H_∞ Norm Computation for Descriptor Symmetric Systems

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***Abstract*— This paper deals with the problem of H_∞ norm computation for general symmetric systems and descriptor symmetric systems. The computation of H_∞ norm for state-space symmetric systems is extended to descriptor symmetric systems. An explicit expression is given based on the Bound Real Lemma (BRL), and the Generalized Bound Real Lemma (GBRL). The results have obvious computational advantages, especially for large scale descriptor symmetric systems. Additionally, two numerical examples are presented to demonstrate the feasibility and effectiveness of the results.**

***Index Terms* — Descriptor Symmetric Systems, the Bound Real Lemma(BRL), the Generalized Bound Real Lemma(GBRL), H_∞ Norm Computation.**

I. INTRODUCTION

Symmetric systems appear quite often in many engineering disciplines. Examples of such systems include electrical and power networks, structural systems, viscoelastic materials and chemical reactions [1]. With the special structure and better control properties, the symmetric systems have received considerable attention. For example, the H_∞ control analysis, the output feedback stabilization and the output feedback H_∞ control synthesis for continues-time and discrete-time state-space symmetric systems have been investigated in [1] and [2]. The stabilizability problem for symmetric systems via using decentralized controllers was addressed in [3]. In [4] and [5], some explicit expressions to computing the H_∞ norm for both of continuous-time and discrete-time state-space symmetric systems have been given. The output feedback stabilization and model reduction, H_∞ control and robust control for symmetric systems have been discussed in [6] to [9]. In this paper, a more general case of continuous-time descriptor symmetric systems will be studied, and some new explicit expressions to computing the H_∞ norm for descriptor state-space symmetric systems is given in the case of coefficient matrix for derivative variables being positive semi-definite and negative semi-definite.

This paper is organized as follows: Section II outlines some basic results concerning symmetric systems and descriptor symmetric systems. Section III presents the main results for some explicit expressions of H_∞ norm for the descriptor state-space symmetric systems and descriptor symmetric systems. In Section IV, we will provide two examples to show the effectiveness of the main results. The conclusion will be given in section V.

This work was supported by National Nature Science Foundation under Grant 60674019 and 60774039

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II. PRELIMINARIES AND PROBLEM STATEMENTS

In order to ensure the integrality of this paper, some basic definitions and lemmas are given as follows.

Consider the following state-space representation of a linear, time invariant (LTI) system:

$$\begin{cases} \dot{x} = Ax + Bw \\ y = Cx + Dw \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is state vector, $w(t) \in \mathbb{R}^m$ is the vector of exogenous inputs, $y(t) \in \mathbb{R}^m$ is the vector of controlled outputs, and $\{A, B, C, D\}$ denote the state-space parameter matrices.

Laplace transform of system (1) yields the transfer function matrix as

$$G(s) = C(sI - A)^{-1}B + D.$$

In this paper, we assume that the notation I is a real identity matrix with appropriate dimensions.

Definition 1: [1] The system (1) is said to be state-space symmetric or internal symmetric if the following conditions hold:

$$A = A^T,\ C^T = B,\ D = D^T. \tag{2}$$

The system (1) is said to be symmetric or external symmetric if $G(s) = G^T(s)$ holds.

Obviously, state-space symmetry implies symmetry, but the converse is not true.

Lemma 1: [10] Let system (1) be a minimal realization of system $G(s)$. Then the system $G(s)$ is symmetric if and only if there exists a real nonsingular symmetric matrix T_s such that

$$A^T T_s = T_s A,\ C^T = T_s B, D^T = D. \tag{3}$$

Moreover, T_s is unique.

Definition 2: The H_∞ norm of the system (1) is defined as

$$||G(s)||_\infty = \sup_{\omega \in \mathbb{R}} \sigma_{\max}\{G(j\omega)\}, \tag{4}$$

where $\sigma_{\max}$ denote the maximum singular value of a matrix.

Remark 1: When $G(s)$ is a scalar function, $||G(s)||_\infty$ is the largest gain of the Bode diagram.

Lemma 2 (BRL): [11] Given a scalar $\gamma > 0$, a stable system (1) has an H_∞ norm less than γ if and only if there exists a symmetric matrix $P > 0$ satisfying

$$\begin{bmatrix} A^TP+PA & PB & C^T \\ B^TP & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0. \quad (5)$$

Lemma 3: (Schur Complement) [11] Consider the block matrix

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix},$$

where S_{11} and S_{22} are symmetric. Then the following statements are equivalent:

(i) $S > 0$;

(ii) $S_{11} > 0$ and $S_{22} - S_{12}^T S_{11}^{-1} S_{12} > 0$;

(iii) $S_{22} > 0$ and $S_{11} - S_{12} S_{22}^{-1} S_{12}^T > 0$.

These conditions can be easily modified to test negative definiteness of a matrix.

Lemma 4: (Finsler's Lemma) [1] Consider matrices M and Q such that M has full column rank and $Q = Q^T$. Then the following statements are equivalent:

(i) There exists a scalar μ such that

$$\mu M M^T - Q > 0. \quad (6)$$

(ii) The following condition holds:

$$M^\perp Q M^{\perp T} < 0. \quad (7)$$

If the above statements hold, then all scalars μ satisfying (6) are given by

$$\mu > \mu_{\min} := \lambda_{\max}[M^+(Q - QM^{\perp T}(M^\perp Q M^{\perp T})^{-1} M^\perp Q)M^{+T}]. \quad (8)$$

where M^+ is the Moore-Penrose generalized inverse of real matrix M, $M^\perp$ is with maximum row rank that satisfies $M^\perp M = 0$ and $M^\perp M^{\perp T} > 0$ is the orthogonal complement of real matrix M.

Remark 2: $M^\perp$ can be computed from the singular value decomposition of M as follows:

$$M^\perp = TU_2^T,$$

where T is an arbitrary nonsingular matrix and U_2 is defined from the singular value decomposition of M

$$M = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \sum_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}.$$

The standard notation $>$ ($<$) is used to denote the positive (negative) definite ordering of symmetric matrices.

Lemma 5: Consider a stable symmetric system (1) and (2). The following statements are equivalent:

(i) System (1) has an H_∞ norm less than γ i.e. $||G(s)||_\infty < \gamma$.

(ii) $P = I$ is a solution for (5), i.e.,

$$\begin{bmatrix} 2A & B & C^T \\ B^T & -\gamma I & D \\ C & D & -\gamma I \end{bmatrix} < 0. \quad (9)$$

(iii)

$$\begin{aligned} &\gamma^2 I - D^2 > 0, \\ &A + B(\gamma I - D)^{-1} B^T < 0. \end{aligned} \quad (10)$$

(iv)

$$\begin{aligned} &\gamma I + D > 0, \\ &D - B^T A^{-1} B < \gamma I. \end{aligned} \quad (11)$$

(v)

$$\begin{bmatrix} 0 & D^T \\ D & 0 \end{bmatrix} - \begin{bmatrix} B^T \\ C \end{bmatrix} (2A)^{-1} \begin{bmatrix} B & C^T \end{bmatrix} < \gamma I.$$

Proof: The proof of the equivalence between (i) and (ii) can be found in [1]. The equivalence among (ii), (iii), (iv) and (v) can be obtained from Lemma 2, Theorem 6 of [1], the Bound Real Lemma (BRL), and the Finsler' Lemma. Here the detail is omitted.

Besides the proof method of [1], there is also another method to prove the equivalence between (ii) and (iii), (ii) and (iv), (ii) and (v).

Using the BRL and the Schur Complement formula, and pre-and post-multiplying (9) by

$$\begin{bmatrix} I & 0 & 0 \\ 0 & \frac{1}{2}I & \frac{1}{2}I \\ 0 & -I & I \end{bmatrix}$$

and its transposition, respectively, we can obtain the following equivalent condition of (9):

$$\begin{bmatrix} 2A & B & 0 \\ B^T & \frac{1}{2}(D-\gamma I) & 0 \\ 0 & 0 & -2(\gamma I + D) \end{bmatrix} < 0.$$

Then, from the Schur Complement formula, we can obtain the further two equivalent conditions:

$$\begin{aligned} &\gamma I + D > 0, \\ &\gamma I - D > 0, \\ &A + B(\gamma I - D)^{-1} B^T < 0. \end{aligned}$$

and

$$\begin{aligned} &2A < 0, \\ &\gamma I + D > 0, \\ &\gamma I - D + B^T A^{-1} B > 0. \end{aligned}$$

Obviously, the above two groups of inequalities are equivalent to (10) and (11), respectively. So, the equivalence between (ii) and (iii), (ii) and (iv) are derived.

Using the Schur Complement formula to (9), we can directly obtained that (ii) is equivalent to (v).

Based on above derivations, Lemma 5 is proved.

□

Consider the following descriptor system:

$$\begin{cases} E\dot{x} = Ax + Bw \\ y = Cx + Dw \end{cases} \quad (12)$$

where $x(t) \in \mathbb{R}^n, w(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^m$ are defined in previous section, and let $\{E, A, B, C, D\}$ denote the state-space parameter matrices with $E \in \mathbb{R}^{n \times n}$ being a real singular or nonsingular matrix.

The statement of descriptor symmetric systems includes general symmetric systems (E is a nonsingular) and singular symmetric systems (E is a singular). The system $E\dot{x} = Ax$, or the pair (E, A) is regular if $det(sE - A)$ is not identically zero. The pair (E, A) is said to be admissible if it is regular, impulse-free and stable.

Definition 3: The descriptor system (12) is said to be state-space symmetric if the following conditions hold:

$$E^T = E,\ A^T = A,\ C^T = B,\ D^T = D. \tag{13}$$

The descriptor system (12) is said to be symmetric if the condition $G_1^T(s) = G_1(s)$ holds, where

$$G_1(s) = C(sE - A)^{-1}B + D.$$

is the transfer function matrix of the system (12).

Lemma 6: [12] The pair (E, A) is admissible if and only if there exists a nonsingular real matrix P such that

$$\begin{array}{l} A^TP + P^TA < 0, \\ E^TP = P^TE \geq 0. \end{array} \tag{14}$$

Lemma 7: (GBRL) [13] Given a scalar $\gamma > 0$, and system (12). The following conditions are equivalent:

(i) (E, A) is admissible and $||G_1(s)||_\infty < \gamma$;

(ii) There exists a real invertible matrix P such that the following LMIs:

$$E^TP = P^TE \geq 0; \tag{15}$$

$$\begin{bmatrix} A^TP + P^TA & P^TB & C^T \\ B^TP & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0. \tag{16}$$

Lemma 8: [1] For any symmetric matrices R and S of same size, then following condition holds:

$$\lambda_{\max}\begin{bmatrix} R & S \\ S & R \end{bmatrix} = \max(\lambda_{\max}(R+S),\ \lambda_{\max}(R-S)).$$

III. MAIN RESULTS

In this section, we deal with the H_∞ norm computation for descriptor state-space symmetric systems and descriptor symmetric systems, which is one of the most important H_∞ control synthesis problems.

Theorem 1: Consider system (12) and (13) with $E \geq 0$ or $E \leq 0$, the following statements are equivalent:

(i) (E, A) is admissible and the system (12) has an H_∞ norm less than γ, i.e., $||G_1(s)||_\infty < \gamma$.

(ii) If $E \geq 0$, $P = I$ is a solution for (16), i.e.,

$$\begin{bmatrix} 2A & B & C^T \\ B^T & -\gamma I & D \\ C & D & -\gamma I \end{bmatrix} < 0, \tag{17}$$

or

If $E \leq 0$, $P = -I$ is a solution for (16), i.e.,

$$\begin{bmatrix} -2A & -B & C^T \\ -B^T & -\gamma I & D \\ C & D & -\gamma I \end{bmatrix} < 0. \tag{18}$$

(iii) If $E \geq 0$, then

$$\begin{array}{l} \gamma^2 I - D^2 > 0 \\ A + B(\gamma I - D)^{-1}B^T < 0, \end{array} \tag{19}$$

or

If $E \leq 0$, then

$$\begin{array}{l} \gamma^2 I - D^2 > 0 \\ -A + B(\gamma I + D)^{-1}B^T < 0. \end{array} \tag{20}$$

(iv) If $E \geq 0$, then

$$\begin{array}{l} \gamma I + D > 0 \\ D - B^TA^{-1}B < \gamma I, \end{array} \tag{21}$$

or

If $E \leq 0$, then

$$\begin{array}{l} \gamma I - D > 0 \\ -D + B^TA^{-1}B < \gamma I. \end{array} \tag{22}$$

(v) If $E \geq 0$, then

$$\begin{bmatrix} 0 & D \\ D & 0 \end{bmatrix} - \begin{bmatrix} B^T \\ C \end{bmatrix}(2A)^{-1}\begin{bmatrix} B & C^T \end{bmatrix} < \gamma I, \tag{23}$$

or

If $E \leq 0$, then

$$\begin{bmatrix} 0 & D \\ D & 0 \end{bmatrix} + \begin{bmatrix} -B^T \\ C \end{bmatrix}(2A)^{-1}\begin{bmatrix} -B & C^T \end{bmatrix}) < \gamma I. \tag{24}$$

Proof: (i) $\Leftrightarrow$ (ii):

Case I: $E \geq 0$.

Sufficiency: Suppose that condition (17) is satisfied for the admissible descriptor state-space symmetric system (12) and (13). Using the Generalized Bound Real Lemma(GBRL) of Lemma 7, obviously, the descriptor symmetric system (12) has an H_∞ norm less than γ.

Necessity: Based on the GBRL of lemma 7, system (12) with (13) has an H_∞ norm less than γ if and only if (15) and (16) hold.

We only need to prove that $P = I$ is a solution of (15) and (16). There is a clue of the proof method from [1]. Since

$$E^T = E \geq 0, EP = P^TE \geq 0,$$

and P is invertible, we can conclude that $P > 0$. Then there exists a orthogonal matrix U satisfies

$$P = U\Sigma_0U^T,\ U^T = U^{-1},$$

$$\Sigma_0 = \begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots & \\ 0 & & & \sigma_n \end{bmatrix} > 0.$$

Let

$$\overline{A} = U^TAU,\ \overline{B} = U^TB,\ \overline{C} = CU,\ \overline{D} = D,$$

then the system $(\overline{A},\ \overline{B},\ \overline{C},\ \overline{D})$ is also state-space symmetric. Pre-and post-multiplying (16) by

$$\begin{bmatrix} \Sigma_0^{-1}U^T & 0 & 0 \\ 0 & \Sigma_0^{-1} & 0 \\ 0 & 0 & I \end{bmatrix}$$

and its transposition, respectively, we obtain the following equivalent condition:

$$\begin{bmatrix} \overline{A}\Sigma_0^{-1}+\Sigma_0^{-1}\overline{A} & \Sigma_0^{-1}\overline{B} & \overline{B} \\ \overline{B}^T\Sigma_0^{-1} & -\gamma I & \overline{D} \\ \overline{B}^T & \overline{D} & -\gamma I \end{bmatrix} < 0. \quad (25)$$

Hence, it is clear that Σ_0^{-1} is also a solution of (25) as long as Σ_0 is a solution. Since $\sigma_1 > 0$, there exists $0 < \lambda_1 < 1$ such that $\lambda_1\sigma_1 + (1-\lambda_1)\sigma_1^{-1} = 1$. Then, we can obtain

$$\Sigma_1 = \begin{bmatrix} \lambda_1\sigma_1 + (1-\lambda_1)\sigma_1^{-1} & & 0 \\ & \ddots & \\ 0 & & \lambda_1\sigma_n + (1-\lambda_1)\sigma_n^{-1} \end{bmatrix} = \begin{bmatrix} 1 & & & 0 \\ & \tilde{\sigma}_2 & & \\ & & \ddots & \\ 0 & & & \tilde{\sigma}_n \end{bmatrix} > 0.$$

By repeating this process, we can construct $\sum_n = I$ satisfying the LMIs of GBRL, that is (17).

Case II: $E \leq 0$.

Sufficiency: Suppose that condition (18) is satisfied for the admissible state-space symmetric system (12) with (13). With the same way, using the GBRL of lemma 7, the descriptor symmetric system has an H_∞ norm less than γ.

Necessity: We only need to show that $P = -I$ is also a solution of (16). The process is similar as the case of $E \geq 0$. Since

$$E^T = E \leq 0, EP = P^TE \geq 0,$$

and P is invertible, we can let $P < 0$. Then there exists an orthogonal matrix U satisfying

$$X = U\Sigma_0U^T,\ U^T = U^{-1},$$

$$\Sigma_0 = \begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots & \\ 0 & & & \sigma_n \end{bmatrix} < 0.$$

Let

$$\overline{A} = U^TAU,\ \overline{B} = U^TB,\ \overline{C} = CU,\ \overline{D} = D,$$

then the system $(\overline{A}, \overline{B}, \overline{C}, \overline{D})$ is also state-space symmetric. Pre-and post-multiplying the above by

$$\begin{bmatrix} \Sigma_0^{-1}U^T & 0 & 0 \\ 0 & \Sigma_0^{-1} & 0 \\ 0 & 0 & I \end{bmatrix}$$

and its transposition, respectively, we obtain the following equivalent condition:

$$\begin{bmatrix} \overline{A}\Sigma_0^{-1}+\Sigma_0^{-1}\overline{A} & \Sigma_0^{-1}\overline{B} & \overline{B} \\ \overline{B}^T\Sigma_0^{-1} & -\gamma I & \overline{D} \\ \overline{B}^T & \overline{D} & -\gamma I \end{bmatrix} < 0. \quad (26)$$

Hence, it is clear that Σ_0^{-1} is also a solution of (26) as long as Σ_0 is a solution. Since $\sigma_1 < 0$, there exists $0 < \lambda_1 < 1$ such that $\lambda_1\sigma_1 + (1-\lambda_1)\sigma_1^{-1} = -1$. Then, we can obtain

$$\Sigma_1 = \begin{bmatrix} \lambda_1\sigma_1 + (1-\lambda_1)\sigma_1^{-1} & & 0 \\ & \ddots & \\ 0 & & \lambda_1\sigma_n + (1-\lambda_1)\sigma_n^{-1} \end{bmatrix} = \begin{bmatrix} -1 & & & 0 \\ & \tilde{\sigma}_2 & & \\ & & \ddots & \\ 0 & & & \tilde{\sigma}_n \end{bmatrix} < 0.$$

By repeating this process, we can construct $\sum_n = -I$ satisfying (26), that is, $P = -I$ is a solution of (16), i.e., (18) holds.

Hence, the equivalence between (i) and (ii) is proved.

The proof method of the equivalence among (ii), (iii), (iv), and (v) is similar to that of Lemma 5, the proof is omitted here.

Then, theorem 1 has been proved.

□

Remark 3: Another proof method that using the Finsler's Lemma can be given to prove Theorem 1, the procedure is similar to [1].

Considering the purpose of this paper, we will give some explicit expressions for H_∞ norm computation in the following corollary.

Corollary 1: Consider the descriptor symmetric system of (12) and (13), with the pair (E, A) being admissible. Then the H_∞ norm of the system (12) can be computed by the following expressions:

(i) If $E \geq 0$, then

$$\begin{aligned} &\|G_1(s)\|_\infty \\ &= \lambda_{\max}\left(\begin{bmatrix} 0 & D \\ D & 0 \end{bmatrix} - \begin{bmatrix} B^T \\ C \end{bmatrix}(2A)^{-1}\begin{bmatrix} B & C^T \end{bmatrix}\right) \\ &= \max\{\lambda_{\max}(-D), \lambda_{\max}(D - B^TA^{-1}B)\}, \end{aligned}$$

or

If $E \leq 0$, then

$$\begin{aligned} &\|G_1(s)\|_\infty \\ &= \lambda_{\max}\left(\begin{bmatrix} 0 & D \\ D & 0 \end{bmatrix} + \begin{bmatrix} -B^T \\ C \end{bmatrix}(2A)^{-1}\begin{bmatrix} -B & C^T \end{bmatrix}\right). \\ &= \max\{\lambda_{\max}(D), \lambda_{\max}(-D + B^TA^{-1}B)\}. \end{aligned}$$

□

In the following we will consider descriptor symmetric systems. It can be easily verified that there is no similar explicit results as Lemma 1, but we can give the following lemma.

Lemma 9: For any given descriptor symmetric system $\{E, A, B, C, D\}$, there exists a real invertible matrix T_s such that $\{T_sE, T_sA, T_sB, C, D\}$ is state-space symmetric.

Proof: If matrix E is singular, the pair (E, A) is regular, then the Weierstrass form can be obtained

$$MEN = \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix},\ MAN = \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix}$$

$$MB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},\ C = \begin{bmatrix} C_1 & C_2 \end{bmatrix},\ D = D^T,$$

where M and N are nonsingular matrices, J is a nilpotent matrix.

In this case, the transfer matrices $C_1(sI-A_1)^{-1}B_1$ and $C_2(sJ-I)^{-1}B_2$ are both symmetric.

From Lemma 1, there exist invertible symmetric matrices T_s^1, T_s^2 satisfying the following conditions:

$$T_s^1A_1 = A_1T_1^s,\ T_s^1B_1 = C_1^T,$$
$$T_s^2J = J^TT_s^2,\ T_s^2B_2 = C_2^T.$$

Moreover, T_s^1 and T_s^2 are unique.

Define:

$$T_s^0 = \begin{bmatrix} T_s^1 & 0 \\ 0 & T_s^2 \end{bmatrix},$$

then the matrix T_s^0 satisfies:

$$\begin{array}{l} T_s^0MEN = N^TE^TM^TT_s^0 \\ T_s^0MAN = N^TA^TM^TT_s^0 \\ T_s^0MB = (CN)^T. \end{array}$$

Let $T_s = N^{-T}T_s^0M$, then we can obtain that:

$$\begin{array}{l} T_sE = E^TT_s^T = (T_sE)^T \\ T_sA = A^TT_s^T = (T_sA)^T \\ T_sB = C^T, D = D^T. \end{array} \qquad (27)$$

i.e., there exists a real invertible matrix T_s such that the system

$$\begin{cases} T_sE\dot{x} = T_sAx + T_sBw \\ y = Cx + Dw. \end{cases}$$

is state-space symmetric.

On the other hand, if matrix E is nonsingular, then $MEN = I, MAN = A_1$. In this case, let $T_s = N^{-T}M$, then (27) still holds.

Therefore, Lemma 9 is proved.

□

Lemma 9 can be more useful for dealing with the descriptor symmetric systems.

Based on Lemma 9, we have a more generalized result as follows.

Theorem 2: Consider the admissible system (12) with $G_1^T(s) = G_1(s)$. If there exists a real invertible matrix T_s such that $\{T_sE, T_sA, T_sB, C, D\}$ is state-space symmetric and $T_sE \geq 0$ or $T_sE \leq 0$, then the H_∞ norm can be given by

(i) If $T_sE \geq 0$, then

$$\begin{array}{l} ||G_1(s)||_\infty \\ = \lambda_{\max}(\begin{bmatrix} 0 & D \\ D & 0 \end{bmatrix} - \begin{bmatrix} C \\ C \end{bmatrix}(2A)^{-1}\begin{bmatrix} B & B \end{bmatrix}) \\ = \max\{\lambda_{\max}(-D), \lambda_{\max}(D - CA^{-1}B)\}, \end{array} \qquad (28)$$

or

(ii) If $T_sE \leq 0$, then

$$\begin{array}{l} ||G_1(s)||_\infty \\ = \lambda_{\max}(\begin{bmatrix} 0 & D \\ D & 0 \end{bmatrix} - \begin{bmatrix} -C \\ C \end{bmatrix}(2A)^{-1}\begin{bmatrix} -B & B \end{bmatrix}) \\ = \max\{\lambda_{\max}(D), \lambda_{\max}(-D + CA^{-1}B)\}. \end{array} \qquad (29)$$

Proof: (i): If $T_sE \geq 0$, from the results of Theorem 1 or Corollary 1, we can obtain

$$\begin{array}{l} ||G_1(s)||_\infty \\ = \lambda_{\max}(\begin{bmatrix} 0 & D \\ D & 0 \end{bmatrix} - \begin{bmatrix} (T_sB)^T \\ C \end{bmatrix}(2T_sA)^{-1}\begin{bmatrix} T_sB & C^T \end{bmatrix}) \\ = \lambda_{\max}(\begin{bmatrix} 0 & D \\ D & 0 \end{bmatrix} - \begin{bmatrix} C \\ C \end{bmatrix}(2A)^{-1}\begin{bmatrix} B & B \end{bmatrix}). \end{array}$$

Using Lemma 8, the (28) holds.

While $T_sE \leq 0$, the proof procedure is similar. So the details are omitted here.

□

Remark 4: For the descriptor state-space symmetric systems in the cases of neither $E \geq 0$ nor $E \leq 0$, there is no similar results as above. In these cases, the problem on computation of H_∞ norm is still open.

Remark 5: If the state-space symmetric descriptor system (12) can be given as follows:

$$E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},\ A = \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix},\ B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$
$$C = \begin{bmatrix} C_1 & C_2 \end{bmatrix},\ D = D.$$

We can give the following derivation:

$$\begin{array}{l} ||G(s)||_\infty \\ = \lambda_{\max}(\begin{bmatrix} 0 & D \\ D & 0 \end{bmatrix} - \begin{bmatrix} B^T \\ C \end{bmatrix}(2A)^{-1}\begin{bmatrix} B & C^T \end{bmatrix}) \\ = \lambda_{\max} \\ (\begin{bmatrix} 0 & D \\ D & 0 \end{bmatrix} - \begin{bmatrix} B_1^T & B_2^T \\ C_1 & C_2 \end{bmatrix}(2A)^{-1}\begin{bmatrix} B_1 & C_1^T \\ B_2 & C_2^T \end{bmatrix}) \\ = \lambda_{\max}(\begin{bmatrix} -B_1^T(2A_1)^{-1}B_1 - \frac{1}{2}B_2^TB_2 & D - B_1^T(2A_1)^{-1}B_1 - \frac{1}{2}B_2^TB_2 \\ D - B_1^T(2A_1)^{-1}B_1 - \frac{1}{2}B_2^TB_2 & -B_1^T(2A_1)^{-1}B_1 - \frac{1}{2}B_2^TB_2 \end{bmatrix}) \\ = \max(\lambda_{\max}(-D),\ \lambda_{\max}(D - B_1^TA_1^{-1}B_1 - B_2^TB_2)). \end{array}$$

On the other hand, from its transfer matrix, we can represent the system as follows:

$$\begin{cases} \dot{x}_1 = A_1x_1 + B_1w \\ y = C_1x_1 + (D - C_2B_2)w, \end{cases}$$

Hence, H_∞ norm of the system can be computed by

$$\begin{array}{l} ||G(s)||_\infty = \lambda_{\max} \\ (\begin{bmatrix} 0 & D - C_2B_2 \\ D - C_2B_2 & 0 \end{bmatrix} - \\ \begin{bmatrix} B_1^T \\ C_1 \end{bmatrix}(2A_1)^{-1}\begin{bmatrix} B_1 & C_1^T \end{bmatrix}) \\ = \lambda_{\max} \\ (\begin{bmatrix} -B_1^T(2A_1)^{-1}B_1 - \frac{1}{2}B_2^TB_2 & D - B_1^T(2A_1)^{-1}B_1 - \frac{1}{2}B_2^TB_2 \\ D - B_1^T(2A_1)^{-1}B_1 - \frac{1}{2}B_2^TB_2 & -B_1^T(2A_1)^{-1}B_1 - \frac{1}{2}B_2^TB_2 \end{bmatrix}) \\ = \max(\lambda_{\max}(-D),\ \lambda_{\max}(D - B_1^TA_1^{-1}B_1 - B_2^TB_2)). \end{array}$$

This shows that the computation formula of H_∞ norm of descriptor systems is consistent with that of the normal systems.

□

IV. NUMERICAL EXAMPLES

In this section we present two numerical examples to illustrate the effectiveness of proposed results of this paper.

Example 1: Consider a general symmetric system (12) and (13) with

$$E = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3/2 & 1/2 \\ 0 & 1/2 & -3/2 \end{bmatrix}$$
$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \ D = 0.$$

It is easy to testify the symmetric system (E, A, B, C, D) is a stable system. By Matlab tool, the H_∞ norm is calculated as $||G(s)||_\infty = 1$.

Using the explicit expression formula of Theorem 1 or Corollary 1, we can also compute the H_∞ norm as following:

$$\begin{aligned} ||G(s)||_\infty &= \max(\lambda_{\max}(-D), \lambda_{\max}(D - B^T A^{-1} B)) \\ &= \max(0.0000,\ 1.0000) \\ &= 1. \end{aligned}$$

□

Example 2: Consider a descriptor symmetric system (12) and (13) with

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$
$$B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \ D = 0.$$

From Lemma 6, it is easy to testify that this descriptor symmetric system (E, A, B, C, D) is admissible. By the Matlab tools, H_∞ norm for above descriptor system can be calculated as $||G(s)||_\infty = 0.6667$.

Based on Theorem 2, we can compute the H_∞ norm as following :

$$\begin{aligned} ||G(s)||_\infty &= \max(\lambda_{\max}(-D), \lambda_{\max}(D - CA^{-1} B)) \\ &= \max(0.6667,\ 0.0000) \\ &= 0.6667. \end{aligned}$$

□

These examples show that the calculated results by using the explicit expression of H_∞ norm computation is the same as the iterative solution, and this provides the applicability of Theorem 1 and Theorem 2.

V. CONCLUSIONS

In this paper we have derived some new explicit expressions of H_∞ norm computation for descriptor symmetric systems. The results are expressed in terms of the state space form without need for iterative solution (see [16] [17]) by solving eigenvalue of Hamilton matrix. The results are more general than the existing ones. Additionally, the proposed results can be extended to discrete-time descriptor symmetric systems which will be investigated in near future.

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