black car travelled away from the tree at 10m/s from a starting position 10m in front of the tree (10+4x10=50). Tonton was not confident in generating a written explanation of the position of the cars or in creating an imaginary script of his own.

Oliver (see Appendix D, Extract 4.B) gave no written response to the first part of the question. He appeared to understand the question as he sketched two lines on the graph sheet similar to those in Figure 4.8 drawn by John. Oliver was not willing to generate a written text to explain the position of the cars or to create an imaginary script of his own.

Figure 4.9 shows details of Task Three involving two main traffic scenarios: First the diagram in Figure 4.9 involving the movement of two cars relative to a tree; Second the italicised question involving the movement of seven cars relative to a telephone box.

Task Three - More Traffic Problems

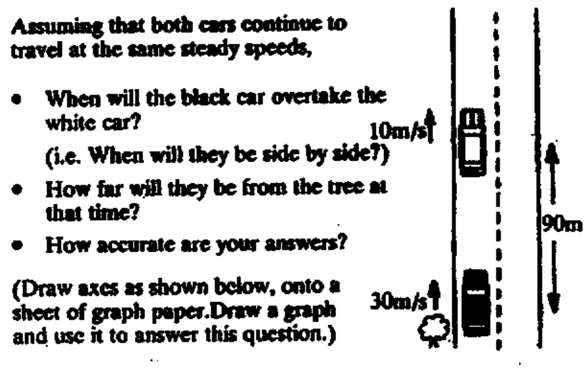


Figure 4.9: Task 3 Constant Velocity
(Shell Centre for Mathematical Education)

The main road joining Nettle Village to Little Huntingford runs North-West. A telephone box stands by the side of the road. The graph was drawn (Appendix C) to show the progress of some traffic along the 200 metre stretch of road beyond the telephone box. All timings were measured from the moment when a green car passed the telephone box.

a) Draw a picture to show the traffic situation after 5 seconds.
b) Which cars were traveling due Northwest? Southeast? East?
c) Write a short story describing what you would have seen if you had been the pilot of the helicopter. (Remember to mention speeds, times, distances and directions in your account.)
d) Write another eyewitness account of the situation from the point of view of the driver of one of the cars. (State clearly which car you choose.) Read and discuss your neighbour's accounts.
e) Make up your own traffic situation and give it to your neighbour to describe.

Welly (see Appendix D, Extract 2.C)

When the black car overtake the white car; when the black car reached 7 seconds. They will be side by side when they reached 6 seconds. 150m they will be from the tree at the time.(Appendix D, p. 332)

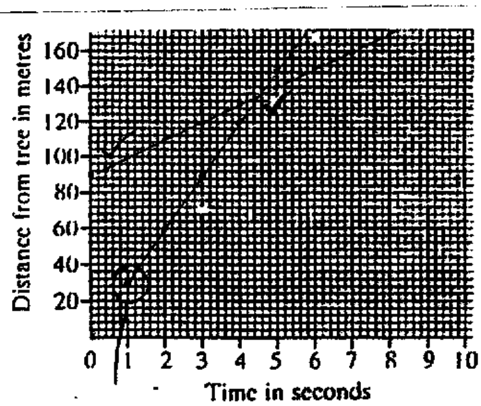


Figure 4.10: Task 3 Welly

Welly appeared to understand the question as he correctly showed the lines and intersection point on a graph (Figure 4.10). It appeared the solution has not been estimated from the graph sheet as the value given for the time of collision did not match the intersection point on the graph. Once again Welly used a time value for each car based on when both cars were at the same distance rather than using the time where they collided. The white car did take 6 seconds to reach 150m (90+6x10=150) but the Black car took only 5 seconds to reach it (30x5=150). It was then assumed that overtaking occurred in the next second. For the second part of the Task (Appendix C) involving the interpretation of a picture, Welly confidently wrote about the position of 7 cars and demonstrated a good understanding of the problem. However, he created a graph rather than a picture as was requested for part of the exercise. Once again he preferred to use the endpoint of 200m as a distance reference point rather than a telephone box at 0m as requested.

John (see Appendix D, Extract 1.C) appeared to understand the question as he correctly translated from the picture to a graph (similar to that in Figure 4.10) showing the lines and intersection point of the path of the cars and found the correct solution. It appeared John estimated the solution from the graph sheet. No other results were provided by John for the remainder of this Task.

Tonton's (see Appendix D, Extract 3.C) response:

> When the black car overtake the white car; when the black car reached 7 seconds. They will be side by side when they reached 6 seconds. 150m they will be from the tree at the time.(Appendix D, p.355)

Tonton appeared to understand the question and correctly showed the lines and intersection point on a graph (as in Figure 4.10) but this was not used for the solution. He preferred to trust another method that eventually led to different solutions. It appeared the solution had not been estimated from the graph sheet he created since the graph contained the correct solution. Tonton used an independent time scale for each car (see the later interview transcript discussion in this Chapter and also Appendix E). He used a similar explanation to Welly in assuming that overtaking occurred in the next second to that where the cars met. Tonton was confident in writing a summary to explain the position of the 7 cars in the second part of the task and had a good understanding of the situation. Once again he preferred to use the endpoint of 200m as a reference point to measure distance when referring to a car coming from the opposite direction, rather than the telephone box at 0m as requested. One aspect of discussion groups is the acceptance of conclusions within the group based on inadequate information or incorrect assumptions. A graphing ambiguity is highlighted: three cars appeared to meet and then follow the same horizontal path as time increased (task details in Appendix C). Tonton said they park but did not consider the possibility of a collision.

Oliver (see Appendix D, Extract 4.C) produced a good graph similar to that of Welly's (see Figure 4.10) but gave no interpretation for the first part of the Task. This was a similar response to earlier Tasks where little written explanation was given to support graphical results.

Task Four - More Traffic Problems

Each question is a description of a situation. You must try to draw a distance-time graph to illustrate each situation.

1. *A heavy lorry is driving along at 30 metres/second. After 8 seconds, it reaches a steep hill and so reduces its speed to 10 metres/second.*
2. *A car is travelling along a country road at 30metres/second. Five seconds after passing a signpost, the driver suddenly completes a swift "U" turn in the road and retraces her journey (still at 30 metres/second).*
3. *While a learner driver is motoring along (at 30 metres/second), his instructor asks him to perform a simple reversing exercise. The driver continues on his way for 6 seconds, then stops the car for 4 seconds (while changing gear), and then reverses at a steady 10 metres/second.*
4. *A motorbike is speeding along a town street at 20 metres/second. After 8 seconds, a small child suddenly steps into the road. Immediately the rider slams on his brakes and screams to a halt. When the child has crossed the road safely, 5 seconds later, the rider continues on his journey again at 20 metres/second.*
5. *Another learner driver is crawling along a road at 10 metres/second. After 5 seconds of frustration, the instructor tells him to stop at the side of the road, where she shouts at him for a very noisy 5 seconds and then tells him to continue. The learner then nervously continues his journey at 20 metres/second.*

A schoolboy has answered some similar problems and has produced the following graphs for his answers.

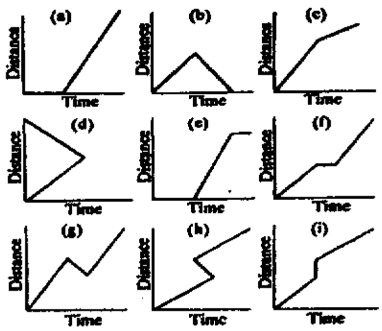


Figure 4.11: Possible Distance-Time Graphs
(Shell Centre for Mathematical Education)

> *i) Some of these graphs cannot possibly be correct! Which graphs can never represent the journey of a single vehicle? Why?*
> *ii) For the rest of the graphs, make up situations which could cause these graphs to be drawn. You needn't worry about the exact values of speed, time or distance - just a rough story will do.*

The aim was to identify universal functional properties of any distance-time graph to lead students to an understanding of generalisation. The distance-time graph then takes on the form of an object with abstract properties (Brown, 1990; Lakoff & Johnson, 1980). The scripts focus on question 2 of the first part of the Task because it introduces the concept of a reversal in direction, which introduced a number of different responses from the four students.

Tonton and **Oliver** (see Appendix D, Extracts 3.D and 4.D) had a very good understanding of translations from written text to a graph, and this was one aspect of their second language background that was well developed. They assumed that the 'U' turn in question 2 was instantaneous and correctly graphed the results. They interpreted "reduces speed to 10 m/s" in question 1 of the Task also as an instantaneous one. A better interpretation would have shown a curved path to indicate a gradual reduction in speed, but this would have required an extension of the ideas covered so far in the instructional material.

These two students had no confidence in describing the distance-time graphs given in the second part of the Task, however. They did not write explanations for the possible origins of each graph, perhaps because they lacked the ability to verbalize them and so give them meaning. This was checked during interviews held later (see page 173). From the nil response given they obviously had no idea which of the given graphs were impossible. This may simply have been an area not emphasised in their mathematical studies in Indonesia and was to become more illuminating when the non-ESL student's results were analysed. This type of skill weakness may simply be a function of the syllabus emphasis in the students' home country, Indonesia, and is an important aspect to consider when preparing instructional material.

Welly (see Appendix D, Extract 2.D) produced some errors in interpretation similar to Tonton and Oliver described earlier - for example, interpreting movement on a distance-time graph at a fixed time as being a suitable representation (see graph (i) in Figure 4.11). Welly had a weak understanding of distance-time graphs: he was aware that time cannot go backwards, but at the same time felt that such a situation represented a car turning around (see graph (d) in Figure 4.11). This implied the use of a direct relationship between the shape of the graph and car movement, and suggested another area where the instructional material should be carefully prepared.

John (see Appendix D, Extract 1.D) successfully completed translations from text to graph for cars that stay in the same direction. As a car made a 'U' turn, however, John suggested the total distance moved must still be increasing, and wanted to represent this on a distance-time graph by showing a curve moving upwards rather than downwards. Thus, John had not identified the y-axis scale as the distance of the car from its starting point (the displacement) but rather as total distance and so the graphs John produced were inappropriate. This concept of increasing distance influenced John when he excluded a number of diagrams from his collection of possible representations. Hence, all supplied graphs with any time interval where the distance decreased John claimed were impossible, like those shown in Figure 4.12 following. John treated a car reversing as having the distance-time graph similar to that of a car traveling in the same direction for the entire journey.

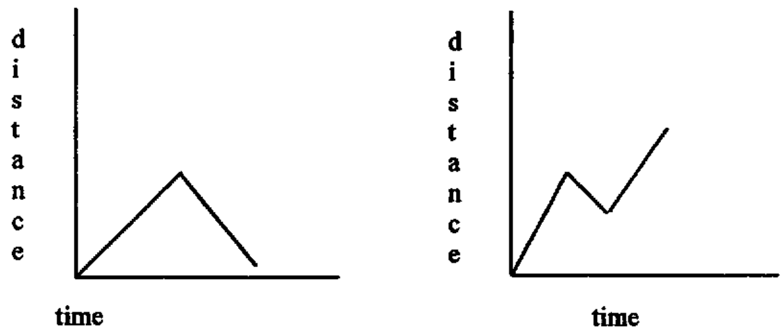


Figure 4.12 Distance-Time Graphs of Reversing Cars

It is important to note that John, the non-ESL student, confidently described in words situations appropriate for these graphs and identified general properties of distance-time graphs as indicated by his translation from graph to text (see Appendix D, Extract 1.D), apart from the situations for reversing cars as indicated above and expressed in his own words below:

> ... cannot represent a journey because you cannot have negative time or go a distance in no time" or " ...because you cannot decrease distance.(Appendix D, p. 313)

In summary, John did not understand the difference between distance and displacement and attempted to use an increasing distance on his graphs, rather than showing a change in the direction of the velocity. Thus, a 'U' turn was interpreted by him as a situation where distance must keep increasing. This is illustrated in Figure 4.13 following:

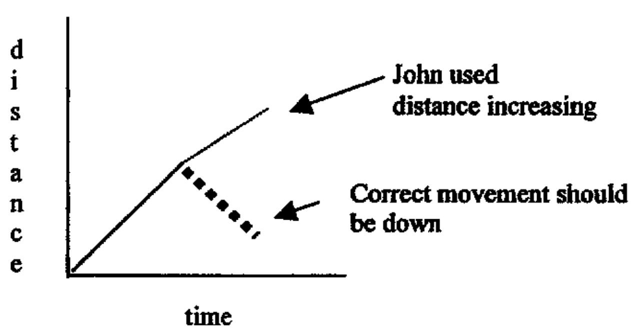
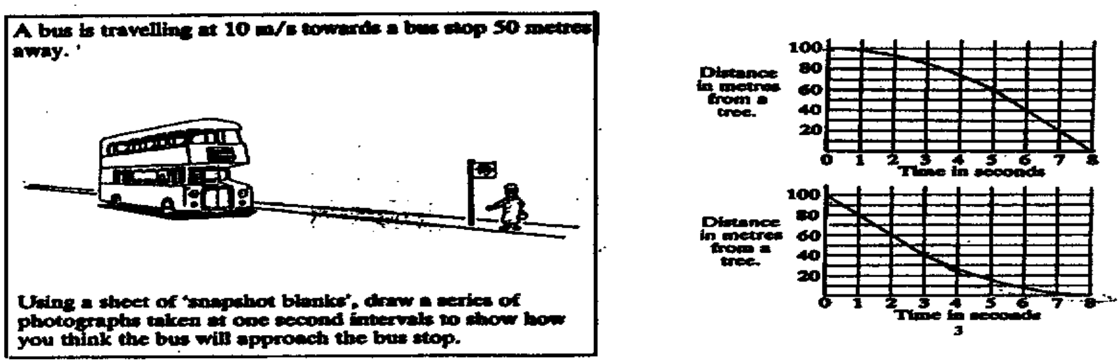


Figure 4.13: Distance-Displacement Illusion

Task Five - Acceleration and Deceleration

> *A bus is traveling at 10 m/s towards a bus stop 50 metres away. Using a sheet of 'snapshot blanks'* (see Appendix D), *draw a series of photographs taken at one second intervals to show how you think the bus will approach the bus stop. In all the examples we have considered so far, vehicles have either been traveling at constant speeds, or they have suddenly changed from one speed to another. Is this realistic? How do vehicles really behave? Study the following two graphs. One shows a car slowing down (decelerating). One shows a car speeding up (accelerating). Which is which? Explain your answer. Can you tell when the drivers begin and stop braking or accelerating?*



(a) (b) and (c)

Figure 4.14: Task 5 Acceleration Problems
(Shell Centre for Mathematical Education)

John had a good knowledge of translating from picture to graph (see Appendix D, Extract 1.E). He used a different reference point and assumed at time zero the position of the bus was at 50m whereas the other students positioned the bus at 0m. His graphs showed no acceleration or deceleration but only straight-line segments. A car reversing direction was graphed incorrectly, with the graph moving backwards so that time was not increasing (see Figure 4.15 (e) following). He understood deceleration as less distance travelled each second, but was not confident enough to make examples of his own. He also had a number of misconceptions (see Table 4.3 following).

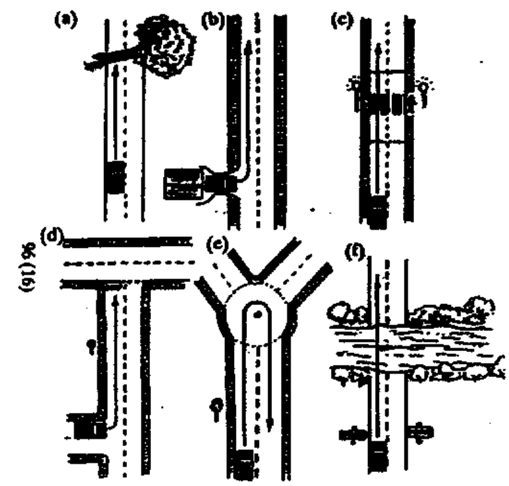
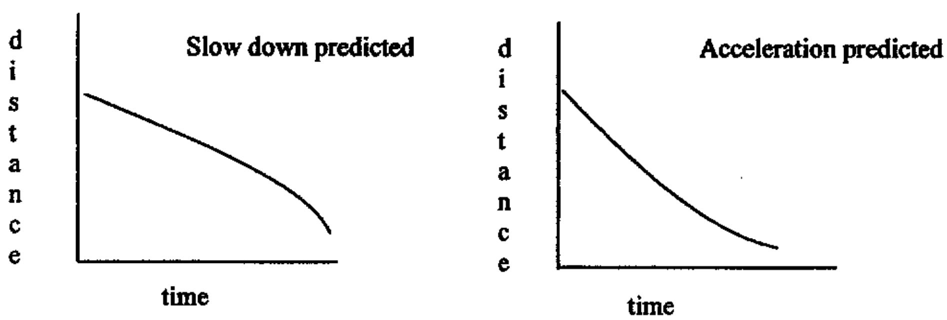


Figure 4.15: Task 5 - Changing Velocity Pictures

Tonton (see Appendix D, Extract 3.E) had a good knowledge of translation from picture to distance-time graph as for John and Oliver. However, given graphs of accelerating cars, misinterpretations were made. Tonton assumed that a downward curve where the distance scale (vertical value) is decreasing implied that the car slowed down. Figure 4.16 (a) and (b) following show the graphs presented to the students, a copy of those in Figure 4.14 (b) and (c) but with an acceleration interpretation.



(a) (b)

Figure 4.16: Changing Velocity Graphs

Table 4.3 following illustrates the ideas from the student scripts where there was confusion about the links between curve shape, whether the car was slowing down or increasing speed, and whether the car was accelerating or decelerating.

Table 4.3: Acceleration Misconceptions

Student	Car Increasing Speed (a)	Car Decreasing Speed (b)	Interpretation
John	Speed up	Speed up	Both speed up as travel more distance
Tonton	Slow down	Slow down	Going down in distance (downward curve)
Oliver	Speed up (accelerate)	Slow down (deceleration)	Can interpret changes per second
Welly	Speed up	Speed up	Cannot interpret changes per second

As well as misconceptions, there were inconsistencies in the students' thinking across different tasks. John, for example, had earlier commented that a curve sloping down represented deceleration or a slowing down, whereas his comments (summarised in Table 4.3 following) show he was willing to assume increasing speed occurred for the situations pictured. Oliver, who interpreted the speed correctly for both graphs, failed to understand the terms *acceleration* and *deceleration* as applied to these situations.

Welly (see Appendix D, Extract 2.E) also created straight-line graphs for the movement of the bus in Figure 4.14 (a) with no non-linear slowing down (deceleration), and no comments to explain the deceleration. He assumed speed was constant around the roundabout shown in Figure 4.15 (e).

As a car moved around a roundabout, Welly drew a distance-time graph with linear velocity illustrated in Figure 4.17 following. This could be due to his lack of knowledge of car movement.

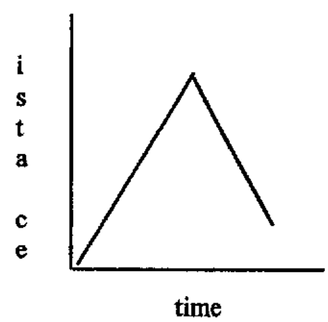


Figure 4.17: Graph of Roundabout Movement

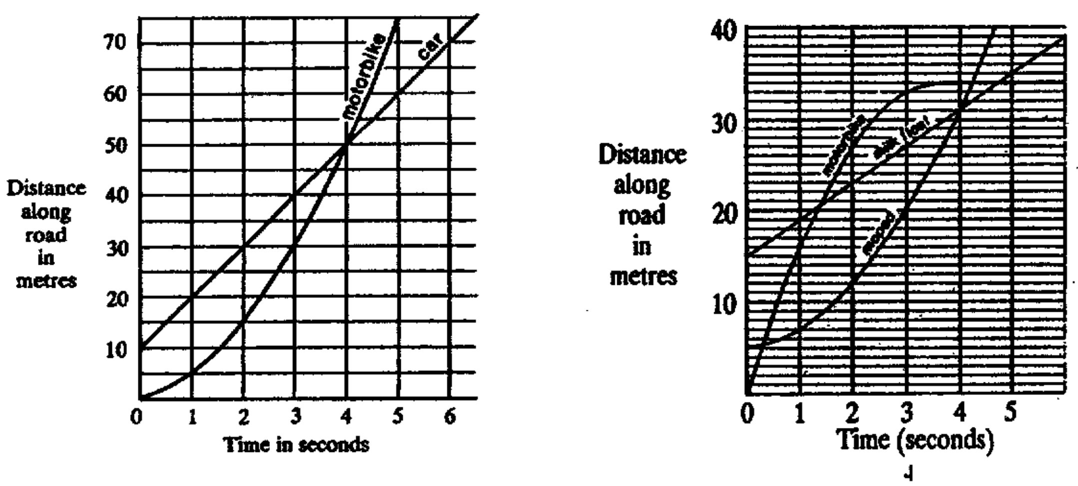
Oliver (see Appendix D, Extract 4.E) had a good knowledge of the use of a picture to graph translation but used only straight-line segments for the situation of the bus decelerating. He made no examples of his own and could not interpret acceleration on a distance-time graph. Speed was also assumed constant around a corner when considering situations such as Figure 4.15 (e).

Task Six - Measuring Speed and Acceleration.

This graph (Figure 4.18 (a)) shows a car and a motorbike, traveling along a country road. Describe what is happening, as fully as possible. Compare their distances apart and their speeds at different times.

- *When are they furthest apart during the first 4 seconds?*
- *When are they traveling at the same speed?*
- *How far does the motorbike travel during the 1st second, 2nd second, ...? (so, what is its average speed during these intervals of time?)*
- *By how much does the average speed increase in each second? (So, what is the acceleration of the motorbike?)*

Now describe what is happening in the following situation (Figure 4.18 (b)). Find speeds and accelerations or decelerations wherever you can.



(a) (b)

Figure 4.18: Task 6-Measuring Speed and Acceleration

(Shell Centre for Mathematical Education)

In Figure 4.18 (a), John could not understand when the speed of the car and the motorbike were the same (Appendix D, Extract 1F). This was based on his inability to use a rate of change, but he was successful using tables to compare the position and velocity of vehicles. Acceleration was introduced as a second difference and velocity as a first difference as part of the task notes. He interpreted a distance of zero at time zero to imply a speed of zero rather than using an appropriate value for the speed at the starting point. This requires identifying a non-zero slope or gradient for the curve at the starting point.

Welly (see Appendix D, Extract 2.F) looked at the curves and concluded:

> They were traveling at the same speed at 4 seconds when they meet.(Appendix D, p. 340)

This was incorrect. In this case, the same position does not correspond to the same speed. Even though the vehicles intersected on the graph the different slope of their curves was not identified by Welly. He failed to use a method such as calculating first and second differences to check the speed and acceleration. Welly believed the motorbike was increasing its speed until 4 seconds but said the moped was increasing speed every second. His first assessment was incorrect, but he correctly interpreted the moped speed. Table 4.4 summarizes the main conclusions made by each student.

Table 4.4: Speed Misconceptions from Figure 4.18 (b)

	Motorbike	**Moped**
Welly	Increases speed until 4s	Increases speed
Tonton	Increases speed until 4s	Decreasing speed
John	Decelerates at $5ms^{-2}$	Accelerates at $3ms^{-2}$
Oliver	No response	No response

Tonton also felt the intersection point in Figure 4.18 (a) corresponded to equal speed for both vehicles. In Figure 4.18 (b) he believed the motorbike was increasing its speed until 4 seconds but said the moped was decreasing its speed, misinterpreting the curve sloping downwards (see interview following). Both were incorrect.

Oliver correctly predicted a motorbike increasing speed in Figure 4.18 (a) by calculating differences in a table, but avoided the second part of the task involving interpretation of Figure 4.18 (b). In a later interview (see page 179), Oliver suggested that if distance was the same for two vehicles then this implied that velocity would also be the same as one overtook the other. He responded well to correction on this issue.

Summary of Main Assessment Results

After the Pre-Test Traffic task analysis just described, there was an extensive range of interventions carried out over the two-semester period of the study, as detailed earlier in Chapter 3 (Tables 3.1 and 3.2, pp. 120-121). Quantitative assessment results are shown in Table 4.5 following. One important assessment early in the study was the Traffic Test running parallel with the Pretest Traffic tasks (and immediately following the first interviews). This test was designed to verify the earlier results observed in the Pretest Traffic tasks. A discussion of the results of a number of the quantitative assessments are included in this chapter, along with a summary of information collected from interviews and a classroom environment survey. Some of the quantitative assessment results have been omitted due to the nature of their content, it being either uninformative relative to the aims of the present study or because of time constraints. Thus a detailed discussion of results was inappropriate for the *speed task* and *speed test*, the *exponential* task and the *semester one examination*. The former assessment tasks showed little variation across students and I felt they added little to the study results. They would have required substantially greater time to incorporate into the study results.

There was a marked improvement in the examination results of the students as shown by the semester one and semester two examination results in Table 4.5. The papers were set by myself and the average mark and individual results improved for each student in the study. Most students improved their semester two-examination results by at least 20% compared to their semester one result. Potential reasons for this change have been detailed in Table 4.18 at the end of the chapter (pp. 204-205) where longitudinal changes in student thinking are summarised. This indicates an overall improvement in the students' calculus problem solving ability over the duration of the present study.

Table 4.5 Assessment Results

Student	PreTest Traffic Task (%)	Traffic Test (%)	Speed Task (%)	Speed Test (%)	Expon Task (%)	Exam Sem 1 (%)	Sem 1 Mark (%)**	Exam Sem 2 (%)
John	75	71	93	51	90	58	68	87
TonTon	65*	50	52	59	90	43	54	79
Oliver	47*	27	17	54	20	32	40	56
Welly	63*	42	70	47	82	38	51	60
Average	**63**	**48**	**58**	**53**	**71**	**43**	**53**	**71**

*Final mark after work redone **Includes other assessments

Details about student understanding of the test and examination questions follow with an analysis of the typology of graphics calculator use and representational type included for both test and examination questions.

The type of tasks used in this study may have made it more difficult for ESL students to be successful because the tasks at times required both instrumental as well as relational thinking, extensive translation of text, and the discussion and interpretation of results – the very aspects of mathematical learning expected to be difficult for ESL students. Certainly, by the end of the first semester, the students and I were somewhat disillusioned with the instrumentation results, and particularly by the results of the Semester 1 examination. Whether this was a result of the interviews utilising too much class time, major differences between the classroom tasks and the test and examination material or other factors needs careful analysis. This was despite the fact students were allowed to repeat *Traffic* tasks if it was clear that they misunderstood the task requirements. Students were only allowed to repeat tasks for the initial *Traffic* worksheets. For other assessments it was assumed students were better prepared in terms of task expectations, having had by then extensive classroom experience with the use of multiple representations as well as the format of the teaching package material.

Table 4.6 following illustrates the fact that repeating tasks had a significant influence on results for some students, but that the performance of students on the extended traffic task did not imply that similar results would occur for the traffic test. The Bonus column in Table 4.6 represents additional marks achieved from redoing tasks. For example, John received a small bonus, due mainly to his already high results. Tonton and Welly took the opportunity to redo and resubmit work. Their results improved greatly, whereas Oliver did not attempt to redo tasks.

Table 4.6: Comparison of Results: Pre-test vs. Test

Student	Initial Traffic Task (136)	Bonus	Final Traffic Task (136)	Traffic Test (%)
John	98½ (72%)	3½	102 (75%)	71
Tonton	68 (50%)	20	88 (65%)	50
Oliver	60½ (44%)	3½	64 (47%)	26
Welly	68 (50%)	17	85 (63%)	42

The Traffic Test results were significantly lower than the ESL students expected based on their initial traffic task results. The choice of test items reflected the nature of the traffic tasks used in the classroom. The choice of test questions highlighted the weaknesses of the ESL students, as they required graph interpretation, description of events and the interpretation and translation of real life situations from pictures. The results of the first test placed further stresses on the use of the study instruments, since the results could have been interpreted as providing evidence that students' results were being adversely affected. The goal of introducing extended tasks during the present study while keeping synchronised with the syllabus was difficult. I decided to reduce the impact of the present study on classroom interactions by minimising constraints caused by the use of interviews, survey data collection and other possible instruments that were seen as interfering with effective instructional time.

Prior to the Traffic test, common misconceptions observed among students during the Pre-Test Traffic Task related to alternative representational understanding for the concepts of speed, distance and acceleration, and included the following:

- Given a calculus word problem, ESL students could create a distance-time graph, but given a graph of a situation, students gave only brief or no textual descriptions without any reference to time comparisons.
- A distance-time graph showing a car reducing distance from a reference point (a downward slope) was often interpreted by students as the car slowing down.
- Changing direction was interpreted by students as the car's distance increasing rather than as displacement decreasing relative to the starting point as reference.
- Velocity increase/decrease was often interpreted by students as acceleration/deceleration without direction of motion being considered.
- All students interpreted a vehicle stopping as being instantaneous rather than reducing velocity over a period of time. This could have been a misinterpretation of the connection between the distance-time graph approaching zero (zero distance) and the slope of that graph approaching zero (zero speed).
- A curve showing increasing velocity was understood to mean that the graph must go upwards.
- Distance zero at the starting point was interpreted by students as meaning speed was zero at the starting point.
- Speed was often interpreted by students as being the same when vehicles overtook rather than using slopes of the distance-time curves as they intersect to determine the velocity (therefore assuming velocity was equal when distance was equal).
- Finding the time for equal position of different vehicles was often interpreted by students as a different time for each vehicle, by looking at a table of data and seeing when distances were the same, rather than treating both cars simultaneously and finding the single time where they both had the same position. This would come under the realm of problem comprehension difficulties.

The results of the test showed a number of improvements had occurred in students' understanding from the earlier traffic tasks and following the interviews. Additionally, some students were changing their representational preferences to ones considered more appropriate to the problem type. However students were still reluctant to use a difference approach to calculate velocity and acceleration values and were having difficulty adapting to velocity-distance graphs rather than distance-time graphs. A summary of these details follows in Table 4.18 at the end of the chapter and further details of the requirements of different assessment items for the Semester One Examination follow in Tables 4.7 and 4.8. A balance of questions were included in this examination based on the criteria of whether graphics calculators were *expected* to be used by all students, some students or no students (not expected to be used). It was essential that graphics calculators could be used in a selection of the questions because the study investigated applications of technology.

Table 4.7: Typology of graphics calculator use: Semester One Examination (Kemp et al. (1996))

Typology	1	2	3	4	5	6
Graphics calculators are expected to be used	No	No	No	No	No	No
Graphics calculators are expected to be used by some students but not by others	Yes	Yes	No	No	No	Yes
Graphics calculators are not expected to be used	No	No	Yes	Yes	Yes	No

In Table 4.8 examination questions have been classified based on their presentation mode and their required result mode (in some cases the results can be given in more than one mode). It was important that questions were presented in a range of modes and not just as word problems.

Table 4.8: Typology of Representational Use: Semester One Examination

Typology	From	To
1	Symbol, Words	Symbol, Graph
2	Picture, Words	Symbol, Graph
3	Symbol	Symbol
4	Table	Graph
5	Graph	Graph
6	Graph	Symbol

Other Assessment Items

Exponential Function Worksheet

This was not very successful as either the students could easily complete tasks such as graphing, or they all required assistance with the limit definition of the rate of change (see Appendix C).

Teacher's Reflections and Fieldnotes

I maintained a record of lesson ideas, students' classroom problems detailing their steps, or scheduling comments relevant to the appropriate sequencing of interventions to enhance my findings in the present study. These ideas and notes were recorded in a diary mainly during classroom lessons. For example, the comment by a student that the *'deceleration was the slowing of the car'* made during one of the audiotaped interviews, involved me in a great deal of reflection about the usefulness of the definitions of accelerating bodies. Observations verified that students asked progressively more content questions and fewer calculator questions during the study, in line with attempts made earlier in the present study to promote the technical knowledge background in the use of the graphics calculator amongst the sample students. These fieldnotes and reflections provided the basis of the inferences to follow.

Audiotaped Interviews

A series of taped interviews were conducted with individual members of the sample. These tapes, their content and the interpretation of the data on them, comprise an important component of the study. Taping each student took about forty minutes during classroom lessons while other students continued with classroom activities, and the recordings were conducted after the first and second quantitative assessment interventions. Earlier in this report the features of interviews were noted (Chapter 3, pp. 123-125). The approach taken with the interviews may be different from other similar studies (they were task-based), and a summary of aspects relevant to the interview process follows. These are presented under the categories of *advantages, disadvantages,* and in regard to *benefits noted for the learning environment.* These are my own ideas based on data collected during the study. In many cases they may reflect the ideas mentioned earlier in Chapter Three (pp. 123-125).

Advantages of interviewing.

- Allowed discussion of topics occurring later in the course, which were relevant at the time, to explain the reason for the approach of particular questions – in particular, a discussion of the concept of discrete change in intervals of time which it became clear the students would need to develop before later topics.
- Enhanced the students' results by allowing them to redeem their work where it was clear that they had fundamental comprehension problems related to literacy skills.
- Developed student-teacher relationships by enabling the teacher to take an interest in understanding students' results at a particular point in time.
- Allowed the explanation of questions where comprehension was a problem for ESL students (such as the term 'eye-witness').
- Contributed to the understanding of students' results, which assisted as a motivating factor in the learning process.
- Allowed students to be instructed and to change their points of view when shown conflict situations.
- Explained to ESL students, who were more used to a different approach, the format of open-ended questions requiring individual responses.
- Enabled students to clarify comprehension and translation of word problems.

- Improved students' results by allowing them to explain what they did, subsequently making suitable adjustments to marking and, in some cases, accepting student corrections.
- Monitored changes in student confidence and representational preferences over a period of time.
- Developed knowledge of common preconceptions (such as picture-graph anomalies), thus providing feedback for lesson development.
- Suggested alternative representational approaches to the student and helped to explain the extra information they supplied.
- Identified favourite representations of students and their level of resistance to resolve conflicts between these representations.
- Indicated areas of instruction that I have poorly explained.
- Isolated areas of my own understanding that need improving, such as the concept of changing velocity and its link to the terms acceleration and deceleration.
- Allowed changes in my viewpoint and perspective to be incorporated by interpreting the interview data some time after the raw data was collected, as in the present study.

Disadvantages of interviewing.

- Time-consuming and interfered with normal lessons as other students were given no assistance with mathematical tasks during the interview.
- Not popular with all students as it put them "on the spot" to justify their earlier responses and exposed them if they were simply copying work.
- Shy students talked softly making interpretations of the tapes difficult.
- Highlighted the interview as a descriptive tool with researcher bias introduced by lengthy time delays in the interpretation process.

Benefits of interviewing for the learning environment.

- Reveals to the teacher the inappropriate interpretations students have made in the context of a given problem.
- Identifies long held beliefs of students and the difficulty of changing them even after open discussion.

- Allows the collection of words and expressions (such as homonyms) that can lead to misunderstandings.
- Demonstrates the varying confidence levels of the students under an interview situation.
- Clarifies why the student obtained a particular answer, and so can lead to better learning and instruction.
- Influences classroom dynamics and identifies difficulties in maintaining the motivation level of the rest of the class as one student is interviewed at a time.
- Directly challenges students prior concepts based on personal experience.
- Shows the different level of detail I have used with different students, often linked to understanding or personality variables.
- Identifies students who are too nervous to talk in class and thus facilitates a more suitable learning environment for them.

Summary of interview results.

Important ideas edited from the interview transcripts are now presented for each student in the sample. Transcripts of the interviews are given in Appendix E and the analysis of a selection of these follows below. Tapes were analysed and an attempt was made by myself to create a picture of the learning process used by each student. Commonly occurring themes are summarised in Chapter Five where recommendations are made for future research. The tape-recorded interviews were based around material the students had completed in earlier assessment items. They form the most original part of this study and were the most difficult to analyse because of their extensive detail and the need to synthesize the data with other data collected during this study.

Tonton:

Tonton was nervous during his two interviews. He didn't like them and seemed uncomfortable when confronted and asked to explain his reasoning when solving problems. This made me respond by being more cautious and perhaps probing in less detail to avoid making the student feel threatened. Conflicts of meaning were left unresolved when the results from a picture translation were expressed in terms of speed or summarised in a table of values and contradicted the results from a distance-time graph. In fact Tonton worked from a different premise in calculating the time each car took to reach a fixed distance of 50 metres, instead of working out the fixed time at which the cars were both at the same distance. The distance-time graph clearly showed that at 3.5 seconds the cars were together, whereas Tonton calculated that one car reached 50 metres after 3 seconds and the other car reached 50m after 4 seconds. Tonton refused to use the graph as his preferred translation and relied on his time calculations from the picture even though there were conflicts between the two results. He simply said 'I was confused'. Choosing the tree as a stationary reference point needed clarifying in Task 2: this allowed one vehicle to increase its distance from the tree as it moved away whereas the other vehicle moved closer to the tree such that its distance measurement from the tree decreased. A distance-time graph with downward slope caused Tonton interpretation problems. In particular, Task 4 with graphs of two curves, one where a car sped up and the other where a car slowed down, were both interpreted by Tonton as situations where the car slows down because in his words (Appendix E):

> ...going down means slower, therefore distance gets smaller so the car slows down.(p. 398)

In particular, Tonton seems to be using the shape of the curve from right to left rather than from left to right as was expected (this would lead to the idea of rising up a hill linked to a car slowing down). His interpretations of the curves where velocity was changing are shown following in Figure 4.19 (a) and (b). These results appear to vary from those earlier in Table 4.3.

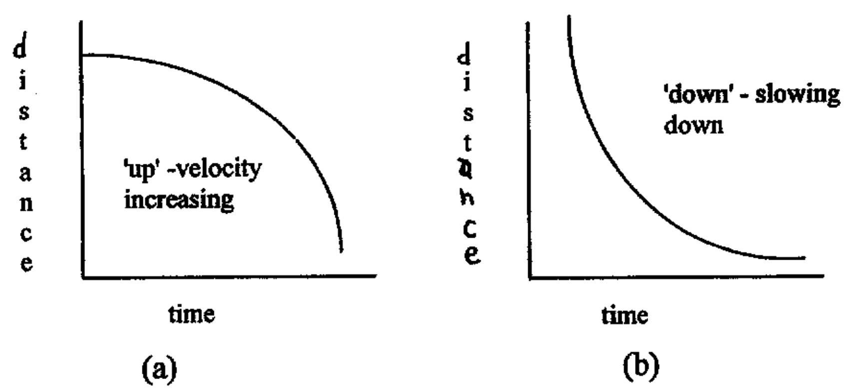


Figure 4.19 Tonton's Interpretations of Acceleration from the Curve Shape

This was a difficult point of the interview, for to create a discussion about the ideas in Figure 4.19 (a) and (b), it was necessary to start from a new direction. One option was to look at the change in distance moved between equal sized time intervals and then compared at different ends of the curves. This would show the change that was taking place. By looking at these discrete changes Tonton agreed that there might be acceleration for at least one of the cars. Similar visual interpretations occur again during Task 6 where the student was asked to comment on whether two vehicles underwent changes in speed. Figures 4.20 (a) and (b) summarise the situation as viewed by Tonton.

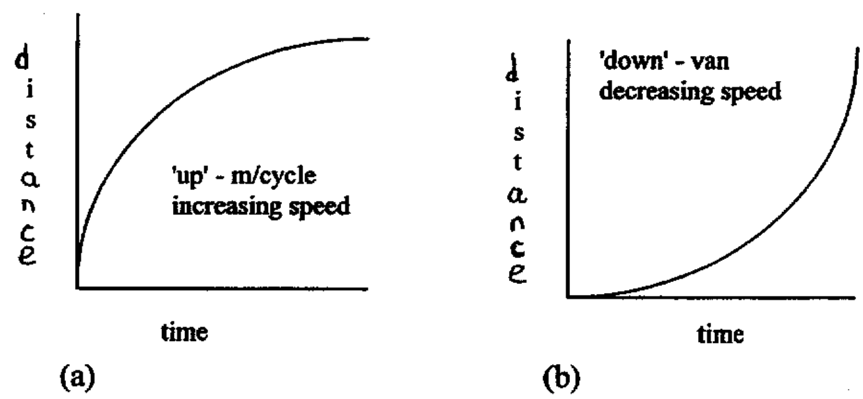


Figure 4.20: Further Curve Interpretations for Acceleration by Tonton

Tonton related the shape of the curve to his own personal view of change. The interview discussion that followed talked about change by looking at small intervals of time and finding the speed whereas Tonton looked at the whole curve shape (a Gestalt view). His approach resulted in a misinterpretation of the situation. There appeared to be some inconsistencies in the responses as Tonton mentioned that the van in Figure 4.20 (b) was initially increasing its speed but then from his point of view because of the 'down' shape Tonton concluded that the van decreased in speed as the curve sloped downward to the left.

Tonton discussed the results he obtained (from Task 6) in interpreting the graphs for that task and illustrated in Figures 4.20 (a) and (b). He appeared to view the overall visual changes, that is a curve moving up or moving down from the left or right, rather than calculating discrete changes in distance that may affect a rate of change.

During an interview discussion with me (the second of the two interviews) Tonton concluded that both graphs were moving down so that they both represented acceleration. Thus by looking at the whole picture this student got the wrong idea. Again, Tonton viewed the graph (for Task 6) and chose the time when the vehicles met, the intersection of two curves, as the time when they have the same speed. Thus he associated catching up as travelling with equal speed. He did not link the steepness of the curve of the cars movement with its speed. Additionally, Tonton viewed the graph from right to left, and interpreted a downward movement as moving down or slowing down, rather than as speeding up. This appeared to be his preferred interpretation of a curve moving down from right to left, and is the exact opposite of what was expected for a time variable moving continuously from left to right. The term "acceleration" also caused confusion in Tonton's mind and alerted me to the difficulties present in the use of this term and the need to use a better expression such as 'speed increasing on an interval' or 'speed decreasing on an interval'. This description would also avoid the situation where a body is increasing in speed in the opposite direction leading to a negative acceleration such as that occurring in Task 4 (Appendix C).

Welly:

Welly was an ESL student who possessed very weak comprehension skills when presented with calculus word problems. As with Tonton, Welly was not used to open-ended problems where students were asked to *create their own situations*. He also failed to comprehend many instructions because his mathematics language facility was inadequate. His interpretations of everyday language in a mathematical context lacked depth, with terms such as *eyewitness* not understood or interpreted as being at a particular position or point in time rather than over a continuum. When asked to consult *neighbours* this term was not understood in the context of the classroom. Welly's belief system that a graph represented a true event or situation and that constant time movement was possible (for example, a car could be placed on a trailer and moved, and that time remained constant during this process) influenced his responses regarding incorrect distance-time graphs (see Task 4, Appendix C and Appendix E transcripts):

> ...the car might be on the back of a truck or someone could push the car. The car could have a break down or service.(Appendix D, p. 338)

Welly was quite open with this response and it was clear he was not quite happy with the continuous time idea and that movement cannot occur at a fixed point in time. Generally, Welly was enthusiastic and happy to be interviewed. His claims of often misunderstanding questions and the extent of details required seemed to overlook inadequacies in his problem solving approach. His answers were often brief and did not relate directly to values on a distance-time graph, the (t, d) pairs, but rather to the independent motion of cars with a stronger reliance shown for the distance variable. His preferred view was that of whole pictures (a Gestalt approach) rather than considering an analysis of discrete changes over small time intervals. In this regard his approach paralleled that of Tonton. His interview response to the Task 5 graphs (Figure 4.19 (a) and (b)) indicated a changed response to that recorded initially, and confirmed his view that any downward movement represents a car increasing speed only if the curve has a concave down shape (Figure 4.19 (a)). His responses also indicated that the method he used did not consider the changes/second aspect of the distance-time graph but rather only used curve shape. In fact in Task 5 (see Figure 4.16 and Table 4.3, pp. 161-162), one graph showed a car slowing down and the other showed it speeding up. Welly interpreted correctly but his use of the terms acceleration and deceleration were reviewed in the interview. When asked (in Task 6), *"when are the vehicles both travelling at the same speed?"*, his response indicated an interpretation based only on observing any straight-line section of the separate paths of the vehicles rather than to use time as a link between them (Appendix E):

> ..constant slope equals same speed so the m/cycle travels same speed when the graph is linear..*when t>4* when its path follows a straight line.(p. 405)

For the second vehicle, Welly did a similar calculation and found the time where that car had constant velocity, a time different from that of the m/cycle. Although correct calculations in themselves, each vehicle had been treated separately and Welly created his new version of the problem rather than indicating when both vehicles had the same speed (by considering, for example, how the slopes of the two curves varied at different times). Welly confused the use of *where* and *what* so that when he was asked *'Where is the car bend'* (when does the car slow down to take the corner) his response was relevant to the question '*What is the bend*'. This annoyed Welly as he felt he was always answering the wrong question and was surprised at the lower than expected marks he received.

John:

John was the non-ESL student in the class who appeared to have other difficulties such as with the translation Task 3 from text to distance-time graph in examples where the car changes direction and returns to its starting point (see Figure 4.21). This student considered only two possible distance-time graphs for this situation as shown below. According to John (Appendix E):

> ...if you go down (on the distance-time graph) this could be mistaken as decelerating or slowing down to a stop.(p. 401)

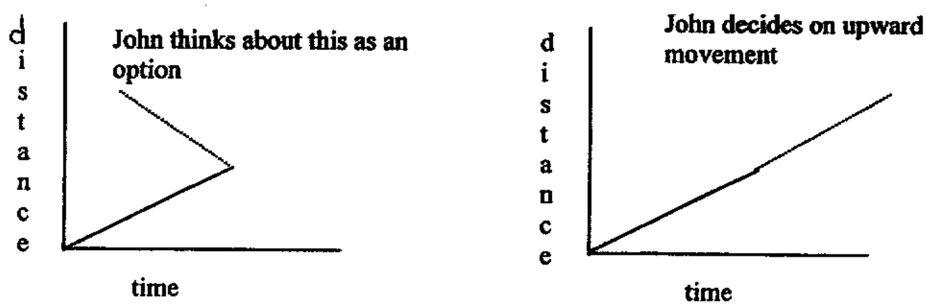


Figure 4.21: John's Ideas on Changing Direction with Constant Speed

John let distance always increase to indicate movement with constant velocity or acceleration whereas he interpreted a downward line as decelerating. He lacked a reference point analogy where the distance axis scale would normally represent displacement from a starting point. This may simply indicate a lack of discussion by me on the distinction between distance and displacement. For John, decelerating was slowing down and accelerating was speeding up. His view (for linear graphs) ignored the situations where the car may change direction and move with a changing negative velocity such as those presented in Task 5 and illustrated in Figures 4.19 (a) and (b). For Task 5 (Appendix C) John contradicted his comments on downward movement indicating a deceleration and indicated that the two curves showing downward movement showed bodies speeding up and therefore accelerating. The interview involved discussions about his use of the term acceleration when cars have an increasing negative velocity rather than an increasing positive velocity and John understood this in relation to the type of questions given. John was uncertain how to start the graph of a car with non-zero speed at time zero (t=0) as was required in Task 6.

Oliver:

Oliver also assumed from prior knowledge that cars met at the same position with the same speed. In the first interview discussion he indicated that during overtaking he assumed cars would have the same speed, but he then correctly changed his thinking after I gave further information. His clarity of thought on changes of velocity between a small interval of time, as illustrated in the notes for Task 6, was superior to that of the other three students. Oliver correctly interpreted acceleration in terms of differences and correctly answered all questions involving this idea, even though he appeared to be the weakest mathematics student overall. He could directly link the distance-time graph with a velocity concept and correctly explain the movement of the vehicles in Task 6. For example, part of the interview discussion included the following statement from him (Appendix E):

> ..the m/cycle decelerates since it is fast early then moved slower and finally stops. (p. 405)

He was the only student to look at the Task 6 accelerations of the motorbike and moped and correctly interpret the link between changes in velocity on the curves and the acceleration/deceleration of them. However Oliver stumbled on Task 5 (Appendix C). The distance-time curves in Task 5 involved a downward slope that seemed to cause trouble with many students. Oliver was not sure whether the car was slowing down or speeding up on those curves. During the interview this was discussed with me from the point of view of velocity increase or decrease over an interval regardless of the direction of motion. Oliver assumed any velocity increase represented acceleration, and a discussion followed highlighting the inadequacies in the current definitions of the terms acceleration and deceleration, in terms of creating a clear and adequate platform for student understanding.

First Test Results

The aim of this test was to identify any changes in student learning over the period since the Pre-Test and interviews. The test, written by myself, was designed to highlight areas where students still had difficulties and indicate any changes in representational preference that may have occurred since the earlier *Traffic* tasks. Most of the problems given to students were designed to suggest the use of a particular representational approach where there was a standard solution method. Problems involving real world situations and with accompanying diagrams were designed to be well structured so that students needed to make a plan before solving them. The ability of the students at this time was unknown, so that I was not aware if the test items were at an appropriate level. This was particularly the case with this group of students, the first of my classes to use the *Traffic* tasks, to undergo interviews in class time as well as to use resources for training in the techniques of the graphics calculator. In many cases, it was the graphing feature of the graphics calculator (the PLOT mode in the case of the HP38G) that was the most commonly used by the students. With this feature, it was possible to handle a variety of calculus problems such as:

- Finding distance, velocity and acceleration for given equations of motion
- Determining maxima of functions
- Checking graph to equation translation problems by entering equations into the calculator.

An example follows where students were asked to translate from $f(x)=x^2$ to the curve moved horizontally two units to the left, with equation $f(x)=(x+2)^2$. Oliver produced his own rule by including an extra term with a negative two coefficient: $f(x)= x^2-2x$. This student, due to time constraints or simply through being careless, did not use the features of the graphics calculator to enter his equation to see the difference in the shapes of the curves. This action would have provided him with reasons to adjust the equation and continue the checking process.

Analysis of test questions.

Question One: This involved the translation of six graphs to equations where each graph involved a different representation of the curve $f(x)=x^2$ (see results in Table

4.11 following and Appendix C to view the question). Tables 4.9 and 4.10 following give a typology of representational and graphics calculator use by question for the test. In Table 4.9 the value of *Yes* occurs only once in each column. In question 2, for example, graphics calculators are expected to be used to assist in problem solution.

Table 4.9: Typology of graphics calculator use: Test One

Typology	1	2	3	4	5
Graphics calculators are expected to be used	No	Yes	No	No	No
Graphics calculators are expected to be used by some students but not by others	Yes	No	No	Yes	No
Graphics calculators are not expected to be used	No	No	Yes	No	Yes

In Table 4.10 it is clear that problem presentation has used a range of modes covering words, pictures, symbols and graphs.

Table 4.10: Typology of Representational Use: Test One

Typology	From	To
1	Graph	Symbol
2 (profit)	Words, Symbols	Table, Graph
3 (roller-coaster)	Picture	Graph
4 (can)	Words, Picture	Symbol, Graph
5 (Task 7)	Graph	Table, Words, Graph

Results for three of the test questions are shown following in Table 4.11. The results in the table are raw scores, for example, Oliver gained 4½ out of 12. For question one Welly and Tonton both used the general form $f(x)=a(x-b)^2+c$ and each had full understanding. They translated directly from graph to equation and obtained full marks whereas John made one error that indicated he did not check by using the graphics calculator. Oliver was consistent in the use of an alternative rule mentioned earlier where terms in the variable x with selected coefficients were added or subtracted. Oliver, in particular, could easily be influenced in his method with the

To find a maximum value of Oliver's equation would give a different solution, dependent on the x-domain used on the graphics calculator, as in his case the maximum depended on the domain used. Oliver's work was marked without realising that a simple transcription error had been made, and therefore more marks should have been allocated. Oliver was confident using his calculator, but in doing so had reduced his chances of a higher mark. Tonton rounded the maximum selling price incorrectly and then calculated the maximum profit manually using the equation without checking on the calculator. None of the students attempted the question on marginal profit, which lowered their overall mark for this question, and each one complained bitterly this topic was not dealt with in class, even though the ideas on changes in distance in small time intervals were introduced in the earlier Pre-Test.

Question Three Roller-coaster: (Bell et al., 1987).

In this question, a roller-coaster car moves along a track, initially at constant speed as shown in Figure 4.24 following. Students were required to translate from the picture to a graph of velocity vs time as well as give a written summary of speed changes as the roller-coaster car travelled along the track between labelled sections. This was the first time that students had been asked to sketch a speed vs distance graph. Labels on the picture represented important features of the movement of the roller-coaster. There is often a tendency among students to assume a strong link between the shape of a picture and its corresponding velocity-time graph, even when the true graph may be very different (Bell, Brekke & Swann, 1987; Goldenberg,1988). In the present study students appeared to have overcome this illusion for this particular situation where only visual cues were given, as no raw data on the velocity change with time was available.

Welly's results (see Appendix D, extract 2.G) indicated a conflict between his velocity-time sketch and the corresponding written answer as verified during the subsequent interview (see Figure 4.23 following). He talked of the roller-coaster moving with constant speed as it moved up the hills (he assumed it must be dragged up by a motor) whereas on the velocity-time sketch he showed the speed decreasing as the car goes uphill.

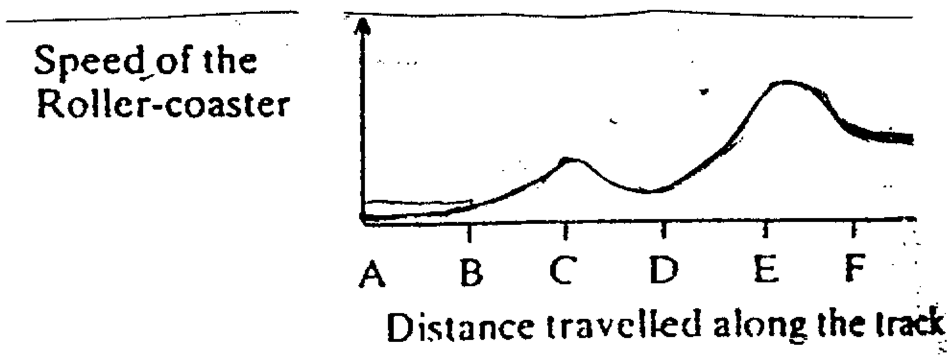


Figure 4.23: Movement on a Roller-Coaster - Welly

His sketch of motion was very accurate and showed non-linear speed changes. At no time did he mention acceleration in the discussion. Welly's sketch followed the roller-coaster picture movement in a reverse pattern with increases in speed shown as the car moved downhill being correctly sketched. This image overrode his understanding of the situation in terms of mechanical aspects, so that the conflict with the written description was not resolved. Again Welly placed little value in the graph compared to other representations.

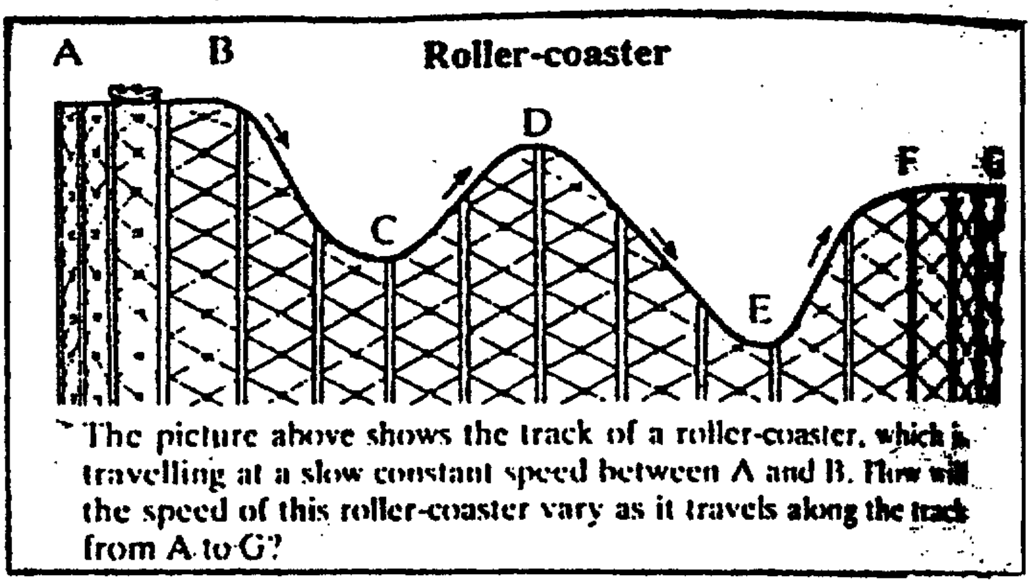


Figure 4.24: Movement on a Roller-Coaster

Oliver and Tonton (Appendix D, extracts 3.G and 4.G) were not as competent as Welly in creating velocity-time graphs and used only linear increases in velocity on their velocity-time graphs even though the roller-coaster track was very curved and so suggested an acceleration/deceleration process. Their graphs looked like a number of straight lines with different gradients distinct from those shown in Figure 4.23. Oliver used his knowledge of the roller-coaster situation to predict it would slow and stop on the last flat section of the ride. No use of the mathematical terms accelerating or decelerating occurred in Oliver's written description. Tonton had a similar view of the roller-coaster movement to Oliver, with movements from constant velocity to acceleration shown by line segments joining sharply, rather than having a smooth transition. However, at the end of the roller-coaster car trip, Tonton described the car as having constant speed on the flat section but sketched the movement of the car as a line with increasing slope indicating a mismatch between his understanding and the graph. Tonton showed the slow constant speed at the start of the car trip as a flat horizontal curve indicating constant speed, but with a value of zero.

In using the terms accelerating and decelerating on the sketch, John used more mathematical language than Tonton, as he correctly interpreted non-linear changes in the speed of the roller-coaster and included them into his sketch. The quality of John's descriptions and sketch were far superior.

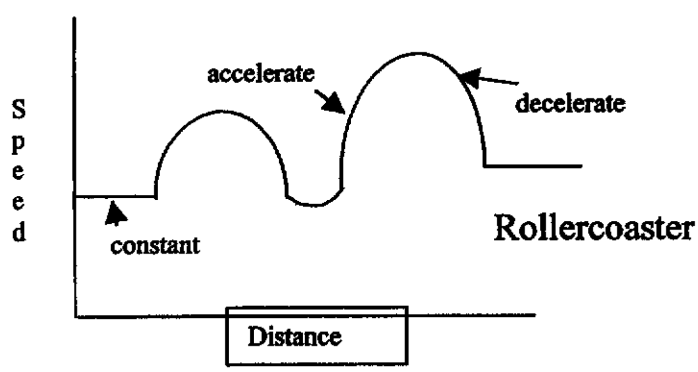
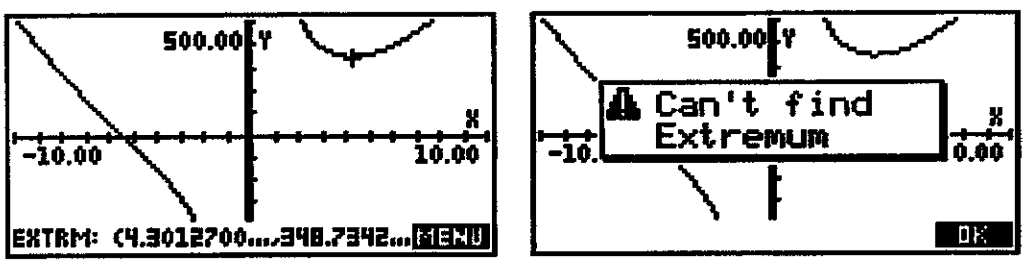


Figure 4.25: Movement on a Rollercoaster (John)

Question Four Tin Can Problem: (Shell Centre for Mathematical Education, 1986). A real world problem presented as a picture of an orange fizzy drink can that must hold half a litre of drink. Students were asked to design a can such that a minimum of aluminium material was needed, and they were given hints as well as access to volume and surface area formulae. The problem required a high degree of reading, comprehension and translation skills. Tonton did not know what to do to get started, as his translation skills from picture and text to equations did not give him ideas on the next step other than guessing one solution. Tonton did not use his graphics calculator as he had no equations to enter. The graphs in Figure 4.26 (a) and (b) following show my calculator solution with surface area versus radius which none of the students achieved. The first graph gives the solution with the cursor (denoted by a + sign) placed correctly on the positive radius axis whereas the second graph shows the result when the cursor was placed on the negative radius axis. There were three important steps to reach this stage using the graphics calculator with different students succeeding to varying degrees. The first was to translate the two equations involving surface area and volume into a single equation, secondly to remove one of the variables, such as height or radius and finally to enter the equations into the graphics calculator and make sure the calculator cursor is on the positive branch of the plot so that an extremum can be found.



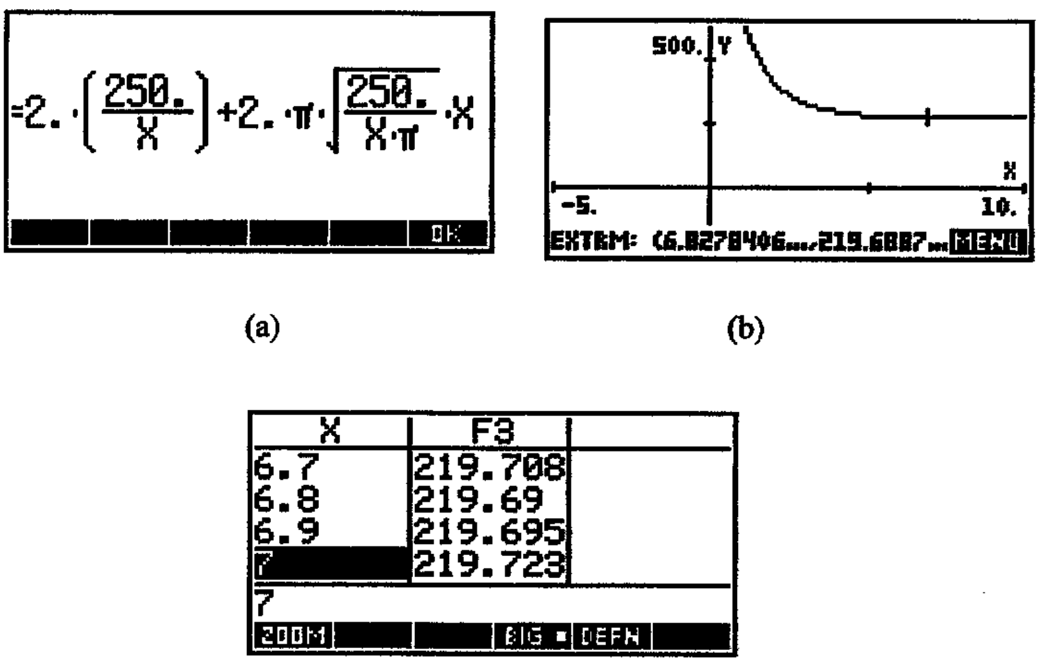
(a) (b)

Figure 4.26: Surface Area vs Radius for a Cylindrical Can

Welly did not seem to understand the covariation properties of height and radius for the can. He predicted that as the radius increased from zero, the surface area would also increase, with a limit to how large the radius can be. In algebraic calculations involving the volume equation, Welly used a term in r mistakenly for r^2, and then no attempt was made to solve the equations. When asked to comment on the effect of making the can taller or wider it became clear Welly had little comprehension of the situation (see Appendix D):

> It would have to be narrow because we want to make the volume of the can 500 cm^3, so it can't be wide or we need a bigger volume of the can. It would have to be short because we must cause it to be 500 cm^3. As the radius increases the volume, the surface area and the height are increasing too.(p. 343)

John used his calculator to come to the conclusions shown in Figure 4.27 (a), (b) and (c), showing respectively the symbolic, graphic and numeric aspects of his solution. A different value for the height results based on a different equation from that used in the graphs of Figure 4.26 (a) and (b). His method was to use a height value to find the radius and surface area using formulae, and because of the difficulty level of the formulae, chose to use a numerical approach. Judging by the small number of iterations he used, John probably used the SOLVE aplet of the calculator, which allows one to enter values to solve for one unknown. In this case, he would need to enter two equations. His technique was valid although there was an algebraic error, consistent with using an incorrect volume formula of $2\pi r^2 h$ rather than $\pi r^2 h$.



(a) (b)

(c)

Figure 4.27: A Second Equation for Surface Area vs Height

Oliver (Appendix D, extract 4.G) understood the inverse relation between height and width but had no idea where to start solving the question from the two given equations. Thus, this student did not have the skills necessary to create an equation with only one variable, enter the equation into the calculator and estimate a solution.

Question 5 (Task seven)

In this task students were asked to interpret a distance-time graph of vehicles as their distance from a telephone box changed. Some vehicles overtook each other as they sped up or slowed down. As part of the same task, students answered questions on a distance-time graph that showed the motion of two cars as they approached and negotiated a bend in the road. Students used time calculations in their discussions of the movements of the cars, something lacking among most students in the pre-test, and which could be seen as a positive result of the series of interviews with each student immediately following the Traffic tasks. Results of both the Can question and Task seven are shown in Table 4.12 following.

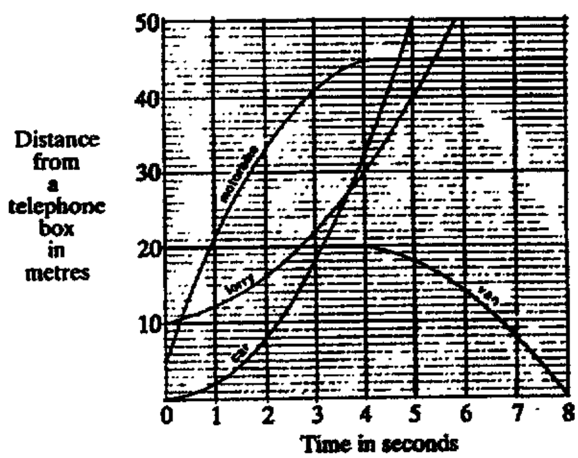


Figure 4.28: Test One Distance-Time Problem

(Shell Centre for Mathematical Education)

Table 4.12: Task 7 Student Results

Student	Can (12)	Task 7 (24)
John	9	9½
Tonton	0	8½
Oliver	½	13½
Welly	½	5½

The results of this final question on test one showed that Oliver performed better than any of the other students. This task had clear visual clues with pictures of the moving cars involved, and Oliver had developed a suitable heuristic approach in constructing a table of values and calculating differences. Oliver's written description (see Appendix D, extract 4.G) of what happens on a distance-time graph as vehicles accelerate, decelerate, slow down or are stationary was restricted to comments on the time variable. No mention was made of velocities or accelerations:

> *..the motorbike takes over the lorry and the parking van in one second before stopping in the fourth second. (p. 385)*

This indicates that his knowledge and use of mathematical language was limited, although he translated information from the graph correctly. When asked directly about the acceleration, Oliver correctly described the situation with all vehicles. As an example (see Figure 4.28), the van moves in the opposite direction to the other vehicles with increasing speed. Oliver interpreted this as acceleration, although he was given no marks. I was not sure how to mark this question, but thought it necessary to indicate to the students that the velocity was becoming more negative. This ESL student experienced interpretation problems when asked '*What is the deceleration of each car?*' and there was an interpretation error. Oliver assumed information about the meaning of deceleration was being asked rather than details about the exact value of the deceleration. When asked '*What do the speed-time graphs look like?*' Oliver simply described the axes of the graph rather than their properties. 'What' questions were apparently too open ended for him, so Oliver attempted his own interpretation of these questions and he created meanings often different to those the teacher expected. However, when directly asked to produce a table or a graph, Oliver responded as expected with the appropriate details. When asked '*How does the distance between the two cars vary? (Draw up a table, sketch a graph, or do both!)*' Oliver created a table and a graph, but gave no written answer to the question, preferring to let the teacher make the interpretations.

The calculation of initial velocity values was not well understood by Oliver either – for example, a starting distance of 5m for the motorbike was interpreted as a starting velocity of 5 units. Oliver's speed-time graph of a decelerating motorbike showed an initial increase in speed before slowing down linearly to zero. No understanding of movement in the opposite direction and the subsequent use of negative values were evident as this concept was beyond the ideas introduced to the students at this stage.

Tonton identified the acceleration of cars going an increasing distance with time but was not sure how to interpret a van moving in the opposite direction (Appendix D, extract 3.G). As part of the Task 6 exercises, the movement of a van (shown by a curve moving down to the right) travels increasing distances with time, but was said by Tonton and Welly to be decelerating. Tonton interpreted distances between two cars as the position of the cars at different points in time (on a horizontally scale), rather than the distance apart at a fixed time (on a vertical scale). This interpretation was carried through to question 5 on Test one, as two cars remain two seconds apart during the entire journey. With a two second interval the cars end up travelling exactly the same distance along the road, but from that frame of reference information is lost about distances apart.

John confidently talked about the position of the cars in terms of time, position and acceleration in Task 6 (see Appendix D, extract 1.G). He saw the van as accelerating, as it moved in the opposite direction and increased its velocity. The notion of a negative increase had not occurred to students at this stage. He expressed uncertainty about the nature of the acceleration, whether it was uniform. The idea of uniform acceleration was not discussed in detail with the students and there was limited accurate data from the simple distance-time graph given to allow a more detailed analysis.

Welly preferred to work without calculators and instead relied, in general, on a symbolic approach where the question allowed (Appendix D, extract 2.G). He did not see the graphics calculator as more effective, quicker or better and did not use it for checking results during the first test. Regarding the roller coaster problem, Welly's previous experience influenced his response. On each upward leg of the coaster trip he assumed that there is a constant velocity dragging the coaster up, as occurs on some roller-coasters at some point of their journey. However there was a technical conflict with his subsequent plot of velocity versus time. Even though Welly said speed was constant on upward legs of the journey, he plotted a line with non-zero slope on these parts of the graph indicating acceleration was occurring. This indicated a text/graph dilemma occurred. Comprehension problems occurred during the surface area minimisation problem with the tin can. Welly could not comprehend the term *SA* (surface area) that limited his understanding of how to tackle the problem. When an example was given as a hint to assist in solving this problem, it was interpreted as being unrelated to the main problem and so was not an advantage.

For Task seven Welly observed the graph but did not consider incremental change in his decision making. When asked *are they accelerating or decelerating?* he considered a simple 'yes' or 'no' as adequate rather than an explanation in more detail. He also considered the bend in the road as the point where the cars slow down rather than involving a reference to the time variable. He viewed the curves with the two cars moving rather than observing the detail of a distance-time graph with an incremental change approach.

The results of the Traffic Test showed a number of improvements had occurred in student understanding following the earlier interviews:

- An improved understanding that time details were appropriate to use in many calculations involving distance-time and velocity-time graphs
- An enhanced self-awareness on the part of the teacher/researcher that the clarity of problem presentation influenced student representational use and problem solving ability

- An improved ability to understand positive velocities, but only to the extent of comprehending and translating from curves showing increasing speed

There were still a number of areas where student understanding was inadequate and little change had occurred. These details are summarised in Table 4.18 (pp. 204-205).

Final Examination (Appendix C)

Special attention was given to a number of factors when choosing questions for the final examination. These included the language used and its combination with other representations in the presentation of a problem, the level of difficulty of the problems and the necessity for the use of graphics calculators for an efficient solution. The examination was trialled on a group of four students during the second semester of the present study. The examination period was three hours and the test consisted of eight questions. Students in the trial found these items difficult, with the two best results scoring around 45%. On an examination of these previous examination results, I considered that there was a suitable balance of questions where the graphics calculator could be used with advantage. This was based on the topology shown in Table 4.13 and was consistent with guidelines from other research (Malone, et al., 2002). The questions from the examination were classified regarding the degree of graphics calculator potential use (Kemp et al., 1996) and these details are summarised in Table 4.13 following. Most questions were classified as items where graphics calculators were expected to be used by all or at least some students. There was only one question where graphics calculators were not expected to be used (question 6).

Table 4.13: Typology of graphics calculator use (Kemp et al. (1996))

Typology	1	2	3	4	5	6	7	8
Graphics calculators are expected to be used	Yes	No	Yes	No	No	No	Yes	Yes
Graphics calculators are expected to be used by some students but not by others	No	Yes	No	Yes	Yes	No	No	No
Graphics calculators are not expected to be used	No	No	No	No	No	Yes	No	No

Table 4.14 summarizes the representational requirements of each question indicating the extensive use of graphs in problem presentation and the common requirement for a symbolic approach.

Table 4.14: Typology of Representational Use: Semester Two Examination

Question	From	To
1	Graph	Symbol
2	Symbol	Symbol, Table
3	Graph, Symbol	Symbol, Table
4	Graph, Symbol	Symbol, Table
5	Graph, Symbol	Symbol
6	Graph	Graph, Symbol
7	Symbol	Graph
8	Graph, Symbol	Symbol

Results of the final examination are shown below in Table 4.15 following. The results show that despite the treatment intended to improve students' results in the introductory calculus course, there were still areas where students showed major misunderstandings. This was highlighted in the results of question 8 involving the velocity-time graph of a moving car, a topic that was covered in extensive detail in the preliminary part of the present study.

Table 4.15: Student Final Examination Results

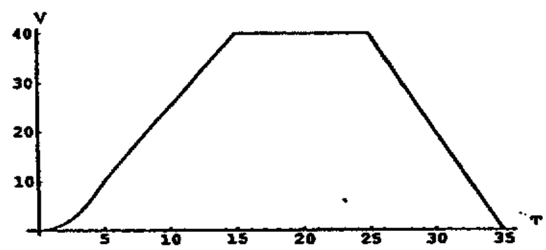
Question / Student	1 (12)	2 (13)	3 (15)	4 (15)	5 (10)	6 (10)	7 (8)	8 (17)	Total (100)
Welly	12	10.5	11.5	15	5	0.5	1	4.5	60
Tonton	9	8.5	15	14	6	10	6	10.5	79
John	12	5	15	15	6	9	8	17	87
Oliver	12	3	13	15	0	0	8	5	56
Average	11.3	6.8	13.6	14.8	4.3	4.9	5.8	9.3	70.5

The results for question 8 (see Figure 4:29) indicate that when the topic of rate of change was placed in new contexts with variations in its representational mode, many of the ESL students were still having difficulty with the concepts involved whereas the non-ESL student scored full marks on this question. A detailed question-by-question analysis follows.

(Year 11) EXAMINATION

8. The traffic police have set up a speed trap. As a speeding car *C* passes through the trap, a police car *P* gives chase. *P* accelerates until it reaches a velocity of 40 ms^{-1}, proceeds at that velocity until it overtakes *C*, then slows down with constant deceleration until it comes to a stop. Meanwhile, *C* continues at a constant velocity until *P* overtakes it, then slows down with constant deceleration until it comes to a stop immediately behind *P*.

The velocity of the police car *P* during the chase is represented by the following velocity-time graph:



where the velocity $v(t)$ in $m\ s^{-1}$ at time t seconds is:

$$v(t) = \begin{cases} pt^2, & 0 \le t < 5 \\ 3t - 5, & 5 \le t < 15 \\ q, & 15 \le t < 25 \\ -rt + s, & 25 \le t \le 35 \end{cases}$$

a) Determine the values of the constants *p*, *q*, *r* and *s*.

b) Find the acceleration of *P* when $t = 4$.

c) How far does *P* travel between $t = 0$ and $t = 35$ seconds?

d) Find the velocity of *C* during the first 25 seconds

e) At what time does *C* stop?

[4, 2, 5, 3 and 3 marks]

END OF EXAM

Figure 4.29: Final Examination Velocity Question

(Secondary Education Authority, 1992)

Question 1 (Final Examination – Appendix C): Excellent results were obtained by the students when finding the area between two curves when the graph is given and graph scales clearly specified (see Appendix C). Techniques varied extensively but only one student, John, used two different calculator methods to check his work.

used the fastest method via the *Home* screen of the graphics calculator, which involves using the symbolic representational form of the question as an integration formula. John used the graphing functions of the calculator to check his Home screen formula approach. Oliver transformed the question into one involving a simple trapezium approximation, comparing this to the result of a calculator *Home* screen integration. He was the only student to take such an approach. The multi-representational approach was a new feature in Oliver's problem solving. Tonton used a symbolic approach without the calculator, and algebraic errors included simple confusion of signs and improper values for limits of integration. This implied an incorrect calculator value may also have been found by this student, as these same errors in problem translation are likely to have been transferred to the graphics calculator. Both Tonton and Welly did not realise the efficiency of the calculator for this problem.

Question 2 (Final Examination – Appendix C): Students were given two definite and two indefinite integrals to calculate (Appendix C). Welly and Tonton did not use calculators when answering the question or when checking results. Both students had difficulty with coefficients, especially for $e^{(x/3)}$, and for Tonton there was a misunderstanding with 2/(1-x). John had major problems understanding the types of functions, and applied standard integration techniques in the wrong context. His integration of $4/x^4$ as $(4/5)/((1/5)x^5)$ and 2/(1-x) as $1/(x-(1/2)x^2)$ show an inappropriate use of rules for the integration of power functions as they were being applied to rational functions. John and Oliver did very poorly on this question because both used the power rule incorrectly as mentioned, and incorrectly entered a power as a product for the indefinite integral calculation where an exponential function was involved. The indefinite integral questions could not be done as efficiently on the graphics calculator.

Question 3 (Final Examination – Appendix C): The translation of this calculus word problem showed the range of abilities in comprehension and translation of text, graphs and formula amongst the sample students. John, Tonton and Oliver had no comprehension or translation problems. Welly interpreted the width of the wake of the boat as the horizontal distance rather than the vertical one, correctly found the approximate trapezium area, knew that the exact area is given by an integral but saw no way to find it.

Question 4 (Final Examination – Appendix C): Students were to find the area between curves shown on a sketch with intersection points not given. Welly and Tonton used algebra without the calculator and successfully solved it (apart from the order of equations for Tonton). Oliver and John chose a symbolic approach using the Home screen of the graphics calculator and successfully found a result. John was the only student to look at two different approaches with the graphics calculator, one as mentioned, the other by entering the functions, graphing and using the area function available on the calculator which then numerically finds the required area. The use of multiple methods with cross-checking is a recommended approach for effective problem solving.

Question 5 (Final Examination – Appendix C): A calculus word problem was presented with a graph showing two sections of a water slide connecting but not joining smoothly in the mathematical sense of being tangential at the connecting point (Appendix C). Students were to verify that the equations given for the slide sections were not tangential, and estimated coefficients for a better slide where the sections do join smoothly. Welly and Tonton used algebra (see Appendix D, extracts 2 and 3, pp. 345, 365), and whereas both understood the term tangential, Tonton successfully differentiated but Welly used the reverse process antidifferentiation. Neither understood how to construct the derivative for a new slide so that it had a smooth connection. Oliver gave no response but John entered the equations into the graphics calculator in an ingenious fashion to find where the curves for the two parts of the slide intersect in more than one point. John then tried other equations that intersected only once but failed to check that they were tangential, in the mathematical sense of equal rates of change at the point where they met. Thus, his initial solution method was not appropriate: to be tangential requires two conditions, and John only checked one of them.

Question 6 (Final Examination – Appendix C): Welly and Oliver interpreted the terms maximum and minimum rate as simply maximum and minimum, so they focused on only part of the question and so did not comprehend completely what was required. Welly interpreted points of inflection incorrectly and used axis intercepts instead. Tonton understood the term points of inflection and used an appropriate formula. John had an excellent understanding of the situation and found the maximum rate of change, but used turning points to locate the minimum rate of change. He seemed uncomfortable with negative values for the rate. It was not clear which methods John used.

Question 7 (Final Examination – Appendix C): This was a question where students could easily be confused by the graphics calculator output and it would take a skilled user to find an effective plot and produce an appropriate solution. The problem involved the sketch of a function over a given domain. There was a vertical asymptote involved, and this often creates difficulties when plotted with a graphics calculator due to the limited resolution quality. Welly sketched the curve required but preferred to use algebra to find the minimum, and then used an inappropriate integration method rather than differentiation. Tonton relied on the graphics calculator completely, so it appears that the textual word *sketch* is one of the suggestions he requires in order to the use the graphics calculator as his preferred approach. His sketch, shown in Figure 4.30 reflected the unclear calculator resolution around asymptotes, and he failed to place the asymptote on the graph. No students relied on a table of values to assist them in finding the sketch. John and Oliver produced excellent results, but John was the only student to place an asymptote on the graph.

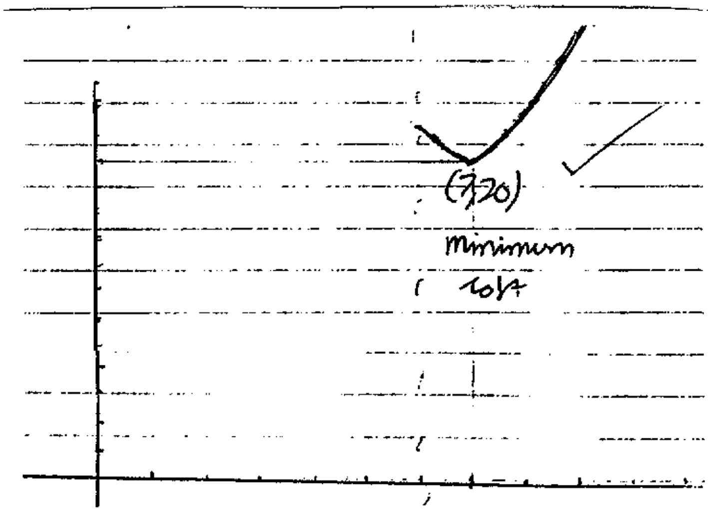


Figure 4.30: Examination Two - Question 7 (Tonton)

Question 8: Calculus word problem showing the (piecewise) velocity-time graph of a moving police car as it caught another speeding car after a 25 second chase and comes to rest again after a total of 35 seconds, as shown following in Figure 4.31.

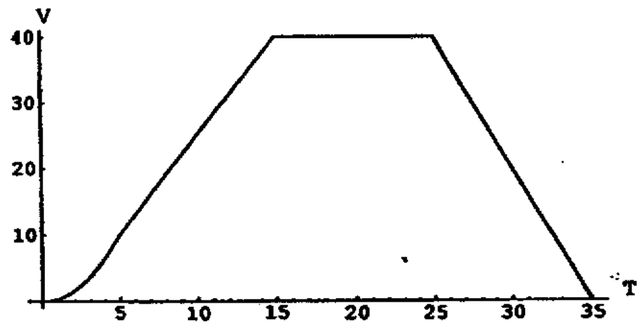


Figure 4.31: Velocity-Time Graph for an Accelerating Car

Students were told a speeding car moves at an unknown constant velocity before decelerating and stopping behind the police car. Welly showed little understanding of the piecewise graph and could not find the acceleration from the equation given for the velocity. He understood that distance is the area under the curve but did not use integration to find the area. The text information about the second car given in the original question could not be interpreted. Tonton did better in finding the coefficients of the velocity equations but failed to correctly find the coefficients for the linear sections of the graph in Figure 4.31. He did not find the acceleration and did not appear to see its connection with the velocity equation, although he understood area and distance relationships and correctly used integration. Tonton read and comprehended the text on the second car and completed the calculations. John had an excellent understanding of the problem, and appeared to find the acceleration using the calculator, without giving details. John incorrectly inferred that the two cars decelerate together at the same rate. As the police car moved faster before the deceleration began, a different stopping distance for the cars resulted. John gave a clear explanation of his working. Oliver found only one coefficient, the difficult one for the quadratic section of the curve. He had no idea about how to find the acceleration or the total distance, although he seemed to link this to the area. Oliver concluded that the two cars were travelling at the same speed when the police car overtook, thus he did not adequately interpret the text of the question and preferred to use his own pre-conceived ideas on car movements. He was not the only student to make this assumption, indicating this problem's level of difficulty.

Graphics Calculator Survey

This survey was designed and conducted during the instrument trials for the main study in order to obtain feedback on students' views of the current learning environment. Necessary improvements could then be made to classroom instructional material and teaching approach before the main study began. Student feelings on the use of the graphics calculator were also collected in the same survey. A number of studies have used a survey on student perceptions of graphics calculator use, including one by Quesada and Maxwell (1994, p. 209) that dealt with the following questions:

> (a) whether the graphing calculator had helped the students to do more exploration and investigation, (b) whether the graphing calculator had helped them to understand the concepts studied in the course, (c) if they would like to take future math courses in sections where the graphing calculator would be used, and (d) their overall rating of the use of the graphing calculator in the course. In addition they were asked (e) if their performance in the course was better than in previous math courses, and (f) if they have spent more time doing mathematics in this class than in previous math courses.

At the end of the first two semesters before the present study commenced, a trial survey was conducted at the end of the introductory calculus course with three ESL students. The survey was based on the Math-CLES, a classroom learning environment survey (Green, 1994) details of which were discussed earlier in Chapter Two (p. 92). The general results were summarised and the main details follow. Students felt that in the environment of the mathematics classroom with student-centred learning and extensive opportunities to use the graphics calculator:

- They generally preferred to use the instrument
- They were not always confident in using it
- The graphics calculators were used too much by the teacher
- Lessons were more enjoyable and interesting
- Problem solution was not always found more quickly.
- The lesson approach helped understanding

- Graphics calculators were not used to check answers
- Their algebra skills improved
- They wished to continue to use graphics calculators

Questions used in the graphics calculator survey were classified into *personal choice*, *mathematical* and *environmental*. These categories were based on earlier work by Green (1994) and have been discussed in more detail in Chapter Three (p. 138). Tables 4.16 and 4.17 detail the number of questions in each of the three categories as well as the average score for two of the students in the trial study, Filda and Jatos.

Table 4.16: Graphics Calculator Survey - Male

Jatos	No. of Questions	Average
Personal Choice	13	3.8
Mathematical	14	3.3
Environmental	9	3.1

For each question students selected from five choices ranging from *Almost Always*, or favourable with a score of five, to *Almost Never*, or unfavourable with a score of one. As an example, the average scores in Table 4.16 for the male student indicate overall favourable outcomes with the *personal choice* category showing the greatest level of student satisfaction with a score of 3.8.

Table 4.17: Graphics Calculator Survey - Female

Filda	No. of Questions	Average
Personal Choice	13	2.1
Mathematical	14	3.4
Environmental	9	3.1

The results of these two students show a diversity of results. This survey had an impact on the main study but was not repeated due to time constraints, the need to concentrate classroom time on regular work and because the survey had achieved its goal as a developmental tool for the instruments and instructional approach of the study.

The average scores shown in Tables 4.16 and 4.17 and the histograms in Figures 4.32 and 4.33 indicate divergent views by Filda and Jatos (both ESL students from Indonesia) on some aspects of graphics calculator use in the classroom. In Figure 4.32, for example, the *Personal Choice* questions had 5 responses in the *Almost Always* category, 3 in the *Often* and *Sometimes* categories, 2 in the *Seldom* category and 0 in the *Almost Never* category. The Personal Choice questions involved items about preference, personal feelings and future thoughts on graphics calculator use. It is clear from the responses that for Jatos the experience was, in general, one that improved confidence and enjoyment, and carried the wish to repeat the use of graphics calculators in future courses. For Filda, the opposite feelings seem to be conveyed, shown by the low average rating in Table 4.17 and that the majority of the histogram values for the *Personal Choice* category have high frequencies for the *Seldom* and *Almost Never* response groups. Questions based on the *mathematical* and *environmental* variables, however, show similar responses, in general, for both students, as indicated by the average marks in Tables 4.16 and 4.17. However, the female student showed considerably more variation in response for the mathematical questions, with the highest frequencies for the female student being four whereas a frequency of six occurred twice for the male student. The *environmental* category indicates a fairly neutral response from both students to the learning style used in the classroom.

Personal Choice | Mathematical | Environmental

Figure 4.32: Response Frequencies for the Male Student

Personal Choice Mathematical Environmental

Figure 4.33: Response Frequencies for the Female Student

Longitudinal Change In Representational Preference

One of the aims of the present study was to compare the student's changing management and use of representations over the period of the study. Table 4.18 synthesises trends noticed in student learning style based on the information summarized in this chapter. The table contains a column for each student in the sample and compares student development by focusing on three important assessment tasks, each at a significant time in the study. Each of these tasks are listed in the first column of Table 4.18.

A summary of the longitudinal change overall for each student follows:

Welly

- Maintained a symbolic mode preference
- Word problems caused confusion and little improvement was made in reaching the symbolic stage of problem solution
- Learnt to plot acceleration as a curved line on a distance-time graph
- Learnt that time calculations were required in distance-time graph analysis
- Trusted calculations in preference to graphical interpretations and maintained this approach over time

Tonton

- Preferred gestalt (overall shape interpretation) instead of a small time interval change approach
- Maintained acceleration as straight line (constant) change rather than using a curved approach
- Learnt that time calculations were required in distance-time graph analysis

- Symbolic preference even for contradictions in answer using different modes
- Velocity-time graph and translation skills weak
- Distance fixation in calculations comparing two cars rather than using time comparisons

Oliver

- Earlier beliefs held strongly – cars overtook at the same speed
- Interpreted curve shape incorrectly as velocity increasing/decreasing but later correctly interpreted the same situation using change in small intervals
- Maintained acceleration as a straight line – no change
- Learnt to use time calculations in small intervals to explain a distance-time curve
- Preferred direct questions giving representations to use rather than "What.." questions
- Preferred graphics calculators for graphing and integration but failed to cross-check using other modes to eliminate errors on negative areas and asymptotes

John

- Preferred a numerical approach e.g. moves 30m/s so 60m in 2 seconds and over time used the graphics calculator to check his work
- Cross-checked integration using both area functions and definite integration formulae both on the graphics calculator. At the same time struggled with indefinite integration
- Interpreted vertical axis on a distance-time graph as distance rather than displacement so required the graph to increase upwards
- Learnt to interpret speed and acceleration/deceleration from a distance-time graph and distance and acceleration from a velocity-time graph

Conclusion

It is clear that the present study has identified a number of situations in calculus problem solving where there are concerns regarding the four ESL students' comprehension and subsequent representational use. Chapter Five reviews the findings from the study, discusses the study's limitations, answers the three research questions, discusses the implications of the findings for teaching and learning calculus and makes suitable recommendations for future research.

Table 4.18: Longitudinal Change

Time \ Student	John	Tonton	Oliver	Welly
Traffic Task March 1997	In Task 1 numerical skills were used eg. moves 30m/s so moves 60m in 2 seconds. Finds intersection points on a graph when requested. Interprets the vertical axis as distance from the start rather than as displacement and has difficulty with change in direction. Has difficulty interpreting speed from a distance-time (d-t) graph.	Curve shape Gestalt fixation instead of change in small intervals. Acceleration drawn as straight line segments (constant change). No use of time calculations when analysing a d-t graph. Symbolic preference when a contradiction occurs. Distance fixation avoiding equal times when looking at the position of 2 cars.	Downward curves cannot interpret speed as increase or decrease. Understands velocity change on a curve using change in small intervals. Acceleration as straight line segments. No use of time calculations when analysing a d-t graph. Good understanding shown of change in velocity in small time intervals.	Prefers symbolic to numerical mode. Curve shape Gestalt fixation instead of change in small intervals. Look at car movement for individual cars rather than finding times when they are together. Acceleration as straight line segments rather than curve. Prefers picture mode and does not trust graphover numerical answer. No use of time calculations when analysing a d-t graph.
Test One April 1997	Interprets acceleration/deceleration. Uses a numerical approach effectively in the graphics calculator.	Acceleration as straight line segments. Use time calculations when analysing a d-t graph. Symbol preference. v-t graph skills weak.	Acceleration as straight line segments. Use time calculations when analysing a d-t graph. 'What' questions cause difficulty. Prefers direct questions which specify exactly the representations that were expected.	Acceleration as curved line segments. Preferred picture and does not trust graph. Use time calculations when analysing a d-t graph. Comprehension of word problems is poor e.g. Can problem.

(Table 4.18: Longitudinal Change)				
Semester Two Examination November 1997	Uses the calculator to check his work. Indefinite integrals cause trouble as g.c. has limited use. Uses the calculator effectively for definite integration. Inputs functions to find area graphically. Interprets velocity-time (v-t) graph to find distance and acceleration.	Does not use calculators for integration but prefers symbols. Uses g-c when asked to sketch. Interprets question details and finds a solution method. Poor v-t graph translation skills.	Translates from a graph to symbolic form and uses calculator integration. Includes negative area answers from the calculator. Prefers g-c but leaves out asymptotes and does not check, e.g. with g-c tables. Cars overtake at same speed dilemma. Double checks.	Does not use calculators for integration but prefers symbols. Calculus word problems cause interpretation problems. Interprets question details poorly and struggles to find solution method.

Chapter Five: Conclusions And Recommendations

Overview Of The Chapter

This chapter highlights the findings of the present study and uses them as a basis for the formulation of a number of conclusions and subsequent recommendations. The section titled *Summary of Findings* discusses the main findings of this study in relation to the three research questions stated in Chapter One. The resulting conclusions have been tabled together separately in a section titled *General Conclusions*. The section that follows, headed *Implications for Teaching*, outlines several recommendations for instruction in mathematics based on the outcomes of this study, and then a section headed *Recommendations for Further Research* is included. The latter details several recommendations for further research in this area of mathematics education. The final section, *Limitations of the Study*, highlights a number of limitations related to the methodology used in the present study. *Concluding Remarks* completes the thesis.

Summary Of Findings

Research Question One

> Can appropriate instructional material introduced over an extended time in an introductory calculus course enhance ESL students' use and management of multiple representations?

One conceivable outcome of approaching the teaching and learning of mathematics with the use of graphics calculators is the development among students of a more positive attitude to learning mathematics. It was argued that this would correlate highly with students' levels of academic achievement. Interviews were used to quantify students' perceptions of their classroom learning environment in relation to the intervention programme, and this information was compared with academic performance. The results, in general, suggested that students used the technology to varying degrees, but that the most important determinants of academic performance were more related to comprehension of problems and general mathematical ability rather than ability and enjoyment in using the calculator and the more open learning environment it encompassed. This was consistent with the views of researchers White and Mitchelmore (1996) with their *abstract-general* theory, and also Anderson

(1997) in her review of student misconceptions when comprehending different aspects of visual cues or particular representations. The outcomes of the present study support the general view of mathematical understanding described by Norman and Prichard (1994) and aligns itself with attempts to stress the instrumental approach encouraged by Anderson (1996). The study's outcomes also support Lauten et al. (1994) findings regarding the need to combine the graphics calculator technology with research on the mathematical inconsistencies observed in student beliefs. At the same time, results from the study compliment the conclusions of research such as that conducted by Grida and Forster (1997), Quesada and Maxwell (1994) and Ruthven (1990) where the graphics calculator created its own advantages in some problem solving situations. These technology tool advantages cannot, however, be separated from the effectiveness of the learning environment created for the present study.

This study set out to explore the notion that through prior experiences and individual preferences, students will approach problem solving using certain representations rather than others. One of the aims of the study was to *enhance knowledge about the use and management of multiple representations in a calculus course*. With a constructivist approach where students were encouraged to use a range of representations, there was marked improvement in the appropriate choice of method. This was greatly influenced by the detail presented in the problem, with improved comprehension resulting from more specific information about representational requirements. This result supports the findings of several researchers – for example Slavit (1997) in his work on the function concept; Shield and Swinson (1997) in their attempts at encouraging writing, and Schwarz and Dreyfus (1995) who further classified the representations students use. Schwarz and Dreyfus's research revealed a new approach where the graphics calculator was an integral part of the classroom instruction and where student conceptual development was integrated with the ability to move between representations. In particular, the findings of the present study will lend further support to the Australian Association of Mathematics Teachers (1996) recommendations on providing a diversity of learning situations to cater for the individual needs of ESL students, and to create in students the ability to adapt to problem situations involving a range of representations.

White and Mitchelmore (1996, p.87) interviewed students who correctly symbolised a problem but had difficulty reconciling the rate of change of volume relation with a fixed value of the volume at a point, a problem related to reconciling graphed information on distance, velocity and acceleration in the present study. The students appeared to have a *manipulation focus* – they could use a given formula but not rearrange it or combine it with another formula in order to eliminate one or more variables. This situation appeared to be present in the case of the earlier test question number four on the tin can (see page 185), although in this example there is no time rate of change:

> As a result, to correctly symbolise the appropriate derivative a student must have an overview of how all the variables fit together and not just zoom in on some visible symbols .(White & Mitchelmore, 1996, p. 89)

Other common misconceptions occurred when the student focused on known variables, the "*x, y syndrome*", rather than considering the meaning of the symbols and creating a suitable function to be used for minimisation. An example from the present study was the development of a formula to minimize the surface area of a can as mentioned above – a problem used in Test one. There was evidence that the process of defining and using new variables, involved different skills that related explicitly given variables:

> It can be argued that defining and using new variables involves forming relationships at a higher level of abstraction than relating those already given. Defining variables in a modelling situation indicates that the solver is making choices with some plan in mind - a plan that has not been laid out in visible cues. Defining new variables therefore involves relationships at a higher level of generality than the concepts they connect, that is, what Hiebert and Lefevre (1986) call "reflective relationships".(White & Mitchelmore, 1996, p. 91)

To answer Research Question One: The instructional material introduced over an extended time in an introductory calculus course definitely appeared to enhance ESL students' use and management of multiple representations.

Research Question Two

> Can ESL students' ability to model calculus word problems be enhanced by using a study environment with small group student-centred learning focussing on skills of comprehension and translation?

The general goal of this study was to investigate the pedagogical implications of a technology-enhanced approach for the teaching of introductory calculus influenced by the encouragement of the diverse use of multiple representations. This linked with one aim of the study to see if it were possible to *increase ESL student classroom communication as a means to improving ability in the modelling of calculus word problems*. Data was collected a number of times over four semesters to ascertain student usage and understanding of alternative methods for the solution of real world applications of calculus.

The reasons for students' misunderstandings were investigated later through a series of interviews conducted in class time in the early part of the study. There was clear evidence of improvement in the level of conceptual understanding and in knowledge of the nuances of the language register over the period of the present study. Table 4.5 earlier shows changes in assessment results, and a comparison of the initial interview results (see pages 171-180) with the final examination analysis (see pages 192-199) clearly shows changes in communication skills. The recognition of patterns in student misunderstandings was consistent with the findings of researchers such as Forster (1997), Frid (1993b), Lauten et al. (1994) and White and Mitchelmore (1996). For example, White and Mitchelmore's research compared the results of students exposed to similar calculus problems presented in different representational frameworks, including the use of calculus word problems. In nearly every case they identified common structural flaws in students' conceptual knowledge. The outcome of the present study supports their findings concerning the important influence of problem presentation, but extends the results to include ESL student achievement. ESL students had extensive comprehension problems but could understand problem requirements when probed in interviews. The importance of problem understanding identified in the present study reflected the research of Ellerton et al. (2000), and the attempt to measure *Readability* of a problem followed the pioneering work of Flesch (1974).

One outcome from approaching the teaching and learning of mathematics from a constructivist perspective is the opportunity to interact with students and identify their prior knowledge as it affects the attainment of their learning outcomes. Additionally, in this study, task-based interviews enabled the identification of learning approaches that linked with ideas carried forward from previous experience. Approaches varied across the four students. The results suggested that the interview process provided the students with feedback and also provided a forum for the discussion of alternative viewpoints. The positive attitude of the students towards the interview process presented an important input to their learning environment. The interview approach conformed with the recommendations of researchers such as Cobb and Whitenack (1996) who developed interview techniques; Schaller and Tobin (1998) and von Glasersfeld (1995). The latter researchers have all advocated the interpretive research approach when it is considered important to understand students' prior knowledge and develop theories of their personal conceptual development.

The use of a constructivist approach to teaching practices was designed to have a positive influence on students' perceptions of their classroom learning environment. The longitudinal study regime, however, highlighted a number of inconsistencies linked closely to student performance so that the overall effect of the learning environment approach was neutral. This finding was consistent with that of Green (1994) who suggested reasons for student disillusionment with the student-centred approach: factors such as the skill and experience of the teacher, the relationships shared between students and teachers and the time constraints associated with the interview approach.

These results were not consistent with the views expressed in the literature in general, however. Malone and Ireland (1996), Schaller and Tobin (1998) and Zevenbergen (1998) all highlighted the benefits of a student-centred learning environment. The present study demonstrated the developmental nature of the teaching process through the close interaction of four students and complements the research of Thompson and Thompson (1994) and Thompson (1994) where both students and teachers interacted in small group environments while studying calculus concepts and coping with the limitations of the approach.

To answer Research Question Two: Certain aspects of ESL students' ability to model calculus word problems can be enhanced by using a study environment with small group student-centred learning focussing on skills of comprehension and translation. The use and benefits of this approach needs further investigation.

Research Question Three

> Can a teaching package utilising a representation mode of study be developed that assists ESL students in calculus learning?

One aim of the present study was to investigate the ability of ESL students to comprehend calculus word problems presented as part of a teaching package. It was necessary to *develop, implement and evaluate a mathematics " teaching package " for ESL students that encourages the use of multiple representations in order to improve conceptual understanding* (see the aims of the study in Chapter One, page 10). The study's outcomes highlight many inadequacies that ESL students faced covering areas such as the application of everyday language in mathematical contexts, the ability to discuss and clarify ideas in a second language, the lack of confidence to ask for assistance, and the influence of the detail in a question to influence comprehension. Highlighted in the present study, were the weak translation skills from distance-time graphs to a written explanation of the phenomenon being observed for the group of Indonesian ESL students.

The study's results were consistent with the findings of Zevenbergen (1998) who stressed the influence of ethnographic factors, also that of Miller (1993) who identified language difficulties of mathematics students in general, and Zepp et al. (1987) in their research on ESL students. The present study attempted to overcome these language problems by the use of a teaching package rich in real world examples as a precursor to the use of more structured mathematical language as suggested by the research of Chapman (1997).

The study's results were also consistent with the findings of Galligan (1993) and the foundation research of Gardner (1974) into the importance of the structure of the mathematics register. Further evidence has been collected to support the proposals of the Australian Association of Mathematics Teachers (1996) recommending an emphasis on the use of a range of mathematical representations in order to assist ESL students. The current research should provide classroom teachers with the confidence to support the use of a structured approach for problem solving such as advocated by Goldin (1993), and encourage teachers to emphasise the importance of written skills consistent with the research of Shepard (1993) and Shield and Swinson (1997). The use of clinical interviews in this study as a support to achieve language competence reflects the findings of Truran and Truran (1998).

One aim of the present study was to develop a teaching package. Incorporated in the teaching package were instructional worksheets to provide students with adequate training in the use of graphics calculators. This was necessary in order to ensure that students possessed adequate competency in their use. Many ESL students failed to develop appropriate skills and an understanding of when to use different representational modes. The attempt to identify areas of student need and to identify the requirements of assessment instruments is consistent with the ongoing research of Forster and Mueller (1999, 2000) into influences of the graphics calculators on calculus assessment, and the research of Geiger (1998) and Haimes and Webster (2000) into the influences of graphics calculators on instruction.

To answer Research Question Three: A teaching package utilising a representation mode of study can be developed that assists ESL students in calculus learning.

General Conclusions

As highlighted in Chapter 1, the general goal of this study was to investigate the potential of a technology-focused multi-representational approach to the teaching of calculus to ESL students, presented as a viable pedagogical alternative to the instrumentalist approach that has dominated the pre-university environment. The achievement of this goal was sought through the design, selection, implementation and evaluation of curriculum materials based upon a multi-representational framework appropriate to the needs of introductory calculus mathematics students in Western Australia. Based on the outcomes of this evaluation, but not withstanding its limitations, it was concluded that a multi-representational approach to the teaching of calculus to ESL students would:

- enhance students' ability to solve problems in the subject
- enhance students' knowledge and understanding of the subject
- and enhance students' confidence and language use in the subject

It general terms it was concluded from this study that a constructivist approach with an emphasis on multi-representations will not necessarily lead students to a more enjoyable mathematical experience. Nor will such an approach lead to better academic achievement in the subject, or to a change in their preferences for particular representational approaches in problem solving. In other words, one cannot assume that a student-focused approach using graphics calculators would significantly change the enjoyment of students for mathematics nor their academic performance beyond previous levels. The learning environment effect was impossible to isolate from other factors present that influenced student performance during the study. Nevertheless, the study shows that individual students can develop their skills in a range of areas when encouraged over an extended period of time.

Implications For Teaching

The following recommendations for teaching are based on the findings of this study.

- Curriculum developers and professional bodies should provide guidelines and resources for teachers to assist them in adopting multi-representational approaches to teaching calculus.

The results of this investigation suggest an improvement in students' knowledge, understanding and application of the mathematics taught as a result of a multi-representational approach. As the single researcher in the study, I was limited by the constraints of the current mathematics curriculum and the need to utilise appropriate real-world applications. Curriculum developers should consider more carefully the benefits of providing teachers, through professional development, with a number of supporting documents which would facilitate a more student-centred approach to the teaching and learning of a variety of mathematical topics.

- Teachers should be encouraged to be flexible in teaching, and to place the emphasis they want into the TEE introductory calculus syllabus.

It was clear throughout the intervention programme that the teaching package approach used in this study increased students knowledge and understanding of mathematics. Despite the fact that the study focused on only four students, this study has highlighted the difficulty in creating curriculum materials that can significantly alter the problem-solving approaches preferred by students when they are introduced to a student-centred classroom environment. Thus there is a need for teachers to use the powers they possess to manipulate the curriculum and develop one with more appropriate guidelines for the introduction of appropriate multi-representational approaches for the teaching and learning of mathematics. Efforts might also be made to influence the Curriculum Council to create a package of resources for high schools by investigating any existing instructional material that would be suitable. The Council could do this through its extensive access to teachers across the state of Western Australia. Next, there is a need to integrate these modules into the existing curriculum in order to make the course sections link closer together.

- Professional bodies should provide teachers with professional development programmes that encourage the application of calculus in real-world situations and supply appropriate resources.

Teachers who are convinced of the benefits of a mathematics curriculum based on constructivist ideas, and teachers who have been involved in studies such as the present one might be used as a resource for developing work programs that encourage and facilitate a technology-based multi-representational approach to classroom teaching. Many of the resources used for this study were developed by myself over a period of time and aided by the use of a pilot study. Discussions on the use of appropriate resources, including the calculator aplets, should form part of any professional development programme for teachers. It is recommended that classroom teachers be made aware of the benefits and nature of audiotaped interviews and be encouraged to use them as an effective component of their teaching approach.

- Initiating a multi-representational approach

It is important to investigate the graphics calculator resources available and prepare the classroom for small group learning. The teacher must choose a topic from the program and prepare hands-on material along with appropriate assessments. Incorporate graphics calculator output directly into the material to assist with student interpretation. Where necessary, get assistance with the writing of graphics calculator aplets or source them from the internet. Evaluate the success of your teaching package, reassess and publish your findings to encourage discussion about your initiative.

- Ensuring students can manipulate the graphics calculator well

A training program needs to be implemented as a prerequisite to any successful teaching or research project involving graphics calculators. The guidelines given in Appendix G, H and I (pp. 404-427) can be used as a foundation for this process.

- Motivating students with a multi-representational approach

Teachers need to learn the representational preference of their students and allow each representational choice to be available in the teaching package. Then develop a new representational skill to challenge each student allowing success at the early stages. This will give exposure to different solution strategies and help the student extend their problem-solving repertoire.

Recommendations For Further Research

The following recommendations for further research are based on the findings of this study.

- The credibility and sustainability of the results might be confirmed through replication of this study.

A replication of this study using other small groups involved with different mathematical topics should provide researchers with some additional insights. For example, a more extensive use of classroom environment instruments may provide additional information concerning the conclusions of this study. In particular, surveys conducted regularly on student' perceptions of the approach used could provide important feedback. More importantly, information like this should assist in confirming the transferability of the results of this and future studies in this area.

- The test instruments used in the study might be examined, revised and refined.

The possibility of developing more sophisticated instruments for measuring a greater range of outcomes needs to be investigated. This would provide a more structured approach to the assessment analysis, thereby providing researchers with other important information on the benefits of a multi-representational student-focused perspective.

- Future studies will need to consider the effects of the treatment on the individual teachers themselves

The use of a teaching package can greatly influence the approach to teaching and learning used in the classroom. The ability of the teacher to adapt to new strategies can influence results and needs further analysis.

- Other researchers might develop distance/velocity test instruments for use in calculus lessons.

The study's exercises used with the four students highlighted the lack of understanding of links between the distance-time and velocity-time concepts, certainly from a graphical perspective. Thus there is a need to teach ideas about velocity from the perspective of small time intervals and not just to train the students in calculator techniques (Ryan, 1994).

- Other researchers might investigate the sequencing of the interviews of students involved in the study.

Implementation of the data collection interviews highlighted a number of procedural problems with introducing a lengthy data collection instrument into a standard classroom. The success of the interview phase of the present study suggests the need to encourage this approach and the need to develop optimal strategies for the use of interviews in classrooms.

- Further studies might investigate the early introduction of graphics calculators at the upper school level.

The degree of difficulty in learning aspects of graphics calculator technical use has prompted calls for changes at the curriculum level (Forster & Mueller, 2000; Swincicky, 1994, p. 109). If students were given more time at achieving such skills as finding a good screen display, this is likely to improve mathematical development. Thus, the use of the graphics calculator earlier in High School would be recommended, as this would allow upper school teachers to concentrate more on course content and the design of effective small group learning environments.

Limitations Of The Study

Several limitations have been identified with respect to this study. The limitations refer specifically to variables such as the sample group, test instruments, comparison measures, researcher/teacher paradox, choice of strategies, choice of mathematical content, package organisation, and time allocation.

- Effectiveness of the sampling procedures used.

The case studies of four students used in this study was drawn from a single classroom from an international college in Western Australia and therefore represented only a very small proportion of the total ESL Year 11 calculus student population in Western Australia. In addition, the need to select the case study subjects from those classes being taught by myself at the college where I was employed meant there was little guarantee of randomness. This together with other possibly unique demographic properties makes it difficult to assume that the sample was representative of all ESL students in Western Australia. More general conclusions might be possible in future studies through larger and more sophisticated sampling techniques. The small sample size suggested that further testing would be necessary before conclusions could be drawn about the discriminant validity of the instruments, therefore whether the scales used measure a unique dimension not covered by other scales in the instrument.

- Use of the interview.

There needs to be a greater variety of instruments available to measure variables related to the use of technology in the classroom environment. Time constraints affected the use of task-based interviews and these could have been more frequent and extensive in their inquiry with better planning. It might be argued, for example, that the interviews would be more effective if used at the start of the study rather than being spread evenly across topics. The use of other instruments by other researchers might provide a greater range of data from which conclusions could be drawn.

- Limited comparison measures used.

Assessment items were compared on an individual basis. With a larger sample, statistical measures such as means and standard deviations could be used more effectively. Interview data were presented in as objective a manner as possible, but the interpretations of the interviewer and their impact on the conduct of the interview will always bias this type of data collection despite efforts to control this variable. The direct comparison of individual student performance for each item in the various test instruments provided additional information, but the effectiveness of this approach depends on the design of suitable test items. Interviews conducted throughout the study after each test or examination would have provided extra data on student understanding.

- Limited teaching strategies used.

The approach used in this study may have led to students spending excessive amounts of time involved in the data collection process, particularly with regard to the time spent on classroom interviews. This time may have been better used on other teaching strategies that could have led to more beneficial learning experiences. Thus the methods advocated in this study for use with the curriculum package need to be studied and adapted to other learning situations where a student-centred approach may be less appropriate.

- Limited mathematical content studied.

One outcome of the study was that student preference for, or dislike of multi-representations was not in general changed by the instructional material. This result needs to be qualified. Only a selection of the coursework was used for data analysis, so that the conclusion might be specific to that particular area of mathematical content. Therefore any conclusions about student representational preference cannot be generalised to include the teaching of other content areas, or even other sections of the same Introductory Calculus course.

- Lack of Graphics Calculator training.

One of the main drawbacks to the use of the graphics calculator in the present study was the need to effectively train students in its use as a graphing tool if the possible graphical misconceptions in its use were to be overcome. When students produce graphs using technology, a range of misconceptions can occur (Goldenberg, 1988, 1991; Quesada & Maxwell, 1994). These include: observing the vertex position rather than the scale when comparing two graphs or finding an equation; choosing an inappropriate scale; and confusing the difference between stretching and shrinking when using zoom features.

- Structure of the teaching package.

The use of multi-representational approaches with ESL students may in fact be a disadvantage for them academically as, in this study, they were required to interpret calculus word problems and justify their methods of solution rather than simply reproducing solution methods for problems presented in an algebraic form. This feature of the teaching package was necessary in order to identify the representational approaches used by ESL students, and so the general conclusions of this study should be interpreted in this light.

- Time frame for the intervention programme.

The extended time frame of the study (2 semesters) was one of its unique features, however much of the data collection occurred in the early phase of the study. The constraints of the curriculum in use at the research site forced this situation onto the study. The outcomes of the study should be viewed in this light.

- Delays in completing the study.

Due to family reasons and work commitments there were major delays in bringing the study to an end. During this time new developments in the field have been identified by keeping abreast of the literature. This is evident in the number of recent articles included in the Reference List and included extensively throughout the study.

- Peer debriefing and member checking.

A number of measures need to be implemented in any study to satisfy credibility requirements. The level of peer debriefing and member checking could have been of a more extensive and structured nature. Many of the conclusions from the study were written down following the conclusion of the introductory calculus course and could not be directly checked with the students concerned.

Concluding Remarks

The general goal of the present study was to investigate the learning implications of the introduction of a multi-representational teaching package to ESL introductory calculus students, as a viable alternative to the traditional algebraic based teacher centred approach. This study illustrated the importance of viewing learning from different representation systems and the advantages of using this approach with ESL students. It also illustrated how teachers might be helped to incorporate conctructivist approaches into the classroom via the use of a range of alternative assessment approaches and the creation of different classroom environments.

The study focussed on a particular area of mathematical content (calculus), while I, as the researcher dealt with the implementation of constructivist ideas about student learning, the use of alternative teaching strategies and the development of new data collection instruments. This was occurring while still relying on traditional assessment procedures in use at the school. This process evolved over time and gave the study added significance by revealing important learning inconsistencies in areas of the calculus curriculum. Throughout the study I experienced new insights into the application of constructivist ideas for use in my classroom.

Calculus is an important subject as it sits at the transition point between high school and university courses in science and engineering. With this important position there is increased responsibility for reform in the presentation, performance and enjoyment students should derive from its study. With the introduction of graphics calculators, it is possible to eliminate tedious and time-consuming procedures such as evaluating functions, solving excessively complex equations, and plotting graphs. It is now possible to investigate relevant and interesting topics that previously were impossible to cover. This study has generally illustrated the advantages of a multi-representational approach to the teaching and learning of calculus for ESL students, and as such it will hopefully form a valuable resource for teachers and researchers.

There is much work to be done in planning new research studies in this area. To start with, there are the misconceptions in the use of graphics calculator technology (Dion, 1990). Next, there are the changes in focus required by the teacher (Tobin, 1993).

Overriding these considerations are the benefits of graphics calculator use for the development of higher-order cognitive skills in calculus (Kissane, 1995a). There is also much to learn about the introduction and optimal use of graphics calculators in the classroom. Kissane comments on future steps necessary:

> We would like to know more about how to help students use sophisticated calculators confidently and efficiently. We would like to know more about the best balance of conceptual and procedural competence. We would like to know more of optimal sequences of developing concepts when such technology is available. We would like more research, we would like more help and we would like more time. But we must make decisions now, as the new age of mathematics will not wait.(Kissane, 1993, p. 240)

This study has attempted to contribute to the literature on using the graphics calculator in a multi-representational approach to the teaching of calculus. The focus on comparisons between ESL students and others means there is the potential to influence research in an area of increasing academic interest, as well as guiding the trend to making technology accessible to all students.

Reference List

Agostinelli, J. (1999, March 4). Calculators figure it out. *The West Australian* (Education Today lift-out), 10-11.

Anderson, J. (1997). Myths and misconceptions in the teaching and learning of mathematics. *Cross Section,* December 1997, 9(4).

Anderson, J. A. (1996). Experiments to investigate instrumental and relational understanding among undergraduates. *International Journal of Mathematical Education, Science and Technology*, 27(6), 813-819.

Andrews, T., & Kissane, B. (Eds.). (1994). *Graphics calculators in the classroom.* Adelaide: Australian Association of Mathematics Teachers.

Aspinwall, L., Shaw, K.L., & Presmeg, N.C. (1997). Uncontrollable mental imagery: Graphical connections between a function and its derivative. *Educational Studies in Mathematics*, 33, 301-317.

Atkins, C.D. (1994). A study to produce guidelines for evaluating calculus reform projects (Doctoral dissertation, North Carolina State University). *Dissertation Abstracts International*, volume unknown.

Atkins, W., & Fergusson, A. (1989a). *Mathematical graphs and concepts - a handbook of blackline masters: Book 1 - standard elementary functions*. Adelaide: Australian Association of Mathematics Teachers.

Atkins, W., & Fergusson, A. (1989b). *Mathematical graphs and concepts - a handbook of blackline masters: Book 2 - calculus.* Adelaide: Australian Association of Mathematics Teachers.

Austin, J.L., & Howson, A.G. (1979). Language and mathematical education. *Educational Studies in Mathematics*, 10, 161-197.

Australian Association of Mathematics Teachers (1996). *Statement on the use of calculators and computers for mathematics in Australian schools.* Adelaide: Australian Association of Mathematics Teachers Inc.

Australian Education Council (1991). *A national statement on mathematics for australian schools.* Carlton, Victoria: The Curriculum Corporation.

Barling, C. (1994). Graphing calculators: potential vs practice. In T. Andrews & B. Kissane (Eds.), *Graphics calculators in the classroom* (pp. 33-50). Adelaide: AAMT.

Barnes, M. (1991). *Investigating change: an introduction to calculus for Australian schools - units 1-10.* Victoria: Curriculum Corporation.

Baumgartner, J., & Shemanske, T. (1990). Dartmouth college: Teaching calculus with true basic. In T.W.Tucker (Ed.), *Priming the calculus pump: Innovations and resources*. U.S.A.: MAA

Beckmann, C.E., & Sundstrom, T.A. (1992). *Exploring calculus with a graphing calculator*. Reading, Mass. : Addison-Wesley.

Beers, T.W., Byrne, D.K., Donnelly, J.A., Jones, R.W., & Yuan, F. (1996). A graphing calculator for mathematics and science classes. *Hewlett-Packard Journal*, June, 1996, Article 6.

Bell, A., Brekke, G., & Swann, M. (1987). Diagnostic teaching (5). Graphical interpretation: Teaching styles and their effects. *Mathematics Teaching*, Sept 1987, 50-57.

Bergamini, D. (1993). Mastering the mysteries of movement. In U. Dudley (Ed.), *Readings for calculus* (pp. 124-130). USA: MAA.

Berger, M. (1998). Graphics Calculators: an interpretive framework. *For the Learning of Mathematics*, 18(2), 13-20.

Bishop, A.J., & Larkin, C. (1994). Institutional and pedagogical support for mathematics teachers working with NESB students. In G.Bell, B.Wright, N.Leeson & J.Geake (Eds.), *Challenges in mathematics education: constraints on construction* (pp. 97-104). *Proceedings of 17th MERGA conference*. Lismore: MERGA.

Bodner, G.M. (1986). Constructivism: A theory of knowledge. *Journal of Chemical Education,* 63 (10), 873-878.

Boers, M.A.M. & Jones, P.L. (1994). Is calculus made easier with the graphics calculator? In T. Andrews & B. Kissane (Eds.), *Graphics calculators in the classroom* (pp. 65-71). Adelaide: AAMT.

Borlaug, V.A. (1993). From algebra to calculus - a tonka toy truck does the trick. *The Mathematics Teacher*, 4, 282-287.

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics,* 23(3), 247-286.

Brown, T. (1990). Active learning within investigational tasks. *Mathematics Teaching*, December, 15-18.

Buecher, C. (1997). Adjusting window variables to make functions come alive! *TI Cares: Educational Support Programs*, Australian Edition, winter 1997, 1(4), 1,7-8.

Buxton, L. (1997). Thinking in three modes. *Mathematics Teacher,* 158, March, 52-53.

Cadby, A. (1993). Computers in the mathematics classroom-why it takes a long time. *Cross Section,* July 1993, 5(2), 4-7.

Carr, W., & Kemmis, S. (1988). *Becoming critical: education, knowledge and action research*. Geelong: Deakin University Press.

Carter, C.R. (1995). A bright new kid in class: the hp 38g. *HP 38G Classnotes*, 1(1), April 1995.

Chapman, A. (1997). Towards a model of language shifts in mathematics learning. *Mathematics Education Research Journal*, 9(2), 152-173.

Chapman, A., & Lee, A. (1990). Rethinking literacy and numeracy. *Australian Journal of Education*, 34(3), 277-289.

Claffey, J. (1996). *The hp 38g graphic calculator beginner's guide*. Victoria Park, W.A.: Park Printing Co.

Cobb, P., & Steffe, L.P. (1983). The constructivist researcher as teacher and model builder. *Journal for Research in Mathematics Education*, 14(2), 83-94.

Cobb, P., & Whitenack, J.W. (1996). A method for conducting longitudinal analyses of classroom videorecordings and transcripts. *Educational Studies in Mathematics*, 30, 213-228.

Cohen, E.G. (1994). Restructuring the classroom: conditions for productive small groups. *Review of Educational Research*, 64(1), 1-35.

Confrey, J. (1991). The concept of exponential functions: a student's perspective. In L.P Steffe (Ed.), *Epistemological foundations of mathematical experience*. New York: Springer-Verlag.

Confrey, J. (1995). How compatible are radical constructivism, sociocultural approaches, and social constructivism. In L.P.Steffe & J.Gale (Eds.), *Constructivism in education*. New Jersey. Lawrence Erlbaum Associates.

Confrey, J., & Smith, E. (1994). Exponential functions, rates of change, and the multiplicative unit. *Educational Studies in Mathematics*, 26, 135-164.

Croft, C. (1998). Some observations regarding the use of graphics calculators in the USA and Australia. *Australian Mathematics Teacher*, 54 (3), 29-33.

Cuevas, G. J. (1984). Mathematics learning in english as a second language. *Journal for Research in Mathematics Education*, 15(2), 134-144.

Cummins, J. (1979). Linguistic interdependence and the educational development of bilingual children. *Review of Educational Research*, 49, 222-251.

Cummins, J. (1986). Empowering minority students: a framework of intervention. *Harvard Educational Review*, 56, 18-36.

Curriculum Council (2001). *Syllabus manual year 11 and 12 accredited subjects 2001*. Volume VII Mathematics. Perth: Curriculum Council.

Davis, B., Maher, C., & Noddings, N. (1990). Introduction: constructivist views on the teaching and learning of mathematics. In R.B.Davis, C.A.Maher & N.Noddings (Eds.), *Constructivist views on the teaching and learning of mathematics*. Reston, VA: NCTM.

Dawe, L. (1983). Bilingualism and mathematical reasoning in english as a second language. *Educational Studies in Mathematics*, 14, 325-353.

Dawe, L., & Anderson, J. (1993). Visual imagery and metacognition in problem solving. In B. Atweh, C. Kanes, M. Carss & G. Booker (Eds.), *Contexts in mathematics education: Proceedings of 16th MERGA conference*. Brisbane: MERGA.

Demana, F., & Waits, B.K. (1993). The particle-motion problem. *The Mathematics Teacher*, 86(4), 288-292.

Denzin, N.K. (1978). *The research act: A theoretical introduction to sociological methods*. New York: Mcgraw-Hill.

Denzin, N.K. (1989). *The research act: A theoretical introduction to sociological methods (3^{rd} ed.).* New York: Mcgraw-Hill.

Denzin, N.K., & Lincoln, Y.S. (2000). The discipline and practice of qualitative research. In N.K.Denzin & Y.S.Lincoln (Eds.), *Handbook of qualitative research* (pp. 1-28). Thousand Oaks, Calif.: Sage Publicationss, Inc.

Dion, G. (1990). The graphics calculator: A tool for critical thinking. *Mathematics Teacher*, 83(7), 564-571.

Donnelly, J. (1995). Focused exploration with hp 38g aplets. Retrieved April 14^{th}, 2000, from *http:// www.freoshs.wa.edu.au/maths/hp38g/aplets/aplet.html.*

Drijvers, P., & Doorman, M. (1996). The graphics calculator in mathematics education. *Journal of Mathematical Behavior*, 15(4), 425-440.

Dubinsky, E. (1994a). A theory and practice of learning college mathematics. In A.H.Schoenfeld (Ed.), *Mathematical thinking and problem solving.* New Jersey: Lawrence Erlbaum Associates.

Dubinsky, E. (1994b). Comments on Kaput. In A.H. Schoenfeld (Ed.), *Mathematical thinking and problem solving.* New Jersey: Lawrence Erlbaum Associates.

Dudley, U. (Ed.). (1993). *Readings for calculus.* U.S.A.: MAA.

Dufour-Janvier, B., Bednarz, N., & Belanger, M. (1987). Pedagogical considerations concerning the problem of representation. In C.Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics.* New Jersey: Lawrence Erlbaum Associates.

Dunham, P.H. (1992). Teaching with graphing calculators: a survey of research on graphing technology. In L.Lum (Ed.), *Proceedings of the fourth annual international conference on technology in collegiate mathematics.* Reading, MA. Addison-Wesley.

Dunham, P.H., & Dick, T.P. (1994). Research on graphing calculators. *The Mathematics Teacher,* 87(6), 440-445.

Durkin, K., & Shire, B.E. (1991). Lexical ambiguity in mathematical contexts. In K.Durkin & B.E. Shire (Eds.), *Language in mathematical education: Research and practice*. Milton Keynes. Open University Press.

Ellerton, N.F., & Clarkson, P.C. (1992). Language factors in mathematics learning. In B.Atweh & J.Watson (Eds.), *Research in Mathematics in Australasia 1988-1991.* Kelvin Grove, Queensland: MERGA.

Ellerton, N.F., & Clements, M.A. (Ken) (1996). Language factors in mathematics learning. In B.Atweh, K.Owens & P.Sullivan (Eds.), *Research in Mathematics in Australasia 1992-1995.* Kelvin Grove, Queensland: MERGA.

Ellerton, N.F., Clements, M.A. (Ken), & Clarkson, P. (2000). Language factors in mathematics education. In K.Owens & J.Mousley (Eds.), *Research in Mathematics in Australasia 1996-1999.* University of Western Sydney, New South Wales: MERGA.

Farrell, A.M. (1996). Roles and behaviors in technology-integrated precalculus classrooms. *Journal of Mathematical Behavior*, 15, 35-53.

Ferrini-Mundy, J., & Graham, K. (1994). Research in calculus learning: Understanding of limits, derivatives and integrals. In J.J. Kaput & E. Dubinsky (Eds.), *Research issues in undergraduate mathematics learning: preliminary analyses and results*. Washington: MAA.

Ferrini-Mundy, J., & Lauten, D. (1993). Teaching and learning calculus. In P.S.Wilson (Ed.), *Research ideas for the classroom: High school mathematics.* New York: Macmillan.

Flesch, R. (1974). *The art of readable writing.* New York: Harper & Row.

Forster, P. (1997). Teaching through technology. *Australian Senior Mathematics Journal*, 11(1), 54-61.

Forster, P.A., & Mueller, U. (1999). Use of graphics calculators in the public examination of calculus: reality and rigour. *Australian Senior Mathematics Journal,* 13(2), 4-20.

Forster, P.A., & Mueller, U. (2000). Diversity and difficulties with graphics calculator usage in the 1999 Calculus tertiary entrance examination in Western Australia. *Australian Senior Mathematics* Journal, 14(1), 4-15.

Forster, P., Mueller, U., Haimes, D., & Malone, J. (In press). Impact on school assessment where use of graphics calculators is mandatory in a related public examination. *International Journal of Mathematical Education in Science and Technology.*

Forster, P., Mueller, U., Malone, J., & Haimes, D. (In press). Assessment in the presence of graphics calculators: random variables and their distributions. *Proceedings of the Third International Conference on Science, Mathematics and Technology Education*: Rhodes University, East London, South Africa.

Fraser, B.J. (1994). Research on classroom and school climate. In D. Gabel (Ed.), *Handbook of Research on Science Teaching and Learning* (pp. 493-541). New York: Macmillan.

Fraser, B.J., Treagust, D.F., Williamson, J.C., & Tobin, K.G. (1987). Validation and application of the college and university classroom environment inventory (CUCEI). In B.J. Fraser (Ed.), *The study of learning environments*, 5, 17-30.

Frid, S.D. (1993a). Communicating mathematics: a social sharing of language and decisions pertaining to truth and validity. In W.M. Stephens (Ed.), *Communicating mathematics: Perspectives from classroom practise and current research* (pp. 177-189). Victoria: ACER.

Frid, S.D. (1993b). Communicating mathematics: How language forms make in/accessible mathematically in/appropriate calculus conceptualizations. In B. Atweh, C. Kanes, M. Carss & G. Booker (Eds.), *Contexts in mathematics education: Proceedings of 16th MERGA confererence*. Brisbane: MERGA.

Galligan, L. (1993). An exploration of the different structure of the mathematics register in english and asian languages: Some consequences for the teaching of ESL students in preparatory programmes. In B. Atweh, C. Kanes, M. Carss & G. Booker (Eds.), *Contexts in mathematics education: Proceedings of 16th MERGA confererence*. Brisbane: MERGA.

Gardner, P.L. (1974). Language difficulties of science students. *The Australian Science Teachers Journal*, 20(1), 63-76.

Geiger, V. (1998). Students' perspectives on using computers and graphing calculators during mathematical collaborative practice. In C. Kanes, M. Goos & E.Warren (Eds.), *Teaching mathematics in new times: Proceedings of the 21st annual conference of the Mathematics Education research Group of Australasia* (pp. 217-224). Gold Coast, Qld: MERGA.

Goldin, G.A. (1987). Levels of language in mathematical problem solving. In C.Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics.* New Jersey: Lawrence Erlbaum Associates.

Goldin, G.A. (1993). Observing mathematical problem solving: perspectives on structured, task-based interviews. In B. Atweh, C. Kanes, M. Carss & G. Booker (Eds.), *Contexts in mathematics education: Proceedings of 16th MERGA conference.* Brisbane: MERGA.

Goldenberg, E.P. (1988). Mathematics, metaphors, and human factors: Mathematical, technical, and pedagogical challenges in the educational use of graphical representation of functions. *Journal of Mathematical Behaviour*, 7, 135-173.

Goldenberg, E.P. (1991). The difference between graphing software and educational graphing software. In W. Zimmerman & S. Cunningham (Eds.), *Visualization in teaching and learning mathematics*. U.S.A.: Mathematical Association of America.

Green, J.M. (1994). The design, implementation and evaluation of a mathematical teaching package based upon a constructivist epistemology. Perth: *Unpublished Doctoral Thesis*, Curtin University of Technology.

Grida, S., & Forster, P. (1997). New approaches through graphics calculators. *Cross Section*, September 1997, 9(3).

Guba, E., & Lincoln, Y. (1989). *Fourth Generation Evaluation*. Beverley Hills, Calif: Sage Publications.

Haimes, D., & Webster, B. (2000). Student perceptions of graphics calculator use in examinations. *Australian Senior Mathematics Journal*, 14(1), 16-23.

Hart, D.K. (1992). Building concept images: supercalculators and students' use of multiple representations in calculus. *Dissertation Abstracts International*, 52, 4254A.

Hewlett-Packard (1995a). *HP38G classnotes*, 1(1), April 1995.

Hewlett-Packard (1995b). *Quick guide to using the hp38g*. New York: Hewlett-Packard.

Holmes, M.H., Ecker, J.G., Boyce, W.E. & Siegmann, W.L. (1993). *Exploring calculus with maple*. Massachusetts: Addison-Wesley.

Hook, C. (1981). *Studying classrooms*. Victoria: Deakin University.

Janvier, C. (1987a). Representation and understanding: the notion of function as an example. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics*. New Jersey: Lawrence Erlbaum Associates.

Janvier, C. (1987b). Translation processes in mathematics education. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics*. New Jersey: Lawrence Erlbaum Associates.

Johnston, E., Mathews, J., Bergman, C., & Heckenbach, A. (1991). Revitalizing engineering calculus at Iowa State University. In L.C. Leinbach, J.R. Hundhausen, A.M. Ostebee, L.J. Senechal & D.B.Small (Eds.), *The laboratory approach to teaching calculus* (pp. 169-185). USA. Mathematical Association of America.

Joiner, K.F. (1998). Trialing and evaluating reform in calculus education. Perth: *Unpublished Doctoral Thesis*, Curtin University of Technology.

Jones, P., & McCrae, B. (1996). Assessing the impact of graphics calculators on mathematics examinations. Retrieved Sept 7, 2000, from *http: // www.edfac.unimelb.edu .au /dsme/tame/docs/tame_res_9.html.*

Kaput, J.J. (1994). Democratizing access to calculus: New routes to old routes. In A.H.Schoenfeld (Ed.), *Mathematical thinking and problem solving*. New Jersey: Lawrence Erlbaum Associates.

Kemp, M., Kissane, B., & Bradley, J. (1996). Graphics calculator use in examinations: Accident or design? *Australian Senior Mathematics Journal*, 10(1), 36-50.

Kerlinger, F.N. (1986). *Foundations of behavioural research: 3rd edition.* Fort Worth: Holt, Rinehart and Winston, Inc.

Kessler, C. (Ed.) (1992). *Cooperative language learning.* New Jersey: Prentice-Hall, Inc.

Kissane, B. (1993). Sophisticated calculators: mathematics for a new age. In A.Herrington (Ed.), *New horizons, new challenges* (pp. 227-240). Perth: AAMT.

Kissane, B. (1995a). Exploring calculus with graphics calculators. *Australian Senior Mathematics Journal*, 9(2), 7-15.

Kissane, B. (1995b). Graphing an equation. *The Australian Mathematics Teacher,* 51(3), 38-42.

Kissane, B. (1995c). Technology in secondary mathematics - the graphics calculator as personal mathematical assistant. In R.P. Hunting, G.E. Fitzsimmons, P.C. Clarkson & A.J. Bishop (Eds.), *Regional Collaboration in Mathematics Education* (pp. 383-392). Melbourne: International Commission on Mathematical Instruction.

Kissane, B., Kemp, M., & Bradley, J. (1995). Student reactions to the use of graphics calculators. In B. Atweh & S. Flavel (Eds.), *Proceedings of the 18th MERGA Conference.* Darwin: MERGA.

Lakoff, G., & Johnson, M. (1980). *Metaphors we live by.* Chicago: University of Chicago Press.

LaTorre, D.R., Fetta, I.B., Kenelly, J.W., Nicholson, J.H., Proctor, T.G., & Reneke, J.A. (1990). Clemson university: Calculator-based calculus. In T.W.Tucker (Ed.), *Priming the calculus pump: innovations and resources* (pp. 11-32). USA: MAA

Lauten, A.D., Graham, K., & Ferrini-Mundy, J. (1994). Student understanding of basic calculus concepts: Interaction with the graphics calculator. *Journal of Mathematical Behavior*, 13, 225-237.

LeCompte, M., & Goetz, J. (1982). Problems of reliability and validity in ethnographic research. *Review of Educational Research*, 52(1), 31-60.

Leder, G., Rowley, G., & Brew, C. (1995). VCE 1992. In R.P. Hunting, G.E. Fitzsimmons, P.C. Clarkson & A.J. Bishop (Eds.), *Regional Collaboration in Mathematics Education.* Melbourne: International Commission on Mathematical Instruction.

Lepik, M. (1990). Algebraic word problems: Role of linguistic and structural variables (short communication). *Educational Studies in Mathematics*, 21, 83-90.

Lesh, R., Post, T., & Behr, M. (1987). Representations and translations among representations in mathematics and problem solving. In C.Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics.* New Jersey: Lawrence Erlbaum Associates.

Lincoln, Y.S., & Guba, E.G. (1985). *Naturalistic inquiry.* Beverley Hills, Calif: Sage Publications.

Magidson, S. (1992). From the laboratory to the classroom: A technology-intensive curriculum for functions and graphs. *Journal of Mathematical Behavior,* 11, 361-376.

Malone, J., Haimes, D., Forster, P., & Mueller, U. (2002). Use of graphics calculators in school tests and examinations. *Proceedings of the twenty-third annual conference of the Mathematics Education Research Group of Australasia.* Auckland: MERGA.

Malone, J., & Ireland, D. (1996). Constructivist research in teaching and learning mathematics. In B.Atweh, K.Owens & P.Sullivan (Eds.), *Research in Mathematics in Australasia 1992-1995.* Kelvin Grove, Queensland: MERGA.

Malone, J., & Miller, D. (1993). Communicating mathematical terms in writing: Some influential variables. In W.M.Stephens (Ed.), *Communicating mathematics: Perspectives from classroom practise and current research* (pp. 177-189). Victoria: ACER.

Mathison, S. (1988). Why triangulate? *Educational Researcher*, March, 13-17.

McCarter, J.H. (1992). *Discovering calculus with graphing calculators.* USA: John Wiley & Sons, Inc.

McNiff, J. (1992). *Action research: Principles and practice.* London. Routledge.

Merriam, S. (1988). *Case study research in education.* Oxford: Jossey-Bass.

Miller, D. (1993). A survey of tertiary students' entry level understanding of mathematics vocabulary. In R.Fox (Ed.), *Teaching and learning: minifellowships.* Perth, Australia: Teaching Learning Group, Curtin University of Technology.

Mochon, S. (1996). What kind of materials are needed to build the structure of calculus? *International Journal of Mathematical Education, Science and Technology,* 27(2), 207-214.

Mori, I., Kojima, M., & Deno, T. (1976). A child's forming the concept of speed. *Science Education*, 60(4), 521-529.

Neydorff, P. (Ed.) (1997). *TI Cares: Educational Support Programs,* Australian Edition, winter 1997, 1(4), 1,7-8.

Niesten, R. (1993). Using chinese cultural patterns as an aid to effective teaching. In R.Fox (Ed.), *Teaching and learning: minifellowships.* Perth, Australia: Teaching Learning Group, Curtin University of Technology.

Norman, F.A., & Prichard, M.K. (1994). Cognitive obstacles to the learning of calculus: a Kruketskiian perspective. In J.J.Kaput & E.Dubinsky (Eds.), *Research issues in undergraduate mathematics learning: Preliminary analyses and results.* Washington: MAA.

Penglase, M., & Arnold, S. (1996). The graphics calculator in mathematics education: A critical review of recent research. *Mathematics Education Research Journal,* 8(1), 58-90.

Pomerantz, H. (1997). The role of calculators in math education. Retrieved July 19th, 2002, from http://www.education.ti.com/t3/resources/therole.html.

Porzio, D.T. (1994). The effects of differing technological approaches to calculus on students' use and understanding of multiple representations when solving problems (problem solving). *Dissertation Abstracts International*, 55, 3128 A.

Presmeg, N.C. (1992). Prototypes, metaphors, metonymies and imaginative rationality in high school mathematics. *Educational Studies in Mathematics*, 23, 595-610.

Quesada, A.R., & Maxwell, M.E. (1994). The effects of using graphing calculators to enhance college students' performance in precalculus. *Educational Studies in Mathematics*, 27, 205-215.

Rogers, P. (1995). Putting theory into practice. In G.Kaiser & P.Rogers (Eds.), *Equity in mathematics education: Influences of feminism and culture* (pp. 175-185). London:The Falmer Press.

Ruthven, K. (1990). The influence of graphic calculator use on translation from graphic to symbolic forms. *Educational Studies in Mathematics*, 21, 431-450.

Ryan, J. (1994). The function concept: making connections within and between representations. *AARE Conference 1994.*

Sadler, A. J. (1992). *Introductory Calculus.* Perth. The Sadler Family Trust.

Sadler, A. J. (1998). *Calculus.* Perth. The Sadler Family Trust.

Salomon, G. (1992). Effects with and of computers and the study of computer-based learning environments. In E.De Corte, M.C.Linn, H.Mandl & L. Verschaffel (Eds.), *Computer-based learning environments and problem solving* (pp. 249-263). Berlin:Springer-Verlag.

Schaller, J.S., & Tobin, K. (1998). Quality criteria for the genres of interpretive research. In J.A.Malone, B.Atweh & J.R.Northfield (Eds.), *Research and supervision in mathematics*. New Jersey: Lawrence Erlbaum Associates, Inc.

Schwarz, B., & Dreyfus, T. (1995). New actions upon old objects: A new ontological perspective on functions. *Educational Studies in Mathematics*, 29, 259-291.

Schoenfeld A.H. (Ed.) (1994). *Mathematical thinking and problem solving.* New Jersey: Lawrence Erlbaum Associates.

Secondary Education Authority (1992). *Introductory calculus- assessment tasks.* Perth: Secondary Education Authority.

Secondary Education Authority (1997a). *Introductory calculus D505- assessment tasks for district seminar workshops.* Perth: Secondary Education Authority.

Secondary Education Authority (1997b). *Calculus curriculum notes.* Perth: Secondary Education Authority.

Senk, S.L., Beckmann, C.E., & Thompson, D.R. (1997). Assessment and grading in high school mathematics classrooms. *Journal for Research in Mathematics Education,* 28(2), 187-215.

Sfard, A. (1991a). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22, 1-36.

Sfard, A. (1991b). The case of the vanishing tuples. *For the Learning of Mathematics*, 11(1), 19-25.

Shell Centre for Mathematical Education (1986). *The language of functions and graphs: Traffic*. Nottingham: JMB/Shell Centre.

Shepard, R.G. (1993). Writing for conceptual development in mathematics. *Journal of Mathematical Behavior*, 12, 287-293.

Shield, M., & Swinson, K. (1997). Encouraging learning in mathematics through writing. *The Australian Mathematics Teacher*, 53(1).

Skemp, R.R. (1987). *The psychology of learning mathematics*. Hillsdale, NJ: L.E.Erlbaum Associates.

Slavit, D. (1997). An alternate route to the reification of function. *Educational Studies in Mathematics*, 33, 259-281.

Smith, (1996). Meta analysis of graphing calculator research. *Dissertation Abstracts International*.

Smith, D.A., & Moore, L.C. (1990). Duke university: Project calc. In T.W.Tucker (Ed.), *Priming the calculus pump: Innovations and resources* (pp. 51-74). U.S.A.: MAA

Solow, A.E. (1993). *Learning by discovery: A lab manual for calculus*. U.S.A.: MAA.

Steele, D. F. (1995). A constructivist approach to mathematics teaching and learning by a fourth grade teacher. *Dissertation Abstracts International*, Unpublished Masters thesis.

Steffe, L.P (Ed.) (1991). *Epistemological foundations of mathematical experience*. New York: Springer-Verlag.

Straffin, P. (Ed.) (1993). *Applications of calculus*. Washington: Mathematical Association of America.

Swincicky, K.B. (1994). An interpretive study of a cooperative learning environment and the use of graphics calculators in a pre-university mathematics class for international students. *Unpublished Masters thesis.* Perth, Western Australia: Curtin University of Technology.

Tall, D. (1991a). Intuition and rigour: the role of visualization in the calculus. In W. Zimmerman & S. Cunningham (Eds.), *Visualization in teaching and learning mathematics.* U.S.A.: Mathematical Association of America.

Tall, D. (1991b). Recent developments in the use of computers to visualize and symbolize calculus concepts. In L.C.Leinbach, J.R.Hundhausen, A.M.Ostebee, L.J.Senechal & D.B.Small (Eds.), *The laboratory approach to teaching calculus.* USA. Mathematical Association of America.

Taylor, P., Fraser, B., & Fisher, D. (1993). Monitoring the development of constructivist learning environments. *Paper presented at the annual convention of the National Science Teachers Association.* Kansas City, MO: National Science Teachers Association.

Thompson, P.W. (1994). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics,* 26, 229-274.

Thompson, P.W., & Thompson, A.G. (1994). Talking about rates conceptually, part 1: A teacher's struggle. *Journal for Research in Mathematics Education,* 25(3), 279-303.

Tobin, K. (1993). Constructivist perspectives on teacher learning. In K.Tobin (Ed.), *The practice of constructivism in science education.* Washington: AAAS.

Tobin, K., & Tippins, D. (1993). Constructivism as a referent for teaching and learning. In K. Tobin (Ed.), *The practice of constructivism in science education.* Washington: AAAS.

Tolias, G. (1993). The effects of using graphing technology in college precalculus. Univ. of Texas, Austin: *Dissertation Abstracts International*, 54, 1274-1275A.

Truran, J.M., & Truran, K.M. (1998). Using clinical interviews in qualitative research. In J.A.Malone, B.Atweh & J.R.Northfield (Eds.), *Research and supervision in mathematics and science education*. New Jersey: Lawrence Erlbaum Associates, Inc.

Tuska, A. (1993). Students' errors in graphing calculator-based precalculus classes. Ohio State Univ.: *Dissertation Abstracts International*, 53, 2725A.

Vinner, S. (1997). The pseudo-conceptual and the pseudo-analytical thought processes in mathematics learning. *Educational Studies in Mathematics*, 34, 97-129.

von Glasersfeld, E. (1990). Questions and answers about radical constructivism. Atlanta: *NARST meeting* (April 1990).

von Glasersfeld, E. (1991). Abstraction, re-presentation, and reflection: An interpretation of experience and piaget's approach. In L.P.Steffe (Ed.), *Epistemological foundations of mathematical experience*. New York: Springer-Verlag.

von Glasersfeld, E. (1995). A constructivist approach to teaching. In L.P.Steffe & J.Gale (Eds), *Constructivism in education* (pp. 3-15). New Jersey. Lawrence Erlbaum Associates.

Vygotsky, L.S. (1962). *Thought and language*. The MIT Press.

Walkerdine, V. (1989). *The mastery of reason*. Routledge.

Wertsch, J.V., & Toma, C. (1995). Discourse and learning in the classroom: A sociocultural approach. In L.P.Steffe & J.E.Gale (Eds.), *Constructivism in education*. New Jersey. Lawrence Erlbaum Associates.

Wheatley, G.H. (1993). The role of negotiation in mathematics learning. In K. Tobin (Ed.), *The practice of constructivism in science education.* Washington: AAAS.

White, L. (1998). Teacher, researcher, collaborator, student: multiple roles and multiple dilemmas. In J.A.Malone, B.Atweh & J.R.Northfield (Eds.), *Research and supervision in mathematics and science education.* New Jersey: Lawrence Erlbaum Associates, Inc.

White, R., & Gunstone, G. (1992). *Probing understanding.* London: The Falmer Press.

White, P., & Mitchelmore, M. (1996). Conceptual knowledge in introductory calculus. *Journal for Research in Mathematics Education,* 27(1), 79-95.

White, P., & Pegg, J. (1996). Recent research in the upper secondary years. In W.Atweh, P.E.Sullivan & K.Owens, (Eds.), *Research in Mathematics in Australasia 1992-1995.* Kelvin Grove, Queensland: MERGA.

Whorf, B. (1956). *Language, thought and reality.* Cambridge, Mass.: MIT Press.

Wong, E.D. (1995). Challenges confronting the researcher/teacher: conflicts of purpose and conduct. *Educational Researcher,* April, 22-28.

Zepp, R., Monin, J., & Lei, C.L. (1987). Common logical errors in English and Chinese. *Educational Studies in Mathematics,* 18, 1-17.

Zevenbergen, R. (1998). Ethnography in the classroom. In J.A.Malone, B.Atweh & J.R.Northfield (Eds.), *Research and supervision in mathematics and science education.* New Jersey: Lawrence Erlbaum Associates, Inc.

Zimmerman, W. (1991). Visual thinking in calculus. In W. Zimmerman & S. Cunningham (Eds.), *Visualization in teaching and learning mathematics* (pp. 127-138). U.S.A.: Mathematical Association of America,.

Appendix A

Graphing Calculator Survey

This instrument was used to obtain information about the trial sample's perceptions of the classroom learning environment.

Graphing Calculator Survey

Directions

1. This questionnaire asks you to describe the Calculus classroom. There are no right or wrong answers. This is not a test. Your opinion is what is wanted.

2. Do not write your name. Your answers are confidential and anonymous.

3. On the next few pages you will find some sentences. For each sentence, circle one number corresponding to your answer.

   For example:

	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
In this class...					
I have a say in how many questions are on the tests.	*5*	*4*	*3*	*2*	*1*

   Circle 5 if this teacher ***Almost Always*** asks you how many test questions you want.
   Circle 1 if this teacher ***Almost Never*** asks you how many test questions you want.
   Or you can choose the number 2, 3 or 4 if this is a more accurate answer.

4. If you want to change your answer, ***cross it out*** and circle a new number, eg. :

   ~~5~~ 4 (3) 2 1

5. Please provide the following details before you commence:

   a. Year Level: ________

   b. Your Gender (please circle): **Male** or **Female**

   c. Secret 4 digit ID number: _ _ _ _
   (Make one up and remember it for use on all surveys).

6. Now turn the page and please give an answer for every question.

bridson96

In this class...					
I prefer to use the graphing calculator	5	4	3	2	1
I find it is faster to use the graphing calculator	5	4	3	2	1
I enjoy this class when we use the graping calculator	5	4	3	2	1
I find the exercises easier with the graphing calculator	5	4	3	2	1
I find it is slower to use the graphing calculator	5	4	3	2	1
I enjoy this class more than any other	5	4	3	2	1

In this class...					
I prefer not to use the graphing calculator in maths	5	4	3	2	1
I am not confident using the graphing calculator	5	4	3	2	1
I ask more questions when using the graphing calculator	5	4	3	2	1
I find the overhead projector calculator displays helpful	5	4	3	2	1
I find the teacher uses the graphing calculator too much	5	4	3	2	1
I enjoy the lesson more when we use the graphing calculator	5	4	3	2	1

In this class...					
I prefer pencil-and-paper than using the graphing calculator	5	4	3	2	1
I am more interested when we use the graphing calculator	5	4	3	2	1
I feel I am cheating when using the graphing calculator	5	4	3	2	1
I know when to use estimation with a graphing calculator	5	4	3	2	1
I understand equations better using a graphing calculator	5	4	3	2	1
I am quicker when using the graphing calculator	5	4	3	2	1

In this class...					
I am too dependent on the graphing calculator	5	4	3	2	1
The graphing calculator helps with understanding	5	4	3	2	1
I check my answers with the graphing calculator	5	4	3	2	1
Long numerical calculations are simpler with a graphing calculator	5	4	3	2	1
I have a good feeling about using a graphing calculator	5	4	3	2	1
I am a better student using a graphing calculator	5	4	3	2	1

In this class...					
Less teaching should use the graphing calculator	5	4	3	2	1
I am happy using the graphing calculator	5	4	3	2	1
My results improved with the graphing calculator	5	4	3	2	1
The graphing calculator is a status symbol	5	4	3	2	1
The graphing calculator is a nuisance	5	4	3	2	1
The graphing calculator is is only used for computations	5	4	3	2	1

In this class...					
My algebra skills improved using a graphing calculator	5	4	3	2	1
My attitude to maths was better using a graphing calculator	5	4	3	2	1
It was difficult using the graphing calculator	5	4	3	2	1
I wish to continue using a graphing calculator	5	4	3	2	1
I found it challenging using a graphing calculator	5	4	3	2	1
I enjoyed using a graphing calculator	5	4	3	2	1

Appendix B

Students' Histories

Details are given of the second-language backgrounds of sample members.

Note: For privacy reasons Appendix B has not been reproduced.

(Co-ordinator, ADT Project (Bibliographic Services), Curtin University of Technology, 30/09/03)

Appendix C

Assessments

Pre-Test Traffic Worksheets
Introductory Calculus Test
First Semester Examination
Second Semester Examination
Exponential Functions Worksheet

Details are given of the assessments used in the present study.

Note: For copyright reasons Appendix C has not been reproduced.

(Co-ordinator, ADT Project (Bibliographic Services), Curtin University of Technology, 30/09/03)

Appendix D

Extracts from Student Scripts

Details are given of student responses based on raw data collected from the classroom.

Extract 1.A: Task One Traffic Situations - John
Extract 1.B: Task Two Traffic Situations - John
Extract 1.C: Task Three Traffic Situations - John
Extract 1.D: Task Four Traffic Situations - John
Extract 1.E: Task Five Traffic Situations - John
Extract 1.F: Task Six Traffic Situations - John
Extract 1.G: Test One - John
Extract 1.H: Final Examination - John

Extract 2.A: Task One Traffic Situations - Welly
Extract 2.B: Task Two Traffic Situations - Welly
Extract 2.C: Task Three Traffic Situations - Welly
Extract 2.D: Task Four Traffic Situations - Welly
Extract 2.E: Task Five Traffic Situations - Welly
Extract 2.F: Task Six Traffic Situations - Welly
Extract 2.G: Test One - Welly
Extract 2.H: Final Examination - Welly

Extract 3.A: Task One Traffic Situations - Tonton
Extract 3.B: Task Two Traffic Situations - Tonton
Extract 3.C: Task Three Traffic Situations - Tonton
Extract 3.D: Task Four Traffic Situations - Tonton
Extract 3.E: Task Five Traffic Situations - Tonton
Extract 3.F: Task Six Traffic Situations - Tonton
Extract 3.G: Test One - Tonton
Extract 3.H: Final Examination - Tonton

Extract 4.A: Task One Traffic Situations - Oliver

Extract 4.B: Task Two Traffic Situations - Oliver

Extract 4.C: Task Three Traffic Situations - Oliver

Extract 4.D: Task Four Traffic Situations - Oliver

Extract 4.E: Task Five Traffic Situations - Oliver

Extract 4.F: Task Six Traffic Situations - Oliver

Extract 4.G: Test One - Oliver

Extract 4.H: Final Examination - Oliver

Note: For privacy reasons Appendix D has not been reproduced.

(Co-ordinator, ADT Project (Bibliographic Services), Curtin University of Technology, 30/09/03)

Appendix E

Interview Data Listed by Experimental Situation

Details are given of the interview data.

Note: For privacy reasons Appendix E has not been reproduced.

(Co-ordinator, ADT Project (Bibliographic Services), Curtin University of Technology, 30/09/03)

Appendix F

Resources for Velocity, Distance and Acceleration

Resources for Velocity, Distance and Acceleration

Details of references and examples follow which may be suitable to include in assessment tasks. They have been listed alphabetically by author(s).

Atkins and Fergusson (1989a, 1989b) have compiled a valuable resource of mathematical graphs formatted as a handbook of blackline masters.

Baumgartner and Shemanske (1990) have detailed an exercise using standard Galilean motion under gravity but with extensions to the inclusion of air resistance:

> '(Alice's Adventure) An exuberant Math 3 student, flying high over Hanover in a small plane one nice day, decided to take a shortcut back to the dorm. She jumped out of the plane. On the way down she naturally wondered how fast she was going.
> "Well," she thought, "since my rate of change of velocity is 32 ft / sec / sec, I need only solve the differential equation dv / dt = 32, subject to the initial conditions v (0) = 0."
>
> (1) What formula did she write down (she had a notebook and pen in her jump suit) to express her velocity as a function of time?
> (2) If the plane was flying at 10,000 feet at the time of her departure from it, how long did it take her to reach her dorm?
> (3) How fast was she moving as she passed the window of her first floor dorm room?
>
> Her roommate exclaimed, "My, you're back early. You couldn't possibly have used the right model. Did you take air resistance into account? Air gives a retarding force proportional to the square of your velocity, you know."
> "Oh my," said Alice, "I should have solved the equation $dv / dt = 32 - kv^2$, $v(0) = 0$, instead. For that I need my Mac, so it's good that I'm home early."
> Alice took the program RungeKuttaPlot and modified it to solve
>
> $$dy / dx = 32 - ky^2, \qquad y(0) = 0$$
>
> On the interval $0 \le x \le 50$. She chose the vertical scale to be $0 \le y \le 200$. She did not know the value of the proportionality constant k, of course, since she has no security clearance, so she decided to experiment. She did this by putting a statement
>
> Input prompt "k = ": k
>
> in the program so that she could enter values of k and observe the result. She tried values k = 1, 0.1, 0.01, 0.001, 0.0001. "My goodness," she expostulated, "I didn't know my velocity did that. How clever." She asked her roommate about the weird behavior and he/she said, "Why, didn't you know, a body reaches a limiting velocity of 185 feet per second in the atmosphere. That's about 125 miles per hour and makes

you really think what it must have been like to be in Charleston when Hugo was huffing and puffing at 135 mph."
"Imagine that," said Alice. "I dare say that I can use my model to estimate the value of k. I've wanted to know its value all my life, and now I realize that Math 3 (and my Mac) allow me to discover what it is. I don't even need a security clearance."
She did. She found the value of k to 2 significant digits. Can you?

Bergamini (1993) gives a detailed historical account of the rate of change and acceleration concepts and their development.

Grida and Forster (1997) have suggested the following:

'Give a visual/numeric introduction to rectilinear motion by first plotting the function below and calculating first and second derivatives, and analysing tables.

$X = t^3 - 8t^2 + 16t - 8$

Holmes, Ecker, Boyce and Siegmann (1993) have provided an example on the trajectories of a baseball, with and without air resistance. This involves two-dimensional motion and the use of trigonometric functions. Their applications to the Maple software package can be adapted for the use of graphic calculators. Their aims seem to match those of this research as far as using technology for creative thinking:

'It is important to stress that, despite its power, the effective use of Maple requires clear thinking and good judgement. The fact is that Maple is a tool and it is up to you to employ it wisely. The benefits of this approach are enormous as Maple can help develop imaginative and critical thinking, rather than having you concentrate on routine mathematical operations' (p.2).

'Suppose that a baseball leaves the bat with an initial velocity of u feet per second at an angle A with the horizontal. We wish to determine the path followed by the ball.
First we consider a simplified version of the problem in which air resistance is neglected. Let v and w be the velocity components in the horizontal (x) and vertical (y) directions, respectively. Then, as a consequence of Newton's law, v and w satisfy the equations of motion

$$dv / dt = 0, \qquad dw / dt = -g$$

where g is the acceleration due to gravity. The most general functions satisfying these equations are v = c1, w = -gt + c2

where c1 and c2 are arbitrary constants of integration.
The following equations are derived (Holmes, et al, 1993, p.137):

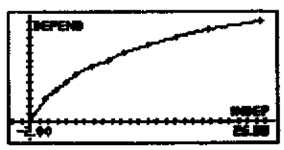
$$x = u\cos(A)t \qquad y = -1/2gt^2 + u\sin(A)t + h$$

where the ball strikes the bat at a height h above home plate and the origin is at the centre of the home plate.

Johnston, Mathews, Bergman and Heckenbach (1991) have suggested a problem related to test results for the movement of a Porsche 911 Carrera 4. The data from this test are:

(0, 0), (1.8, 0), (2.8,4 0), (3.9, 50), (5.1, 60), (6.8, 70)
(8.6,80), (10.5,90), (13.0,100), (16.2,110), (19.7,120)
(25.6, 130)

as reported in Car and Driver 35(1989), p43. A typical data point has the form (t,v) where v is the velocity of the Porsche at time t. The point (2.8, 40), for example, contains the information that 2.8 seconds after the acceleration test began; the Porsche was travelling at 40 miles per hour.



Above is a graph of v against t. You will be asked to find the position x(t) of the Porsche at times t = 0, 1.8, ..., 25.6, interpret the computations geometrically and compute the average accelerations on intervals used.

Smith and Moore (1990) give an example used at Duke University of what they term "prototype problems" from other disciplines. The example refers to motion of an object falling under gravity with air resistance.

'This data reflects the force of gravity, the retarding force and a small random measurement error. We reproduce half of the data here.

Time (in seconds)	distance (in meters)
0	0
1	4.8
2	18.5
3	40
4	69
5	104

6	146
7	193
8	244
9	300
10	361

We want to find a function s (t) that approximates this data. In the case without air resistance, we found that the rate of change of the velocity, the acceleration, was essentially constant. If we assume that dv / dt is constant we are led to an approximating function that is a quadratic polynomial. In the present case we'll begin the same way, by trying to approximate the velocity, v = ds / dt. Since we data at a number of discrete points, it is reasonable to approximate the derivative by difference quotients.

Suppose we label the times at which we have the data as t_0, t_1, ..., t_{20} and try to estimate the derivative at t_1. Our first idea is to use the *forward difference quotient*

$s(t_2) - s(t_1) / (t_2 - t_1)$

since it is the limit of difference quotients of this type (as t_2 approaches t_1) that defines the derivative. We argue that a better estimate to the derivative is obtained by using the *symmetric difference quotient*

$s(t_2) - s(t_0) / (t_2 - t_0)$

Appendix G

The Nature of the Graph on the HP38G

Features of the HP38G
Finding a Graph
Good Plotting Techniques
Good Plotting Techniques Two

Features of the HP38G

The HP 38G is the first HP calculator with the power of a high-end graphic calculator made easy for high school math. View equations numerically, graphically or symbolically on its large graphic display. And with aplets learning math has never been more easy and fun. These small applications, packaged as electronic lessons can be adapted from textbooks and other sources such as the Hewlett Packard Internet site (http://www.hp.com/calculators/graphic/38g_info.html).

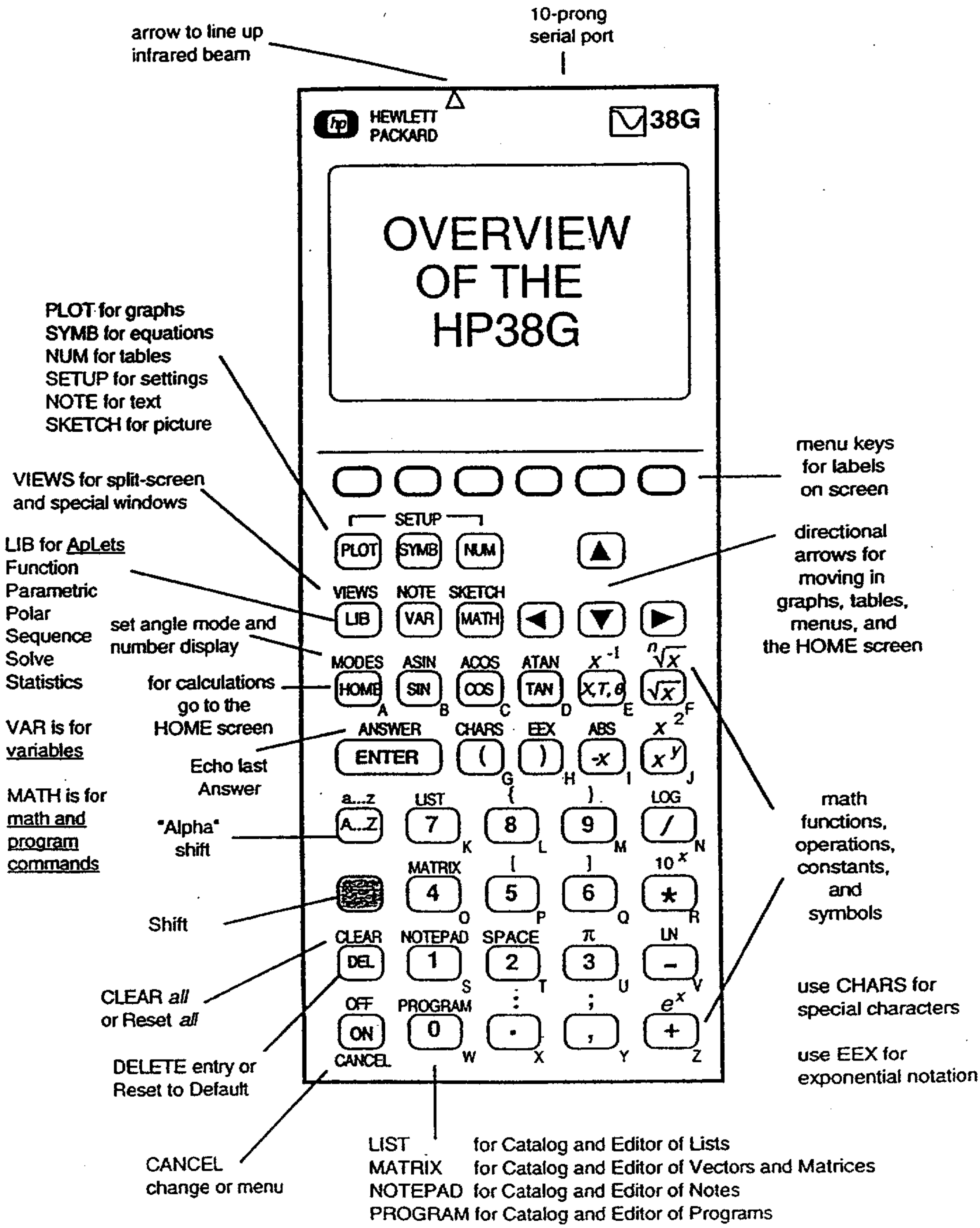
Add variables, pictures and graphs, then interact and work in new ways at your own pace. Dynamic Split Screen View can look at math relationships in two displays on one screen for side-by-side comparison. An interactive history of calculations is kept up to available RAM menu so you can recall and reuse solutions previously created. The impact-resistant sliding hard cover protects the display and keys.

In line with a description by LaTorre et al. (1990) of the HP-28S calculator, the display screen of the HP-38G is a 131 by 64 rectangular grid of picture element dots called "pixels". When the PLOT command is activated, the entire screen is used to display the graph of the expression stored in the storage area designated as FN(X). On the graphing screen, with the default plotting parameters, the pixels in the leftmost column have x coordinates -6.5 and the pixels in the rightmost column have x coordinates 6.5. The highest row of pixels has y coordinates 3.2 and the lowest row of pixels has y coordinates -3.1. Two adjacent columns of pixels have x coordinates differing by 0.1 units and two adjacent rows of pixels have y coordinates differing by 0.1 units.

Table G.1: Technical Data for the HP38G

Product name	HP 38G
Model Number	F1200A
Physical Features	
Display	8 line x 22 char. LCD
Entry system logic	Algebraic
Menus, prompts, alpha messages, softkeys	Yes
RAM/no of variables or registers	32 KB, unlimited within available memory
Built-in functions	Over 600
Redefineable keyboard and menu keys	Yes
Math and Statistical Features	
+, -, x, /, sq. root, 1/x, +/-, lnx, e^x, n!, y^x, logx, 10^x, x^2, %, pi	Yes
Fractions	Yes
Trig., Hyperbolics, HP Solve (root finder)	Yes
Numeric integration/ Complex number functions	Yes
Statistical analysis	Multivariate/stat
SUM x, SUM x2, SUM y, SUM y2, SUM xy	Yes
Sample standard deviation/ mean/ weighted mean	Yes + Population standard
Linear regression, combinations, permutations	Yes
Curve fit (LIN ,LOG, EXP, POW)	Yes
Normal, X, t, F distribution	No
Scientific Features	
Decimal hrs/hrs.min.sec. conversions	No
Polar/ rectangular and angle conversions	Yes
Base conversions and arithmetic	No
Unit conversions	No
Bit, Boolean operations	No
Vector and Matrix Operations	
Matrix operations, rectangular and polar	Yes
Cylindrical, spherical	No
HP matrix writer, row and column operations	Yes
APLETS (built-in or added applications including Note and Sketch menus)	Yes
Notepad	Yes (built-in)
Graphic Functions with interactive graphics	2D
Customizing Features	
Method	HP Solve
No. of steps/formulas or programs/ formulas	unlimited within available memory
Peripherals	
Optional Infra-red Printer, HP 82240B	Yes

Figure G.1: HP38G Screen Display



The calculator technical details have also been discussed extensively by Croft on an Internet site (http://athena.sthildas.wa.edu.au/~ccroft/).

> ‘ The default screen is -6.5 to 6.5 on the x axis. If you include the dot for zero, this means that there are 131 dots with each dot on the screen being 0.1 on the x axis. When you are planning your choice of scale in PLOT SETUP it is a good idea to use multiples of this default scale. For example, rather than using –10 to 10, use -13 to 13 (double 6.5). If you need to see 0 to 20, then choose –2 to 24 instead (which is -13 to 13 moved right by 11). If you don't like having your scale end at "nasty" numbers like 6.5 then add 0.5 to both ends, changing -6.5 to 6.5 into -6 to 7. An ApLet called FUNCTION PLUS is one of the ApLets in my set of Maths ApLets. It is identical to the normal Function ApLet but includes an extra option on the VIEWS menu under Autoscale called ‘Window Fix’. If you choose this option then the PLOT SETUP will be altered to find the closest scale to the one you have chosen which will has ‘nice’ jump sizes.’

When you activate the PLOT command for a function f stored in FN(X), the HP-38G calculates values of y = f(x) for each of the 131 values of x, 0.1 units apart, from x = -6.5 to x = 6.5. If the size of the y coordinate puts the point on the display, that is, if $-3.1 \leq y \leq 3.2$, the appropriate pixel is energized and appears as a point on the graph. So the "graph" of a function on the HP-38G is actually a set of 131, or less, distinct points.

Try these problems:

- $2^x=x^4$ as $2^x/ x^4-1 = 0$ or $x^4/2^x - 1 = 0$ or $2^x- x^4 = 0$ or $2^x= x^{10}$ or try 69
- Some special curves include y = sinx + | sin100x | / 100 (Tall, 1991a)

Buecher (1997) has discussed the finding of cubic and quadratic curves. Before this can be done an understanding of the value of a suitable choice of Xmin and Ymin values is necessary. The default plot setup has values in the domain from $-6.5\leq X\leq 6.5$ and range $-3.1\leq Y\leq 3.2$. These choices give a fixed X increment each time the arrow keys are pressed from the PLOT view window and the (x,y) trace has been turned on. Following Buecher (1997) the size of each increment in X can be found from Change = (xmax-xmin)/130:

'A user-friendly viewing window is one where the value of 'x increment' is a multiple or divisor of 1, 2 or 5' (Buecher, 1997).

For example if Xmin = -6.5 and Xmax = 6.5 then

X increment = change = (6.5-(-6.5))/130

= 13.0/130

= 0.1

Thus the following Table G.1 shows examples of suitable values that would create user-friendly windows:

Table G.2: Working With Friendly Windows.

Xmin	Xmax	Increment
-6.5	6.5	0.1
0	13.0	0.1
-13.0	13.0	0.2
-32.5	32.5	0.5
0	65.0	0.5
-65.0	65.0	1
0	130.0	1
-1300	1300	20

Go to the function library and start the Function aplet. Press the SYM key and enter the function F1(X)=x2 –8x –20. Go to PLOT setup and set Xmin = -13.0 Xmax = 13.0. Check that the axes are labelled. Choose Xtick = 2 and Ytick = 2. Now press PLOT to view the graph. Press the menu key that is positioned as the 6th key on the right. Select (x,y) to give the cursor position. Each time the right arrow is pressed the value of the x-coordinate increases by 0.2.

Try Y = x2 – 64
Y = -x2 –23x – 120
Y = x3 -2x2 –5x +6
Y = -x3 +6x2 +13x – 42
Y = x3 -3x2 –61x +63
Y = x2 –5x – 19

$Y = x2 - 7x + 40$
$Y = -x3 - 19x2 - 1997x - 97$

As you move to the right edge of the calculator screen the scale automatically floats to beyond the range and the x-scale is updated. However, as you move to the left from the right boundary, the scale will not change and will only do so if you move to the other side of the screen. The base settings change e.g. from –7.5 ↔ 5.5 becomes –6.5 ↔ 6.5, -4.5 ↔ 8.5, -3.5 ↔ 9.5, -2.5 ↔ 10.5. Unfortunately, the final settings will be transferred to all future graphs drawn. It would be useful to draw a graph in one step. How can the default setting be returned immediately? VIEWS → AUTOSCALE does a good job of finding a graph but does not use default x domain values. It only changes the Y-scale, keeping -7.5 ↔ 5.5. One option may be to use LIB → FUNCTION → RESET.

Swincicky (1994, p.87) has discussed the need for a 'good' screen display and given a definition:

> '...I refer to a good screen display which is the image that is displayed on the screen of the graphics calculator that allows a given problem to be solved. For example: Find the maximum value of y = -x2 + 5x. in this instance, a good screen display is one that shows the turning point of the graph.'

It is also noted by Swincicky (1994, p.100) that the RANGE function key on the Sharp calculator caused considerable problems with lower achieving business mathematics students, when they tried to produce a good screen display. The importance of range selection was emphasized to students by a sequence of steps:

> '(1) getting the students to focus on the concept of the range and how the range could be used to view points of intersection (this initial phase did not require students to use a calculator); (2) introducing students to the entry of equations into the graphics calculator and the adjustment of the range (using the RANGE function key) using given end points of the screen'

This process becomes less a problem in this study since the HP 38G calculator allows labels on axes so that plot setup values and the graph can be shown together on the screen. An interview with two low-ability international students is particularly relevant for this study. The students were asked the question "what topic was hardest when using the calculator":

'Me: Graphing?
Mary: Yes, graphing and the range.
Julie: Yes, the range.
Julie: Yes, just finding the range.
Me: Are you getting better at it?
Mary: No.
Julie: I don't know how much of the range you should use. Some people can tell roughly what range to use. I can't. AUTOSCALE! Then from there I work my own [range].
Me: Do you eventually get there?
Julie: Yes, but not all the time.
Mary: AUTO SCALE doesn't always help.
Julie: It only shows half a diagram, and you don't realise it's only half.

A number of resources are available to assist in the development and use of programs. In the case of the HP-38G, programs can be designed as interactive lessons or ApLets (applications packaged as electronic lessons):

> 'An ApLet is like a handout - complete with variables, pictures, graphs and notes. Your students can use it to explore math concepts freely on the HP 38G - without your guidance - and without fear of losing either their work or the original lesson. During their explorations, they can save their work, try new approaches, or just start over if things don't work out (Carter, 1995)'.

With the addition of an overhead projector panel linking the calculator, it is possible for the first time to have truly interactive teaching. Combining this feature with an easy to use interface gives two important reasons why this model has been so popular.

The calculator itself has six main "super" settings, called ApLets: Function, Parametric, Polar, Sequence, Solve and Statistics. For working in each of these ApLets, the HP38G has important tools:

> 'graphical tools (in PLOT) to plot and analyze graphs of graphs of functions, equations, and data;
> symbolic tools (in SYMB) to enter and view formulas and fit equations to data;
> numerical tools (in NUMB) to make tables of function values or data, or to analyze equations' (Hewlett-Packard, 1995a, 1995b)

To prepare worksheets it is an advantage to prepare screen dumps from the calculator as illustrated by Kissane (1995b). These screens require careful interpretation, especially with graphs, where scales and graph chunkiness created new images which students need practice with. The graph scales may lack text, screens have lower resolution than computers, and the images are small. Solutions given may be approximate rather than exact but this can be seen as an advantage if the aim is to develop student thoughtfulness and insight with the use of calculators:

> 'Although exact solutions of equations are important, it is also important that students learn when an approximate solution is adequate for a practical purpose, and how close is close enough. This involves learning decision-making rather than rule-following skills, which have long been emphasized in this part of the curriculum (Kissane, 1995b, p. 42)'.

Steele (1995), in an unpublished masters thesis, noted that graphics calculators need emphasis on scaling, as graphs, which omit important features, can be misleading.

A Program Example:

```
DISP 2; "By J. Bridson":
BEEP 897.96963; .5:
BEEP 599.32283; .2:
WAIT .5:
MSGBOX " Happy Birthday ":
BEEP 534; .1:
BEEP 600; .1:
BEEP 673; .1:
BEEP 712; .1:
BEEP 800; .1:
BEEP 900; .1:
BEEP 1008; .1:
BEEP 1070; .1:
BEEP 1200; .1:
BEEP 1350; .1:
BEEP 1425; .1:

BEEP 1600; .1:
BEEP 1800; .1:
BEEP 2015; .1:
BEEP 2135; .1:
WAIT .5:
```

Finding a Graph

The algebra of polynomial functions for the HP38-G has been covered extensively in a book by Claffey (1996, p. 66-74). By using the POLYROOT function you could determine the roots of any polynomial and then use these to choose a suitable minimum and maximum for the horizontal x - axis. This would imply that you wanted the roots to appear in the plot. An example taken from the Texas Instruments newsletter, TI Cares, and edited by Neydorff (1997) follows:

Find a friendly viewing window which shows all of the essential features (intercepts and turning points) of the function $y = x^2 + 99.5x - 50$. The use of POLYROOT([1,99.5,-50]) returns the result [.5,-100]. Thus the roots of this function (or polynomial, in this case) are .5 and -100. These values can then be used to create a suitable viewing window, as once the horizontal scale is set, the calculator has an automatic feature to select a suitable vertical scale for the plot, using the VIEWS and AUTO SCALE options.

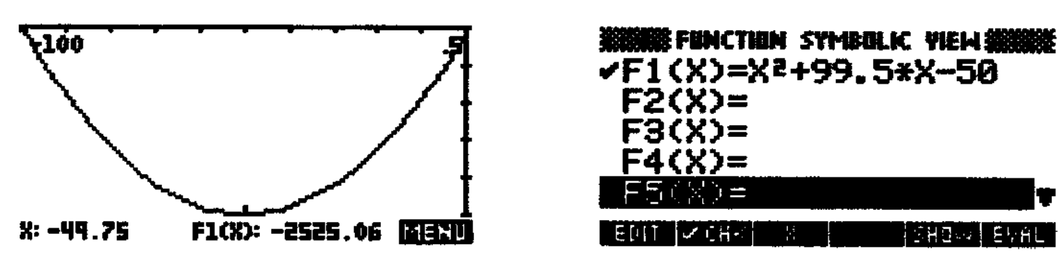


Figure G.2: Using Polynomial Roots to Find a Graph

This gives the student a good idea of the plot. To include the positive axes in the plot it is necessary to include a greater horizontal range, and, in particular, to include more positive x and y values. How much to include will depend on the aim of the exercise as well as the calculator limitations. For the HP 38G, it is necessary to include both positive and negative values on both axes if you wish labels to be shown. Following the suggestions in the TI article the values used are as follows:

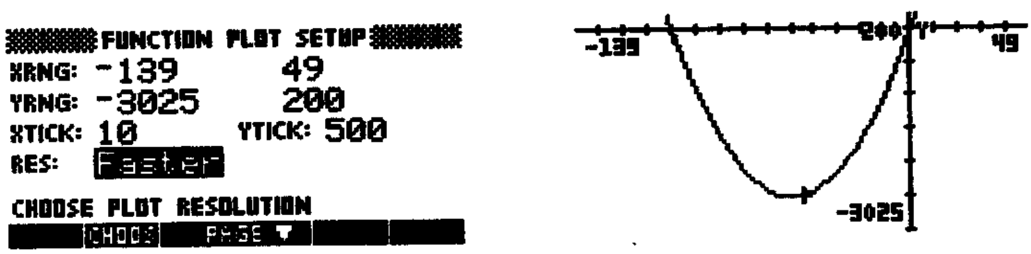


Figure G.3: How to Obtain Axis Labels

Once again the screen needs careful analysis, as the labels on the axes are not placed directly below the appropriate scale divisions. Offcourse, here the x and y axes have very different scales. Students' prior knowledge of steepness and relative speed and knowledge of slope can be combined on a graph. An example from Swincicky (1994) involves the curves: $Y_1 = - 0.5\ (x - 200)^2\ + 50$ *and* $Y_2 = 20\ x - 3910$ which could be used to assess students' zooming abilities.

In a study of students at Swinburne University of Technology, Boers and Jones reported on the use of the graphics calculator as a tool (Penglase & Arnold, 1996, p. 78):

> 'Also as a result of this survey, it was found that students tended to feel that the introduction of this tool had caused them to adopt mathematical behaviour which is considered to aid learning: specifically, using more exploration in the solution of problems, using graphs as an aid to solving problems, and using the graphics calculator to check algebraic solutions.... The five benefits that the students felt were most important were: the ease of sketching and obtaining information from graphs; being able to check quickly the correctness of derivatives, integrals and solutions; being able to understand and interpret graphs and derivatives more easily; the ease of calculation and checking procedures regarding difficult formulae; and the increase in confidence and enthusiasm associated with the use of the tool.'

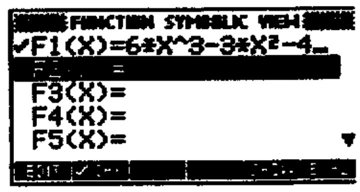
Good Plotting Techniques	**May**
1997	

Objectives:

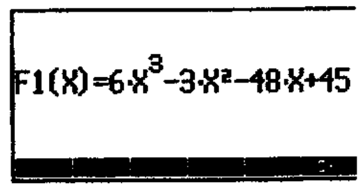
Using the graphics calculator, you will investigate the behaviour of a cubic polynomial taken from Dion (1990, p.569). Find a good screen display using the zoom features of the graphics calculator. Choosing an appropriate viewing screen is crucial when exploring graphs. You will compare the default screen with a number of others.

Functionality:

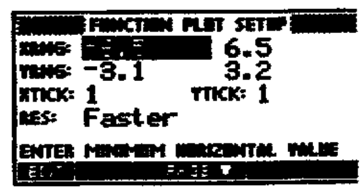
Select the [LIB] key and choose the Function aplet.
Clear all functions and enter the cubic below:
$F(x) = 6x^3 - 3x^2 - 48x + 45$
This curve can be factorised as :
$F(x) = 3(x + 3)(x - 1)(2x - 5)$
Tick F1



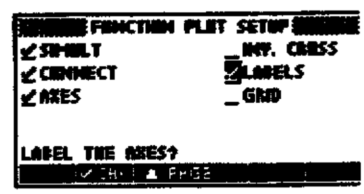
[SHOW] allows you to check your entry and displays the function as you would see it in a textbook.



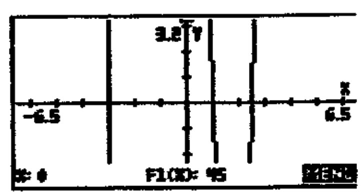
There are 130 pixels on the HP38G screen. The range chosen on the x-axis is usually a multiple or factor of 130 in order to give integer increments when using the (x,y) menu item, in this case we use the default 130/20 = 6.5. The other details are as shown.



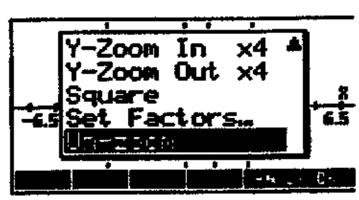
Press the [PAGE] key and select the options shown. It is useful to include the axes and labels and to connect the points.



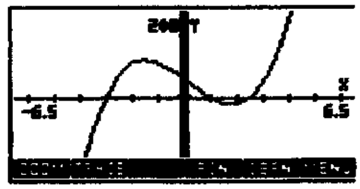
The [PLOT] gives very poor information about stationary points and the goal is to produce a better result. Because we cannot see the shape of the curve it is necessary to expand the vertical y-axis using a [ZOOM] option. A Y-ZOOM OUT can be used. You can use the Set Factors and Un-zoom for better control.



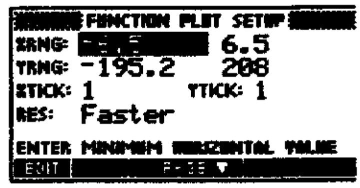
This process can be repeated a number of times. As you zoom out the ytick value will need to be adjusted.



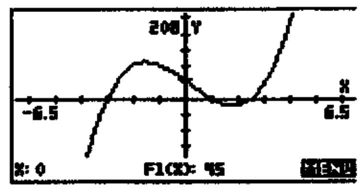
This solution comes from using the Y-ZOOM OUT with a factor of 4 three times. To see the overall effect on the y-axis scale we go back to the plot setup screen to check the final values.



The factor of 4x4x4 applied would give the values 204.8 and -198.4 and with a recenter feature these are changed by the original yrng value of 3.2 to the values shown.

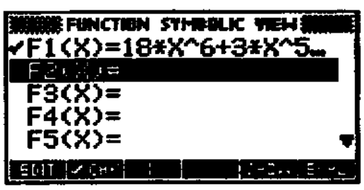


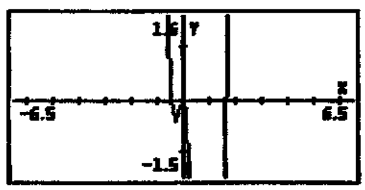
The final good plot is shown opposite with a ytick of 40.

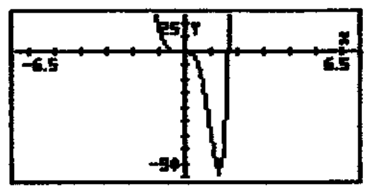


Additional Exploration:

Plot the function $y = 18x^6 + 3x^5 - 25x^4 -41x^3 -15x^2$ and use a similar technique to above to locate at least three stationary points (Dion, 1990, p.569):







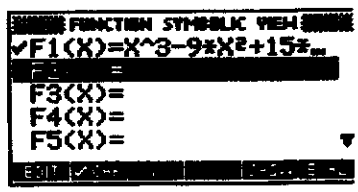
Good Plotting Techniques Two	May
1997	

Objectives:

Using the graphics calculator, you will investigate the behaviour of a cubic polynomial. Find a good screen display using the zoom features of the graphics calculator. Choosing an appropriate viewing screen is crucial when exploring graphs. You will compare the default screen with a number of others.

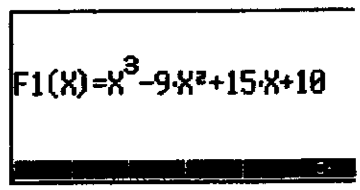
Functionality:

Select the LIB key and choose the Function aplet. Clear all functions and enter the cubic below:
$F(x) = x^3 - 9x^2 + 15x + 10$

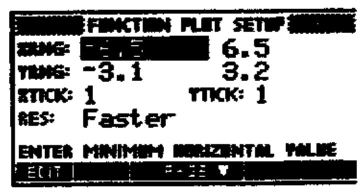


Tick F1

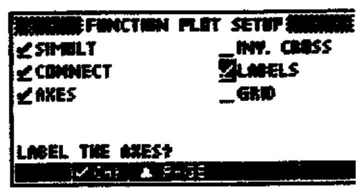
SHOW allows you to check your entry and displays the function as you would see it in a textbook.



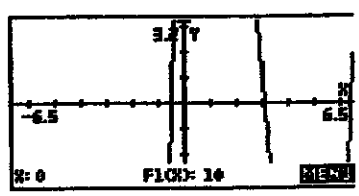
There are 130 pixels on the HP38G screen. The range chosen on the x-axis is usually a multiple or factor of 130 in order to give integer increments when using the (x,y) menu item, in this case we use the default 130/20 = 6.5. The other details are as shown.



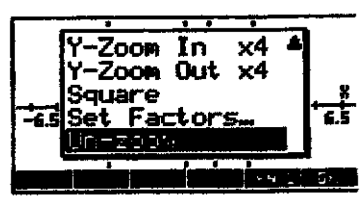
Press the PAGE key and select the options shown. It is useful to include the axes and labels and to connect the points.



The PLOT gives very poor information about stationary points and the goal is to produce a better result. Because we cannot see the shape of the curve it is necessary to expand the vertical y-axis using a ZOOM option. A Y-ZOOM OUT can be used. Also the x-axis needs extending to the right..

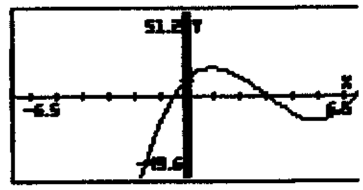


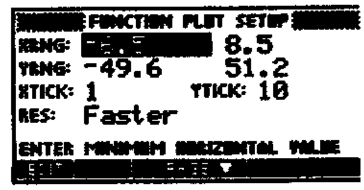
This process can be repeated a number of times. As you zoom out the ytick value will need to be adjusted.



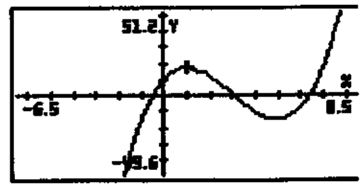
This solution comes from using the Y-ZOOM OUT with a factor of 4 twice. To see the overall effect on the y-axis scale we go back to the plot setup screen to check the final values.

The factor of 4x4 applied would give the values 51.2 and -49.6, simple multiplying the original range values by the factor of 16 (making sure the recenter feature in the set values is turned off).



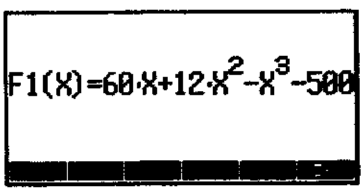


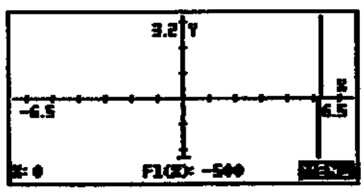
The final good plot is shown opposite with a ytick of 10 and an upper x value of 8.5.

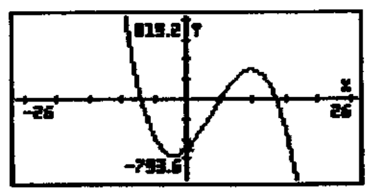


Additional Exploration:

Plot the function $y = 60x - 500 - x^3 + 12x^2$ and use a similar technique to above to locate the stationary points. A combination of Y-ZOOM OUT and X-ZOOM OUT will give best results:







Appendix H

Programming the HP38G

Introduction

Learning to use the Graphics Calculators

Introduction

The HP38G displays two graphs or a graph and table on a vertical split-screen; draws and annotates diagrams; saves and shares equations and machine settings; offers full programmability. The HP38G calculator design team formed an Education Advisory Committee consisting of representatives from high schools, community colleges and universities. One aspect of the design was for it to operate like a child's sandbox (Beers, Byrne, Donnelly, Jones & Yuan, 1996, p. 15):

> 'the child is given toys for playing and exploration, but within a protected, specialized environment. Thus one of our main goals with the calculator software design was to encourage exploration by limiting choices. This led us to the concept of aplets:an aplet is a small application that focuses on a particular problem.'

The design of aplets can use the National Council of Teachers of Mathematics, USA, "three views": graphic, symbolic and numeric. In this way, a function can be expressed as a graph (Figure H.1), in symbolic form (Figure H.2), or as a table of numbers (Figure H.3).

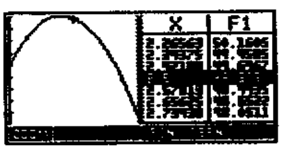


Figure H.1: Split Graph/Table

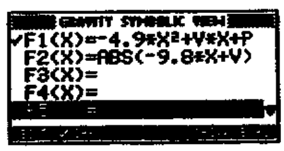


Figure H.2: Symbolic

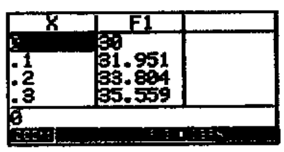


Figure H.3: Numeric

Learning to Use the Graphics Calculators

Jim Donnelly (1995) from Hewlett-Packard has explained the potential teaching packages available with the HP38G:

> 'The HP 38G is designed to provide focused exploration using ``aplets". An aplet is an encapsulation of all the information needed to explore a small subject. Two very important guiding principles of the HP 38G design are:

1. The observation that in a classroom setting there is (and should be) a large discrepancy between the amount of information entered into a graphing calculator vs. the amount of information produced by the calculator. We seek to minimize the calculator input required for the student and the teacher and maximize the returned information.

2. Exclude irrelevant possibilities. By reducing irrelevant choices you focus the user's attention on the subject of interest and avoid distractions from things that don't matter. Many choices made during the design of the HP 38G were based on this goal.'

Table H.1: <u>Selected Programming Commands</u>

PIXON X;Y	Turn on or light up a point or pixel. Good for producing a quick sketch usually using a loop command such as FOR TO STEP and a very small step size. Try PIXON 1;3. A short example is: FOR J=0 TO Xmax STEP (Xmax-Xmin)/65; PIXON J;F1(J) END: DISPLAY→ Page: FREEZE This plots a sketch of the function F1(X). For the values of Xmax=6.5 and Xmin=-6.5 this gives a step size of 0.2. No axes are drawn
QUOTE (X)	Gives the variable as a result rather than its value. QUOTE(F(X))=F1(QUOTE(X)) will give the value F(X)=8-2*X if that is the function stored in variable F1
→GROB	Transfers an expression into a graphics file with a given font size, e.g. →GROB G4;F1(QUOTE(X));2: REPLACE G1;(-6.5,3.1);G4: DISPLAY→ Page: FREEZE places the expression 8-2*X into the top left corner of the screen
IF THEN END	Branch command, which executes only if true. IF (N<0) THEN MSGBOX" N must be greater than zero": END:
DO UNTIL END	A loop which executes repeatedly until true. DO INPUT V;"Velocity";"Starting velocity";"Give the initial velocity m/s" UNTIL (V>0) OR (V=0) END:
STO (▶)	Enter values. To give room on the display to allow for the menu use: Ymin-.125*(Ymax-Ymin) ▶Ymin:

Appendix I

Graphics Calculator Aptitude Worksheets

Cubic Functions Worksheet
Good Screen Display
Rates of Change

Cubic Functions Worksheet

In September 1996 the researcher gave the following exercise to the Introductory Calculus class:

> 'Sketch $y = x^3 - 9x^2 + 15x + 10$ with guidelines that only one graph should be selected and that Autoscale may be unsuccessful. It was suggested that students try $x \varepsilon (0,10)$; $y \varepsilon (-40,40)$. There are options of using dy/dx as a symbolic approach using the HP 38G SYM key or looking at numbers via the NUM key.'

An example from August 1996 is $f(x) = 3x^2 - (2/x)$, a very complex curve with the use of Box Zoom to find asymptotes and then to find turning points as $f(x) \rightarrow \infty$ as $x \rightarrow \pm \infty$. Does the calculator give "true" results? Since a cubic is involved in this case. The derivative has the form $F^1 (x) = 6x + (2/x^2)$

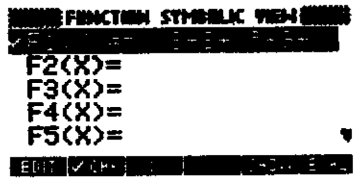
Using a September 1996 exercise $y = (1/8) . X . (X-12)^2$ instead of $y = (1/8) . X. (X-12)^2$ where a multiplication sign has been left out (.) gives a situation where the calculator cannot understand the equation. Change y tick interval to 5 allows a scale to show up on the screen. Press menu on and off shows up the axis scale. Allow negative axis domain values if you want a scale to show up and the x-axis to be seen. Notice that the second derivative goes from – , 0 , + around a point of inflection. Sketching this curve use Home Modes to use a fixed (0) number of decimal places to give a simple scale. The scale will be reset automatically but only when plot is pressed twice. The use of d^2y/dx^2 is limited to the NUM option as the evaluation of derivatives uses an unfriendly extended product rule notation.

Objectives:

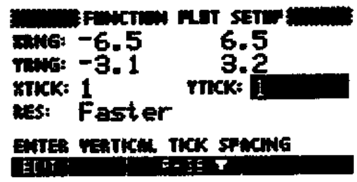
Using the graphics calculator, you will find a good screen display for the function $y=-x^3 + 3x^2 + 9x$ so that the intervals where the function is increasing or decreasing can be determined (Swincicky, 1994, p. 105-106).

Functionality:

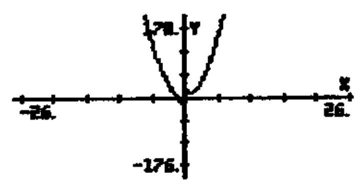
Select the LIB key and choose the Function aplet. Clear all functions and enter as shown



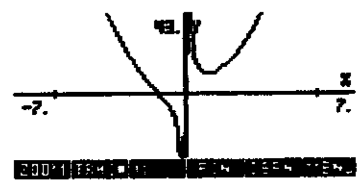
PLOT SETUP allows you to choose a suitable x and y range. The default values are shown.



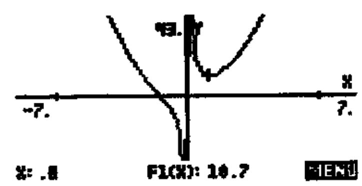
PLOT gives you a sketch of functions with a tick (✓) next to them. Tick F1. The range on the x-axis is usually a multiple of 130, the number of pixels on the screen. In this case 130/20 = 6.5. The value 6.5 on the axis refers to the endpoint. Each x interval (xtick) is 1 unit. Use the y range shown.



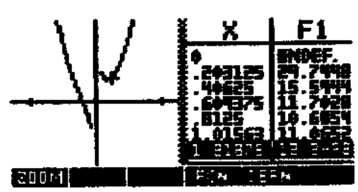
ZOOM will make the display clearer so that stationary points can be found. A simple 4x4 ZOOM In clearly shows the local minimum. Notice the x and y ranges have been divided by 4.



Use TRACE and (x,y) to approximate the position of the local minimum. In this case the x values will increase by .1 at a time

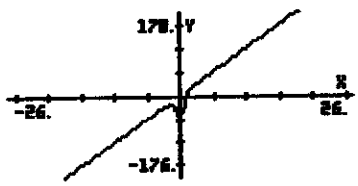


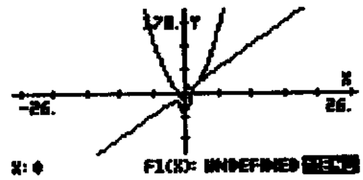
From VIEWS choose Plot-Table.
The left and right arrows will shift the trace along one of the curves. Zoom In to find the local minimum to two decimal places. Check with the FCN key. Which method do you prefer?

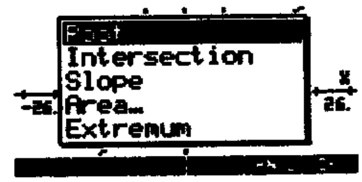


Additional Exploration:

Plot the rate of change function F3. The same ranges as above have been used. Next, plot both functions together, find their intersection points and discuss.







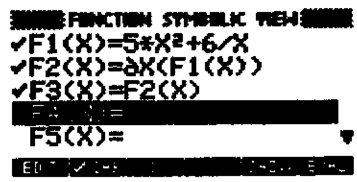
Rates of Change August 1996

Objectives:

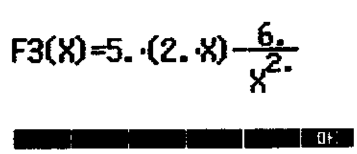
Using the graphics calculator, you will investigate the approximate rate of change of the function in example 1 of the text, page 108 (Sadler, 1992). Find a good screen display and plot the function and its rate of change together on a split screen.

Functionality:

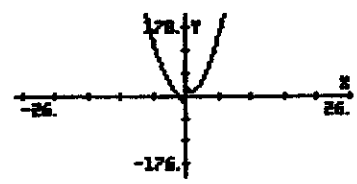
Select the LIB key and choose the Function aplet. Clear all functions and enter as shown



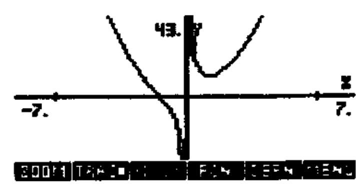
EVAL allows you to calculate function F3 and then use the SHOW key. This is the exact form of the rate of change and we will use it to produce faster plots.



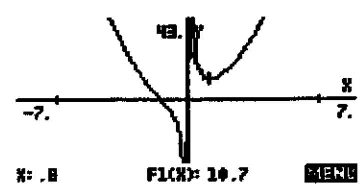
PLOT gives you a sketch of all functions with a tick (✓) next to them. Only tick F1. The range on the x-axis is usually a multiple of 130, the number of pixels on the screen. In this case 130/5 = 26. The value 26 on the axis refers to the endpoint. Each x interval (xtick) is 5 units. Use the y range shown.



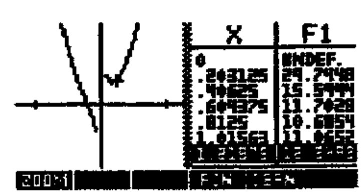
ZOOM will make the display clearer so that stationary points can be found. A simple 4x4 ZOOM In clearly shows the local minimum. Notice the x and y ranges have been divided by 4.



Use TRACE and (x,y) to approximate the position of the local minimum. In this case the x values will increase by .1 at a time

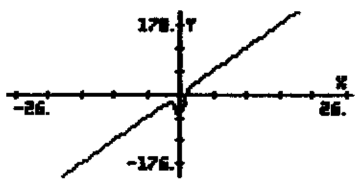


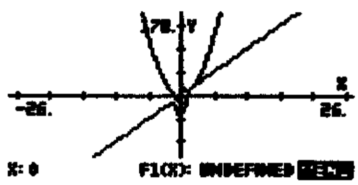
From VIEWS choose Plot-Table.
The left and right arrows will shift the trace along one of the curves. Zoom In to find the local minimum to two decimal places. Check with the FCN key. Which method do you prefer?

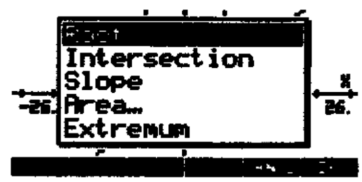


Additional Exploration:

Plot the rate of change function F3. The same ranges as above have been used. Next, plot both functions together, find their intersection points and discuss.







Appendix J

Gravity Aplets for the Hp 38g

Gravity Aplets for the HP38G

Since students were observed to view overall visual change rather than discrete change and there were many misconceptions occurring in visual modes such as distance-time graphs in particular, it was decided to complement the lessons with a few optional calculator programs or aplets. These would have the following features:

- A graph display combined with a numerical table of values to encourage students to look at discrete change
- Compare the values of the speed of cars when they catch up by using distance functions and creating speed functions from the derivatives. This would help overcome the 'velocity same: distance same' misconception
- Have examples of concave downward curves (inverted hills) and illustrate that speed increases as you move further up the curve, and not slowing down as for a hill. Again a table of velocities would be useful
- Use the term speed increasing or decreasing rather than directly using acceleration and develop the concept of acceleration gradually
- Compare speed on a velocity-distance graph as well as a distance-time graph to improve knowledge of new situations
- Include examples with concave downward distance-time curves to illustrate that downward movement does not imply slowing down or speeding up

Components of the Aplet

- Reset x and y axis ranges each time a new plot is drawn as otherwise any graph will use the current settings the student selected during the previous exercise

With this aplet, you can explore the effect of gravity on the vertical position and velocity of a falling object.

EQUATIONS:
F1(X)=Position (m)
F2(X)=Velocity (m/sec)

VARIABLES:
"X" represents time, NOT position. X=0 is the moment you release the object.

"P" is the initial position of the object (in metres) at the moment of release.

"V" is the initial velocity, which can be positive or negative to represent a throw directly upwards or directly downwards. For instance, V=-20 indicates a 20 m/sec downward throw.

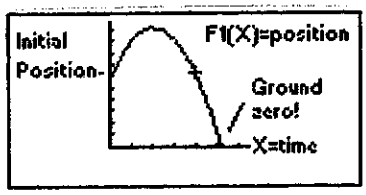
Press the [VIEWS] button to start exploring!

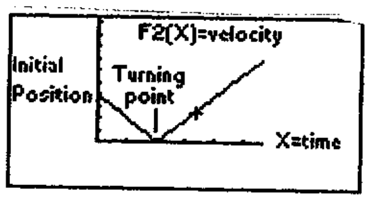
```
@-------------------------------
@ Check the velocity and position
@ functions.

CHECK 1:
CHECK 2:
```

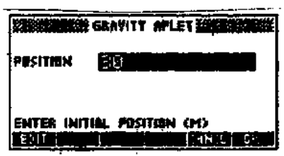
```
@-------------------------------
@ Get valid vals from user

ERASE:
INPUT P;"GRAVITY APLET";"Position";"Enter initial position (m)";P:
ABS(P)▶P:
INPUT V;"GRAVITY APLET";"Velocity";"Enter initial velocity (m/sec)";V:

@-------------------------------
@ Find the positive root of F1
@ which is the time at which the
@ object strikes the ground.

POLYROOT([-4.9,V,P])▶M1:
MAX(M1(1),M1(2))▶R:

@-------------------------------
@ Find the peak point but record
@ only if it happens when t>0.

POLYROOT([-9.8,-V])▶M1:
IF M1(1)>0 THEN
   F1(M1(1))▶B:
ELSE
   0▶B:
END:

@-------------------------------
@ Store the highest value of A,
@ B and R in D
```

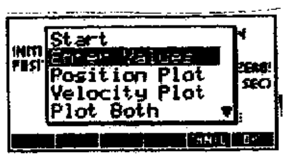
```
@--------------------------------
@ Check the position function
@ only.

CHECK 1:
UNCHECK 2:
```

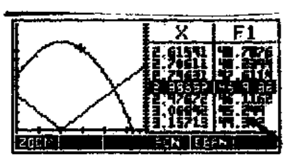
Gravity Aplet











```
@------------------------
@ Set up functions

'-4.9*X█+V*X+P'àF1(X):
'ABS(-9.8*X+V)'àF2(X):

@------------------------
@ and plot parameters.

20àV:  30àP:
0àXMIN:  -10àXMAX:
-10àYMIN:  60àYMAX:
5àYTICK: 1àXTICK:
```

```
@--------------------------------
@ Check the velocity function
@ only.

UNCHECK 1:
CHECK 2:
```

Appendix K

Traffic Aplets for the HP38G

- Carrera aplet

Beers, Byrne, Donnelly, Jones & Yuan (1996) have provided guidelines for creating aplets. These notes encourage researchers to first develop aplets directly on the graphics calculators rather than use any secondary software package such as the Aplet Development Kit supplied by Hewlett-Packard. This approach was found useful in the current study, particularly for developing the first few aplets.

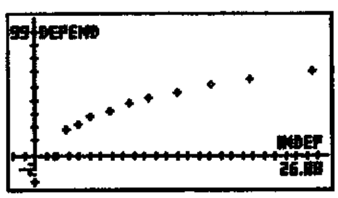
Carrera Aplet (see Appendix F, page 402)

The SKETCH view shows a graph of velocity against time. You will be asked to find the position of the Porsche at times t=0,1.8,...,25.6, interpret the results geometrically, and compute the average accelerations on intervals used.

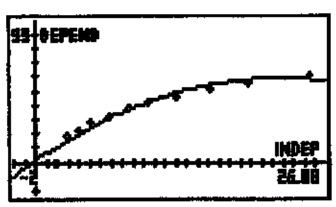
INPUT A;"Enter the starting time";"A= ";"Enter time in seconds";A:
INPUT B;"Enter the finishing time";"B= ";"Enter time in seconds";B:

$\int_A^B$ (-0.111X²+5.06X+0.954)dx →P:

MSGBOX "Position from time "A" to "B" is approx.: "P:
WAIT 1:
FOR I=1 TO 300 STEP 30;
BEEP 800+I;.5:
END:



SETVIEWS
"New Interval";"CA.NI";7;
"Find Position";"CA.PO";7;
"Start";;8;
" ";"CA.SV";0:



Equation of fitted quadratic used for calculations given by:
$F(x) = -.111X^2 + 5.06X + 0.954$

Appendix M

Readability and Human Interest Charts

Note: For copyright reasons Appendix M has not been reproduced.

Flesch, Rudolf (1949). *Readability and human interest charts.*

(Co-ordinator, ADT Project (Bibliographic Services), Curtin University of Technology, 30/09/03)