Global existence and temporal decay for the 3D compressible Hall-magnetohydrodynamic system *

Fuyi Xu[a†,d] Xinguang Zhang[b,c] Yonghong Wu[c] Lishan Liu[c,d]

[a] School of Science, Shandong University of Technology, Zibo 255049, Shandong, China

[b] School of Mathematics, Yantai University, Yantai 264005, Shandong, China

[c] Department of Mathematics and Statistics, Curtin University, Perth 6845, WA, Australia

[d] Department of Mathematics Science , Qufu normal University, Qufu 263516, Shandong, China

Abstract In this paper, we are concerned with the 3D compressible Hall-magnetohydrodynamic system in the whole space. We prove the global existence and temporal decay rates of the solutions to the system when the initial data are close to a stable equilibrium state by using a pure energy method.

Key words: energy estimates; compressible Hall-magnetohydrodynamic system; temporal rates.

1 Introduction and main results

In this paper, we consider the following 3D compressible Hall-magnetohydrodynamic equations [1]:

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mu\Delta u - (\lambda+\mu)\nabla\mathrm{div}u + \nabla p = (\nabla\times B)\times B, \\ \partial_t B - \nabla\times(u\times B) + \nu\nabla\times(\nabla\times B) + \nabla\times\Big(\dfrac{(\nabla\times B)\times B}{\rho}\Big) = 0, \\ \mathrm{div}B = 0, \end{cases} \tag{1.1}$$

where $\rho(t,x)$, $u(t,x)$, $B(t,x)$ denote, respectively, the density, velocity, and magnetic field. $p = p(\rho)$ is pressure satisfying $p'(\rho) > 0$ and for all $\rho > 0$. The Lamé coefficients μ and λ

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[1†]Corresponding author.

[2]E-mail addresses: zbxufuyi@163.com (F.Xu).

satisfy the physical conditions

$$\mu > 0, \quad 2\mu + 3\lambda > 0, \tag{1.2}$$

which ensures that the operator $-\mu\Delta - (\lambda + \mu)\nabla\mathrm{div}$ is a strongly elliptic operator and $\nu > 0$ is the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field. In this paper, we are concerned with the Cauchy problem of the system (1.1) in $\mathbb{R}^+ \times \mathbb{R}^3$ subject to the initial data

$$(\rho, u, B)|_{t=0} = (\rho_0, u_0, B_0). \tag{1.3}$$

In many current physics problems, Hall-MHD is required. The first systematic study of Hall-MHD is due to Lighthill [25] followed by Campos [3]. The Hall-MHD is indeed needed for such problems as magnetic reconnection in space plasmas [19, 22], star formation [2], and neutron stars [27]. A physical review on these questions can be found in [26]. Mathematical derivations of Hall-MHD equations from either two-fluids or kinetic models can be found in [1] and in this paper, the first existence result of global weak solutions is given. Comparing with the usual MHD equations, the Hall-MHD equations have the Hall term $\nabla \times \left(\frac{(\nabla\times B)\times B}{\rho}\right)$, which plays an important role in magnetic reconnection. When $\rho = const$, system (1.1) becomes the incompressible Hall-MHD system, which has received many studies, see [1, 4, 5, 6, 13, 14, 17, 36, 37, 38]. When $\mathrm{div}u = 0$, system (1.1) becomes the density-dependent Hall-MHD system, which has been investigated by many authors, and for more details, see [12, 18].

When the Hall effect term $\nabla \times \left(\frac{(\nabla\times B)\times B}{\rho}\right)$ is neglected, system (1.1) reduces to the well-known MHD system. The blow-up criterion, issues of well-posedness and dynamical behaviors of the solution to the MHD system are rather complicated to investigate because of the strong coupling and interplay interaction between the fluid motion and the magnetic field. In spite of these, important progress has been achieved in recent years on the mathematical analysis of these topics for the MHD system. For incompressible MHD equations, many problems have been investigated including the blow-up criterion, the uniqueness of weak solutions and the well-posedness of the smooth solutions, and for more details, see [7, 8, 9, 10, 15, 16, 21, 39, 40] and the references therein. On the other hand, there are also many results regarding the global existence of the solutions and the decay of the smooth solutions to the compressible MHD equations, see [23, 24, 28, 35]. It is well known that the study for optimal decay rates of the solutions to the fluid dynamics equations is interesting in mathematical analysis. Indeed, the decay rates of the solutions are very important topic in the study of the fluid dynamics equations for the purpose of scientific computation. There are many fruitful results on the optimal decay rates of the solutions close to a constant state for all sorts of fluid dynamics equations, and for details see for example [11, 28, 29, 30, 32]. On the study of decay rates for compressible MHD equations, Umeda, Kawashima and Shizuta [28] studied the global existence and time decay rates of smooth solutions to the linearized 3D compressible MHD equations. Recently, the global existence and optimal decay estimates of smooth solutions to the 3D compressible MHD system were obtained in [24] when the initial data are close to an equilibrium state and belong to $H^3(\mathbb{R}^3_x) \cap L^1(\mathbb{R}^3_x)$. Tan and Wang [35] obtained the optimal decay rates of the 3D compressible MHD system by pure energy method. However, to the best of our knowledge, very few results have been established on the dynamics of the global solutions to the 3D compressible Hall-MHD system, especially on the temporal decay of the solutions. Very recently, Fan et al. [13] first obtained the global existence and the optimal decay rates for the 3D compressible Hall-MHD equations (1.1) where the initial data are close to an equilibrium state and belong to $H^3(\mathbb{R}^3_x) \cap L^1(\mathbb{R}^3_x)$. In these known results mentioned above (See, [13, 24, 28]), $L^1(\mathbb{R}^3_x)$ integrability plays an important role in the proof of the optimal decay rates based on the spectral analysis of the semigroup. In general, for evolution equations in which $L^2(\mathbb{R}^3_x)$ based norms can be propagated by the solution, it is common to make a bounded assumption on the $L^1(\mathbb{R}^3_x)$ norm of the initial data and combine this with $L^2(\mathbb{R}^3_x)$ type estimates in order to obtain large time decay estimates. Unfortunately, it is often the case that propagating bounds on $L^1(\mathbb{R}^3_x)$ norms is difficult along the time evolution. This can cause severe difficulties in applications because one could improve existing theories by showing that an $L^1(\mathbb{R}^3_x)$ type norm is small or bounded after a finite but large time $T > 0$, and then applying the aforementioned decay theory. A nature question is what may happen about the temporal decay rates of the global solutions to the 3D compressible Hall-MHD equations (1.1) if the initial data belong to an $L^2(\mathbb{R}^3_x)$ based spaces. The goal of this paper is to give a answer to the questions mentioned above. Our main ideas are based on a pure energy method recently developed by Guo and Wang. Compared with the known results of the optimal decay rates for the compressible Navier-Stokes equations [20], the main difficulties are much more complicate nonlinear terms and the Hall effect term in the system (1.1).

Now we state our main results as follows:

Theorem 1.1 *Assume that* $(\rho_0 - 1, u_0, B_0) \in H^N$ *for an integer* $N \geq 3$. *Then there exists a constant* $\delta_0 > 0$ *such that if*

$$\|(\rho_0 - 1, u_0, B_0)\|_{H^3} \leq \delta_0, \tag{1.4}$$

then the Cauchy problem (1.1)-(1.3) *admits a unique global solution* $(\rho(t), u(t), B(t))$ *satisfy-*

ing that for all $t \geq 0$,

$$\begin{aligned}&\|\rho(t)-1\|_{H^N}^2++\|(u,B)(t)\|_{H^N}^2+\int_0^t \|\nabla\rho(\tau)\|_{H^{N-1}}^2+\|\nabla(u,B)(\tau)\|_{H^N}^2\,d\tau\\&\leq C\left(\|\rho_0-1\|_{H^N}^2+\|(u_0,B_0)\|_{H^N}^2\right).\end{aligned}\tag{1.5}$$

Theorem 1.2 *Under all the assumptions of Theorem 1.1, moreover, if $(\rho_0-1,u_0,B_0)\in \dot{H}^{-s}$ for some $s\in[0,3/2)$, then for all $t\geq 0$,*

$$\|\rho(t)-1\|_{\dot{H}^{-s}}^2+\|(u,B)(t)\|_{\dot{H}^{-s}}^2\leq C_0,\tag{1.6}$$

and

$$\|\nabla^\ell\rho(t)\|_{H^{N-\ell}}+\|\nabla^\ell(u,B)(t)\|_{H^{N-\ell}}\leq CC_0^{\frac{1}{2}}(1+t)^{-\frac{(\ell+s)}{2}}\quad \text{for } -s<\ell\leq N-1.\tag{1.7}$$

Remark 1.3 *Note that $L^p\hookrightarrow \dot{H}^{-s}$ with $s=3(\frac{1}{p}-\frac{1}{2})\in[0,\frac{3}{2})$. Then by Theorem 1.1, we have the following L^p-L^2 type of the temporal decay rates. Under the assumptions of the Theorem 1.1, if we replace $\dot{H}^{-s}$ assumption by $\rho_0-1,u_0,B_0\in L^p$ for some $p\in(1,2]$, then the following decay result holds*

$$\|\nabla^\ell\rho(t)\|_{H^{N-\ell}}+\|\nabla^\ell(u,B)(t)\|_{H^{N-\ell}}\leq CC_0^{\frac{1}{2}}(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)+\frac{\ell}{2}}\quad \text{for } -s<\ell\leq N-1.\tag{1.8}$$

Remark 1.4 *Notice that we only assume that the H^3 norm of initial data is small, which the higher order Sobolev norm can be arbitrarily large.*

Remark 1.5 *Compared with the known result [13, 24], our results only need the conditions that the initial data are close to a constant equilibrium state in H^3 framework and remove $L^1(\mathbb{R}_x^3)$ integrability. Furthermore, we also obtain temporal decay rates for the higher-order spatial derivatives of the solutions.*

Remark 1.6 *Compared with the known result [24, 35], our result need deal with much more complicated Hall effect term $\nabla\times\left(\frac{(\nabla\times B)\times B}{\rho}\right)$.*

Notaions. We denote by L^p, $W^{m,p}$ the usual Lebesgue and Sobolev spaces on $\mathbb{R}^3$ and $H^m=W^{m,2}$, with norms $\|\cdot\|_{L^p}$, $\|\cdot\|_{W^{m,p}}$ and $\|\cdot\|_{H^m}$ respectively. For the sake of conciseness, we do not distinguish functional space when scalar-valued or vector-valued functions are involved. We denote $\nabla=\partial_x=(\partial_1,\partial_2,\partial_3)$, where $\partial_i=\partial_{x_i}$, $\nabla_i=\partial_i$ and put $\partial_x^l f=\nabla^l f=\nabla(\nabla^{l-1}f)$. We assume C be a positive generic constant throughout this paper that may vary at different places.

2 Preliminaries and Lemmas

Before we present the energy estimates method, we recall the following useful Lemmas which we will use extensively in this paper.

First, we will review the Sobolev interpolation of the Gagliardo-Nirenberg inequality.

Lemma 2.1 *[33] Let* $0 \leq m, \alpha \leq \ell, p > 1$*, then we have*

$$\|\nabla^\alpha f\|_{L^p} \lesssim \|\nabla^m f\|_{L^2}^{1-\theta} \|\nabla^\ell f\|_{L^2}^{\theta} \tag{2.1}$$

where $0 \leq \theta \leq 1$ *and* α *satisfy*

$$\frac{1}{p} - \frac{\alpha}{3} = \left(\frac{1}{2} - \frac{m}{3}\right)(1-\theta) + \left(\frac{1}{2} - \frac{\ell}{3}\right)\theta. \tag{2.2}$$

Lemma 2.2 *[35] Assume that* $\|\rho\|_{L^p} \leq 1$ *and* $p > 1$*. Let* $g(\rho)$ *be a smooth function of* ρ *with bounded derivatives, then we have for any integer* $m \geq 1$

$$\|\nabla^m (g(\rho))\|_{L^p} \leq C\|\nabla^m \rho\|_{L^p}.$$

Lemma 2.3 *[31] Let* $m \geq 1$ *be an integer and define the commutator*

$$[\nabla^m, f]g = \nabla^m (fg) - f\nabla^m g.$$

Then we have

$$\|[\nabla^m, f]g\|_{L^p} \leq C\big(\|\nabla f\|_{L^{p_1}} \|\nabla^{m-1} g\|_{L^{p_2}} + \|\nabla^m f\|_{L^{p_3}} \|g\|_{L^{p_4}}\big),$$

where $p, p_2, p_3 \in (1, +\infty)$ *and*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

In order to establish the negative Sobolev estimates, we should review the following useful Lemmas related to the negative Sobolev norms. Here, we first introduce some necessary definitions.

Definition 2.4 *The operator* $\Lambda^s, s \in \mathbb{R}$ *by*

$$\Lambda^s f(x) = \int_{\mathbb{R}^3} |\xi|^s \hat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi, \tag{2.3}$$

where $\hat{f}$ *is the Fourier transform of* f*.*

Definition 2.5 *The homogeneous Sobolev space* $\dot{H}^s$ *of all* f *for which* $\|f\|_{\dot{H}^s}$ *is finite, where*

$$\|f\|_{\dot{H}^s} := \|\Lambda^s f\|_{L^2} = \||\xi|^s \hat{f}\|_{L^2}.$$

We will use the non-positive index s. For convenience, we will change the index to be "$-s$" with $s \geq 0$. We will employ the following special Sobolev interpolation:

Lemma 2.6 *[20] Let* $s \geq 0$ *and* $\ell \geq 0$*, then we have*

$$\|\nabla^\ell f\|_{L^2} \leq \|\nabla^{\ell+1} f\|_{L^2}^{1-\theta} \|\Lambda^{-s} f\|_{L^2}^{\theta}, \ \textit{where } \theta = \frac{1}{\ell+1+s}. \tag{2.4}$$

Lemma 2.7 *[34] Let* $0 < s < 3,\ 1 < p < q < \infty,\ 1/q + s/3 = 1/p$*, then we have*

$$\|\Lambda^{-s} f\|_{L^q} \leq C\|f\|_{L^p}. \tag{2.5}$$

3 Reformulation of the Original System (1.1)

In this section, we first reformulate the original system (1.1) into a different form. For the magnetic field B, we have the following identities:

$$\nabla(|B|^2) = 2(B\cdot\nabla)B + 2(\nabla\times B)\times B,$$

$$\nabla\times(\nabla\times B) = \nabla \mathrm{div} B - \Delta B,$$

and

$$\nabla\times(u\times B) = u(\mathrm{div}B) - B(\mathrm{div}u) + B\cdot\nabla u - u\cdot\nabla B = -B(\mathrm{div}u) + B\cdot\nabla u - u\cdot\nabla B,$$

for $\mathrm{div}B = 0$.

Without loss of generality, we will assume that $\bar{\rho} = 1$, and denote that $c = \rho - 1$. Then, in term of the new variables (c, u, B), system (1.1)-(1.3) becomes

$$\begin{cases} \partial_t c + \mathrm{div}u = f, \\ \partial_t u - \mathcal{A}u + \nabla c = g, \\ \partial_t B - \nu\Delta B = h, \\ \mathrm{div}B = 0, \\ (c,\, u,\, B)|_{t=0} = \big(c_0,\, u_0,\, B_0\big), \end{cases} \tag{3.1}$$

where

$$\begin{aligned} &f = -c\mathrm{div}u - u\cdot\nabla c, \\ &g = -u\cdot\nabla u - L_1(c)\mathcal{A}u + L_2(c)\nabla c - L_3(c)\Big(\frac{1}{2}\nabla|B|^2 - B\cdot\nabla B\Big), \\ &h = -B(\mathrm{div}u) + B\cdot\nabla u - u\cdot\nabla B - \nabla\times\Big(L_3(c)(\nabla\times B)\times B\Big), \end{aligned}$$

in which

$$\mathcal{A}=\mu\Delta+(\lambda+\mu)\nabla\mathrm{div},\quad L_1(c)=\frac{c}{1+c},\quad L_2(c)=\frac{P'(1+c)}{1+c}-1,\quad L_3(c)=\frac{1}{c+1}.$$

4 Energy estimates

As a classical argument, the global existence of solutions will be obtained by combining the local existence result with a priori estimates. Since the local strong solutions can be proven by [13], global solutions will follow in a standard continuity argument after we establish a priori estimate (1.5). We assume that that

$$\|(\rho-1,u,B)\|_{H^3}\le\delta_0\ll 1, \tag{4.1}$$

which is equivalent to

$$\|(c,u,B)\|_{H^3}\le\delta\ll 1. \tag{4.2}$$

Here $\delta_0\sim\delta$ is small enough. This, together with Sobolev's inequalities, implies in particular that

$$\|(c,u,B)\|_{L^\infty}\le C\delta. \tag{4.3}$$

Furthermore, we have

$$|L_1(c)|,|L_2(c)|\le C|c|,\quad |L_1'(c)|,|L_2'(c)|,|L_3(c)|,|L_3'(c)|\le C. \tag{4.4}$$

Lemma 4.1 *Under the priori assumption* (4.2)*, then for* $k=0,1,\cdots,N-1$*, we have*

$$\frac{1}{2}\frac{d}{dt}\|\nabla^k(c,u,B)\|_{L^2}^2+C\|\nabla^{k+1}(u,B)\|_{L^2}^2\le C\delta\|\nabla^{k+1}(c,u,B)\|_{L^2}^2. \tag{4.5}$$

Proof. Applying ∇^k to the first three equations of (3.1) and multiplying them by $\nabla^k c$,$\nabla^k u$, $\nabla^k B$ respectively, and then integrating them over $\mathbb{R}^3$, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla^k(c,u,B)\|_{L^2}^2+\mu\|\nabla^{k+1}u\|_{L^2}^2+(\mu+\lambda)\|\nabla^k\mathrm{div}u\|_{L^2}^2+\nu\|\nabla^{k+1}B\|_{L^2}^2\\&=\langle\nabla^k f,\nabla^k c\rangle+\langle\nabla^k g,\nabla^k u\rangle+\langle\nabla^k h,\nabla^k B\rangle\\&\equiv I_1+I_2+I_3,\end{aligned} \tag{4.6}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner-product in $L^2(\mathbb{R}^3)$. We first bound the second term and the third term in left-hand side of (4.6) as follows

$$\mu\|\nabla^{k+1}u\|_{L^2}^2+(\mu+\lambda)\|\nabla^k\mathrm{div}u\|_{L^2}^2\ge C\|\nabla^{k+1}u\|_{L^2}^2, \tag{4.7}$$

since the constraint (1.2).

Then, we shall estimate each term in the right-hand side of (4.6) term by term. The key point is that we will carefully interpolate the spatial derivatives between the higher-order derivatives and the lower-order ones to bound these nonlinear terms in the right-hand side of (4.6).

Firstly, we should consider one special situation when $k = 0$. By Hölder's inequality, Young's inequality together with Sobolev's inequality, we obtain

$$\begin{aligned} I_1 &\le \|c\|_{L^3}\|u\|_{L^6}\|\nabla c\|_{L^2} + \|c\|_{L^6}\|\nabla u\|_{L^2}\|c\|_{L^3} \\ &\le C\delta\big(\|\nabla c\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\big). \end{aligned} \tag{4.8}$$

Similarly, for I_2, we have

$$\begin{aligned} I_2 &\le \|u\|_{L^3}\|u\|_{L^6}\|\nabla u\|_{L^2} + |\langle L_1(c)\mathcal{A}u, u\rangle| + |\langle L_2(c)\nabla c, u\rangle| + |\langle L_3(c)\big(B\cdot\nabla B\big), u\rangle| \\ &\le C\delta\|\nabla u\|_{L^2}^2 + \Big|\langle\nabla(\frac{c}{1+c})\nabla u, u\rangle\Big| + \Big|\langle\frac{c}{1+c}\nabla u, \nabla u\rangle\Big| + \Big|\langle(\frac{P'(c+1)}{(1+c)} - 1)\nabla c, u\rangle\Big| \\ &\quad + \Big|\langle\frac{1}{1+c}\big(B\cdot\nabla B\big), u\rangle\Big| \\ &\le C\delta\|\nabla u\|_{L^2}^2 + C\|\nabla c\|_{L^3}\|\nabla u\|_{L^2}\|u\|_{L^6} + C\|c\|_{L^\infty}\|\nabla u\|_{L^2}^2 + C\|c\|_{L^6}\|u\|_{L^3}\|\nabla c\|_{L^2} \\ &\quad + C\|B\|_{L^3}\|\nabla B\|_{L^2}\|u\|_{L^6} \\ &\le C\delta\big(\|\nabla c\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla B\|_{L^2}^2\big). \end{aligned} \tag{4.9}$$

For I_3, integrating by parts, and then employing Hölder's inequality, Young's inequality together with Sobolev's inequality, we get

$$\begin{aligned} I_3 &\le \Big|\Big\langle\Big(-u\cdot\nabla B - \mathrm{div}uB - \nabla\times\big(L_3(c)(\nabla\times B)\times B\big)\Big), B\Big\rangle\Big| \\ &\le \|u\|_{L^6}\|\nabla B\|_{L^2}\|B\|_{L^3} + \|B\|_{L^6}\|\nabla u\|_{L^2}\|B\|_{L^3} + \Big|\Big\langle\big(L_3(c)(\nabla\times B)\times B\big), \nabla\times B\Big\rangle\Big| \\ &\le C\Big(\|u\|_{L^6}\|\nabla B\|_{L^2}\|B\|_{L^3} + \|B\|_{L^6}\|\nabla u\|_{L^2}\|B\|_{L^3} + \|\nabla B\|_{L^2}\|B\|_{L^\infty}\|\nabla B\|_{L^2}\Big) \\ &\le C\delta\big(\|\nabla u\|_{L^2}^2 + \|\nabla B\|_{L^2}^2\big). \end{aligned} \tag{4.10}$$

Hence, putting (4.8)-(4.10) into (4.6), which immediately yields (4.5) for $k = 0$.

When $k \ge 1$, we estimate these nonlinear terms by the right-hand side of (4.6) as follows. First, we bound the term I_1 by Hölder's inequality and Leibniz's formula,

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^3} -\nabla^k(c\mathrm{div}u)\nabla^k c dx - \int_{\mathbb{R}^3} \nabla^k(u\cdot\nabla c)\nabla^k n dx \\ &= \int_{\mathbb{R}^3} -\nabla^k \mathrm{div}(cu)\nabla^k c dx = \int_{\mathbb{R}^3} \nabla^{k-1}\mathrm{div}(cu)\nabla^{k+1} c dx \\ &\le \sum_{0\le l\le k} C_k^l\|\nabla^l c\nabla^{k-l}u\|_{L^2}\|\nabla^{k+1}c\|_{L^2} \\ &\le C\sum_{0\le l\le k}\|\nabla^l c\nabla^{k-l}u\|_{L^2}\|\nabla^k c\|_{L^2}. \end{aligned} \tag{4.11}$$

If $0 \le l \le [\frac{k}{2}]$, using Hölder's inequality, Young's inequality and Lemma 2.1, we obtain

$$\begin{aligned} I_1 &\le C \sum_{0 \le l \le k} \|\nabla^l c\|_{L^3} \|\nabla^{k-l} u\|_{L^6} \|\nabla^{k+1} c\|_{L^2} \\ &\le C \|\nabla^\alpha c\|_{L^2}^{1-\frac{l}{k}} \|\nabla^{k+1} c\|_{L^2}^{\frac{l}{k}} \|\nabla u\|_{L^2}^{\frac{l}{k}} \|\nabla^{k+1} u\|_{L^2}^{1-\frac{l}{k}} \|\nabla^{k+1} c\|_{L^2} \\ &\le C\delta\big(\|\nabla^{k+1} c\|_{L^2}^2 + \|\nabla^{k+1} u\|_{L^2}^2\big), \end{aligned} \tag{4.12}$$

where α is defined by

$$\begin{aligned} &\frac{l}{3} - \frac{1}{3} = \Big(\frac{\alpha}{3} - \frac{1}{2}\Big)\Big(1 - \frac{l}{k}\Big) + \Big(\frac{k+1}{3} - \frac{1}{2}\Big)\frac{l}{k} \\ &\Longrightarrow \alpha = 1 - \frac{k}{2(k-l)} \in \Big[0, \frac{1}{2}\Big], \quad \text{since} \quad 0 \le l \le \Big[\frac{k}{2}\Big]. \end{aligned}$$

If $[\frac{k}{2}] + 1 \le l \le k$, by Hölder's inequality, Young's inequality and Lemma 2.1, we conclude that

$$\begin{aligned} I_1 &\le C \sum_{0 \le l \le k} \|\nabla^l c\|_{L^6} \|\nabla^{k-l} u\|_{L^3} \|\nabla^{k+1} c\|_{L^2} \\ &\le C \|c\|_{L^2}^{1-\frac{l+1}{k+1}} \|\nabla^{k+1} c\|_{L^2}^{\frac{l+1}{k+1}} \|\nabla^\alpha u\|_{L^2}^{\frac{l+1}{k+1}} \|\nabla^{k+1} u\|_{L^2}^{1-\frac{l+1}{k+1}} \|\nabla^{k+1} c\|_{L^2} \\ &\le C\delta\big(\|\nabla^{k+1} c\|_{L^2}^2 + \|\nabla^{k+1} u\|_{L^2}^2\big), \end{aligned} \tag{4.13}$$

where α is defined by

$$\begin{aligned} &\frac{k-l}{3} - \frac{1}{3} = \Big(\frac{\alpha}{3} - \frac{1}{2}\Big)\frac{l+1}{k+1} + \Big(\frac{k+1}{3} - \frac{1}{2}\Big)\Big(1 - \frac{l+1}{k+1}\Big) \\ &\Longrightarrow \alpha = \frac{k+1}{2(l+1)} \in \Big[\frac{1}{2}, 1\Big], \quad \text{since} \quad \Big[\frac{k}{2}\Big] + 1 \le l \le k. \end{aligned}$$

Combining (4.12) with (4.13), we obtain

$$I_1 \le C\delta\big(\|\nabla^{k+1} c\|_{L^2}^2 + \|\nabla^{k+1} u\|_{L^2}^2\big). \tag{4.14}$$

Next, we bound the terms I_2 and I_3. By integrating by parts and Hölder's inequality, we have

$$\begin{aligned} &\int_{\mathbb{R}^3} -\nabla^k (u \cdot \nabla B) \nabla^k B dx = \int_{\mathbb{R}^3} \nabla^{k-1} (u \cdot \nabla B) \nabla^{k+1} B dx \\ &\le \|\nabla^{k-1}(u \cdot \nabla B)\|_{L^2} \|\nabla^{k+1} B\|_{L^2} \le \sum_{0 \le l \le k-1} C_{k-1}^l \|\nabla^l u \nabla^{k-l} B\|_{L^2} \|\nabla^{k+1} B\|_{L^2} \\ &\le C \sum_{0 \le l \le k-1} \|\nabla^l u \nabla^{k-l} B\|_{L^2} \|\nabla^{k+1} B\|_{L^2}. \end{aligned}$$

If $0 \le l \le [\frac{k-1}{2}]$, by Hölder's inequality and Lemma 2.1, we get

$$\begin{aligned} \|\nabla^l u \nabla^{k-l} B\|_{L^2} \|\nabla^{k+1} B\|_{L^2} &\le \|\nabla^l u\|_{L^3} \|\nabla^{k-l} B\|_{L^6} \|\nabla^{k+1} B\|_{L^2} \\ &\le \|\nabla^\alpha u\|_{L^2}^{1-\frac{l}{k}} \|\nabla^{k+1} u\|_{L^2}^{\frac{l}{k}} \|\nabla B\|_{L^2}^{\frac{l}{k}} \|\nabla^{k+1} B\|_{L^2}^{1-\frac{l}{k}} \|\nabla^{k+1} B\|_{L^2} \\ &\le C\delta\big(\|\nabla^{k+1} u\|_{L^2}^2 + \|\nabla^{k+1} B\|_{L^2}^2\big), \end{aligned}$$

where α is defined by

$$\begin{aligned}
&\frac{l}{3}-\frac{1}{3}=\Big(\frac{\alpha}{3}-\frac{1}{2}\Big)\Big(1-\frac{l}{k}\Big)+\Big(\frac{k+1}{3}-\frac{1}{2}\Big)\frac{l}{k}\\
&\Longrightarrow \alpha=1-\frac{k}{2(k-l)}\in\Big(0,\frac{1}{2}\Big],\quad \text{since}\quad 0\le l\le\Big[\frac{k-1}{2}\Big].
\end{aligned}$$

If $[\frac{k-1}{2}]+1\le l\le k-1$, by Hölder's inequality and Lemma 2.1, we have

$$\begin{aligned}
\|\nabla^l u\nabla^{k-l}B\|_{L^2}\|\nabla^{k+1}B\|_{L^2}&\le \|\nabla^l u\|_{L^6}\|\nabla^{k-l}B\|_{L^3}\|\nabla^{k+1}B\|_{L^2}\\
&\le \|u\|_{L^2}^{1-\frac{l+1}{k+1}}\|\nabla^{k+1}u\|_{L^2}^{\frac{l}{k}}\|\nabla^{\alpha}B\|_{L^2}^{\frac{l}{k}}\|\nabla^{k+1}B\|_{L^2}^{1-\frac{l}{k}}\|\nabla^{k+1}B\|_{L^2}\\
&\le C\delta\big(\|\nabla^{k+1}u\|_{L^2}^2+\|\nabla^{k+1}B\|_{L^2}^2\big),
\end{aligned}$$

where α is defined by

$$\begin{aligned}
&\frac{k-l}{3}-\frac{1}{3}=\Big(\frac{\alpha}{3}-\frac{1}{2}\Big)\frac{l+1}{k+1}+\Big(\frac{k+1}{3}-\frac{1}{2}\Big)\Big(1-\frac{l+1}{k+1}\Big)\\
&\Longrightarrow \alpha=\frac{k+1}{2(l+1)}\in\Big(\frac{1}{2},1\Big],\quad \text{since}\quad \Big[\frac{k-1}{2}\Big]+1\le l\le k-1.
\end{aligned}$$

Combining the above estimates, we conclude that

$$\int_{\mathbb{R}^3}-\nabla^k(u\cdot\nabla B)\nabla^k B dx\le C\delta\big(\|\nabla^{k+1}u\|_{L^2}^2+\|\nabla^{k+1}B\|_{L^2}^2\big). \tag{4.15}$$

Similarly, we have

$$\int_{\mathbb{R}^3}-\nabla^k(u\cdot\nabla u)\nabla^k u dx\le C\delta\|\nabla^{k+1}u\|_{L^2}^2, \tag{4.16}$$

$$\int_{\mathbb{R}^3}-\nabla^k(\mathrm{div} u B)\nabla^k B dx\le C\delta\big(\|\nabla^{k+1}u\|_{L^2}^2+\|\nabla^{k+1}B\|_{L^2}^2\big). \tag{4.17}$$

By doing the approximation to simplify the presentations, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3}-\nabla^k\big(L_1(c)(\mu\Delta u+(\lambda+\mu)\nabla\mathrm{div}u)\big)\nabla^k u dx\approx\int_{\mathbb{R}^3}\nabla^{k-1}(L_1(c)\nabla^2 u)\nabla^{k+1}u dx\\
&\le \|\nabla^{k-1}(L_1(c)\nabla^2 u)\|_{L^2}\|\nabla^{k+1}u\|_{L^2}\le \sum_{0\le l\le k-1}C_{k-1}^l\|\nabla^l L_1(c)\nabla^{k-l+1}u\|_{L^2}\|\nabla^{k+1}u\|_{L^2}\\
&\le C\sum_{0\le l\le k-1}\|\nabla^l L_1(c)\nabla^{k-l+1}u\|_{L^2}\|\nabla^{k+1}u\|_{L^2}.
\end{aligned}$$

In order to bound the above term, we shall discuss it in the following cases:

i) For $l=0$, since $|L_1(c)|\le C|c|$, by Höder's inequality and Sobolev's inequality, we have

$$\begin{aligned}
\|L_1(c)\nabla^{k+1}u\|_{L^2}\|\nabla^{k+1}u\|_{L^2}&\le\|L_1(c)\|_{L^\infty}\|\nabla^{k+1}u\|_{L^2}\|\nabla^{k+1}u\|_{L^2}\\
&\le C\|c\|_{L^\infty}\|\nabla^{k+1}u\|_{L^2}\|\nabla^{k+1}u\|_{L^2}\\
&\le C\delta\|\nabla^{k+1}u\|_{L^2}^2.
\end{aligned}$$

ii) For $l=1$, since $|L_1^{'}(c)|\leq C$, by Höder's inequality and Sobolev's inequality, we obtain

$$\begin{aligned}\|\nabla L_1(c)\nabla^{k-l+1}u\|_{L^2}\|\nabla^{k+1}u\|_{L^2}&\leq \|L_1^{'}(c)\nabla c\nabla^k u\|_{L^2}\|\nabla^{k+1}u\|_{L^2}\\ &\leq \|L_1^{'}(c)\nabla c\|_{L^3}\|\nabla^k u\|_{L^6}\|\nabla^{k+1}u\|_{L^2}\\ &\leq \|L_1^{'}(c)\|_{L^\infty}\|\nabla c\|_{L^3}\|\nabla^{k+1}u\|_{L^2}\|\nabla^{k+1}u\|_{L^2}\\ &\leq C\delta\|\nabla^{k+1}u\|_{L^2}^2.\end{aligned}$$

iii) For $2\leq l\leq k-1$. If $2\leq l\leq [\frac{k}{2}]$, by Höder's inequality, Young's inequality, Lemma 2.1 and Lemma 2.2, we get

$$\begin{aligned}&\|\nabla^l L_1(c)\nabla^{k-l+1}u\|_{L^2}\|\nabla^{k+1}u\|_{L^2}\\ &\leq \|\nabla^l L_1(c)\|_{L^\infty}\|\nabla^{k-l+1}u\|_{L^2}\|\nabla^{k+1}u\|_{L^2}\\ &\leq C\|\nabla^\alpha c\|_{L^2}^{1-\frac{l}{k}}\|\nabla^{k+1}c\|_{L^2}^{\frac{l}{k}}\|\nabla u\|_{L^2}^{\frac{l}{k}}\|\nabla^{k+1}u\|_{L^2}^{1-\frac{l}{k}}\|\nabla^{k+1}u\|_{L^2}\\ &\leq C\delta\big(\|\nabla^{k+1}c\|_{L^2}^2+\|\nabla^{k+1}u\|_{L^2}^2\big),\end{aligned}$$

where α is defined by

$$\begin{aligned}&\frac{l}{3}=\Big(\frac{\alpha}{3}-\frac{1}{2}\Big)\Big(1-\frac{l}{k}\Big)+\Big(\frac{k+1}{3}-\frac{1}{2}\Big)\frac{l}{k}\\ &\Longrightarrow \alpha=1+\frac{k}{2(k-l)}\in\Big(\frac{3}{2},2\Big],\quad\text{since}\quad 2\leq l\leq\Big[\frac{k}{2}\Big].\end{aligned}$$

If $[\frac{k}{2}]+1\leq l\leq k-1$, by Höder's inequality, Young's inequality, Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}&\|\nabla^l L_1(c)\nabla^{k-l+1}u\|_{L^2}\|\nabla^{k+1}u\|_{L^2}\\ &\leq \|\nabla^l L_1(c)\|_{L^2}\|\nabla^{k-l+1}u\|_{L^\infty}\|\nabla^{k+1}u\|_{L^2}\\ &\leq C\|\nabla c\|_{L^2}^{1-\frac{l-1}{k}}\|\nabla^{k+1}c\|_{L^2}^{\frac{l-1}{k}}\|\nabla^\alpha u\|_{L^2}^{\frac{l-1}{k}}\|\nabla^{k+1}u\|_{L^2}^{1-\frac{l-1}{k}}\|\nabla^{k+1}u\|_{L^2}\\ &\leq C\delta\big(\|\nabla^{k+1}c\|_{L^2}^2+\|\nabla^{k+1}u\|_{L^2}^2\big),\end{aligned}$$

where α is defined by

$$\begin{aligned}&\frac{k-l+1}{3}=\Big(\frac{\alpha}{3}-\frac{1}{2}\Big)\frac{l-1}{k}+\Big(\frac{k+1}{3}-\frac{1}{2}\Big)\Big(1-\frac{l-1}{k}\Big)\\ &\Longrightarrow \alpha=1+\frac{k}{2(l-1)}\in\Big(\frac{3}{2},2\Big],\quad\text{since}\quad \Big[\frac{k}{2}\Big]+1\leq l\leq k-1.\end{aligned}$$

Thus,

$$\begin{aligned}&\int_{\mathbb{R}^3}-\nabla^k\big(L_1(c)(\mu\Delta u+(\lambda+\mu)\nabla\mathrm{div}u)\big)\nabla^k u dx\\ &\leq C\delta\big(\|\nabla^{k+1}c\|_{L^2}^2+\|\nabla^{k+1}u\|_{L^2}^2\big).\end{aligned}\tag{4.18}$$

Next, by integrating by parts and Höder's inequality, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3} -\nabla^k\big(L_2(c)\nabla c\big)\nabla^k uv dx = \int_{\mathbb{R}^3} \nabla^{k-1}(L_2(c)\nabla c)\nabla^{k+1}u dx\\
&\le \|\nabla^{k-1}(L_2(c)\nabla c)\|_{L^2}\|\nabla^{k+1}u\|_{L^2} \le \sum_{0\le l\le k-1} C_{k-1}^l \|\nabla^l L_2(c)\nabla^{k-l}c\|_{L^2}\|\nabla^{k+1}u\|_{L^2}\\
&\le C\sum_{0\le l\le k-1}\|\nabla^l L_2(c)\nabla^{k-l}c\|_{L^2}\|\nabla^{k+1}u\|_{L^2}.
\end{aligned}$$

Similar to the estimate (4.15), we get

$$\int_{\mathbb{R}^3} -\nabla^k\big(L_2(c)\nabla c\big)\nabla^k u dx \le C\delta\big(\|\nabla^{k+1}c\|_{L^2}^2 + \|\nabla^{k+1}u\|_{L^2}^2\big). \tag{4.19}$$

Next, we bound the term $\int_{\mathbb{R}^3} -\nabla^k\big(L_3(c)(\frac{1}{2}\nabla|B|^2 - B\cdot\nabla B)\big)\nabla^k u$ as follows.

$$\begin{aligned}
&\int_{\mathbb{R}^3} -\nabla^k\big(L_3(c)(B\cdot\nabla B)\big)\nabla^k u dx = \int_{\mathbb{R}^3} \nabla^{k-1}\big(L_3(c)B\cdot\nabla B\big)\nabla^{k+1}u dx\\
&\le \sum_{0\le l\le k-1} C_{k-1}^l\|\nabla^l L_3(c)\nabla^{k-l-1}(B\cdot\nabla B)\|_{L^2}\|\nabla^{k+1}u\|_{L^2}\\
&\le C\sum_{0\le l\le k-1}\|\nabla^l L_3(c)\nabla^{k-l-1}(B\cdot\nabla B)\|_{L^2}\|\nabla^{k+1}u\|_{L^2}.
\end{aligned}$$

We shall discuss it in the following cases:

If $0\le l\le[\frac{k}{2}]$, by Hölder's inequality, Lemma 2.1 and Lemma 2.3, we obtain

$$\begin{aligned}
\|\nabla^l L_3(c)\nabla^{k-l-1}(B\cdot\nabla B)\|_{L^2} &\le \|\nabla^l L_3(c)\|_{L^3}\|\nabla^{k-l-1}(B\cdot\nabla B)\|_{L^6}\\
&\le C\|\nabla^l c\|_{L^3}\|\nabla^{k-l}(B\cdot\nabla B)\|_{L^2}\\
&\le C\|\nabla^\alpha c\|_{L^2}^{1-\frac{l}{k}}\|\nabla^{k+1}c\|_{L^2}^{\frac{l}{k}}\big(\|B\cdot\nabla^{k-l}\nabla B\|_{L^2} + \|[\nabla^{k-l},B]\nabla B\|_{L^2}\big)\\
&\le C\|\nabla^\alpha c\|_{L^2}^{1-\frac{l}{k}}\|\nabla^{k+1}c\|_{L^2}^{\frac{l}{k}}\big(\|B\|_{L^\infty} + \|\nabla B\|_{L^3}\big)\|\nabla^{k-l+1}B\|_{L^2}\\
&\le C\|\nabla^\alpha c\|_{L^2}^{1-\frac{l}{k}}\|\nabla^{k+1}c\|_{L^2}^{\frac{l}{k}}\|\nabla B\|_{L^2}^{\frac{l}{k}}\|\nabla^{k+1}B\|_{L^2}^{1-\frac{l}{k}}\\
&\le C\delta\big(\|\nabla^{k+1}c\|_{L^2} + \|\nabla^{k+1}B\|_{L^2}\big),
\end{aligned}$$

where α is defined by

$$\begin{aligned}
&\frac{l-1}{3} = \Big(\frac{\alpha}{3}-\frac{1}{2}\Big)\Big(1-\frac{l}{k}\Big) + \Big(\frac{k+1}{3}-\frac{1}{2}\Big)\frac{l}{k}\\
&\Longrightarrow \alpha = 1-\frac{k}{2(k-l)} \in \Big[0,\frac{1}{2}\Big], \quad \text{since} \quad 0\le l\le\Big[\frac{k}{2}\Big].
\end{aligned}$$

If $[\frac{k}{2}]+1\le l\le k-1$, we have

$$\begin{aligned}
\|\nabla^l L_3(c)\nabla^{k-l-1}(B\cdot\nabla B)\|_{L^2} &= \|\nabla^{l-1}(L_3'(c)\nabla c)\nabla^{k-l-1}(B\cdot\nabla B)\|_{L^2}\\
&= \|\sum_{0\le m\le l-1} C_{l-1}^m\nabla^m L_3'(c)\nabla^{l-m}c\nabla^{k-l-1}(B\cdot\nabla B)\|_{L^2}\\
&\le C\sum_{0\le m\le l-1}\|\nabla^m L_3'(c)\nabla^{l-m}c\nabla^{k-l-1}(B\cdot\nabla B)\|_{L^2}.
\end{aligned}$$

For $m=0$, from Lemma 2.1 and Lemma 2.3, we get

$$\begin{aligned}\|L_3'(c)\nabla^l c\nabla^{k-l-1}(B\cdot\nabla B)\|_{L^2}&\le C\|\nabla^l c\|_{L^3}\|\nabla^{k-l-1}(B\cdot\nabla B)\|_{L^6}\\&\le C\delta\big(\|\nabla^{k+1}c\|_{L^2}+\|\nabla^{k+1}B\|_{L^2}\big).\end{aligned}$$

For $1\le m\le l-1$, from Lemma 2.1 and Lemma 2.2, we obtain

$$\begin{aligned}&\|\nabla^m L_3'(c)\nabla^{l-m}c\nabla^{k-l-1}(B\cdot\nabla B)\|_{L^2}\\&\le C\|\nabla^m L_3'(c)\|_{L^\infty}\|\nabla^{l-m}c\|_{L^\infty}\|\nabla^{k-l-1}(B\cdot\nabla B)\|_{L^2}\\&\le C\|\nabla^m c\|_{L^\infty}\|\nabla^{l-m}c\|_{L^\infty}\|\nabla^{k-l-1}(B\cdot\nabla B)\|_{L^2}\\&\le C\|\nabla^2 c\|_{L^2}^{1-\frac{m+\frac12}{k-1}}\|\nabla^{k+1}c\|_{L^2}^{\frac{m+\frac12}{k-1}}C\|\nabla^2 c\|_{L^2}^{1-\frac{l-m-\frac12}{k-1}}\|\nabla^{k+1}c\|_{L^2}^{\frac{l-m-\frac12}{k-1}}\|\nabla^{k-l-1}(B\cdot\nabla B)\|_{L^2}\\&\le C\delta\|\nabla^{k+1}c\|_{L^2}^{\frac{l}{k-1}}\big(\|B\cdot\nabla^{k-l}B\|_{L^2}+\|[\nabla^{k-l-1},B]\nabla B\|_{L^2}\big)\\&\le C\delta\|\nabla^{k+1}c\|_{L^2}^{\frac{l}{k-1}}\big(\|\nabla^{k-l}B\|_{L^2}+\|\nabla^{k-l-1}B\|_{L^2}\big)\\&\le C\delta\|\nabla^{k+1}c\|_{L^2}^{\frac{l}{k-1}}\big(\|\nabla^2 B\|_{L^2}^{\frac{l}{k-1}}+\|\nabla^\alpha B\|_{L^2}^{\frac{l}{k-1}}\big)\|\nabla^{k+1}B\|_{L^2}^{1-\frac{l}{k-1}}\\&\le C\delta\big(\|\nabla^{k+1}c\|_{L^2}+\|\nabla^{k+1}B\|_{L^2}\big),\end{aligned}$$

where α is defined by

$$\begin{aligned}&\frac{k-l}{3}-\frac12=\Big(\frac{\alpha}{3}-\frac12\Big)\frac{l}{k-1}+\Big(\frac{k+1}{3}-\frac12\Big)\Big(1-\frac{l}{k-1}\Big)\\&\Longrightarrow\alpha=2-\frac{k-1}{l}\in\Big(0,1\Big],\quad\text{since}\quad\Big[\frac{k}{2}\Big]+1\le l\le k-1.\end{aligned}$$

Thus

$$\int_{\mathbb{R}^3}-\nabla^k\big(L_3(c)(B\cdot\nabla B)\big)\nabla^k u dx\le C\delta\big(\|\nabla^{k+1}c\|_{L^2}^2+\|\nabla^{k+1}u\|_{L^2}^2+\|\nabla^{k+1}B\|_{L^2}^2\big).\tag{4.20}$$

Finally, we bound the term $\int_{\mathbb{R}^3}-\nabla^k\big(\nabla\times(L_3(c)((\nabla\times B)\times B)\big)\nabla^k B dx$ as follows

$$\begin{aligned}&\int_{\mathbb{R}^3}-\nabla^k\big(\nabla\times(L_3(c)((\nabla\times B)\times B)\big)\nabla^k B dx\\&=\int_{\mathbb{R}^3}-\nabla^k\big(L_3(c)(\nabla\times B)\times B\big)\nabla^{k+1}B dx\\&\le\sum_{0\le l\le k}C_k^l\|\nabla^l L_3(c)\nabla^{k-l}(B\cdot\nabla B)\|_{L^2}\|\nabla^{k+1}B\|_{L^2}\\&\le C\sum_{0\le l\le k}\|\nabla^l L_3(c)\nabla^{k-l}(B\cdot\nabla B)\|_{L^2}\|\nabla^{k+1}B\|_{L^2}.\end{aligned}$$

We shall discuss it in the following cases.

If $0 \le l \le [\frac{k}{2}]$, by Hölder's inequality, Lemma 2.1, Lemma 2.2 and Lemma 2.3, we get

$$
\begin{aligned}
\|\nabla^l L_3(c)\nabla^{k-l}(B\cdot\nabla B)\|_{L^2} &\le \|\nabla^l L_3(c)\|_{L^\infty}\|\nabla^{k-l}(B\cdot\nabla B)\|_{L^2}\\
&\le C\|\nabla^l c\|_{L^\infty}\|\nabla^{k-l}(B\cdot\nabla B)\|_{L^2}\\
&\le C\|\nabla^\alpha c\|_{L^2}^{1-\frac{l}{k}}\|\nabla^{k+1}c\|_{L^2}^{\frac{l}{k}}\big(\|B\cdot\nabla^{k-l}\nabla B\|_{L^2}+\|[\nabla^{k-l},B]\nabla B\|_{L^2}\big)\\
&\le C\|\nabla^\alpha c\|_{L^2}^{1-\frac{l}{k}}\|\nabla^{k+1}c\|_{L^2}^{\frac{l}{k}}\big(\|B\|_{L^\infty}+\|\nabla B\|_{L^3}\big)\|\nabla^{k-l+1}B\|_{L^2}\\
&\le C\|\nabla^\alpha c\|_{L^2}^{1-\frac{l}{k}}\|\nabla^{k+1}c\|_{L^2}^{\frac{l}{k}}\|\nabla B\|_{L^2}^{\frac{l}{k}}\|\nabla^{k+1}B\|_{L^2}^{1-\frac{l}{k}}\\
&\le C\delta\big(\|\nabla^{k+1}c\|_{L^2}+\|\nabla^{k+1}B\|_{L^2}\big),
\end{aligned}
$$

where α is defined by

$$
\begin{aligned}
&\frac{l}{3}=\Big(\frac{\alpha}{3}-\frac{1}{2}\Big)\Big(1-\frac{l}{k}\Big)+\Big(\frac{k+1}{3}-\frac{1}{2}\Big)\frac{l}{k}\\
&\Longrightarrow \alpha=\frac{1}{2}+\frac{k}{2(k-l)}\in\Big[1,\frac{3}{2}\Big],\quad \text{since}\quad 0\le l\le\Big[\frac{k}{2}\Big].
\end{aligned}
$$

If $[\frac{k}{2}]+1\le l\le k$, by Hölder's inequality, Lemma 2.1 and Lemma 2.3, we obtain

$$
\begin{aligned}
&\|\nabla^l L_3(c)\nabla^{k-l}(B\cdot\nabla B)\|_{L^2}\\
&\le \|\nabla^l L_3(c)\|_{L^6}\|\nabla^{k-l}(B\cdot\nabla B)\|_{L^3}\\
&\le C\|\nabla^l c\|_{L^6}\|\nabla^{k-l}(B\cdot\nabla B)\|_{L^3}\\
&\le C\|\nabla c\|_{L^2}^{1-\frac{l}{k}}\|\nabla^{k+1}c\|_{L^2}^{\frac{l}{k}}\big(\|B\cdot\nabla^{k-l+1}B\|_{L^3}+\|[\nabla^{k-l+1},B]\nabla B\|_{L^3}\big)\\
&\le C\|\nabla c\|_{L^2}^{1-\frac{l}{k}}\|\nabla^{k+1}c\|_{L^2}^{\frac{l}{k}}\big(\|B\|_{L^6}\|\nabla^{k-l+1}B\|_{L^6}+\|\nabla B\|_{L^6}\|\nabla^{k-l}B\|_{L^6}\big)\\
&\le C\|\nabla c\|_{L^2}^{1-\frac{l}{k}}\|\nabla^{k+1}c\|_{L^2}^{\frac{l}{k}}\big(\|B\|_{L^6}\|\nabla^\alpha B\|_{L^2}^{\frac{l}{k}}\|\nabla^{k+1}B\|_{L^2}^{1-\frac{l}{k}}\\
&\quad+\|\nabla B\|_{L^6}\|\nabla B\|_{L^2}^{\frac{l}{k}}\|\nabla^{k+1}B\|_{L^2}^{1-\frac{l}{k}}\big)\\
&\le C\delta\big(\|\nabla^{k+1}c\|_{L^2}+\|\nabla^{k+1}B\|_{L^2}\big),
\end{aligned}
$$

where α is defined by

$$
\begin{aligned}
&\frac{k-l+2}{3}-\frac{1}{2}=\Big(\frac{\alpha}{3}-\frac{1}{2}\Big)\frac{l}{k}+\Big(\frac{k+1}{3}-\frac{1}{2}\Big)\Big(1-\frac{l}{k}\Big)\\
&\Longrightarrow \alpha=1+\frac{k}{l}\in\Big[2,3\Big),\quad \text{since}\quad \Big[\frac{k}{2}\Big]+1\le l\le k.
\end{aligned}
$$

Thus,

$$
\int_{\mathbb{R}^3}-\nabla^k\big(\nabla\times(L_3(c)((\nabla\times B)\times B)\big)\nabla^k B\le C\delta\big(\|\nabla^{k+1}c\|_{L^2}^2+\|\nabla^{k+1}B\|_{L^2}^2\big). \tag{4.21}
$$

From (4.15)-(4.21), we have

$$
I_2+I_3\le C\delta\big(\|\nabla^{k+1}c\|_{L^2}^2+\|\nabla^{k+1}u\|_{L^2}^2+\|\nabla^{k+1}B\|_{L^2}^2\big). \tag{4.22}
$$

Combining (4.6), (4.7) and (4.14) with (4.22), which immediately yields (4.5).

Lemma 4.2 *Under the priori assumption (4.2), then for $k = 0, 1, \cdots, N-1$, we have*

$$\frac{1}{2}\frac{d}{dt}\|\nabla^{k+1}(c,u,B)\|_{L^2}^2 + C\|\nabla^{k+2}(u,B)\|_{L^2}^2 \le C\delta\big(\|\nabla^{k+1}(c,u,B)\|_{L^2}^2 + \|\nabla^{k+2}(u,B)\|_{L^2}^2\big). \tag{4.23}$$

Proof. Applying ∇^{k+1} to the first three equations of (3.1) and multiplying them by $\nabla^{k+1}c$, $\nabla^{k+1}u$, $\nabla^{k+1}B$ respectively, and then integrating them over $\mathbb{R}^3$, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla^{k+1}(c,u,B)\|_{L^2}^2 + \mu\|\nabla^{k+2}u\|_{L^2}^2 + (\mu+\lambda)\|\nabla^{k+2}\mathrm{div}u\|_{L^2}^2 + \nu\|\nabla^{k+2}B\|_{L^2}^2\\ &= \langle\nabla^{k+1}f, \nabla^{k+1}c\rangle + \langle\nabla^{k+1}g, \nabla^{k+1}u\rangle + \langle\nabla^{k+1}h, \nabla^{k+1}B\rangle\\ &\equiv II_1 + II_2 + II_3.\end{aligned} \tag{4.24}$$

We first bound the second term and the third term in left-hand side of (4.24) as follows

$$\mu\|\nabla^{k+2}u\|_{L^2}^2 + (\mu+\lambda)\|\nabla^{k+1}\mathrm{div}u\|_{L^2}^2 \ge C\|\nabla^{k+2}u\|_{L^2}^2 \tag{4.25}$$

due to the condition (1.2).

We shall estimate each term in the right-hand side of (4.24). First, we bound the term II_1 as follows,

$$\begin{aligned}II_1 &= -\int_{\mathbb{R}^3}\nabla^{k+1}(u\cdot\nabla c)\nabla^{k+1}c dx - \int_{\mathbb{R}^3}\nabla^{k+1}(c\mathrm{div}u)\nabla^{k+1}c dx\\ &\equiv II_{11} + II_{12}.\end{aligned} \tag{4.26}$$

Integrating by parts and using Hölder's inequality, Young's inequality and Lemma 2.3, we get

$$\begin{aligned}II_{11} &= -\int_{\mathbb{R}^3}[\nabla^{k+1}, u]\cdot\nabla c\nabla^{k+1}c dx - \int_{\mathbb{R}^3}u\cdot\nabla\nabla^{k+1}c\nabla^{k+1}c dx\\ &\le C\big(\|\nabla u\|_{L^\infty}\|\nabla^{k+1}c\|_{L^2} + \|\nabla c\|_{L^\infty}\|\nabla^{k+1}u\|_{L^2}\big)\|\nabla^{k+1}c\|_{L^2}\\ &\quad - \frac{1}{2}\int_{\mathbb{R}^3}u\cdot\nabla|\nabla^{k+1}c|^2 dx\\ &\le C\big(\|\nabla u\|_{L^\infty}\|\nabla^{k+1}c\|_{L^2} + \|\nabla c\|_{L^\infty}\|\nabla^{k+1}u\|_{L^2}\big)\|\nabla^{k+1}c\|_{L^2}\\ &\quad + \frac{1}{2}\int_{\mathbb{R}^3}\mathrm{div}u|\nabla^{k+1}c|^2 dx\\ &\le C\delta\big(\|\nabla^{k+1}c\|_{L^2}^2 + \|\nabla^{k+1}u\|_{L^2}^2\big).\end{aligned}$$

$$\begin{aligned}II_{12} &= -\int_{\mathbb{R}^3}[\nabla^{k+1}, c]\mathrm{div}u\nabla^{k+1}c dx - \int_{\mathbb{R}^3}c\cdot\nabla\nabla^{k+1}u\nabla^{k+1}n dx\\ &\le C\big(\|\nabla c\|_{L^\infty}\|\nabla^{k+1}u\|_{L^2} + \|\nabla u\|_{L^\infty}\|\nabla^{k+1}c\|_{L^2}\big)\|\nabla^{k+1}c\|_{L^2}\\ &\quad + \|c\|_{L^\infty}\|\nabla^{k+1}c\|_{L^2}\|\nabla^{k+2}u\|_{L^2}\\ &\le C\delta\big(\|\nabla^{k+1}c\|_{L^2}^2 + \|\nabla^{k+1}u\|_{L^2}^2 + \|\nabla^{k+2}u\|_{L^2}^2\big),\end{aligned}$$

Combining the above estimates with (4.26), we conclude that

$$II_1 \leq C\delta\big(\|\nabla^{k+1}c\|_{L^2}^2 + \|\nabla^{k+1}u\|_{L^2}^2 + \|\nabla^{k+2}u\|_{L^2}^2\big). \tag{4.27}$$

Next, we shall bound the term II_2. Integrating by parts and using Hölder's inequality, Young's inequality and Lemma 2.3, we have

$$\begin{aligned}
II_{21} &= \int_{\mathbb{R}^3} \nabla^{k+1}(u\cdot\nabla u)\nabla^{k+1}u dx\\
&= -\int_{\mathbb{R}^3}[\nabla^{k+1}, u]\cdot\nabla u\nabla^{k+1}udx - \int_{\mathbb{R}^3} u\cdot\nabla\nabla^{k+1}u\nabla^{k+1}udx\\
&\leq C\big(\|\nabla u\|_{L^\infty}\|\nabla^{k+1}u\|_{L^2} + \|\nabla u\|_{L^\infty}\|\nabla^{k+1}u\|_{L^2}\big)\|\nabla^{k+1}u\|_{L^2}\\
&\quad -\frac{1}{2}\int_{\mathbb{R}^3} u\cdot\nabla|\nabla^{k+1}u|^2dx\\
&\leq C\big(\|\nabla u\|_{L^\infty}\|\nabla^{k+1}u\|_{L^2} + \|\nabla u\|_{L^\infty}\|\nabla^{k+1}u\|_{L^2}\big)\|\nabla^{k+1}u\|_{L^2}\\
&\quad +\frac{1}{2}\int_{\mathbb{R}^3} \mathrm{div}u|\nabla^{k+1}u|^2dx\\
&\leq C\delta\|\nabla^{k+1}u\|_{L^2}^2.
\end{aligned}$$

Employing the approximation to simplify the presentations, Hölder's inequality and Leibniz'sformula, we have

$$\begin{aligned}
&II_{22} \approx \int_{\mathbb{R}^3} \nabla^{k+1}(L_1(c)\nabla^2u)\nabla^{k+1}udx = \int_{\mathbb{R}^3}\nabla^k(L_1(c)\nabla^2u)\nabla^{k+2}udx\\
&\leq \|\nabla^k(L_1(c)\nabla^2u)\|_{L^2}\|\nabla^{k+2}u\|_{L^2} \leq \sum_{0\leq l\leq k} C_k^l\|\nabla^l L_1(c)\nabla^{k-l+2}u\|_{L^2}\|\nabla^{k+2}u\|_{L^2}\\
&\leq C\sum_{0\leq l\leq k}\|\nabla^l L_1(c)\nabla^{k-l+2}u\|_{L^2}\|\nabla^{k+2}u\|_{L^2}.
\end{aligned}$$

To bound the above term, we divide it into the following three cases:

i) For $l=0$, since $|L_1(c)|\leq C|c|$, by Höder's and Sobolev's inequalities, we have

$$\begin{aligned}
\|L_1(c)\nabla^{k+2}u\|_{L^2}\|\nabla^{k+2}u\|_{L^2} &\leq \|L_1(c)\|_{L^\infty}\|\nabla^{k+2}u\|_{L^2}\|\nabla^{k+2}u\|_{L^2}\\
&\leq C\|c\|_{L^\infty}\|\nabla^{k+2}u\|_{L^2}^2\\
&\leq C\delta\|\nabla^{k+2}u\|_{L^2}^2.
\end{aligned}$$

ii) For $l=1$, since $|L_1'(c)|\leq C$, we obtain

$$\begin{aligned}
\|\nabla L_1(c)\nabla^{k-l+2}u\|_{L^2}\|\nabla^{k+2}u\|_{L^2} &\leq \|L_1'(n)\nabla n\nabla^{k+1}v\|_{L^2}\|\nabla^{k+2}v\|_{L^2}\\
&\leq \|L_1'(c)\nabla c\|_{L^3}\|\nabla^{k+1}u\|_{L^6}\|\nabla^{k+2}u\|_{L^2}\\
&\leq \|L_1'(c)\|_{L^\infty}\|\nabla c\|_{L^3}\|\nabla^{k+2}u\|_{L^2}\|\nabla^{k+2}u\|_{L^2}\\
&\leq C\delta\|\nabla^{k+2}u\|_{L^2}^2.
\end{aligned}$$

iii) For $2 \le l \le k$. If $2 \le l \le [\frac{k}{2}]$, by Höder's inequality, Young's inequality, Lemma 2.1 and Lemma 2.2, we get

$$
\begin{aligned}
&\|\nabla^l L_1(c)\nabla^{k-l+2}u\|_{L^2}\|\nabla^{k+2}u\|_{L^2}\\
&\le \|\nabla^l L_1(c)\|_{L^\infty}\|\nabla^{k-l+2}u\|_{L^2}\|\nabla^{k+2}u\|_{L^2}\\
&\le C\|\nabla^\alpha c\|_{L^2}^{\frac{k-l+1}{k+1}}\|\nabla^{k+1}c\|_{L^2}^{1-\frac{k-l+1}{k+1}}\|\nabla u\|_{L^2}^{1-\frac{k-l+1}{k+1}}\|\nabla^{k+2}u\|_{L^2}^{\frac{k-l+1}{k+1}}\|\nabla^{k+2}u\|_{L^2}\\
&\le C\delta\big(\|\nabla^{k+1}c\|_{L^2}^2+\|\nabla^{k+2}u\|_{L^2}^2\big),
\end{aligned}
$$

where α is defined by

$$
\begin{aligned}
&\frac{l}{3}=\Big(\frac{\alpha}{3}-\frac{1}{2}\Big)\frac{k-l+1}{k+1}+\Big(\frac{k+1}{3}-\frac{1}{2}\Big)\Big(1-\frac{k-l+1}{k+1}\Big)\\
&\Longrightarrow \alpha=\frac{3(k+1)}{2(k-l+1)}\in\Big(\frac{3}{2},3\Big),\quad \text{since}\quad 2\le l\le\Big[\frac{k}{2}\Big].
\end{aligned}
$$

If $[\frac{k}{2}]+1 \le l \le k$, by Höder's inequality, Young's inequality, Lemma 2.1 and Lemma 2.2, we have

$$
\begin{aligned}
&\|\nabla^l L_1(c)\nabla^{k-l+2}u\|_{L^2}\|\nabla^{k+2}u\|_{L^2}\\
&\le \|\nabla^l L_1(c)\|_{L^2}\|\nabla^{k-l+2}u\|_{L^\infty}\|\nabla^{k+2}u\|_{L^2}\\
&\le C\|\nabla c\|_{L^2}^{1-\frac{l-1}{k}}\|\nabla^{k+1}c\|_{L^2}^{\frac{l-1}{k}}\|\nabla^\alpha u\|_{L^2}^{\frac{l-1}{k}}\|\nabla^{k+2}u\|_{L^2}^{1-\frac{l-1}{k}}\|\nabla^{k+2}u\|_{L^2}\\
&\le C\delta\big(\|\nabla^{k+1}c\|_{L^2}^2+\|\nabla^{k+2}u\|_{L^2}^2\big),
\end{aligned}
$$

where α is defined by

$$
\begin{aligned}
&\frac{k-l+2}{3}=\Big(\frac{\alpha}{3}-\frac{1}{2}\Big)\frac{l-1}{k}+\Big(\frac{k+1}{3}-\frac{1}{2}\Big)\Big(1-\frac{l-1}{k}\Big)\\
&\Longrightarrow \alpha=\frac{k}{2(l-1)}+2\in\Big(\frac{5}{2},3\Big],\quad \text{since}\quad \Big[\frac{k}{2}\Big]+1\le l\le k.
\end{aligned}
$$

Thus,

$$
II_{22}\le C\delta\big(\|\nabla^{k+1}c\|_{L^2}^2+\|\nabla^{k+2}u\|_{L^2}^2\big).
$$

Next, we bound the term II_{23}. Integrating by parts and employing Höder's inequality and Leibniz's formula, we conclude that

$$
\begin{aligned}
II_{23}&=\int_{\mathbb{R}^3}-\nabla^{k+1}\big(L_2(c)\nabla c\big)\nabla^{k+1}u\,dx=\int_{\mathbb{R}^3}\nabla^k(L_2(c)\nabla c)\nabla^{k+2}u\,dx\\
&\le\|\nabla^k(L_2(c)\nabla c)\|_{L^2}\|\nabla^{k+2}u\|_{L^2}\le\sum_{0\le l\le k}C_k^l\|\nabla^l L_2(c)\nabla^{k-l+1}c\|_{L^2}\|\nabla^{k+2}u\|_{L^2}\\
&\le C\sum_{0\le l\le k}\|\nabla^l L_2(c)\nabla^{k-l+1}c\|_{L^2}\|\nabla^{k+2}u\|_{L^2}.
\end{aligned}
$$

In order to obtain the estimate of the above term, we shall deal with it in the following cases:

i) For $l=0$, since $|L_2(c)|\le C|c|$, by Höder's inequality , Sobolev's inequality and Young's inequality, we have

$$
\begin{aligned}
\|L_2(c)\nabla^{k+1}c\|_{L^2}\|\nabla^{k+2}u\|_{L^2} &\le \|L_1(c)\|_{L^\infty}\|\nabla^{k+1}c\|_{L^2}\|\nabla^{k+2}u\|_{L^2}\\
&\le C\|c\|_{L^\infty}\|\nabla^{k+1}c\|_{L^2}\|\nabla^{k+2}u\|_{L^2}\\
&\le C\delta\big(\|\nabla^{k+1}c\|_{L^2}^2+\|\nabla^{k+2}u\|_{L^2}^2\big).
\end{aligned}
$$

ii) For $l=1$, since $|L_2^{'}(c)|\le C$, we obtain

$$
\begin{aligned}
\|\nabla L_2(c)\nabla^{k}c\|_{L^2}\|\nabla^{k+2}u\|_{L^2} &\le \|L_2^{'}(c)\nabla c\nabla^{k}c\|_{L^2}\|\nabla^{k+2}u\|_{L^2}\\
&\le \|L_2^{'}(c)\nabla c\|_{L^3}\|\nabla^{k}c\|_{L^6}\|\nabla^{k+2}u\|_{L^2}\\
&\le \|L_2^{'}(c)\|_{L^\infty}\|\nabla c\|_{L^3}\|\nabla^{k+1}c\|_{L^2}\|\nabla^{k+2}u\|_{L^2}\\
&\le C\delta\big(\|\nabla^{k+1}c\|_{L^2}^2+\|\nabla^{k+2}u\|_{L^2}^2\big).
\end{aligned}
$$

iii) For $2\le l\le k$. If $2\le l\le [\frac{k}{2}]$, by Höder's inequality and Young's inequality, Lemma 2.1 and Lemma 2.2, we get

$$
\begin{aligned}
&\|\nabla^{l}L_2(c)\nabla^{k-l+1}c\|_{L^2}\|\nabla^{k+2}u\|_{L^2}\\
&\le \|\nabla^{l}L_2(c)\|_{L^\infty}\|\nabla^{k-l+1}c\|_{L^2}\|\nabla^{k+2}u\|_{L^2}\\
&\le C\|\nabla^{\alpha}c\|_{L^2}^{\frac{k-l}{k}}\|\nabla^{k+1}c\|_{L^2}^{1-\frac{k-l}{k}}\|\nabla c\|_{L^2}^{1-\frac{k-l}{k}}\|\nabla^{k+1}c\|_{L^2}^{\frac{k-l}{k}}\|\nabla^{k+2}u\|_{L^2}\\
&\le C\delta\big(\|\nabla^{k+1}c\|_{L^2}^2+\|\nabla^{k+2}u\|_{L^2}^2\big),
\end{aligned}
$$

where α is defined by

$$
\begin{aligned}
&\frac{l}{3}=\Big(\frac{\alpha}{3}-\frac{1}{2}\Big)\frac{k-l}{k}+\Big(\frac{k+1}{3}-\frac{1}{2}\Big)\Big(1-\frac{k-l}{k}\Big)\\
&\Longrightarrow \alpha=1+\frac{k}{2(k-l)}\in\Big(\frac{3}{2},2\Big],\quad \text{since}\quad 2\le l\le\Big[\frac{k}{2}\Big].
\end{aligned}
$$

If $[\frac{k}{2}]+1\le l\le k$, by Höder's inequality and Young's inequality, Lemma 2.1 and Lemma 2.2, we have

$$
\begin{aligned}
&\|\nabla^{l}L_2(c)\nabla^{k-l+1}c\|_{L^2}\|\nabla^{k+2}u\|_{L^2}\\
&\le \|\nabla^{l}L_2(c)\|_{L^2}\|\nabla^{k-l+1}c\|_{L^\infty}\|\nabla^{k+2}u\|_{L^2}\\
&\le C\|\nabla c\|_{L^2}^{1-\frac{l-1}{k}}\|\nabla^{k+1}c\|_{L^2}^{\frac{l-1}{k}}\|\nabla^{\alpha}c\|_{L^2}^{\frac{l-1}{k}}\|\nabla^{k+1}c\|_{L^2}^{1-\frac{l-1}{k}}\|\nabla^{k+2}u\|_{L^2}\\
&\le C\delta\big(\|\nabla^{k+1}c\|_{L^2}^2+\|\nabla^{k+2}u\|_{L^2}^2\big),
\end{aligned}
$$

where α is defined by

$$
\begin{aligned}
&\frac{k-l+1}{3}=\Big(\frac{\alpha}{3}-\frac{1}{2}\Big)\frac{l-1}{k}+\Big(\frac{k+1}{3}-\frac{1}{2}\Big)\Big(1-\frac{l-1}{k}\Big)\\
&\Longrightarrow \alpha=\frac{k}{2(l-1)}+1\in\Big(\frac{3}{2},2\Big],\quad \text{since}\quad \Big[\frac{k}{2}\Big]+1\le l\le k.
\end{aligned}
$$

Thus,

$$II_{23} \leq C\delta\big(\|\nabla^{k+1}c\|_{L^2}^2 + \|\nabla^{k+2}u\|_{L^2}^2\big).$$

For the term II_{24}, we have

$$\begin{aligned}
II_{24} &= \int_{\mathbb{R}^3} -\nabla^{k+1}\big(L_3(c)(B\cdot\nabla B)\big)\nabla^{k+1}udx = \int_{\mathbb{R}^3} \nabla^{k}\big(L_3(c)B\cdot\nabla B\big)\nabla^{k+2}udx \\
&\leq \sum_{0\leq l\leq k} C_k^l \|\nabla^l L_3(c)\nabla^{k-l}(B\cdot\nabla B)\|_{L^2}\|\nabla^{k+2}u\|_{L^2} \\
&\leq C\sum_{0\leq l\leq k} \|\nabla^l L_3(c)\nabla^{k-l}(B\cdot\nabla B)\|_{L^2}\|\nabla^{k+2}u\|_{L^2}.
\end{aligned}$$

Using Hölder's inequality, Lemma 2.1 and Lemma 2.3, we obtain

$$\begin{aligned}
&\|\nabla^l L_3(c)\nabla^{k-l}(B\cdot\nabla B)\|_{L^2} \\
&\leq \|\nabla^l L_3(c)\|_{L^6}\|\nabla^{k-l}(B\cdot\nabla B)\|_{L^3} \\
&\leq C\|\nabla^l c\|_{L^6}\|\nabla^{k-l}(B\cdot\nabla B)\|_{L^3} \\
&\leq C\|\nabla c\|_{L^2}^{1-\frac{l}{k}}\|\nabla^{k+1}c\|_{L^2}^{\frac{l}{k}}\big(\|B\cdot\nabla^{k-l+1}B\|_{L^3} + \|[\nabla^{k-l+1},B]\nabla B\|_{L^3}\big) \\
&\leq C\|\nabla c\|_{L^2}^{1-\frac{l}{k}}\|\nabla^{k+1}c\|_{L^2}^{\frac{l}{k}}\big(\|B\|_{L^6}\|\nabla^{k-l+1}B\|_{L^6} + \|\nabla B\|_{L^6}\|\nabla^{k-l}B\|_{L^6}\big) \\
&\leq C\|\nabla c\|_{L^2}^{1-\frac{l}{k}}\|\nabla^{k+1}c\|_{L^2}^{\frac{l}{k}}\big(\|B\|_{L^6}\|\nabla^2 B\|_{L^2}^{\frac{l}{k}}\|\nabla^{k+2}B\|_{L^2}^{1-\frac{l}{k}} \\
&\quad + \|\nabla B\|_{L^6}\|\nabla B\|_{L^2}^{\frac{l}{k}}\|\nabla^{k+1}B\|_{L^2}^{1-\frac{l}{k}}\big) \\
&\leq C\delta\big(\|\nabla^{k+1}c\|_{L^2} + \|\nabla^{k+1}B\|_{L^2} + \|\nabla^{k+2}B\|_{L^2}\big),
\end{aligned}$$

which implies that

$$II_{24} \leq C\delta\big(\|\nabla^{k+1}c\|_{L^2}^2 + \|\nabla^{k+2}u\|_{L^2}^2 + \|\nabla^{k+2}B\|_{L^2}^2 + \|\nabla^{k+1}B\|_{L^2}^2\big). \tag{4.28}$$

From the above estimates, we have

$$II_2 \leq C\delta\big(\|\nabla^{k+1}c\|_{L^2}^2 + \|\nabla^{k+1}u\|_{L^2}^2 + \|\nabla^{k+2}u\|_{L^2}^2 + \|\nabla^{k+1}B\|_{L^2}^2 + \|\nabla^{k+2}B\|_{L^2}^2\big). \tag{4.29}$$

Finally, we bound the term II_3. Here, we first deal with the term $\int_{\mathbb{R}^3} -\nabla^{k+1}\big(\nabla\times(L_3(c)((\nabla\times B)\times B)\big)\nabla^{k+1}Bdx$. We shall discuss it in the following cases:

If $0 \le l \le [\frac{k+1}{2}]$, using Hölder's inequality, Lemma 2.1, Lemma 2.2 and Lemma 2.3, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^3} -\nabla^{k+1}\big(\nabla \times (L_3(c)((\nabla \times B) \times B)\big)\nabla^{k+1}Bdx \\
&= \int_{\mathbb{R}^3} -\nabla^{k+1}\big(L_3(c)(\nabla \times B) \times B\big)\nabla^{k+2}Bdx \\
&\le \sum_{0\le l\le k} C_{k+1}^l \|\nabla^l L_3(c)\nabla^{k-l+1}(B\cdot\nabla B)\|_{L^2}\|\nabla^{k+2}B\|_{L^2} \\
&\le C \sum_{0\le l\le k+1} \|\nabla^l L_3(c)\|_{L^\infty}\|\nabla^{k-l+1}(B\cdot\nabla B)\|_{L^2} \\
&\le C \sum_{0\le l\le k+1} \|\nabla^l c\|_{L^\infty}\|\nabla^{k-l+1}(B\cdot\nabla B)\|_{L^2}\|\nabla^{k+2}B\|_{L^2} \\
&\le C\|\nabla^\alpha c\|_{L^2}^{1-\frac{l}{k}}\|\nabla^{k+1}c\|_{L^2}^{\frac{l}{k}}\big(\|B\cdot\nabla^{k-l+1}\nabla B\|_{L^2} + \|[\nabla^{k-l}, B]\nabla B\|_{L^2}\big)\|\nabla^{k+2}B\|_{L^2} \\
&\le C\|\nabla^\alpha c\|_{L^2}^{1-\frac{l}{k}}\|\nabla^{k+1}c\|_{L^2}^{\frac{l}{k}}\big(\|B\|_{L^\infty} + \|\nabla B\|_{L^3}\big)\|\nabla^{k-l+2}B\|_{L^2}\|\nabla^{k+2}B\|_{L^2} \\
&\le C\|\nabla^\alpha c\|_{L^2}^{1-\frac{l}{k}}\|\nabla^{k+1}c\|_{L^2}^{\frac{l}{k}}\|\nabla^2 B\|_{L^2}^{\frac{l}{k}}\|\nabla^{k+2}B\|_{L^2}^{1-\frac{l}{k}}\|\nabla^{k+2}B\|_{L^2} \\
&\le C\delta\big(\|\nabla^{k+1}c\|_{L^2}^2 + \|\nabla^{k+2}B\|_{L^2}^2\big),
\end{aligned}
$$

where α is defined by

$$
\begin{aligned}
&\frac{l}{3} = \Big(\frac{\alpha}{3} - \frac{1}{2}\Big)\Big(1 - \frac{l}{k}\Big) + \Big(\frac{k+1}{3} - \frac{1}{2}\Big)\frac{l}{k} \\
&\Longrightarrow \alpha = \frac{1}{2} + \frac{k}{2(k-l)} \in \Big[1, \frac{3}{2}\Big), \quad \text{since} \quad 0 \le l \le \Big[\frac{k+1}{2}\Big].
\end{aligned}
$$

If $[\frac{k+1}{2}] + 1 \le l \le k+1$, from Hölder's inequality, Lemma 2.1, Lemma 2.2 and Lemma 2.3, we get

$$
\begin{aligned}
&\int_{\mathbb{R}^3} -\nabla^{k+1}\big(\nabla \times (L_3(c)((\nabla \times B) \times B)\big)\nabla^{k+1}Bdx \\
&= \int_{\mathbb{R}^3} -\nabla^{k+1}\big(L_3(c)(\nabla \times B) \times B\big)\nabla^{k+2}Bdx \\
&\le \sum_{0\le l\le k} C_{k+1}^l \|\nabla^l (B\cdot\nabla B)\nabla^{k-l+1}L_3(c)\|_{L^2}\|\nabla^{k+2}B\|_{L^2} \\
&\le C \sum_{0\le l\le k+1} \|\nabla^{k-l+1}c\|_{L^\infty}\|\nabla^l(B\cdot\nabla B)\|_{L^2}\|\nabla^{k+2}B\|_{L^2} \\
&\le C\|\nabla^\alpha c\|_{L^2}^{\frac{l}{k+1}}\|\nabla^{k+1}c\|_{L^2}^{1-\frac{l}{k+1}}\big(\|B\cdot\nabla^{l+1}B\|_{L^2} + \|[\nabla^l, B]\nabla B\|_{L^2}\big)\|\nabla^{k+2}B\|_{L^2} \\
&\le C\|\nabla^\alpha c\|_{L^2}^{\frac{l}{k+1}}\|\nabla^{k+1}c\|_{L^2}^{1-\frac{l}{k+1}}\big(\|B\|_{L^\infty}\|\nabla^{l+1}B\|_{L^2} + \|\nabla B\|_{L^3}\|\nabla^l B\|_{L^6}\big)\|\nabla^{k+2}B\|_{L^2} \\
&\le C\|\nabla^\alpha c\|_{L^2}^{\frac{l}{k+1}}\|\nabla^{k+1}c\|_{L^2}^{1-\frac{l}{k+1}}\big(\|B\|_{L^\infty} + \|\nabla B\|_{L^3}\big)\|\nabla^{l+1}B\|_{L^2}\|\nabla^{k+2}B\|_{L^2} \\
&\le C\|\nabla^\alpha c\|_{L^2}^{\frac{l}{k+1}}\|\nabla^{k+1}c\|_{L^2}^{1-\frac{l}{k+1}}\big(\|B\|_{L^\infty} + \|\nabla B\|_{L^3}\big)\|\nabla B\|_{L^2}^{1-\frac{l}{k+1}}\|\nabla^{k+2}B\|_{L^2}^{\frac{l}{k+1}}\|\nabla^{k+2}B\|_{L^2} \\
&\le C\delta\big(\|\nabla^{k+1}c\|_{L^2}^2 + \|\nabla^{k+1}B\|_{L^2}^2\big),
\end{aligned}
$$

where α is defined by

$$\begin{aligned}
&\frac{k-l+1}{3}-\frac{1}{2}=\Big(\frac{\alpha}{3}-\frac{1}{2}\Big)\frac{l}{k+1}+\Big(\frac{k+1}{3}-\frac{1}{2}\Big)\Big(1-\frac{l}{k+1}\Big)\\
&\Longrightarrow \alpha=\frac{3(k+1)}{2l}\in\Big[\frac{3}{2},3\Big],\quad \text{since}\quad \Big[\frac{k+1}{2}\Big]\le l\le k+1.
\end{aligned}$$

The estimates of the others in II_3 are similar to the argument of the term II_2, we omit it. Thus,

$$II_3\le C\delta\big(\|\nabla^{k+1}c\|_{L^2}^2+\|\nabla^{k+1}u\|_{L^2}^2+\|\nabla^{k+2}u\|_{L^2}^2+\|\nabla^{k+1}B\|_{L^2}^2+\|\nabla^{k+2}B\|_{L^2}^2\big). \tag{4.30}$$

In light of (4.24),(4.25), (4.27), (4.29) and (4.30), we deduce (4.33) for $0\le k\le N-1$.

In the following lemma we give the dissipation on c.

Lemma 4.3 *Under the priori assumption* (4.2)*, then for* $k=0,1,\cdots,N-1$*, we have*

$$\frac{d}{dt}\int_{\mathbb{R}^3}\nabla^k u\nabla^{k+1}c dx+C\|\nabla^{k+1}c\|_{L^2}^2\le C\big(\|\nabla^{k+1}(u,B)\|_{L^2}^2+\|\nabla^{k+2}(u,B)\|_{L^2}^2\big). \tag{4.31}$$

Proof. Applying ∇^k to the second equation of the system (3.1) and multiplying it by $\nabla^{k+1}c$, and then integrating it over $\mathbb{R}^3$, we obtain that

$$\begin{aligned}
&\|\nabla^{k+1}c\|_{L^2}^2\\
&=\langle-\nabla^k u_t,\nabla^{k+1}c\rangle-\langle\nabla^k\mathcal{A}u,\nabla^{k+1}c\rangle+\langle\nabla^k g,\nabla^{k+1}c\rangle\\
&\equiv III_1+III_2+III_3.
\end{aligned} \tag{4.32}$$

Notice that the first term III_1 in the right-hand side of (4.32) involves the time derivative; thus, by the first equation in the system (3.1) and integrating by parts for both the $t-$ and $x-$ variables, we conclude that

$$\begin{aligned}
III_1=\langle-\nabla^k u_t,\nabla^{k+1}c\rangle&=-\int_{\mathbb{R}^3}\nabla^k u_t\nabla^{k+1}c dx\\
&=-\frac{d}{dt}\int_{\mathbb{R}^3}\nabla^k u\nabla^{k+1}c dx-\int_{\mathbb{R}^3}\nabla^k \mathrm{div}u\nabla^k c_t dx\\
&=-\frac{d}{dt}\int_{\mathbb{R}^3}\nabla^k u\nabla^{k+1}c dx+\int_{\mathbb{R}^3}\nabla^k \mathrm{div}u\nabla^k\big(\mathrm{div}u+\mathrm{div}(cu)\big)dx\\
&=-\frac{d}{dt}\int_{\mathbb{R}^3}\nabla^k u\nabla^{k+1}c dx+\|\nabla^{k+1}u\|_{L^2}^2+\int_{\mathbb{R}^3}\nabla^k \mathrm{div}u\nabla^k\mathrm{div}(cu)dx.
\end{aligned} \tag{4.33}$$

By similar argument for the proof of the estimate (4.14), we bound the last term of the right side of (4.33) as follows

$$\begin{aligned}
\int_{\mathbb{R}^3}\nabla^k \mathrm{div}u\nabla^k\mathrm{div}(cu)dx&\le\|\nabla^k(cu)\|_{L^2}\|\nabla^{k+2}u\|_{L^2}\\
&\le C\sum_{0\le l\le k+1}\|\nabla^l c\nabla^{k-l+1}u\|_{L^2}\|\nabla^{k+2}u\|_{L^2}\\
&\le C\delta\big(\|\nabla^{k+1}c\|_{L^2}^2+\|\nabla^{k+1}u\|_{L^2}^2+\|\nabla^{k+2}u\|_{L^2}^2\big).
\end{aligned} \tag{4.34}$$

Combining the above estimates with (4.33)-(4.34), we have

$$III_1 \le -\frac{d}{dt}\int_{\mathbb{R}^3} \nabla^k u \nabla^{k+1} c dx + C\|\nabla^{k+1}u\|_{L^2}^2 + C\delta\|\nabla^{k+2}u\|_{L^2}^2 + C\delta\|\nabla^{k+1}c\|_{L^2}^2. \tag{4.35}$$

By integrating by parts, Hölder's inequality and Cauchy's inequality, we have

$$III_2 \le C\varepsilon\|\nabla^{k+1}c\|_{L^2}^2 + C\|\nabla^{k+2}u\|_{L^2}^2. \tag{4.36}$$

Finally, similar to the estimate of the term I_2, we have

$$III_3 \le C\delta\big(\|\nabla^{k+1}c\|_{L^2}^2 + \|\nabla^{k+1}u\|_{L^2}^2 + \|\nabla^{k+1}B\|_{L^2}^2 + \|\nabla^{k+2}B\|_{L^2}^2\big). \tag{4.37}$$

Putting the estimates (4.35), (4.36) and (4.37) into (4.32), we conclude (4.31) since δ and ε are small.

5 Negative Sobolev estimates

In this section, we will derive the evolution of the negative Sobolev norms of the solution.

Lemma 5.1 *For $s \in (0, 1/2]$, we have*

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3} \big(|\Lambda^{-s}c|^2 + \Lambda^{-s}u|^2 + |\Lambda^{-s}B|^2\big)\,dx + C\|\nabla\Lambda^{-s}(u,B)\|_{L^2}^2 \\
&\le C\|(\nabla c, \nabla u, \nabla B)\|_{H^1}^2\big(\|\Lambda^{-s}c\|_{L^2} + \|\Lambda^{-s}u\|_{L^2} + \|\Lambda^{-s}B\|_{L^2}\big);
\end{aligned} \tag{5.1}$$

and for $s \in (1/2, 3/2)$, we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3} \big(|\Lambda^{-s}c|^2 + |\Lambda^{-s}u|^2 + |\Lambda^{-s}B|^2\big)\,dx + C\|\nabla\Lambda^{-s}(u,B)\|_{L^2}^2 \\
&\le C\|(c,u,B)\|_{L^2}^{s-1/2}\|(\nabla c, \nabla u, \nabla B)\|_{H^1}^{5/2-s}\big(\|\Lambda^{-s}c\|_{L^2} + \|\Lambda^{-s}u\|_{L^2} + \|\Lambda^{-s}B\|_{L^2}\big) \\
&\quad + C\|\nabla B\|_{L^2}^{s-1/2}\|\nabla^2 B\|_{L^2}^{3/2-s}\|\nabla B\|_{L^2}\|\Lambda^{-s}B\|_{L^2}.
\end{aligned} \tag{5.2}$$

Proof. Applying Λ^{-s} to the first three equations of (3.1), and multiplying the resulting by $\Lambda^{-s}c, \Lambda^{-s}u, \Lambda^{-s}B$ respectively, summing up and then integrating over $\mathbb{R}^3$ by parts, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} \big(|\Lambda^{-s}c|^2 + |\Lambda^{-s}u|^2 + |\Lambda^{-s}B|^2\big)\,dx + \int_{\mathbb{R}^3}\big((\mu+\nu)|\nabla\Lambda^{-s}(u,B)|^2 + (\mu+\lambda)|\mathrm{div}\Lambda^{-s}u|^2\big)\,dx \\
&= \int_{\mathbb{R}^3}\big(\Lambda^{-s}f\Lambda^{-s}c - \Lambda^{-s}g\Lambda^{-s}u + \Lambda^{-s}h\Lambda^{-s}B\big)\,dx.
\end{aligned} \tag{5.3}$$

Due to the condition (1.2), we first obtain the second term in left-hand side of (5.3) as follows

$$\int_{\mathbb{R}^3}\big(\mu|\nabla\Lambda^{-s}u|^2 + (\mu+\lambda)|\mathrm{div}\Lambda^{-s}u|^2\big)\,dx \ge C\|\nabla\Lambda^{-s}u\|_{L^2}^2. \tag{5.4}$$

In order to estimate the nonlinear terms in the right-hand side of (5.3), we shall use the estimate (2.5) of Riesz potential in Lemma 2.7. This forces us to require that $s \in (0, 3/2]$. If $s \in (0, 1/2]$, then $1/2 + s/3 < 1$ and $3/s \geq 6$. Then using the estimate (2.5) and the Sobolev interpolation of Lemma 2.1, together with Hölder's inequality and Young's inequality, we get

$$
\begin{aligned}
-\int_{\mathbb{R}^3} \Lambda^{-s}(c\mathrm{div}u)\Lambda^{-s}c\,dx &\leq C\|\Lambda^{-s}(c\mathrm{div}u)\|_{L^2}\|\Lambda^{-s}c\|_{L^2} \\
&\leq C\|c\mathrm{div}u\|_{L^{\frac{1}{1/2+s/3}}}\|\Lambda^{-s}c\|_{L^2} \leq C\|c\|_{L^{3/s}}\|\nabla u\|_{L^2}\|\Lambda^{-s}c\|_{L^2} \\
&\leq C\|\nabla c\|_{L^2}^{1/2-s}\|\nabla^2 c\|_{L^2}^{1/2+s}\|\nabla u\|_{L^2}\|\Lambda^{-s}c\|_{L^2} \\
&\leq C\big(\|\nabla c\|_{H^1}^2 + \|\nabla u\|_{L^2}^2\big)\|\Lambda^{-s}c\|_{L^2}.
\end{aligned} \tag{5.5}
$$

$$
\begin{aligned}
-\int_{\mathbb{R}^3} \Lambda^{-s}\big(L_3(c)(B\cdot\nabla B)\Lambda^{-s}u\,dx &\leq C\|\Lambda^{-s}\big(L_3(c)(B\cdot\nabla B)\|_{L^2}\|\Lambda^{-s}u\|_{L^2} \\
&\leq C\|L_3(c)\|_{L^\infty}\|B\cdot\nabla B\|_{L^{\frac{1}{1/2+s/3}}}\|\Lambda^{-s}c\|_{L^2} \\
&\leq C\|B\|_{L^{3/s}}\|\nabla B\|_{L^2}\|\Lambda^{-s}u\|_{L^2} \\
&\leq C\|\nabla B\|_{L^2}^{1/2-s}\|\nabla^2 B\|_{L^2}^{1/2+s}\|\nabla B\|_{L^2}\|\Lambda^{-s}c\|_{L^2} \\
&\leq C\big(\|\nabla B\|_{H^1}^2 + \|\nabla u\|_{L^2}^2\big)\|\Lambda^{-s}c\|_{L^2}.
\end{aligned} \tag{5.6}
$$

Similarly, we can bound the remaining terms by

$$-\int_{\mathbb{R}^3} \Lambda^{-s}(u\cdot\nabla c)\Lambda^{-s}c\,dx \leq C\big(\|\nabla u\|_{H^1}^2 + \|\nabla c\|_{L^2}^2\big)\|\Lambda^{-s}c\|_{L^2}, \tag{5.7}$$

$$-\int_{\mathbb{R}^3} \Lambda^{-s}\big(u\cdot\nabla u\big)\cdot\Lambda^{-s}u\,dx \leq C\|\nabla u\|_{H^1}^2\|\Lambda^{-s}u\|_{L^2}, \tag{5.8}$$

$$\int_{\mathbb{R}^3} \Lambda^{-s}\big(L_1(c)\nabla^2 u\big)\Lambda^{-s}u\,dx \leq C\big(\|\nabla c\|_{H^1}^2 + \|\nabla^2 u\|_{L^2}^2\big)\|\Lambda^{-s}u\|_{L^2}, \tag{5.9}$$

$$-\int_{\mathbb{R}^3} \Lambda^{-s}\big(L_2(c)\nabla c\big)\cdot\Lambda^{-s}u\,dx \leq C\|\nabla c\|_{H^1}^2\|\Lambda^{-s}u\|_{L^2}, \tag{5.10}$$

$$-\int_{\mathbb{R}^3} \Lambda^{-s}h\cdot\Lambda^{-s}B\,dx \leq C\big(\|\nabla B\|_{H^1}^2 + \|\nabla u\|_{H^1}^2\big)\|\Lambda^{-s}B\|_{L^2}. \tag{5.11}$$

Hence, plugging the estimates (5.5)-(5.11) into (5.3), we deduce (5.1).

For $s \in (1/2, 3/2)$, we shall estimate the right hand side of (5.3) in a different way. Since $s \in (1/2, 3/2)$, we have that $1/2 + s/3 < 1$ and $2 < 3/s < 6$. Then using Sobolev's interpolation, we have

$$
\begin{aligned}
-\int_{\mathbb{R}^3} \Lambda^{-s}(c\mathrm{div}u)\Lambda^{-s}c\,dx &\leq C\|c\|_{L^{3/s}}\|\nabla u\|_{L^2}\|\Lambda^{-s}c\|_{L^2} \\
&\leq C\|c\|_{L^2}^{s-1/2}\|\nabla c\|_{L^2}^{3/2-s}\|\nabla u\|_{L^2}\|\Lambda^{-s}c\|_{L^2},
\end{aligned} \tag{5.12}
$$

$$-\int_{\mathbb{R}^3} \Lambda^{-s}(u\cdot\nabla c)\Lambda^{-s}c\,dx \leq C\|u\|_{L^2}^{s-1/2}\|\nabla u\|_{L^2}^{3/2-s}\|\nabla c\|_{L^2}\|\Lambda^{-s}c\|_{L^2}, \tag{5.13}$$

$$-\int_{\mathbb{R}^3} \Lambda^{-s}\big(L_3(c)(B\cdot\nabla B)\big)\cdot\Lambda^{-s}u\,dx \leq C\|B\|_{L^2}^{s-1/2}\|\nabla B\|_{L^2}^{3/2-s}\|\nabla B\|_{L^2}\|\Lambda^{-s}u\|_{L^2}, \tag{5.14}$$

$$-\int_{\mathbb{R}^3} \Lambda^{-s}\big(u\cdot\nabla u\big)\cdot\Lambda^{-s}u\,dx \le C\|u\|_{L^2}^{s-1/2}\|\nabla u\|_{L^2}^{3/2-s}\|\nabla u\|_{L^2}\|\Lambda^{-s}u\|_{L^2}, \tag{5.15}$$

$$-\int_{\mathbb{R}^3} \Lambda^{-s}\big(L_1(c)\nabla^2 u)\big)\Lambda^{-s}u\,dx \le C\|c\|_{L^2}^{s-1/2}\|\nabla c\|_{L^2}^{3/2-s}\|\nabla^2 u\|_{L^2}\|\Lambda^{-s}u\|_{L^2}, \tag{5.16}$$

$$-\int_{\mathbb{R}^3} \Lambda^{-s}\big(L_2(c)\nabla c\big)\cdot\Lambda^{-s}u\,dx \le C\|c\|_{L^2}^{s-1/2}\|\nabla c\|_{L^2}^{3/2-s}\|\nabla c\|_{L^2}\|\Lambda^{-s}c\|_{L^2}. \tag{5.17}$$

$$\begin{aligned}\int_{\mathbb{R}^3} \Lambda^{-s}h\cdot\Lambda^{-s}B\,dx \le{}& C\|u\|_{L^2}^{s-1/2}\|\nabla u\|_{L^2}^{3/2-s}\|\nabla B\|_{L^2}\|\Lambda^{-s}B\|_{L^2}\\ &+C\|B\|_{L^2}^{s-1/2}\|\nabla B\|_{L^2}^{3/2-s}\|\nabla u\|_{L^2}\|\Lambda^{-s}B\|_{L^2}\\ &+C\|B\|_{L^2}^{s-1/2}\|\nabla B\|_{L^2}^{3/2-s}\|\nabla B\|_{L^2}\|\Lambda^{-s}B\|_{L^2}\\ &+C\|B\|_{L^2}^{s-1/2}\|\nabla B\|_{L^2}^{3/2-s}\|\nabla^2 B\|_{L^2}\|\Lambda^{-s}B\|_{L^2}\\ &+C\|\nabla B\|_{L^2}^{s-1/2}\|\nabla^2 B\|_{L^2}^{3/2-s}\|\nabla B\|_{L^2}\|\Lambda^{-s}B\|_{L^2}.\end{aligned} \tag{5.18}$$

Hence, plutting the estimates (5.12)-(5.18) into (5.3), we conclude (5.2).

6 The proof of Theorems 1.1 and 1.2

In this section, we shall combine all the energy estimates that we have derived in the previous two sections to prove Theorems 1.1 and 1.2.

The proof of Theorem 1.1 In order to prove (1.5), we need to close the energy estimates at each $l-th$ level in weak sense. Let $N\ge 3$ and $0\le l\le m-1$ with $1\le m\le N$. Summing up the estimates (4.5) for from $k=l$ to $m-1$, and then adding the estimate (4.23) for $k=m-1$, by changing the index and since δ is sufficiently small, we obtain

$$\begin{aligned}&\frac{d}{dt}\sum_{\ell\le k\le m}\big(\|\nabla^k c\|_{L^2}^2+\|\nabla^k u\|_{L^2}^2+\|\nabla^k B\|_{L^2}^2\big)+C_1\sum_{\ell+1\le k\le m+1}\|\nabla^k(u,B)\|_{L^2}^2\\ &\le C_2\delta\sum_{\ell+1\le k\le m}\|\nabla^k c\|_{L^2}^2.\end{aligned} \tag{6.1}$$

Summing up the estimates (4.31) for from $k=\ell$ to $m-1$, we have

$$\begin{aligned}&\frac{d}{dt}\sum_{\ell\le k\le m-1}\int_{\mathbb{R}^3}\nabla^k u\cdot\nabla\nabla^k c\,dx+C_3\sum_{\ell+1\le k\le m}\|\nabla^k c\|_{L^2}^2\\ &\le C_4\sum_{\ell+1\le k\le m+1}\|\nabla^k(u,B)\|_{L^2}^2.\end{aligned} \tag{6.2}$$

Multiplying (6.2) by $2C_2\delta/C_3$, adding it with (6.1), since $\delta > 0$ is small, we deduce that there exists a constant $C_5 > 0$ such that for $0 \le \ell \le m-1$,

$$
\begin{aligned}
&\frac{d}{dt}\Big\{ \sum_{\ell\le k\le m} \big(\|\nabla^k c\|_{L^2}^2 + \|\nabla^k u\|_{L^2}^2 + \|\nabla^k B\|_{L^2}^2\big) \\
&\quad + 2C_2\delta/C_3 \sum_{\ell\le k\le m-1} \int_{\mathbb{R}^3} \nabla^k u\cdot\nabla\nabla^k c\,dx\Big\} \\
&\quad + C_5\Big(\sum_{\ell+1\le k\le m} \|\nabla^k c\|_{L^2}^2 + \sum_{\ell+1\le k\le m+1} \|\nabla^k (u,B)\|_{L^2}^2\Big) \le 0.
\end{aligned}
\tag{6.3}
$$

Define

$$
\begin{aligned}
\mathcal{E}_\ell^m(t) = &\sum_{\ell\le k\le m} \big(\|\nabla^k c\|_{L^2}^2 + \|\nabla^k u\|_{L^2}^2 + \|\nabla^k B\|_{L^2}^2\big) \\
&+ 2C_2\delta/C_3 \sum_{\ell\le k\le m-1} \int_{\mathbb{R}^3} \nabla^k u\cdot\nabla\nabla^k c\,dx.
\end{aligned}
$$

Since δ is so small that $\mathcal{E}_\ell^m(t)$ can be equivalent to $\|\nabla^\ell c(t)\|_{H^{m-\ell}}^2+\|\nabla^\ell u(t)\|_{H^{m-\ell}}^2+\|\nabla^\ell B(t)\|_{H^{m-\ell}}^2$.

Then, we may write (6.3) as that for $0 \le \ell \le m-1$,

$$
\frac{d}{dt}\mathcal{E}_\ell^m(t) + \|\nabla^{\ell+1} c(t)\|_{H^{m-\ell-1}}^2 + \|\nabla^{\ell+1} u(t)\|_{H^{m-\ell}}^2 + \|\nabla^{\ell+1} B(t)\|_{H^{m-\ell}}^2 \le 0. \tag{6.4}
$$

Now, let $l = 0$ and $m = 3$ in (6.4), and then integrating the equation directly in time, we get

$$
\|\rho(t)-1\|_{H^3}^2 + \|u(t)\|_{H^3}^2 + \|B(t)\|_{H^3}^2 \le C\mathcal{E}_0^3 \le C(\|\rho_0-1\|_{H^3}^2 + \|u_0\|_{H^3}^2 + \|B_0\|_{H^3}^2). \tag{6.5}
$$

By a standard continuity argument, this closes the priori estimates (4.2) if we assume that $\|\rho_0 - 1\|_{H^3} + \|u_0\|_{H^3} + \|B_0\|_{H^3} \le \delta$ is sufficiently small. This in turn allows us to take $l = 0$ and $m = N$ in (6.4) to get

$$
\begin{aligned}
&\|\rho(t)-1\|_{H^N}^2 + \|u(t)\|_{H^N}^2 + \|B(t)\|_{H^N}^2 + \int_0^t \|\nabla\rho(\tau)\|_{H^{N-1}}^2 + \|\nabla^\ell (u,B)(t)\|_{H^N}^2 \\
&\le C\big(\|\rho_0-1\|_{H^N}^2 + \|u_0\|_{H^N}^2 + \|B_0\|_{H^N}^2\big).
\end{aligned}
\tag{6.6}
$$

This proves (1.5). This completes the proof of the Theorem 1.1

The proof of Theorem 1.2 Now we turn to prove (1.6)-(1.7). However, we are not able to prove them for all $s \in [0, 3/2)$ at this moment. We shall first prove them for $s \in [0, 1/2]$. We define $\mathcal{E}_{-s}(t)$ to be the expression under the time derivative in the estimates (5.1)–(5.2) of Lemma 5.1 , which is equivalent to $\|\Lambda^{-s}c(t)\|_{L^2}^2 + \|\Lambda^{-s}u(t)\|_{L^2}^2 + \|\Lambda^{-s}B(t)\|_{L^2}^2$. Then, integrating in time (5.1) , by (1.5), we obtain that for $s \in (0, 1/2]$,

$$
\begin{aligned}
\mathcal{E}_{-s}(t) &\le \mathcal{E}_{-s}(0) + C\int_0^t \|\nabla(c,u,B)(\tau)\|_{H^1}^2 \sqrt{\mathcal{E}_{-s}(\tau)}\,d\tau \\
&\le C_0\Big(1 + \sup_{0\le\tau\le t} \sqrt{\mathcal{E}_{-s}(\tau)}\Big),
\end{aligned}
\tag{6.7}
$$

which implies that

$$\|\Lambda^{-s}c(t)\|_{L^2}^2 + \|\Lambda^{-s}u(t)\|_{L^2}^2 + \|\Lambda^{-s}B(t)\|_{L^2}^2 \le C_0 \quad \text{for } s \in [0,1/2]. \tag{6.8}$$

From Lemma 2.6 we have

$$\|\nabla^{\ell+1} f\|_{L^2} \ge C\|\Lambda^{-s}f\|_{L^2}^{-\frac{1}{\ell+s}}\|\nabla^\ell f\|_{L^2}^{1+\frac{1}{\ell+s}} \quad \text{for } \ell = 1,\dots,N-1. \tag{6.9}$$

By (6.8) and (6.9), we obtain

$$\|\nabla^{\ell+1}c\|_{L^2}^2 + \|\nabla^{\ell+1}u\|_{L^2}^2 + \|\nabla^{\ell+1}B\|_{L^2}^2 \ge C_0\left(\|\nabla^\ell c\|_{L^2}^2 + \|\nabla^\ell u\|_{L^2}^2 + \|\nabla^\ell B\|_{L^2}^2\right)^{1+\frac{1}{\ell+s}}. \tag{6.10}$$

This together with (1.5) implies in particular that for $\ell = 1,\dots,N-1$,

$$\begin{aligned} &\|\nabla^{\ell+1}c\|_{H^{N-\ell-1}}^2 + \|\nabla^{\ell+1}u\|_{H^{N-\ell-1}}^2 + \|\nabla^{\ell+1}B\|_{H^{N-\ell-1}}^2 \\ &\ge C_0\left(\|\nabla^\ell c\|_{H^{N-\ell}}^2 + \|\nabla^\ell u\|_{H^{N-\ell}}^2 + \|\nabla^\ell B\|_{H^{N-\ell}}^2\right)^{1+\frac{1}{\ell+s}}. \end{aligned} \tag{6.11}$$

In view of (6.11) and (6.4), we deduce the following time differential inequality

$$\frac{d}{dt}\mathcal{E}_\ell^N + C_0\left(\mathcal{E}_\ell^N\right)^{1+\frac{1}{\ell+s}} \le 0 \quad \text{for } \ell = 1,\dots,N-1. \tag{6.12}$$

Solving this inequality directly gives, together with (6.6),

$$\mathcal{E}_\ell^N(t) \le C_0(1+t)^{-(\ell+s)} \quad \text{for } \ell = 1,\dots,N-1. \tag{6.13}$$

Consequently, we obtain that for $s \in [0,1/2]$,

$$\|\nabla^\ell c(t)\|_{H^{N-\ell}}^2 + \|\nabla^\ell u(t)\|_{H^{N-\ell}}^2 + \|\nabla^\ell B(t)\|_{H^{N-\ell}}^2 \le C_0(1+t)^{-(\ell+s)} \quad \text{for } \ell = 1,\dots,N-1. \tag{6.14}$$

Thus, by (6.14), (1.5) and the interpolation, we deduce (1.7) for $s \in [0,1/2]$.

For $s \in (1/2,3/2)$, noticing that $c_0, u_0, B_0 \in \dot{H}^{-1/2}$ since $\dot{H}^{-s} \cap L^2 \subset \dot{H}^{-s'}$ for any $s' \in [0,s]$, we then deduce (1.6) and (1.7) with $s = 1/2$ and the following decay result holds:

$$\|\nabla^\ell c(t)\|_{H^{N-\ell}}^2 + \|\nabla^\ell u(t)\|_{H^{N-\ell}}^2 + \|\nabla^\ell B(t)\|_{H^{N-\ell}}^2 \le C_0(1+t)^{-(\ell+1/2)} \quad \text{for } -\frac{1}{2} \le \ell \le N-1. \tag{6.15}$$

Therefore, for $s \in (1/2,3/2)$, from (6.15) and (5.2), we have

$$\begin{aligned} \mathcal{E}_{-s}(t) &\le \mathcal{E}_{-s}(0) + C\int_0^t \left(\|(c,u,B)\|_{L^2}^{s-1/2}\|\nabla(c,u,B)\|_{H^1}^{5/2-s}\right)\sqrt{\mathcal{E}_{-s}(\tau)}\,d\tau \\ &\quad + C\int_0^t \left(\|\nabla B\|_{L^2}^{s+1/2}\|\nabla^2 B\|_{H^1}^{3/2-s}\right)\sqrt{\mathcal{E}_{-s}(\tau)}\,d\tau \\ &\le C_0 + C_0\int_0^t (1+\tau)^{-(7/4-s/2)}\,d\tau \sup_{0\le\tau\le t}\sqrt{\mathcal{E}_{-s}(\tau)} \\ &\quad + C_0\int_0^t (1+\tau)^{-(9/4-s/2)}\,d\tau \sup_{0\le\tau\le t}\sqrt{\mathcal{E}_{-s}(\tau)} \\ &\le C_0\left(1 + \sup_{0\le\tau\le t}\sqrt{\mathcal{E}_{-s}(\tau)}\right). \end{aligned} \tag{6.16}$$

This implies (1.6) for $s \in (1/2, 3/2)$, that is,

$$\|\Lambda^{-s} c(t)\|_{L^2}^2 + \|\Lambda^{-s} u(t)\|_{L^2}^2 + \|\Lambda^{-s} B(t)\|_{L^2}^2 \leq C_0 \quad \text{for } s \in (1/2, 3/2). \tag{6.17}$$

Repeat the similar argument (6.9)-(6.14), we can prove (1.7) for $s \in (1/2, 3/2)$. This completes the proof of the Theorem 1.2.

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