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TRACKING CONTROL OF LINEAR SWITCHED SYSTEMS

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Abstract

This paper deals with the optimal tracking problem for switched systems, where the control input, the switching times and the switching index are all design variables. We propose a three-stage method for solving this problem. First, we fix the switching times and switching index sequence, which leads to a linear tracking problem, except different subsystems are defined in their respective time intervals. The optimal control and the corresponding cost function obtained depend on the switching signal. This gives rise to an optimal parameter selection problem for which the switching instants and the switching index are to be chosen optimally. In the second stage, the switching index is fixed. A reverse time transformation followed by a time scaling transform are introduced to convert this subproblem into an equivalent standard optimal parameter selection problem. The gradient formula of the cost function is derived. Then the discrete filled function is used in the third stage to search for the optimal switching index. On this basis, a computational method, which combines a gradient-based method, a local search algorithm and a filled function method, is developed for solving this problem. A numerical example is solved, showing the effectiveness of the proposed approach.

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1. Introduction

Switched systems, which include variable structure systems and multi-modal systems, are an important class of hybrid systems. They have many practical applications arising in areas such as the control of mechanical systems, the automotive industry, aircraft

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and air traffic control, and switching power converters. For further details, see [14]. With regard to these, the question on the determination of optimal control laws for switched systems has been widely investigated in recent years and many results are now available in the control and computer science literature. In [12, 13, 15], some necessary and sufficient conditions for optimality are obtained. In [2], a method for solving a hybrid optimal control problem is formulated for a two-switched system. The two-switched system is embedded in a larger family of systems for which an optimization problem is formulated. In [6], convex dynamic programming is used to approximate optimal switching control laws and to compute lower and upper bounds of the optimal cost. In [18], a class of optimal control problems for the switched systems with a pre-specified sequence of active subsystems is considered. This optimal control problem is first transformed into an equivalent optimal parameter selection problem parameterized by the switching instants. The gradient formula of the cost function is then derived and a gradient-based method is thus developed. However, how to determine the switching index sequence is still a challenge. For this, a discrete filled function method can be used. The concept of a filled function was first introduced in [5] for global optimization with continuous variables. A discrete filled function method is developed in [11] to solve discrete global optimization problems. In [4, 17], efficient discrete filled function methods are developed to solve the optimal control problem of discrete-time switched systems and impulsive optimal control problem, respectively.

Tracking problems are a special class of optimal control problems. The theory of optimal tracking controllers for linear systems is now fundamental to the subject and is covered in most standard control text books. See, for example, [1] and [10]. For nonlinear dynamical systems, results obtained using the differential geometric approach are summarized in [7] - an outstanding book by Isidori, where clear connections linking the concept of the inverse system and the zero dynamics are established.

For fixed multi-variable linear or nonlinear continuous dynamic plants, internal model theory has been developed to realize good tracking. For switched and hybrid systems, there are very few results on tracking problems. In [3], an approach is proposed for exact output-tracking of switched systems, which is also applicable to non-minimum-phase systems. They present necessary and sufficient conditions for the solvability of the inversion problem for linear systems with switches, where the inverse is used to track the desired output. In [19], a multi-contact hand manipulation problem in a hybrid system field is considered and an MLD model, which encapsulates switching between types of motion and phases of continuous motion, is proposed. Then the tracking problem of a hybrid system to follow a family of reference signals produced by an external signal generator is considered, and the existing internal model theory for continuous systems is extended to deal with the tracking problem for the hybrid system.

The rest of the paper is organized as follows. In Section 2, we provide the exact problem formulation. In Section 3, we decompose our problem into three subproblems and then propose a method for solving the first subproblem. In Section 4, we suppose the switching index sequence is given. Then, a reverse time transformation followed by a time scaling transformation is introduced. We convert this problem into an equivalent optimal parameter selection problem in standard form. Then a gradient-based algorithm is developed. In Section 5, a discrete filled function is used to search for the optimal switching index sequence. A computational procedure is developed in the same section. A numerical example is solved in Section 6 so as to illustrate the effectiveness of the method. Section 7 concludes the paper.

2. Problem formulation

We consider the switched linear system

$$\dot{x}(t) = A_{k_i}(t)x(t) + B_{k_i}(t)u(t), \quad t \in I_i,\ i = 1, 2, \ldots, N, \tag{2.1}$$

$$y(t) = C(t)x(t), \tag{2.2}$$

with the initial condition and intermediate conditions

$$x(t_0) = x^0, \tag{2.3}$$

$$x(t_i^+) = x(t_i), \quad i = 1, \ldots, N-1, \tag{2.4}$$

where $I_i = (t_{i-1}, t_i]$, $i = 1, 2, \ldots, N$, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ denote, respectively, the state and the control vectors, and $y \in \mathbb{R}^r$ is the output vector, while $A_i(t)$, $B_i(t)$, $i = 1, 2, \ldots, M$, and $C(t)$ are time-varying matrices with appropriate dimensions. Here, $\hat{t}^+$ and $\hat{t}^-$ are, respectively, defined by

$$\hat{t}^+ = \lim_{t \downarrow \hat{t}} t \quad \text{and} \quad \hat{t}^- = \lim_{t \uparrow \hat{t}} t.$$

The set of the switching signal sequences in $t \in [t_0, t_f]$ is defined as:

$$\Sigma = \left\{\sigma \,|\, \sigma = \big((t_1, k_1), \ldots, (t_i, k_i), \ldots, (t_N, k_N) \big)\right\},$$

where $t_0 \le t_1 \le \cdots \le t_N = t_f$, while for each $i = 1, \ldots, N$, $k_i \in \{1, 2, \ldots, M\}$ and (t_i, k_i) indicates that the subsystem k_i is active during the time interval $(t_{i-1}, t_i]$.

Define

$$\Theta = \left\{ t \,\middle|\, t = (t_1, t_2, \ldots, t_{N-1})^T \in \mathbb{R}^{N-1} \right\}, \tag{2.5}$$

where $0 = t_0 \le t_1 \le \cdots \le t_{N-1} \le t_N = t_f$. Let $\mathscr{U}$ be the set of all piecewise continuous functions defined on $[t_0, t_f]$ with values in $\mathbb{R}^m$.

Define

$$\Gamma = \{\gamma \mid \gamma = (k_1, k_2, \ldots, k_i, \ldots, k_N),\ k_i \in \{1, 2, \ldots, M\}\},$$

where γ denotes the switching index sequence.

We assume throughout that the following condition is satisfied.

ASSUMPTION. The elements of the matrix functions $A_i(t)$ and $B_i(t)$ are continuous for each $i \in \{1, \ldots, M\}$, while $C(t)$, $Q(t)$ and $R(t)$ are continuous in $t \in [t_0, t_f]$.

The optimal tracking problem for the switched system may now be formulated as follows.

PROBLEM 1. Given the switched dynamical system (2.1)–(2.4), find a piecewise continuous control input $u(t) \in \mathscr{U}$ and a switching signal $\sigma \in \Sigma$ such that the cost function

$$\begin{aligned} J(\sigma, u) = {} & \frac{1}{2}\big[\bar{y}(t_f) - y(t_f)\big]^T S \big[\bar{y}(t_f) - y(t_f)\big] \\ & + \frac{1}{2}\int_{t_0}^{t_f} \Big[\big(\bar{y}(t) - y(t)\big)^T Q(t)\big(\bar{y}(t) - y(t)\big) + u^T(t)R(t)u(t)\Big]\,dt \end{aligned} \tag{2.6}$$

is minimized, where $\bar{y}(t)$ is a given r-dimensional continuous trajectory vector to be tracked, $Q(t)$ and $R(t)$ are continuous matrix-valued functions with appropriate dimensions in $t \in [t_0, t_f]$, and for each $t \in [t_0, t_f]$, they are symmetric positive semi-definite, while S is a symmetric positive semi-definite constant matrix with proper dimension.

3. A three-stage optimization approach

In Problem 1, there are three types of decision variables: The first one is a piecewise continuous control input, the second one is the switching time sequence and the third one is the switching index sequence. Problem 1 is, in fact, a mixed-integer programming problem. We can't solve this problem by using conventional linear quadratic optimal control theory due to the presence of the varying switching times and the switching index sequence. If we suppose that the switching times and the switching index sequence are fixed, then the optimal tracking problem becomes a linear tracking problem. However, it is not a standard linear tracking problem, as the active subsystem is different during different subintervals. We decompose this problem into three subproblems: Problem P1, Problem P2 and Problem P3 to be defined in the following.

PROBLEM P1. Given a fixed switching signal, find an optimal control input $u \in \mathscr{U}$ such that the cost function (2.6) is minimized subject to system (2.1)-(2.4).

Problem P1 is a tracking problem but with different subsystems being used in different time intervals. The following theorem gives the parameterized optimal control and the corresponding cost function.

THEOREM 3.1. *The optimal control input and the optimal cost function for Problem* P1 *are, respectively,*

$$u^*(t|\sigma) = -R^{-1}(t)B_{k_i}^T(t)P(t)x(t) - R^{-1}(t)B_{k_i}^T(t)b(t), \quad t \in I_i,\ i = 1, \dots, N, \tag{3.1}$$

and

$$J(\sigma, u^*(\sigma)) = \frac{1}{2}x^T(t_0)P(t_0)x(t_0) + x^T(t_0)b(t_0) + \frac{1}{2}g(t_0), \tag{3.2}$$

where $b(t)$, $g(t)$ and $P(t)$ are, respectively, determined by

$$\dot{b}(t) = -\left(A_{k_i}(t) - B_{k_i}(t)R^{-1}(t)B_{k_i}^T(t)P(t)\right)^T b(t) + C^T(t)Q(t)\bar{y}(t), \quad t \in I_i,\ i = 1, \dots, N, \tag{3.3}$$

$$b(t_f) = -C^T(t_f)S\bar{y}(t_f), \tag{3.4}$$

$$b(t_i^+) = b(t_i), \quad i = 1, \dots, N-1, \tag{3.5}$$

$$\dot{g}(t) = b^T(t)B_{k_i}(t)R^{-1}(t)B_{k_i}^T(t)b(t) - \bar{y}^T(t)Q(t)\bar{y}(t), \quad t \in I_i,\ i = 1, \dots, N, \tag{3.6}$$

$$g(t_f) = \bar{y}^T(t_f)S\bar{y}(t_f), \tag{3.7}$$

$$g(t_i^+) = g(t_i), \quad i = 1, \dots, N-1 \tag{3.8}$$

and

$$-\dot{P}(t) = P(t)A_{k_i}(t) + A_{k_i}^T(t)P(t) - P(t)B_{k_i}(t)R^{-1}(t)B_{k_i}^T(t)P(t) + C^T(t)Q(t)C(t), \quad t \in I_i,\ i = 1, \dots, N, \tag{3.9}$$

$$P(t_f) = C^T(t_f)SC(t_f), \tag{3.10}$$

$$P(t_i^+) = P(t_i), \quad i = 1, \dots, N-1. \tag{3.11}$$

PROOF. Define the Hamiltonian function as

$$H(t, x, u, \lambda) = \frac{1}{2}\left[\left(\bar{y}(t) - y(t)\right)^T Q(t)\left(\bar{y}(t) - y(t)\right) + u^T(t)R(t)u(t)\right] + \sum_{i=1}^{N} \lambda^T(t)\left(A_{k_i}(t)x(t) + B_{k_i}(t)u(t)\right)\chi_{I_i}(t), \tag{3.12}$$

$$\text{where} \quad \chi_{I_i}(t) = \begin{cases} 1, & \text{if } t \in I_i, \\ 0, & \text{otherwise.} \end{cases} \tag{3.13}$$

Applying the necessary condition for optimality, we take the partial differentiation of H with respect to u and then set it equal to zero. This gives

$$R(t)u^*(t) + \sum_{i=1}^{N} B_{k_i}^T(t)\lambda(t)\chi_{I_i}(t) = 0, \quad t \in I_i,\ i = 1, \dots, N, \tag{3.14}$$

where $\lambda(t)$ is the solution of the co-state system:

$$\begin{aligned} \dot{\lambda}(t) = -C^T(t)Q(t)\bar{y}(t) + C^T(t)Q(t)C(t)x(t) + A_{k_i}^T(t)\lambda(t),& \\ t \in I_i,\ i = 1, \dots, N,& \end{aligned} \tag{3.15}$$

with the terminal condition and intermediate conditions

$$\lambda(t_f) = C^T(t_f)S^T\bar{y}(t_f) + C^T(t_f)SC(t_f)x(t_f) \quad \text{and}$$
$$\lambda(t_i^+) = \lambda(t_i), \quad i = 1, \dots, N-1.$$

Since $R(t)$ is positive definite, (3.14) yields

$$u^*(t) = -R^{-1}(t)B_{k_i}^T(t)\lambda(t), \quad t \in I_i,\ i = 1, \dots, N.$$

We assume that the state and the co-state satisfy

$$\lambda(t) = P(t)x(t) + b(t). \tag{3.16}$$

Differentiating (3.16) and comparing the results with (3.15), we obtain

$$\begin{aligned} &\left[\dot{P}(t) + P(t)A_{k_i}(t) + A_{k_i}^T(t)P(t) + C^T(t)Q(t)C(t) - P(t)B_{k_i}(t)R^{-1}(t)B_{k_i}^T(t)P(t)\right]x(t) \\ &\quad + \left[\dot{b}(t) - P(t)B_{k_i}(t)R^{-1}(t)B_{k_i}^T(t)b(t) + A_{k_i}^T(t)b(t) - C^T(t)Q(t)\bar{y}(t)\right] = 0, \\ &\qquad t \in I_i,\ i = 1, \dots, N. \end{aligned}$$

Thus

$$\begin{aligned} -\dot{P}(t) = P(t)A_{k_i}(t) + A_{k_i}^T(t)P(t) - P(t)B_{k_i}(t)R^{-1}(t)B_{k_i}^T(t)P(t)& \\ + C^T(t)Q(t)C(t), \quad t \in I_i,\ i = 1, \dots, N& \end{aligned} \tag{3.17}$$

with the terminal condition and intermediate conditions

$$\begin{aligned} P(t_f) &= C^T(t_f)SC(t_f) \\ P(t_i^+) &= P(t_i), \quad i = 1, \dots, N-1, \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} \dot{b}(t) = -\left(A_{k_i}(t) - B_{k_i}(t)R^{-1}(t)B_{k_i}^T(t)P(t)\right)^T b(t) + C^T(t)Q(t)\bar{y}(t),& \\ t \in I_i,\ i = 1, \dots, N& \end{aligned} \tag{3.19}$$

with the terminal condition and intermediate conditions

$$b(t_f) = -C^T(t_f)S\bar{y}(t_f) \quad \text{and} \tag{3.20}$$

$$b(t_i^+) = b(t_i), \quad i = 1, \dots, N-1. \tag{3.21}$$

The closed loop control law is

$$u^*(t) = -R^{-1}(t)B_{k_i}^T(t)P(t)x(t) - R^{-1}(t)B_{k_i}^T(t)b(t), \quad t \in I_i,\ i = 1, \ldots, N.$$

By virtue of (2.2), (3.17) and (3.19), we can show, on substituting u^* for u in (2.6) and using integration by parts, that

$$\begin{aligned} J(\sigma, u^*(\sigma)) = \frac{1}{2}\big[&\bar{y}^T(t_f)S\bar{y}(t_f) - \bar{y}^T(t_f)SC(t_f)x(t_f) - x^T(t_f)C^T(t_f)S\bar{y}(t_f) \\ &+x^T(t_f)C^T(t_f)Sx(t_f)\big] - \frac{1}{2}x^T(t)P(t)x(t)\Big|_{t=t_0}^{t=t_f} - x^T(t)b(t)\Big|_{t=t_0}^{t=t_f} \\ &- \left[b^T(t)B_{k_i}(t)R^{-1}(t)B_{k_i}^T(t)b(t) - \bar{y}^T(t)Q(t)\bar{y}(t)\right]\Big|_{t=t_0}^{t=t_f}. \end{aligned} \tag{3.22}$$

Let

$$\dot{g}(t) = b^T(t)B_{k_i}(t)R^{-1}(t)B_{k_i}^T(t)b(t) - \bar{y}^T(t)Q(t)\bar{y}(t) \tag{3.23}$$

with the terminal condition and the intermediate conditions

$$\begin{aligned} g(t_f) &= \bar{y}^T(t_f)S\bar{y}(t_f) \quad \text{and} \\ g(t_i{}^+) &= g(t_i), \quad i = 1, \ldots, N-1. \end{aligned} \tag{3.24}$$

Then, by (3.18), (3.20), (3.23) and (3.24), we obtain

$$J(\sigma, u^*(\sigma)) = \frac{1}{2}x^T(t_0)P(t_0)x(t_0) + x^T(t_0)b(t_0) + \frac{1}{2}g(t_0).$$

This completes the proof. □

From Theorem 3.1, we see that we have obtained the optimal control input and the corresponding form of the cost function. Now, let the switching index sequence be fixed. Then we have Problem P2.

PROBLEM P2. Suppose that the switching index sequence is given. Find a switching times vector $t \in \Theta$ such that (3.2) is minimized subject to the systems (3.3)–(3.11).

4. Problem transformation

Problem P2 is an optimal parameter selection problem governed by systems of differential equations with final time and intermediate time conditions. We shall transform it into the form involving initial value systems by using a simple reverse time transformation.

Denote

$$
\begin{aligned}
\tau = t_f - t = t(\tau), \qquad & \tau_i = t_{N-i}, \qquad i = 0, 1, \dots, N \\
\hat{A}_i(\tau) = A_i(t(\tau)), \qquad & \hat{B}_i(\tau) = B_i(t(\tau)), \quad i = 1, \dots, M, \\
\hat{C}(\tau) = C(t(\tau)), \quad \hat{Q}(\tau) = Q(t(\tau)), \quad & \hat{R}(\tau) = R(t(\tau)), \quad \hat{y}(\tau) = \bar{y}(t(\tau)), \\
\hat{P}(\tau) = P(t(\tau)), \quad \hat{b}(\tau) = b(t(\tau)), \quad & \hat{g}(\tau) = g(t(\tau)).
\end{aligned}
$$

Let $\theta = (\tau_1, \dots, \tau_{N-1})^T$. Then, Θ, which is defined in (2.5), is also the set of all such θ.

Problem P2 becomes:

PROBLEM $\hat{\text{P}}2$. Suppose that the switching index sequence γ is given. Find a switching time vector $\theta \in \Theta$ such that the cost function

$$
\begin{aligned}
\mathscr{L}(\theta, \gamma) &= \mathscr{L}(\sigma) = J(\sigma, u^*(\sigma)) \\
&= \frac{1}{2} x^T(t_0) \hat{P}(t_f) x(t_0) + x^T(t_0) \hat{b}(t_f) + \frac{1}{2} \hat{g}(t_f)
\end{aligned} \tag{4.1}
$$

is minimized subject to the systems

$$
\begin{aligned}
\dot{\hat{P}}(\tau) = {}& \hat{P}(\tau) \hat{A}_{k_{N+1-i}}(\tau) + \hat{A}^T_{k_{N+1-i}}(\tau) \hat{P}(\tau) + \hat{C}^T(\tau) \hat{Q}(\tau) \hat{C}(\tau), \\
& - \hat{P}(\tau) \hat{B}_{k_{N+1-i}}(\tau) \hat{R}^{-1}(\tau) \hat{B}^T_{k_{N+1-i}}(\tau) \hat{P}(\tau),
\end{aligned} \tag{4.2}
$$

$$\tau \in [\tau_{i-1}, \tau_i), \quad i = 1, \dots, N,$$

$$\hat{P}(0) = \hat{C}^T(0) S \hat{C}(0), \tag{4.3}$$

$$\hat{P}(\tau_i^-) = \hat{P}(\tau_i), \quad i = 1, \dots, N-1, \tag{4.4}$$

$$
\begin{aligned}
\dot{\hat{b}}(\tau) = {}& \left[\hat{A}_{k_{N+1-i}}(\tau) - \hat{B}_{k_{N+1-i}}(\tau) \hat{R}^{-1}(\tau) \hat{B}^T_{k_{N+1-i}}(\tau) \hat{P}(\tau) \right]^T \hat{b}(\tau) \\
& - \hat{C}^T(\tau) \hat{Q}(\tau) \hat{y}(\tau), \quad \tau \in [\tau_{i-1}, \tau_i), \; i = 1, \dots, N,
\end{aligned} \tag{4.5}
$$

$$\hat{b}(0) = -\hat{C}^T(0) S \hat{y}(0), \tag{4.6}$$

$$\hat{b}(\tau_i^-) = \hat{b}(\tau_i), \quad i = 1, \dots, N-1 \tag{4.7}$$

and

$$
\begin{aligned}
\dot{\hat{g}}(\tau) = {}& -\hat{b}^T(\tau) \hat{B}_{k_{N+1-i}}(\tau) \hat{R}^{-1}(\tau) \hat{B}^T_{k_{N+1-i}}(\tau) \hat{b}(\tau) + \hat{y}^T(\tau) \hat{Q}(\tau) \hat{y}(\tau), \\
& \tau \in [\tau_{i-1}, \tau_i), \quad i = 1, \dots, N,
\end{aligned} \tag{4.8}
$$

$$\hat{g}(0) = \hat{y}^T(0) S \hat{y}(0), \tag{4.9}$$

$$\hat{g}(\tau_i^-) = \hat{g}(\tau_i), \quad i = 1, \dots, N-1. \tag{4.10}$$

Accordingly, the optimal control input (3.1) becomes

$$u(t_f - \tau, \sigma) = \hat{R}^{-1}(\tau)\hat{B}^T_{N-k_i}(\tau)\hat{P}(\tau)x(t_f - \tau) - \hat{R}^{-1}(\tau)\hat{B}^T_{k_{N+1-i}}(\tau)\hat{b}(\tau),$$
$$t \in [\tau_{i-1}, \tau_i), \quad i = 1, \dots, N. \tag{4.11}$$

Note that the switching instants in the cost function are decision variables. Now we introduce a time scaling transform, which is called the control parametrization enhancing transform (CPET) in [9], given below:

$$\dot{\tau}(s) = v(s) \tag{4.12}$$

with the initial condition

$$\tau(0) = 0 \tag{4.13}$$

and the terminal condition

$$\tau(N) = t_f, \tag{4.14}$$

where $v(s)$ is a piecewise constant function with possible discontinuity points at $s = 1, \dots, N-1$. It is a time scaling control given by

$$v(s) = \sum_{i=1}^{N} \delta_i \chi_{[i-1,i)}(s). \tag{4.15}$$

Let δ_i, $i = 1, \dots, N$, satisfying $\delta_i \geq 0$, be referred to collectively as δ, and let Δ be the set of all such δ. Then, the new state $\tau(s)$ is given by

$$\tau(s) = \int_0^s v(\xi)d\xi = \sum_{j=1}^{i-1} \delta_j + \delta_i(s - i + 1), \quad s \in [i-1, i). \tag{4.16}$$

Denote

$$\tilde{A}_i(s) = \hat{A}_i(\tau(s)), \quad \tilde{B}_i(s) = \hat{B}_i(\tau(s)), \quad i = 1, \dots, M,$$
$$\tilde{C}(s) = \hat{C}(\tau(s)), \quad \tilde{Q}(s) = \hat{Q}(\tau(s)), \quad \tilde{R}(s) = \hat{R}(\tau(s)), \quad \tilde{y}(s) = \hat{y}(\tau(s)),$$
$$\tilde{P}(s) = \hat{P}(\tau(s)), \quad \tilde{b}(s) = \hat{b}(\tau(s)), \quad \tilde{g}(s) = \hat{g}(\tau(s)).$$

Then, (4.1)-(4.10) is written as

$$\mathcal{L}(\theta, \gamma) = \frac{1}{2}x^T(t_0)\tilde{P}(N)x(t_0) + x^T(t_0)\tilde{b}(N) + \frac{1}{2}\tilde{g}(N), \tag{4.17}$$

where $\tilde{P}$, $\tilde{b}$ and $\tilde{g}$ are, respectively, determined by

$$\dot{\tilde{P}}(s) = \Big[\tilde{P}(s)\tilde{A}_{k_{N+1-i}}(s) + \tilde{A}^T_{k_{N+1-i}}(s)\tilde{P}(s) - \tilde{P}(s)\tilde{B}_{k_{N+1-i}}(s)\tilde{R}^{-1}(s)\tilde{B}^T_{k_{N+1-i}}(s)\tilde{P}(s)$$
$$+\tilde{Q}(s)\Big]v(s), \quad s \in [i-1, i), \ i = 1, \dots, N, \tag{4.18}$$

$$\tilde{P}(0) = \tilde{C}^T(0)S\tilde{C}(0), \tag{4.19}$$

$$\tilde{P}(i^-) = \tilde{P}(i), \quad i = 1, \ldots, N-1, \tag{4.20}$$

$$\begin{aligned}\dot{\tilde{b}}(s) = -\Big[&-\left(\tilde{A}_{k_{N+1-i}}(s) - \tilde{B}_{k_{N+1-i}}(s)\tilde{R}^{-1}(s)\tilde{B}^T_{k_{N+1-i}}(s)\tilde{P}(s)\right)^T\tilde{b}(s)\\ &+\tilde{C}^T(s)\tilde{Q}(s)\tilde{y}(s)\Big]v(s), \quad s \in [i-1, i),\ i = 1, \ldots, N,\end{aligned} \tag{4.21}$$

$$\tilde{b}(0) = -\tilde{C}^T(0)S\tilde{y}(0), \tag{4.22}$$

$$\tilde{b}(i^-) = \tilde{b}(i), \quad i = 1, \ldots, N-1 \tag{4.23}$$

and

$$\begin{aligned}\dot{\tilde{g}}(s) = -\left[\tilde{b}^T(s)\tilde{B}_{k_{N+1-i}}(s)\tilde{R}^{-1}(s)\tilde{B}^T_{k_{N+1-i}}(s)\tilde{b}(s) - \tilde{y}^T(s)\tilde{Q}(s)\tilde{y}(s)\right]v(s),\\ s \in [i-1, i),\ i = 1, \ldots, N,\end{aligned} \tag{4.24}$$

$$\tilde{g}(0) = \tilde{y}^T(0)S\tilde{y}(0), \tag{4.25}$$

$$\tilde{g}(i^-) = \tilde{g}(i), \quad i = 1, \ldots, N-1. \tag{4.26}$$

For brevity, we set

$$\tilde{P}(s) = \begin{bmatrix} \tilde{P}_{11}(s) & \tilde{P}_{12}(s) & \cdots & \tilde{P}_{1n}(s) \\ \tilde{P}_{12}(s) & \tilde{P}_{22}(s) & \cdots & \tilde{P}_{2n}(s) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{P}_{1n}(s) & \tilde{P}_{2n}(s) & \cdots & \tilde{P}_{nn}(s) \end{bmatrix} \quad \text{and} \quad \tilde{b}(s) = \left[\tilde{b}_1(s), \tilde{b}_2(s), \ldots, \tilde{b}_n(s)\right]^T.$$

Define

$$\tilde{x}(s) = \left[\tilde{x}_1^T(s), \tilde{x}_2^T(s), \tilde{x}_3^T(s), \tilde{x}_4^T(s)\right]^T, \quad \text{where}$$

$$\tilde{x}_1(s) = \left[\tilde{P}_{11}(s), \ldots, \tilde{P}_{1n}(s), \tilde{P}_{22}(s), \ldots, \tilde{P}_{2n}(s), \ldots, \tilde{P}_{nn}(s)\right]^T,$$

$$\tilde{x}_2(s) = \left[\tilde{b}_1(s), \ldots, \tilde{b}_n(s)\right]^T, \quad \tilde{x}_3(s) = \tilde{g}(s) \quad \text{and} \quad \tilde{x}_4(s) = \tau(s).$$

Then, (4.12), (4.18)–(4.26) is written collectively as:

$$\dot{\tilde{x}}(s) = \tilde{f}\left(s, \tilde{x}(s), \delta\right), \quad s \in [0, N] \tag{4.27}$$

$$\tilde{x}(0) = \tilde{x}_0, \tag{4.28}$$

where $\tilde{f}$, $\tilde{x}_0$ are obtained from (4.18)-(4.26). The cost function (4.17) takes the form

$$\tilde{\mathscr{L}}(\delta, \gamma) = \Phi\left(\tilde{x}(N)\right), \tag{4.29}$$

where Φ is still a function of $\tilde{P}$, $\tilde{b}$ and $\tilde{g}$. Problem $\hat{\mathrm{P}}2$ becomes:

PROBLEM $\tilde{P}2$. Suppose that the switching index sequence γ is given. Find a $\delta \in \Delta$ such that the cost function (4.29) is minimized, subject to the system (4.27)-(4.28).

Note that in the transformed problem $\tilde{P}2$, all the locations of the switchings of the state differential equation are known and fixed. To solve the problem as a mathematical programming problem, we require the gradient formula of the cost function (4.29). Let H be the Hamilton function defined by

$$H(s, \tilde{x}(s), \mu(s), \delta) = \mu^T(s)\tilde{f}(s, \tilde{x}(s), \delta), \tag{4.30}$$

where $\mu(s)$ is the co-state, which satisfies

$$\dot{\mu}^T(s) = -\frac{\partial H(s, \tilde{x}(s), \mu(s), \delta)}{\partial \tilde{x}(s)} \tag{4.31}$$

with the terminal condition

$$\mu^T(N) = \frac{\partial \Phi(\tilde{x}(N))}{\partial \tilde{x}(N)}. \tag{4.32}$$

The following theorem gives the gradient formula of the cost function with respect to δ.

THEOREM 4.1. *The gradient of the cost function* (4.29) *is given by*

$$\frac{\partial \tilde{\mathscr{L}}(\delta, \gamma)}{\partial \delta} = \int_0^N \frac{\partial H(s, \tilde{x}(s), \mu(s), \delta)}{\partial \delta} ds. \tag{4.33}$$

PROOF. The proof is similar to that given for Theorem 5.2.1 in [16]. □

To solve Problem $\tilde{P}2$, any gradient-based method can be used, such as the sequential quadratic programming method (see, for example, [16]). For this, we need the value of the cost function (4.29) for each $\delta \in \Delta$ as well as the gradient formula of the cost function (4.33) for each $\delta \in \Delta$. They are calculated by:

ALGORITHM 1. Step 1. Construct the expressions of the cost function (4.17), the control input (4.11) and the state function (4.18)-(4.26).
Step 2. For a given $\delta \in \Delta$, compute the solution $\tilde{x}(\cdot|\delta)$ of the system (4.27)-(4.28) forward in time from $s = 0$ to $s = 1$ to obtain $\tilde{x}(\cdot|\delta)$, $s \in [0, 1)$. Then use $\tilde{x}(1|\delta)$ as the initial condition, solve the system from $s = 1$ to $s = 2$ to obtain $\tilde{x}(s|\delta)$, $s \in [1, 2)$. This process is repeated until $\tilde{x}(\cdot|\delta)$ are obtained for $s \in [N-1, N)$. Then, $\tilde{x}(\cdot|\delta)$ is known for $s \in [0, N]$.
Step 3. Compute the corresponding value of $\tilde{\mathscr{L}}(\delta, \gamma)$ from (4.29) for the given $\delta \in \Delta$.
Step 4. Similar to Step 2, we solve the co-state differential equation (4.31) backward in time with the terminal condition (4.32) from $s = N$ to $s = 0$. This gives rise to the solution $\mu(\cdot|\delta)$ of the co-state system.
Step 5. Compute the gradient of the cost function with respect to δ according to formula (4.33).

5. Discrete filled function method

Suppose that for each switching index vector γ, through solving Problem $\tilde{\text{P}}2$, we obtain an optimal switching time vector, denoted by $\delta^*(\gamma)$. Then we obtain Problem P3.

PROBLEM P3. Find a switching index vector $\gamma \in \Gamma$, such that the cost function

$$\bar{\mathscr{L}}(\gamma) = \tilde{\mathscr{L}}(\delta^*(\gamma), \gamma) \tag{5.1}$$

is minimized.

For all $\gamma^1, \gamma^2 \in \Gamma$, let ρ be a nonnegative function from $\Gamma \times \Gamma$ into $\mathbb{R}$ defined by

$$\rho\left(\gamma^1, \gamma^2\right) = \sum_{t=1}^{N} \varrho\left(\gamma^1(t), \gamma^2(t)\right),$$

where ϱ denotes the characteristic function given by

$$\varrho(i, j) = \begin{cases} 0, & \text{if } i = j, \\ 1, & \text{if } i \neq j. \end{cases}$$

Here ρ is a metric in the space Γ. Then, the space Γ equipped with the metric ρ is a metric space. Let it be denoted by (Γ, ρ).

Before proceeding further, we introduce the following definitions [4].

DEFINITION 1. For all $\gamma \in (\Gamma, \rho)$, the neighbourhood of γ is defined by

$$\mathscr{N}(\gamma) = \{\gamma\} \cup \left\{\gamma' \in \Gamma : \rho(\gamma, \gamma') = 1\right\}.$$

DEFINITION 2. A point γ^* is called a local minimizer of $\bar{\mathscr{L}}$ over (Γ, ρ) if $\bar{\mathscr{L}}(\gamma^*) \leq \bar{\mathscr{L}}(\gamma), \forall \gamma \in \mathscr{N}(\gamma^*)$. If, in addition, $\bar{\mathscr{L}}(\gamma^*) < \bar{\mathscr{L}}(\gamma), \forall \gamma \in \mathscr{N}(\gamma^*) \backslash \{\gamma^*\}$, then γ^* is called a strict local minimizer of $\bar{\mathscr{L}}$ over (Γ, ρ).

Based on Definitions 1 and 2, we can easily find a local minimizer by using the following algorithm.

ALGORITHM 2. Step 1. Set the initial switching index vector γ^0 and compute the value of (5.1).

Step 2. Compute the value of $\bar{\mathscr{L}}(\gamma)$, $\forall \gamma \in \mathscr{N}(\gamma^0)$. If there exists γ^1, such that $\bar{\mathscr{L}}(\gamma^1) < \bar{\mathscr{L}}(\gamma^0)$, then go to Step 3. Otherwise, γ^0 is a local minimizer of $\bar{\mathscr{L}}(\gamma)$. Stop.

Step 3. Set $\gamma^0 = \gamma^1$ as the initial switching index vector. Go to Step 2.

In order to find a global solution, we shall use the discrete filled function method. The following definition is needed.

DEFINITION 3. Given that γ^* is a local minimizer of $\bar{\mathscr{L}}$ over (Γ, ρ), $B(\gamma^*)$ is said to be a discrete basin of $\bar{\mathscr{L}}$ at γ^* over (Γ, ρ) if $B(\gamma^*) \subset (\Gamma, \rho)$ and the descent trajectory from any initial point in $B(\gamma^*)$ converges to γ^*, but the descent trajectory from any initial point in $\Gamma \setminus B(\gamma^*)$ does not converge to γ^*.

We introduce the following function

$$F_{\mu_1,\mu_2}(\gamma;\gamma^*) = \mu_1\left[\bar{\mathscr{L}}(\gamma) - \bar{\mathscr{L}}(\gamma^*)\right]^2 - \mu_2\rho^2(\gamma,\gamma^*), \quad \text{if } \bar{\mathscr{L}}(\gamma) \geq \bar{\mathscr{L}}(\gamma^*). \tag{5.2}$$

When $\mu_2 > 0$, $\mu_1 < \mu_2/L$ (L is a sufficiently large real number), (5.2) is called a discrete filled function (for details, see [11]).

It follows from [11] that γ^* is a local maximizer of F and the optimization process applied to F can jump out of the current discrete basin by using any descent method.

Using the current local minimizer as an initial vector and considering the discrete filled function as the the objective function, we can find a better minimizer, if it exists.

The solution procedure for Problem 1 may now be summarized as follows.

Step 1. Choose an initial switching index sequence $\gamma^0 \in \Gamma$ and an initial switching time sequence $t^0 \in \Theta$. Regard Problem 1 as a linear tracking problem.

Step 2. Obtain the optimal control by using (3.1), and substitute it into the cost function (2.6). This gives rise to a new cost function (3.2).

Step 3. Use the reverse time transformation to convert Problem P2 into Problem $\hat{\text{P}}2$.

Step 4. Use the time scaling transformation (4.12) to (4.14) to convert Problem $\hat{\text{P}}2$ into Problem $\tilde{\text{P}}2$.

Step 5. Problem $\tilde{\text{P}}2$ is an optimal parameter selection problem, which is solved as a constrained optimization problem. The optimal control software package, MISER 3.3 [8] is used, where the optimization technique used is sequential quadratic programming (see [16]). The value of the cost function (4.29) and its gradient for each $\delta \in \Delta$ are computed by Algorithm 1. The optimal solution obtained is then used to construct the function $\bar{\mathscr{L}}$.

Step 6. Apply Algorithm 2 to find a local minimizer for the cost function (5.1), where for each switching index vector, its corresponding value of $\bar{\mathscr{L}}$ is computed from Step 2 to Step 5. Let γ^* denote the minimizer.

Step 7. Set γ^* as the initial point and apply Algorithm 2 to search for a point better than the current local minimizer γ^*, with the discrete filled function F_{μ_1,μ_2} used as the objective function. If a point γ better than γ^* is found during the search, then stop searching and set $\gamma^0 = \gamma$. Go to Step 6. Else if a point better than the current minimizer υ^* is not found when the local minimizer γ^{**} of the discrete filled function has been determined, go to Step 8.

Step 8. Return the optimal solution γ^* and the corresponding cost function value $\bar{\mathscr{L}}^*$.

6. Illustrative example

EXAMPLE 1. Consider a second-order switched linear system. Its dynamics are chosen from a finite set $\{(A_1, B_1), (A_2, B_2), (A_3, B_3)\}$ in 9 time intervals, and its output matrix is C, where

$$A_1 = \begin{bmatrix} -0.5 & -0.1 \\ 0 & -0.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.7 & 0 \\ -0.3 & -0.7 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -0.9 & 0 \\ 0 & -0.9 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.3 & 0 \\ 0 & 1.0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1.2 & 0 \\ 0 & 0.4 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1.0 & 0.3 \\ 0.3 & 1.0 \end{bmatrix}$$

$$\text{and} \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Here, $t_0 = 0$, $t_f = 9$ and $x(0) = [2, 1]^T$.

We need to find the optimal control input, the optimal switching time sequence and the optimal switching index sequence such that the cost function

$$J = \frac{1}{2}\left[\bar{y}(t_f) - y(t_f)\right]^T S \left[\bar{y}(t_f) - y(t_f)\right] + \frac{1}{2}\int_{t_0}^{t_f} \left[(\bar{y}(t) - y(t))^T Q(t)\,(\bar{y}(t) - y(t)) + u^T(t)R(t)u(t)\right] dt$$

is minimized, where

$$S = \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix}, \quad Q(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R(t) = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.25 \end{bmatrix},$$

while $\bar{y}(t) = [2\sin(t), 2\cos(t)]^T$ is a given two-dimensional reference vector to be tracked.

We summarize the computational stages as follows.

Choose an initial switching sequence

$$\gamma^0 = (1, 2, 3, 1, 2, 3, 1, 2, 3)$$

with the switching time instants

$$(1.0, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0).$$

The objective function value is 12.8033933. Apply the time scaling transform and then MISER 3.3 is used to solve the corresponding version of the optimal parameter selection problem. The objective function value obtained is 10.182974. Apply Algorithm 2 to obtain a local optimal switching sequence

$$\gamma^* = (1, 2, 3, 1, 2, 3, 1, 2, 1).$$

The objective function value is 9.58141745. The discrete filled function method is used in conjunction with MISER 3.3 to jump out from the local optimal switching sequence, giving rise to a new switching sequence

$$\gamma^* = (1, 2, 3, 1, 2, 1, 3, 2, 1)$$

and a new sequence of switching time instants

$$(0.53593, 2.02713, 3.4693, 3.4693, 4.6707, 5.67035, 6.56754, 7.77026).$$

The objective function value obtained is 9.26002408. This process is continued. Finally, we obtain the optimal switching sequence

$$\gamma^* = (1, 2, 1, 3, 2, 1, 3, 2, 1),$$

the optimal sequence of switching time instants

$$(0.55572, 1.52709, 2.51956, 3.42688, 4.66945, 5.67101, 6.56765, 7.77028)$$

and the optimal objective function value 8.95108976.

The trajectory of the system output using the optimal switching index sequence and the optimal sequence of switching time instants versus the reference signal are depicted in Figure 1, where the trajectory of the system output and the reference signal with the optimal switching index sequence are displaced.

7. Conclusions

We considered a class of optimal tracking problems involving switched systems, where the switching signal and the control input are considered as decision variables. We developed an efficient method for solving this optimal tracking problem involving switched systems as a three-stage optimization problem. Through an illustrative example we can see the effectiveness of the method developed.

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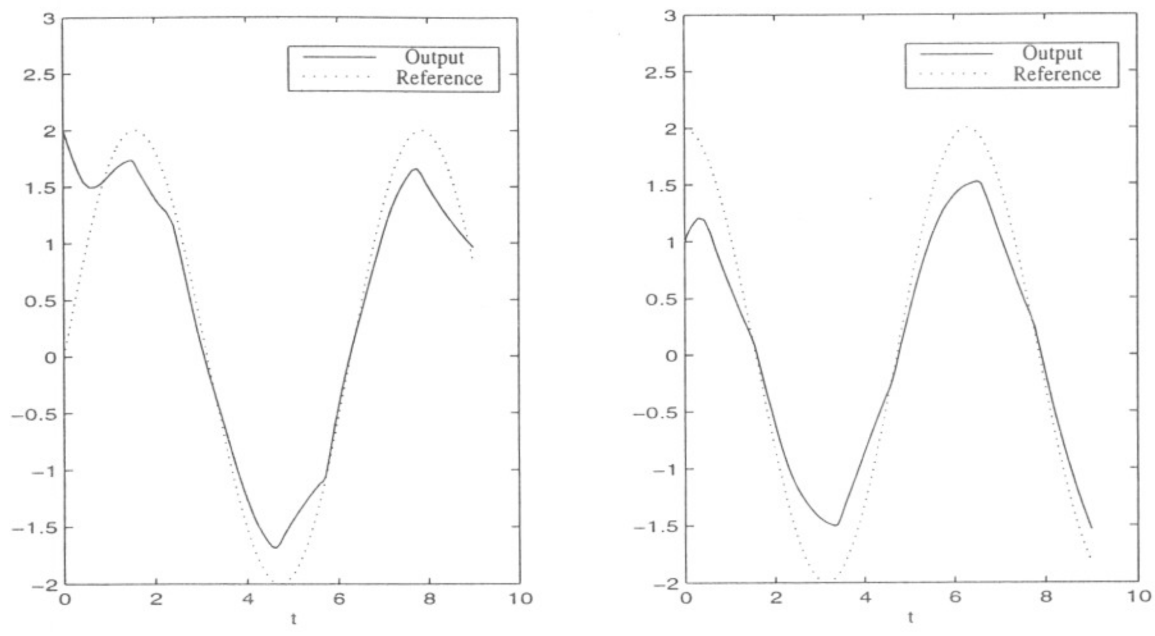

FIGURE 1. The trajectories of the optimal system output and the reference signal.

References

[1] B. D. Anderson and J. B. Moore, *Optimal control: Linear quadratic methods* (Prentice Hall, Englewood Cliffs, NJ, 1990).

[2] S. Bengea and R. DeCarlo, "Optimal control of switching systems", *Automatica* **41** (2005) 11–27.

[3] S. Devasia, B. Paden and C. Rossi, "Exact-output tracking theory for systems with parameter jumps", *Int. J. Control* **67** (1997) 11–131.

[4] Z. G. Feng, K. L. Teo and V. Rehbock, "A discrete filled function method for the optimal control of switched systems in discrete time", Submitted.

[5] R. P. Ge, "A filled function method for finding a global minimizer of a function of several variables", *Math. Program.* **46** (1990) 191–204.

[6] S. Hedlund and A. Rantzer, "Convex dynamic programming for hybrid systems", *IEEE Trans. Automat. Contr.* **47** (2002) 1536–1540.

[7] A. Isidori, *Nonlinear control systems*, 2nd ed. (Springer-Verlag, Berlin, 1989).

[8] L. S. Jennings, M. E. Fisher, K. L. Teo and C. J. Goh, *MISER 3.3-Optimal control software: Theory and user manual*, 2004.

[9] H. W. J. Lee, K. L. Teo, V. Rehbock and L. S. Jennings, "Control parametrization enhancing technique for time optimal control problems", *Dyn. Syst. Appl.* **6** (1997) 243–261.

[10] F. L. Lewis, *Optimal control*, 2nd ed. (John Wiley, New York, 1995).

[11] C. K. Ng, L. S. Zhang, D. Li and W. W. Tian, "Discrete filled function method for discrete global optimization", *Comput. Optim. Appl.* **31** (2005) 87–115.

[12] B. Piccoli, "Necessary conditions for hybrid optimization", in *Proc. 38th IEEE CDC*, (1999), 410–415.

[13] P. Riedinger, C. Iung and F. Kratz, "An optimal control approach for hybrid systems", *European J. Contr.* **49** (2003) 449–458.

[14] Z. Sun and S. S. Ge, *Switched linear systems– control and design* (Springer, Berlin, 2004).

[15] H. Sussman, "A maximum principle for hybrid optimal control problems", in *Proc. 38th IEEE*

CDC, Phoenix, USA, (1999), 425–430.

[16] K. L. Teo, C. J. Goh and K. H. Wong, *A unified computational approach to optimal control problems* (Longman Scientific and Technical, United Kingdom, 1991).

[17] C. Z. Wu and K. L. Teo, "Global impulsive optimal control computation", *J. Ind. Manag. Optim.* **2** (2006) 435–450.

[18] X. Xu and P. Antsaklis, "Optimal control of switched systems based on parameterization of the switching instants", *IEEE Trans. Automat.Contr.* **49** (2004) 2–16.

[19] Y. Yin and S. Hosoe, "Tracking control of discrete and continuous hybrid systems: modeling and servoing problem of dextrous hand manipulation", in *Proc. 2004 IEEE Int. Conf. Control Applications, 2–4 Sept. 2004*, (2004), 860–865.