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A Hamiltonian approach to the H_2 decoupling of previewed input signals

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Abstract— **This paper addresses the H_2 optimal decoupling of previewed inputs. In particular, the synthesis of a decoupling filter exploiting the preview is carried out by recasting the problem as a single finite-horizon LQ regulator, with a generalized quadratic terminal cost. The present approach is based on a computationally attractive parametrization of the solutions of the associated Hamiltonian differential equation, and can also be applied for the solution of the dual problem, the H_2 optimal fixed-lag smoothing problem.**

I. INTRODUCTION

The decoupling problem, whose objective is to minimize the influence of an input signal on the output of a given system, is a deeply investigated topic in control theory, both in an H_∞ and in an H_2 context, e.g. [10], [15], [3], [14].

If the signal to be decoupled is known in advance with a finite amount of time, the compensation scheme can take advantage of the pre-knowledge of this signal through a feedforward action. The first pioneering contribution on this topic is [13], where the H_2 (LQ) tracking with preview was formulated in a stochastic setting as the problem of making the outputs of the plant follow some previewed command signals, that were assumed to be generated by a shaping filter fed by a Gaussian white process.

The LQ with preview was also tackled in [4], where the problem formulation, differently from that considered herein, involves a disturbance which is supposed to appear in a finite time interval: this choice is motivated by the aim of solving an optimal servosystem problem with predictive action as a decoupling problem.

A fundamental contribution on the optimal control problem with finite preview in an H_∞ context, hence quite different from the problem herein dealt with as far as its formulation and solution are concerned, has been recently provided by Tadmor and Mirkin in [12]. The analysis of the dual problem, the optimal fixed-lag smoothing, has also been considered.

In [7], the H_2 decoupling problem with preview was solved by splitting the overall time interval into two subintervals, so that the H_2 optimization problem consisted of a finite-horizon LQ regulator with parametric terminal state and of an infinite-horizon LQ with parametric initial state, and a linear algebraic equation linking the two parameters. These two LQ subproblems were tackled by exploiting a closed-form formula parametrizing the set of solutions of the Hamiltonian differential equation associated with the LQ problem for stabilizable pairs, thus extending the parametrization presented in [2] which required the controllability of the underlying system. In this way it was possible to impose the boundary conditions to the parametrized state and costate functions so as to determine the parameters ensuring optimality. The drawback was that this extension was based on a change of coordinates in the state space, which inevitably led to heavy theoretical and computational burden.

In this paper, the solution presented in [7] is improved in twofold directions:

1) The H_2 decoupling problem with preview can be recast as a *single* finite-horizon LQ problem, where the quadratic cost to be minimized involves a quadratic term weighting the difference between the terminal state and an assigned target state.
2) The formula parametrizing the solutions of the Hamiltonian differential equation which is herein employed for the solution of the H_2-preview decoupling does not require any change of coordinates. Moreover, in the present solution, all the matrices appearing in the parametrized expression of the optimal state and control functions are easily obtainable through standard and currently available software routines (see for example the MATLAB® functions `care.m` and `lyap.m`), and are well-conditioned and robust even when the preview interval is large.

This approach is therefore constructive, and the control scheme consists of a dynamical unit whose inner structure is herein analyzed in detail. This framework encompasses the case where the preview time is zero, i.e. when the signal to be decoupled is accessible for measurement but not previewed, which is a good resort for tackling the decoupling when the geometric conditions for total rejection presented in [1] are not satisfied.

Notation. The image, the null-space and the trace of matrix A are denoted by $\mathrm{im}A$, $\ker A$ and $tr A$, respectively, whereas $A^\top$ and A^H denote the transpose and the transpose conjugate of A, respectively. The symbol I_k stands for the $k \times k$ identity matrix.

II. PROBLEM STATEMENT

Consider the LTI system Σ

$$\dot{x}(t) = A\,x(t) + B_1\,u(t) + B_2\,w(t), \qquad (1)$$

$$y(t) = C\,x(t) + D_1\,u(t) + D_2\,w(t), \qquad (2)$$

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where, for all $t \geq 0$, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, $w(t) \in \mathbb{R}^r$ is the input signal to be decoupled from the output $y(t) \in \mathbb{R}^p$, A, B_1, B_2, C, D_1 and D_2 are real constant matrices of proper dimensions. The signal $w(t)$ is supposed to be zero in $[0,T)$ and known in advance with a preview time $T>0$. Define the previewed input $w_p(t) := w(t+T)$, $t \geq 0$.

Problem 1: Find an LTI feedforward compensator Σ_c connected as in Figure 1 such that the transfer function matrix $\widehat{G}(s)$ of the overall system $\widehat{\Sigma}$, from the previewed input $w_p(t)$ to the output $y(t)$, is internally stable and its H_2-norm $||\widehat{G}||_2$ is minimized.

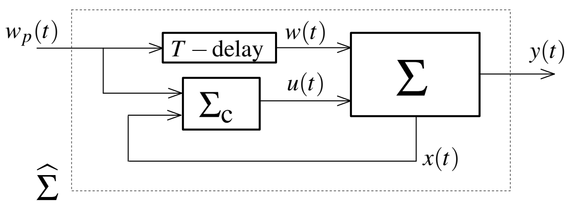

Fig. 1. H_2-optimal decoupling scheme of a previewed signal $w(t)$.

The T-delay stage accounts for the pre-knowledge of the signal $w(t)$. In fact, as shown in Figure 1, the compensator Σ_c exploits the preview information on $w(t)$ represented by $w_p(t)$.

III. Preliminary results

For the readers' convenience, we recall that the H_2-norm of the transfer function matrix $G(s)$ is defined as

$$||G||_2 := \sqrt{\frac{1}{2\pi} \text{trace} \int_{\mathbb{R}} G^H(i\omega)\, G(i\omega)\, d\omega}.$$

In view of the Parseval identity, the latter can be written in the time-domain by means of the Laplace anti-transform of $G(s)$, which is the impulse response matrix $\mathscr{G}(t)$, $t \geq 0$:

$$||G||_2^2 = \text{trace} \int_0^\infty \mathscr{G}^\top(t)\mathscr{G}(t)\, dt. \tag{3}$$

Now, let $(e_i)_{i\in\{1,\ldots,r\}}$ denote the canonical basis of $\mathbb{R}^r$. By (3), the H_2 norm of $\widehat{G}(s)$ can be expressed in terms of the impulse response matrix $\widehat{\mathscr{G}}(t)$, $t \geq 0$, of $\widehat{\Sigma}$ as

$$||\widehat{G}||_2^2 = \sum_{i=1}^r \int_0^\infty \widehat{\mathscr{G}}_i^\top(t)\widehat{\mathscr{G}}_i(t)\, dt \tag{4}$$

where $\widehat{\mathscr{G}}_i(t)$ denotes the i-th column of $\widehat{\mathscr{G}}(t)$, i.e., is the response of $\widehat{\Sigma}$ with zero initial state to the input $w_p(t) = e_i\, \delta(t)$, here $\delta(t)$ denoting the Dirac impulse, [3, p.265].

Let $Q := C^\top C$, $S := C^\top D_1$, $R := D_1^\top D_1$. The first result that we need recall consists of a closed-formula parametrizing the set of solutions of the Hamiltonian differential equation

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} = H \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix}, \tag{5}$$

where the matrix

$$H := \begin{bmatrix} A - BR^{-1}S^\top & -BR^{-1}B^\top \\ -Q + SR^{-1}S^\top & -A^\top + SR^{-1}B^\top \end{bmatrix} \tag{6}$$

is usually referred to as the Hamiltonian matrix; system (5) is obtained by extending the state $x(t)$ of Σ with the costate $\lambda(t) \in \mathbb{R}^n$, $t \geq 0$, [6].

Lemma 1: Let the pair (A,B) be stabilizable and let H have no eigenvalues on the imaginary axis. Let $P_+ = P_+^\top \geq 0$ be the stabilizing solution of the ARE

$$PA + A^\top P - (S + PB)\, R^{-1}\, (S^\top + B^\top P) + Q = 0, \tag{7}$$

let A_+ denote the corresponding closed-loop matrix

$$A_+ := A - BR^{-1}\, (S^\top + B^\top P_+), \tag{8}$$

and let W be the solution of the closed-loop Lyapunov equation

$$A_+ W + W A_+^\top + BR^{-1}B^\top = 0. \tag{9}$$

The set of trajectories solving the Hamiltonian system (5) is parametrized in $t \in [0,T)$ in terms of $p, q \in \mathbb{R}^n$ as

$$\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = \begin{bmatrix} I_n \\ P_+ \end{bmatrix} e^{A_+ t}\, p + \begin{bmatrix} W \\ P_+ W - I_n \end{bmatrix} e^{-A_+^\top (t-T)}\, q. \tag{10}$$

A complete proof of Lemma 1 can be found in [9]. Here, we only observe that, since the pair (A,B) is stabilizable and H has no eigenvalues on $i\mathbb{R}$, the ARE (7) admits a stabilizing solution $P_+ = P_+^\top \geq 0$, i.e., such that all the eigenvalues of the closed-loop matrix A_+ are in the open left-half complex plane, [8, p.354]. As a consequence, the Lyapunov equation (9) admits a unique solution $W = W^\top \geq 0$, [5, Theorem 5.2.2]. In (10), the expressions of the optimal state and costate functions are given in terms of the matrix exponentials $\exp[A_+ t]$ and $\exp[A_+^\top (T-t)]$. Hence, the solutions of the Hamiltonian differential equation involve exponentials of strictly stable matrices in the overall time interval $[0,T]$, thus ensuring that their computation is numerically robust even for large time horizons. Furthermore, the matrices P_+, A_+ and W may be computed by standard and reliable algorithms available in any control package (see e.g. the MATLAB® routines `care.m` and `lyap.m`).

IV. Derivation of the optimal controller

Let $Q := C^\top C$, $S := C^\top D_1$, $R := D_1^\top D_1$. Assume that

(A1) the pair (A, B_1) is stabilizable;

(A2) (A, B_1, C, D_1) has no invariant zeros on $i\mathbb{R}$ and D_1 is full-column rank;

(A3) $\operatorname{im} D_2 \subseteq \operatorname{im} D_1$.

The injectivity of D_1 following from assumption (A2) ensures that $R = R^\top > 0$, while the absence of invariant zeros of the quadruple (A, B_1, C, D_1) on the imaginary axis guarantees that the corresponding Hamiltonian matrix (6) has no eigenvalues on the imaginary axis, [15, Theorem 13.7, Lemma 13.9, Corollary 13.10]. Hence, assumptions (A1)-(A2) ensure that the stabilizing solution $P_+ = P_+^\top \geq 0$ of the ARE (7), with $B := B_1$, exists.

The following theorem is the main result of this paper, and provides the solution to Problem 1 when the feedthrough matrix D_2 is zero. In Remark 1 it will be shown how to deal with the case where $D_2 \neq 0$.

Theorem 1: Consider Problem 1, with $D_2 = 0$. Let assumptions (A1)-(A2) hold. Let $P_+ = P_+^\top \geq 0$ be the stabilizing solution of the ARE (7) with $B = B_1$, let A_+ be given by (8). The optimal compensator Σ_c for Problem 1 is described by the following input-output relation:

$$u(t) = \int_0^T \Phi(\tau)\, w_p(t-\tau)\, d\tau - K_+ x(t), \tag{11}$$

where $\Phi(t) := -R^{-1} B_1^\top e^{A_+^\top (T-t)} P_+ B_2$ and $K_+ := R^{-1}(S^\top + B^\top P_+)$.

Proof: Let $i \in \{1,\ldots,r\}$. Consider the problem of minimizing the i-th term of the sum in (4). As already observed, this is equivalent to finding $u_i(t)$, $t \in [0,+\infty)$, that minimizes $\int_0^\infty y_i^\top(t) y_i(t) dt$, where $y_i(t)$ is the output of (1)-(2) with $w_i(t) = e_i\, \delta(t-T)$ as input, whose effect is the instantaneous transition of the state $x_i(t)$ at $t = T$:

$$x_i(T) = \int_0^T e^{A(T-\tau)} \Big(B_1 u_i(\tau) + B_2 e_i\, \delta(\tau - T)\Big) d\tau. \tag{12}$$

Hence, $x_i(T) = B_2^i + x_i(T_-)$, where B_2^i is the i-th column of B_2 and $x_i(T_-) := \int_0^T e^{A(T-\tau)} B_1 u_i(\tau) d\tau$ is the state at $t = T$ obtained by the application of the sole input $u_i(t)$, $t \in [0,T)$. The optimal control $u_i(t)$, $t \in [T,+\infty)$, is then obtained by solving an infinite-horizon LQ problem consisting of the minimization of the quadratic index $\int_T^\infty y_i^\top(t) y_i(t) dt$ with initial condition $x_i(T)$, and can be expressed by the algebraic state feedback $u_i(t) = -K_+ x_i(t) = -K_+ e^{A_+(t-T)} x_i(T)$, $t \geq T$. The corresponding optimal cost is given by the quadratic form $x_i^\top(T) P_+ x_i(T)$, [5, Theorem 16.3.3]. Hence, the contribution of the infinite-horizon part can be expressed by the end-point penalty term $x_i^\top(T) P_+ x_i(T) = (x_i(T_-) + B_2^i)^\top P_+ (x_i(T_-) + B_2^i)$, which is added to the finite-horizon part. Hence, the optimal control law $u_i(t)$, $t \in [0,T)$, is the solution of the minimization of the functional

$$J_i := \int_0^T y_i^\top(t)\, y_i(t)\, dt + \Big(x_i(T_-) + B_2^i\Big)^\top P_+ \Big(x_i(T_-) + B_2^i\Big). \tag{13}$$

The minimization of this performance index can be achieved by exploiting Lemma 1. In fact, the necessary and sufficient conditions for the optimal solution consist of the Hamiltonian differential equation (5), the trasversality condition

$$u(t) = -R^{-1}\Big(S^\top x(t) + B^\top \lambda(t)\Big). \tag{14}$$

and the two boundary conditions

$$x_i(0) = 0 \qquad \text{and} \qquad \lambda_i(T) = P_+ (x_i(T_-) + B_2^i). \tag{15}$$

Since the set of solutions of the Hamiltonian differential equation in $[0,T)$ is parametrized by (10), in order to obtain the optimal solution minimizing J_i we have to impose the boundary conditions (15) on (10), so as to obtain the parameters p and q corresponding to a trajectory satisfying both the Hamiltonian differential equation and the boundary conditions:

$$x_i(0) = p + W e^{A_+^\top T} q = 0, \tag{16}$$

$$\begin{aligned}&\lambda_i(T) - P_+ (x_i(T_-) + B_2^i) \\ &\quad = P_+ e^{A_+ T} p + (P_+ W - I_n) q - P_+ (e^{A_+ T} p + W q + B_2^i) = 0\end{aligned} \tag{17}$$

where W is the solution of (9) with $B = B_1$. The values of the parameters thus obtained are:

$$p_i = W e^{A_+^\top T} P_+ B_2^i \quad \text{and} \quad q_i = -P_+ B_2^i, \tag{18}$$

By virtue of (14), the optimal $u_i(t)$, $t \in [0,T)$ is obtained by replacing (18) in

$$u_i(t) = -K_+ e^{A_+ t} p_i - (K_+ W - R^{-1} B_1^\top) e^{A_+^\top (T-t)} q_i. \tag{19}$$

On the other hand, as already observed, for $t \geq T$ the optimal control law is $u_i(t) = -K_+ e^{A_+(t-T)} x_i(T)$, where, by virtue of (12), (10) and (18), $x_i(T) = B_2^i + e^{A_+ T} p_i + W q_i$. It is easily checked that the control scheme in the statement, which does not depend on i, gives rise to this input function when fed by the input $w_i(t) = e_i \delta(t-T)$, so that it minimizes each J_i simultaneously. ■

The inner structure of the optimal compensator Σ_c involves a finite impulse response system, whose impulse response matrix is $\Phi(t)$ for $t \in [0,T)$ and zero elsewhere, and an algebraic state feedback, as shown in Figure 2. The transfer

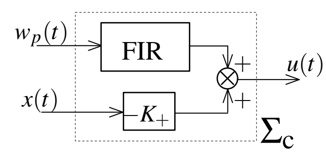

Fig. 2. Inner structure of the LTI compensator Σ_c.

function matrix $G_{FIR}(s)$ of the finite impulse response system is given by the $\mathscr{L}$-transform of $\Phi(t)$, that leads to

$$\begin{aligned}G_{FIR}(s) = R^{-1} B_1^\top e^{A_+^\top T} (A_+^\top + s I_n)^{-1} \\ \cdot \Big(e^{-(A_+^\top + s I_n) T} - I_n\Big) P_+ B_2.\end{aligned}$$

One may wonder how the input-output relation of this FIR system can be reproduced by using MATLAB®, or any other software tool oriented to control which handles LTI realizations. The structure of $G_{FIR}(s)$ suggests that the input-output behaviour of the FIR system can be simulated by means of the parallel of the two tranfer function matrices

$$\begin{aligned}G_1(s) &= -R^{-1} B_1^\top e^{A_+^\top T} (A_+^\top + s I_n)^{-1} P_+ B_2, \\ G_2(s) &= R^{-1} B_1^\top e^{A_+^\top T} (A_+^\top + s I_n)^{-1} e^{-(A_+^\top + s I_n) T} P_+ B_2,\end{aligned}$$

since $G_{FIR}(s) = G_1(s) + G_2(s)$. However, although the FIR system is itself a stable system, the transfer function matrices $G_1(s)$ and $G_2(s)$ corresponding respectively to the realization $(-A_+^\top, P_+ B_2, -R^{-1} B_1^\top e^{A_+^\top T}, 0)$ and $(-A_+^\top, e^{-A_+^\top T} P_+ B_2, R^{-1} B_1^\top e^{A_+^\top T}, 0)$ (the latter being adjusted by the introduction of the delay due to the exponential $e^{-s I_n T}$) are anti-stable. It follows that from a computational point of view the parallel of the two realizations herein presented for $G_1(s)$ and $G_2(s)$ provide an acceptable simulation of the input-output relation of $G_{FIR}(s)$ only for small preview intervals $[0,T]$. On the contrary, very good results can be achieved by a discretization of the integral

$$\int_0^T -R^{-1} B_1^\top e^{A_+^\top (T-\tau)} P_+ B_2\, w_p(t-\tau)\, d\tau$$

and by computing its value by means, for example, of the Simpson's rule.

Remark 1: If D_2 differs from zero, Problem 1 can still be solved, provided that the condition $\operatorname{im} D_2 \subseteq \operatorname{im} D_1$ is satisfied. In fact, if D_2 differs from zero, by taking (2) into account we easily see that the impulse response matrix $\widehat{\mathscr{G}}(t)$ involves a term $D_2\,\delta(t-T)$, such that the integral in (4) diverges unless its contribution is directly canceled by a part of the control input $u(t)$. In other words, the H_2 norm of the overall system is finite if and only if $\operatorname{im} D_2 \subseteq \operatorname{im} D_1$ and the control input has the following structure:

$$u(t) = -D_1^{+} D_2\, w(t) + u_c(t),$$

where the first part $-D_1^{+} D_2\, w(t)$ cancels the feedthrough term $D_2\, w(t)$ from the output since $\operatorname{im} D_2 \subseteq \operatorname{im} D_1$, while the term $u_c(t)$ is the control that follows from the procedure presented in Theorem 1. In fact, we have recast the problem as one of the type dealt with in Theorem 1, i.e. in which the feedthrough matrix from $w(t)$ to $y(t)$ is zero, where $u(t)$ is now replaced by $u_c(t)$ and $B_2\, w(t)$ is replaced by $(B_2 - B_1 D_1^{+} D_2)\, w(t)$.

V. Feedforward decoupling filters

In the case where Σ is stable, or in the case where the internal stability of the overall system is not required in the formulation of Problem 1 and the pair (A, C) is detectable (so that P_+ is the only positive semidefinite solution of the ARE), the optimal solution can be implemented via a pure feedforward action, as shown in Figure 3.
In this case, the inner structure of Σ_c involves a finite impulse

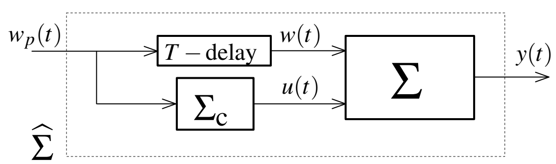

Fig. 3. Pure feedforward compensation scheme.

response term and a copy of the dynamics of Σ with the optimal infinite-horizon state feedback, see Figure 4. The state of system Σ_1 is described by the model

$$\dot{z}(t) \;=\; A_+ z(t) + B_1\, v(t) + B_2(t)\, w(t),$$

and its output equals $z(t)$. If the state of Σ is not accessible

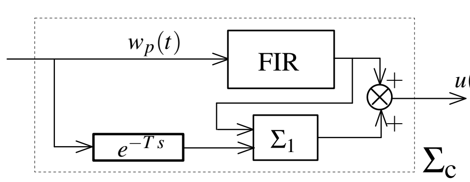

Fig. 4. Inner structure of a pure feedforward compensator.

for measurement, under the assumption of detectability of the pair (A, C) the solution presented in Theorem 1 can still be performed after a preliminary stabilization. In fact, an output-feedback unit Σ_s can be introduced in order to prestabilize Σ, see Figure 5. In this case, the optimal solution of Problem 1 referred to the stabilized system thus obtained does not change with respect to that referred to the original system. To

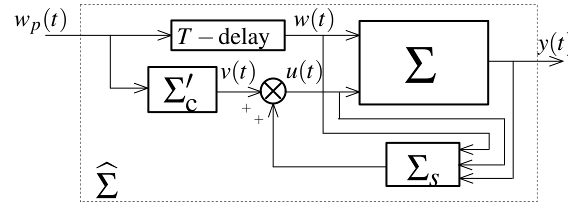

Fig. 5. H_2-optimal decoupling scheme with stabilization.

prove this fact, consider Figure 3. Let $G_1(s)$ and $G_2(s)$ denote the transfer function matrices between the inputs $u(t)$, $w(t)$ and the output $y(t)$, respectively, and by $G_c(s)$ and $D(s) := e^{-T I_t s}$ the transfer function matrices of Σ_c and of the T-delay, respectively. A straightforward computation shows that the transfer function matrix of the overall system $\widehat{G}_1(s)$ from the input $w_p(t)$ to the output $y(t)$ is

$$\widehat{G}_1(s) = G_1(s)\, G_c(s) + G_2(s)\, D(s).$$

Now, consider the scheme in Figure 5. Concerning the stabilizing unit Σ_s, let $S_1(s)$, $S_2(s)$ and $S_3(s)$ denote the transfer function matrices between the inputs $w(t)$, $u(t)$, $y(t)$, and the output of Σ_s, respectively, and denote by $G_c'(s)$ the transfer function matrix of Σ_c'. The transfer function matrix of the overall system $\widehat{G}_2(s)$ from $w_p(t)$ to $y(t)$ in the new scheme is

$$\widehat{G}_2(s) = G_2(s)\, D(s) + G_1(s)\,\Big(I_p - S_2(s) - S_3(s)\, G_1(s)\Big)^{-1} \cdot \Big(G_c'(s) + S_1(s)\, D(s) + S_3(s)\, G_2(s)\, D(s)\Big).$$

By comparing the expressions of $\widehat{G}_1(s)$ and $\widehat{G}_2(s)$, it is easily seen that a compensator Σ_c' can be found such that $\widehat{G}_1(s) = \widehat{G}_2(s)$. In fact, the following equality holds:

$$G_1(s)\, G_c(s) = G_1(s)\,\Big(I_p - S_2(s) - S_3(s)\, G_1(s)\Big)^{-1} \cdot \Big(G_c'(s) + S_1(s)\, D(s) + S_3(s)\, G_2(s)\, D(s)\Big),$$

that can be solved in $G_c'(s)$, since by virtue of assumption (A2) the transfer function matrix $G_1(s)$ is left-invertible, so that

$$G_c'(s) = \Big(I_p - S_2(s) - S_3(s)\, G_1(s)\Big)\, G_c(s) - \Big(S_1(s) - S_3(s)\, G_2(s)\Big)\, D(s).$$

As a result, the stabilizer Σ_s ensures the internal stability of the overall system, but the optimum does not change.

VI. H_2 FIXED-LAG SMOOTHING

The approach presented for the decoupling of previewed input functions provides a closed-form solution to another very well known H_2 optimization problem, the *fixed-lag smoothing*, which is dual to the H_2 preview decoupling. Consider the LTI system Σ ruled by

$$\dot{x}(t) = A\,x(t) + B\,u(t), \qquad (20)$$

$$y(t) = C_1\,x(t) + D_1\,u(t), \qquad (21)$$

$$z(t) = C_2\,x(t) + D_2\,u(t), \qquad (22)$$

where, for all $t \geq 0$, $y(t) \in \mathbb{R}^{p_1}$ is a measurable output that is exploited to reconstruct the output $z(t) \in \mathbb{R}^{p_2}$, with a latency $T > 0$. The purpose is that of finding an LTI system Σ_o

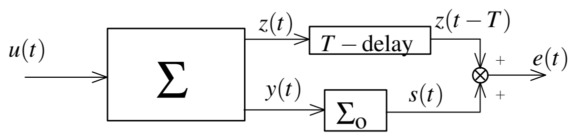

Fig. 6. Fixed-lag smoothing scheme.

connected as in Figure 6 such that the H_2 norm of the transfer function matrix $\widehat{G}(s)$ of the overall system $\widehat{\Sigma}$, from the input $u(t)$ to the output $e(t) := z(t-T) - s(t)$, is minimized. If (A,B) is stabilizable and (A,C_1) is detectable, D_1 is full-row rank, the quadruple (A,B,C_1,D_1) has no invariant zeros on $i\mathbb{R}$ and the geometric condition $\ker D_2 \supseteq \ker D_1$ holds, the structure of the LTI compensator Σ_c presented in Theorem 1 can be dualized in order to obtain the optimal LTI observer Σ_o. Clearly, the transfer function matrix $G_o(s)$ of Σ_o is the transpose of that of Σ_c, i.e., $G_o(s) = G_c^\top(s)$.

VII. AN EXAMPLE AND CONCLUDING REMARKS

In this section, we present a simple example of H_2 decoupling with preview, and we briefly discuss the results presented in this paper. Further details and examples will be presented in a forthcoming journal paper.

A. An illustrative example

Consider a system (1)-(2), where

$$A = \begin{bmatrix} 0.5 & 1 & -0.4 & 0 \\ 0.1 & 0.7 & 0 & -0.5 \\ 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0.6 \end{bmatrix}, \; B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \; B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \; D_1 = \begin{bmatrix} 1.2 & 0.3 \\ -0.1 & 0.8 \end{bmatrix}, \; D_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix},$$

subject to the scalar input $w(t)$ depicted in Figure 7, which is a clock-type function with values in $\{0,2\}$ and duty cycle 0.5. By following the design procedure outlined in Section II, aimed at deriving a compensator Σ_c for the rejection of the previewed signal $w(t)$ from the output $y(t)$ of Σ, in Figure 8 we compare the different output functions that are obtained by varying the value of the preview time T.

In fact, in this case assumptions (A1)-(A3) hold true. The optimum compensator Σ_c can be designed as shown

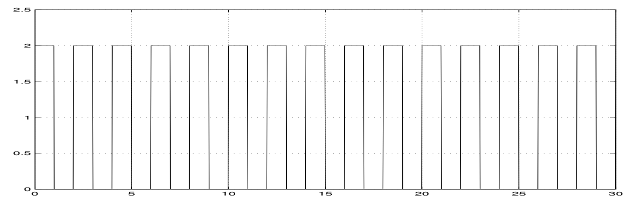

Fig. 7. Input function $w(t)$.

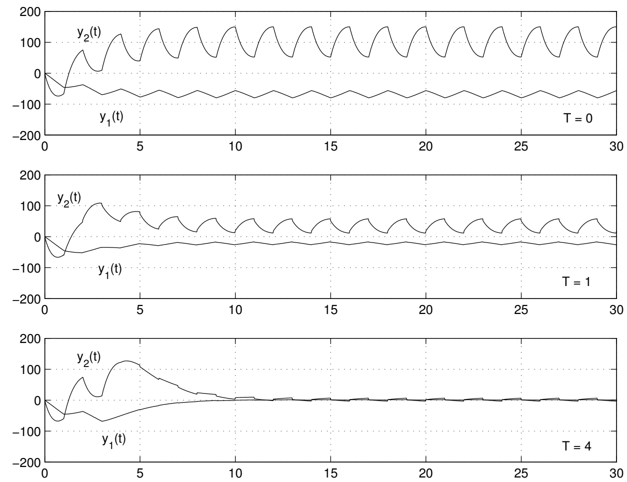

Fig. 8. Output functions obtained with the preview time $T = 0$, $T = 1$ and $T = 4$, respectively.

in Theorem 1, working jointly with the algebraic unit described in Remark 1, accounting for the feedforward action $-D_1^+ D_2\,w(t)$. The first subplot in Figure 8 shows the rejection that can be achieved when the preview time T is zero, i.e., when the signal $w(t)$ to be decoupled is accessible for measurement but not previewed. The second and the third subplots present the output $y(t)$ for increasing values of T, i.e., for $T = 1$ and $T = 4$. As we could expect, the rejection achieved considerably improves as the preview interval increases. Note also that in this interval there is an evident transient, which is due to the input function $w(t)$ not being zero in $[0,T]$ in the case considered. Hence, during this transient, the compensator has no previewed information on the actual value of $w(t)$, and no rejection is possible except for that determined by the feedforward term $-D_1^+ D_2\,w(t)$. Even if all the outputs in Figure 8 seem to be equal in $[0,T]$, this is not true: during this time interval the compensator takes into account the future information on $w(t)$, so as to minimize the effect of $w(t)$ on $y(t)$ when $t > T$.

B. Conclusions

In this paper the H_2 optimal decoupling problem of previewed signals/references has been considered. The method proposed for its solution is very simple from both a theoretical and computational viewpoint. In fact, it does not involve heavy variational tools or dynamic programming techniques, but consists of a direct application of a general result of LQ theory to a slightly generalized quadratic performance index.

The structure of the decoupling filter involves a feedforward unit accounting for the pre-knowledge of the input to be rejected, and a static state-feedback. It has been shown how the present solution can be easily dualized to solve the H_2 optimal fixed-lag smoothing problem.

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