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THE NON–LINEAR 2D SYMMETRIC HELMERT TRANSFORMATION : AN EXACT NON–LINEAR LEAST–SQUARES SOLUTION

Abstract

*In this paper a particular class of non-linear least-squares problems for which it is possible to take advantage of the special structure of the non-linear model, is discussed. The non-linear models are of the ruled-type (*Teunissen, *1985a). The proposed solution strategy is applied to the* 2D *non-linear Symmetric Helmert transformation which is defined in the paper. An exact non-linear least-squares solution, using a rotational invariant covariance structure is given.*

1. Introduction

The aim of the present paper is to derive an exact non-linear least-squares solution for the 2D non-linear Symmetric Helmert transformation. In section two we discuss a particular class of ***non-linear*** least-squares problems for which it is possible to take advantage of the special structure of the non-linear model. The non-linear models are manifolds of the ***ruled-type*** (see Teunissen, 1985a). We show that for this class of non-linear least-squares problems a two-step procedure can be devised. The first step consists of a ***linear*** least-squares problem, while the second step consists of a non-linear least-squares problem of a ***reduced*** dimension. In general the second step has to be solved through the use of linearization and iteration techniques, such as Gauss'method or variations thereof. A ***theorem*** is given which justifies the proposed two-step procedure.

In section three we generalize the stochastic model of the classical linear 2D Helmert transformation to ***rotational-invariant*** covariance matrices. The linear least-squares solution is given.

In section four we introduce our new non-linear 2D Symmetric Helmert transformation. A rotational-invariant covariance structure is assumed. The non-linear least-squares solution is derived with the proposed two-step procedure. We show that the product of the scale estimators $\hat{\lambda}_{SH}$ and $\hat{\lambda}'_{SH}$ of the Symmetric Helmert transformation and its inverse satisfies $\hat{\lambda}_{SH} \cdot \hat{\lambda}'_{SH} = 1$. We also show that in general one systematically underestimates the scale when using the classical Helmert transformation.

The appendix contains a proof of an expression for the derivative of an orthogonal projector. This result is useful in itself for perturbation analysis and is needed when one wants to apply Gauss'iteration method to the second step of the proposed two-step procedure.

2. A particular class of Non-Linear Least-Squares Problems

We will study a method that takes advantage of a special structure of an optimization problem, which is expressed so that the optimization with respect to some of the variables is easier than with respect to the others.

Example : Orthogonal projection onto a ruled surface (see Teunissen, 1985a)

A ***ruled surface*** is a surface which has the property that through every point of the surface there passes a straight line which lies entirely in the surface. Thus the surface is covered by straight lines, called ***rulings*** which form a family depending on one parameter.

In order to find a parametrization of a ruled surface choose on the surface a curve transversal to the rulings. Let this curve be given by $c(v)$, $v \in R$. At any point of this curve take a vector t of the ruling which passes through this point. This vector obviously depends on v. Thus we have $t(v)$. Now we can write the equation of the surface as

$$a(u, v) = c(v) + ut(v), \quad u, v \in R, \quad a, c, t \in R^3 . \tag{2.1}$$

The parameter v indicates the ruling on the surface and the parameter u shows the position on the ruling.

Now let us assume that we have to solve for the following non-linear least-squares problem :

$$\underset{u,v}{\mathit{min.}} \; \| y - a(u, v) \|^2 , \tag{2.2}$$

with the norm $\| . \|^2 = (.)^* Q_y^{-1} (.)$.

Since the ruled surface is flat in the directions of the rulings, whilst curved in the directions transversal to it, it becomes advantageous to perform the adjustment in two steps. In the first step one would then solve for a ***linear*** least-squares adjustment problem, and in the second step for a non-linear adjustment problem of a ***reduced dimension.*** That is, one first solves for

$$\underset{u}{\mathit{min.}} \; \| (y - c(v)) - t(v) u \|^2 , \tag{2.3}$$

which gives

$$u(v) = [t^*(v) Q_y^{-1} t(v)]^{-1} t^*(v) Q_y^{-1} (y - c(v)) . \tag{2.4}$$

Then in the second step one solves for the non-linear problem;

$$\underset{v}{\mathit{min.}} \; \| y - (c(v) + t(v) u(v)) \|^2 . \tag{2.5}$$

As a generalization of the foregoing example, we are interested in solving the non-linear model

$$E\{y\} = A(z)\,x \ , \ \ Cov.\{y\} = Q_y \ , \tag{2.6}$$

in a least-squares sense; where y is the m dimensional vector of observational variates, $E\{.\}$ stands for the mathematical expectation, $A(z)$ is a mxn_1 matrix, Q_y is the mxm positive definite covariance matrix of y, and x and z are respectively the n_1– and n_2 dimensional vectors of unknown parameters. We will assume that matrix $A(z)$ has constant full rank for all z of interest. We can write (2.6) in index notation as

$$E\{y^i\} = A^i_\alpha(z)\,x^\alpha \ , \ \ Cov.\{y^i\} = g^{ij} \ . \tag{2.6'}$$

We will assume that the mn_1 functions $A^i_\alpha(z)$ are continuously differentiable. We define

$$\begin{cases} \text{a)} \quad f(x,z) \triangleq \| y - A(z)\,x \|^2 \\ \text{b)} \quad f_1(z) \triangleq \| P^\perp_{A(z)}\, y \|^2 \\ \text{c)} \quad x(z) \triangleq A^-(z)\, y \end{cases} \tag{2.7}$$

where $\|.\|^2 = (.)^* Q_y^{-1} (.)$, $P_{A(z)}$ is the orthogonal projector projecting onto the rangespace of $A(z)$, $P^\perp_{A(z)} = I - P_{A(z)}$ is the orthogonal projector projecting onto the orthogonal complement of the rangespace of $A(z)$ and $A^-(z)$ is the least-squares inverse of $A(z)$.

Since $x(z)$ is the solution of $\underset{x}{min.}\, f(x,z)$ we have that

$$\begin{cases} \text{a)} \quad f_1(z) = f(x(z),z) = \underset{x}{min.}\, f(x,z) \quad \forall \ z \\ \text{b)} \quad f_1(z) \leqq f(x,z) \qquad \forall \ \ x, z \ . \end{cases} \tag{2.8}$$

From (2.7) also follows that

$$\begin{cases} \text{a)} \quad \partial_x f(x,z) = -2\,(y - A(z)\,x)^* Q_y^{-1} A(z) \\ \text{b)} \quad \partial_z f(x,z) = -2\,(y - A(z)\,x)^* Q_y^{-1} \partial_z A(z)\,x \\ \text{c)} \quad \partial_z f_1(z) = -2\,(P^\perp_{A(z)}\,y)^* Q_y^{-1} \partial_z P_{A(z)}\, y \ . \end{cases} \tag{2.9}$$

In the appendix it is proved that

$$\partial_z P_{A(z)} = (I - P_{A(z)})\partial_z A(z) A^-(z) + [Q_y^{-1}(I - P_{A(z)})\partial_z A(z) A^-(z) Q_y]^* . \quad (2.10)$$

Since

$$\begin{cases} \text{a)} \quad P^{\perp *}_{A(z)} Q_y^{-1} P^{\perp}_{A(z)} = Q_y^{-1} P^{\perp}_{A(z)} \\ \text{b)} \quad A^-(z) P^{\perp}_{A(z)} = 0 \ , \end{cases} \qquad (2.11)$$

substitution of (2.10) into (2.9c) gives

$$\partial_z f_1(z) = -2 y^* Q_y^{-1} P^{\perp}_{A(z)} \partial_z A(z) A^-(z) y \ . \qquad (2.12)$$

We are now ready to proof the following theorem, which gives a justification for the discussed two-step procedure.

Theorem

(i). If $\hat{x}$ and $\hat{z}$ are such that

$$\text{a)} \quad \partial_z f_1(\hat{z}) = 0 , \qquad \text{b)} \quad \hat{x} = A^-(\hat{z}) y \ , \qquad (2.13)$$

then

$$\text{a)} \quad f_1(\hat{z}) = f(\hat{x}, \hat{z}) , \quad \text{b)} \quad \partial_x f(\hat{x}, \hat{z}) = 0 , \quad \text{c)} \quad \partial_z f(\hat{x}, \hat{z}) = 0 \ . \qquad (2.14)$$

(ii). If $\hat{x}$ and $\hat{z}$ are such that

$$\text{a)} \quad f_1(\hat{z}) \leqq f_1(z) \ \forall \ z , \qquad \text{b)} \quad \hat{x} = A^-(\hat{z}) y \ , \qquad (2.15)$$

then

$$f(\hat{x}, \hat{z}) \leqq f(x, z) \ \forall \ x, z \ . \qquad (2.16)$$

(iii). If $\hat{x}$ and $\hat{z}$ are such that

$$f(\hat{x}, \hat{z}) \leqq f(x, z) \ \forall \ x, z \ , \qquad (2.17)$$

then

$$\text{a)} \quad f_1(\hat{z}) = f(\hat{x}, \hat{z}) , \qquad \text{b)} \quad f_1(\hat{z}) \leqq f_1(z) \ \forall \ z \ . \qquad (2.18)$$

proof of (i) :

(2.14a) follows from (2.7c), (2.13b) and (2.8a).

(2.14b) follows from (2.13b), (2.9a), (2.11a) and the fact that $P^{\perp}_{A(z)} A(z) = 0$.

(2.14c) follows from (2.13a), (2.13b), (2.9b), (2.11a) and (2.12).

Thus if $\hat{z}$ is a stationary point of $f_1(z)$ and $\hat{x}$ is defined by (2.13b) then $(\hat{x}, \hat{z})$ forms a stationary point of $f(x, z)$.

proof of (ii) :

We will give the proof by contradiction. Assume that a $\bar{x}$ and $\bar{z}$ exist such that $f(\bar{x}, \bar{z}) < f(\hat{x}, \hat{z})$. With (2.8b) this gives : $f_1(\bar{z}) \leqq f(\bar{x}, \bar{z}) < f(\hat{x}, \hat{z})$. With (2.15b), (2.7c) and (2.8a) this gives : $f_1(\bar{z}) \leqq f(\bar{x}, \bar{z}) < f(\hat{x}, \hat{z}) = f(x(\hat{z}), \hat{z}) = f_1(\hat{z})$. But this contradicts our assumption that $f_1(\hat{z}) \leqq f(z) \ \forall \ z$. Hence no $\bar{x}$ and $\bar{z}$ exist such that $f(\bar{x}, \bar{z}) < f(\hat{x}, \hat{z})$.

Thus if $\hat{z}$ is a global minimum of $f_1(z)$ and $\hat{x}$ is defined by (2.15b) then $(\hat{x}, \hat{z})$ is a global minimum of $f(x, z)$.

proof of (iii) :

First we will proof (2.18a).

From (2.8b) follows that $f_1(\hat{z}) \leqq f(\hat{x}, \hat{z})$. Now let $\bar{x} = A^-(\hat{z})y$. With (2.7c) and (2.8a) follows then that $f_1(\hat{z}) = f(\bar{x}, \hat{z}) \leqq f(\hat{x}, \hat{z})$. Since $(\hat{x}, \hat{z})$ is a global minimum of $f(x, z)$ we must have equality, i.e. $f_1(\hat{z}) = f(\bar{x}, \hat{z}) = f(\hat{x}, \hat{z})$. We will proof (2.18b) by contradiction.

Assume that a $\bar{z}$ exists such that $f_1(\bar{z}) < f_1(\hat{z})$. Now let $\bar{x} = A^-(\bar{z})y$. With (2.7c) and (2.8a) this gives : $f_1(\bar{z}) = f(x(\bar{z}), \bar{z}) = f(\bar{x}, \bar{z}) < f_1(\hat{z})$. According to (2.18a) we have $f_1(\hat{z}) = f(\hat{x}, \hat{z})$ and thus $f_1(\bar{z}) = f(x(\bar{z}), \bar{z}) = f(\bar{x}, \bar{z}) < f_1(\hat{z}) = f(\hat{x}, \hat{z})$. But this contradicts our assumption that $f(\hat{x}, \hat{z}) \leqq f(x, z) \ \forall \ x, z$. Hence no $\bar{z}$ exists such that $f_1(\bar{z}) < f_1(\hat{z})$.

Thus if $(\hat{x}, \hat{z})$ is a global minimum of $f(x, z)$ then $\hat{z}$ is a global minimum of $f_1(z)$. This concludes the proof of the theorem.

From (ii) and (iii) of the Theorem follows that if the global minimum $(\hat{x}, \hat{z})$ of $f(x, z)$ is unique, then also the global minimum of $f_1(z)$ is unique and is given by $\hat{z}$. Conversely, if the global minimum $\hat{z}$ of $f_1(z)$ unique then the global minimum of $f(x, z)$ is unique and is given by $(\hat{x}, \hat{z})$. The uniqueness of the x-component follows from the uniqueness of the least-squares inverse $A^-(z)$, since $A(z)$ is assumed to be of constant full rank.

When one applies the above described two-step procedure one still has to solve for the non-linear problem $\underset{z}{min.} \| P^{\perp}_{A(z)} y \|^2$. This can be done by Gauss'iteration method or variations thereof. In this paper we will not discuss the application of the Gauss'iteration method to the above problem, but see e.g. (**Teunissen**, 1984, 1985a and b) for more details. Instead we will use the described two-step procedure to solve for the 2 dimensional non-linear ***Symmetric*** Helmert transformation in an analytical way.

3. The 2D Helmert transformation with a rotational invariant covariance structure

The linear model of the 2D Helmert transformation reads

$$E\left\{\begin{bmatrix} x_i \\ y_i \end{bmatrix}\right\} = \lambda \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u_i \\ v_i \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} \quad , \quad i = 1, \ldots, n \ , \tag{3.1}$$

where $(x_i \, , \, y_i)$ are the observed cartesian coordinates, $(u_i \, , \, v_i)$ are the fixed given coordinates and λ , θ , t_x and t_y are respectively the four unknown scale, orientation and translation parameters.

We can write (3.1) in a more convenient form by making use of the Kronecker product $\otimes$, for which the following four properties hold (see e.g. Rao, 1973) :

$$\begin{cases} (A \otimes B)^* = A^* \otimes B^* \quad (A \otimes B)^- = A^- \otimes B^- \quad \text{using any inverse} \\ A_1 A_2 \otimes B_1 B_2 = (A_1 \otimes B_1)(A_2 \otimes B_2) \ , \\ (A + B) \otimes C = A \otimes C + B \otimes C \end{cases} \tag{3.2}$$

Take therefore the definitions :

$$\begin{cases} x \triangleq (\ldots x_i \ldots)^* \ , \quad y \triangleq (\ldots y_i \ldots)^* \ , \quad z \triangleq (x^* \ y^*)^* \ , \\ u \triangleq (\ldots u_i \ldots)^* \ , \quad v \triangleq (\ldots v_i \ldots)^* \ , \quad w \triangleq (u^* \ v^*)^* \ , \\ S \triangleq \lambda \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \quad e \triangleq (1 \ldots 1)^* \ , \quad t \triangleq (t_x \ t_y)^* \ , \end{cases} \tag{3.3}$$

and write (3.1) as

$$E\{z\} = (S \otimes I_n \quad I_2 \otimes e) \begin{pmatrix} w \\ t \end{pmatrix} \tag{3.4}$$

We assume the covariance matrix of z to be ***rotational invariant,*** i.e.

$$Cov.\{z\} = I_2 \otimes Q_z \ , \tag{3.5}$$

where Q_z is an arbitrary $n \times n$ positive definite matrix. The least-squares solution of the linear model (3.4) – (3.5) of the 2D Helmert transformation with a rotational invariant covariance structure was given in (Teunissen, 1986) and reads :

$$\left\{\begin{aligned} \hat{\lambda} &= \frac{[\,[\bar{u}^* Q_z^{-1}\bar{x} + \bar{v}^* Q_z^{-1}\bar{y}]^2 + [\bar{v}^* Q_z^{-1}\bar{x} - \bar{u}^* Q_z^{-1}\bar{y}]^2\,]^{\frac{1}{2}}}{[\bar{u}^* Q_z^{-1}\bar{u} + \bar{v}^* Q_z^{-1}\bar{v}]} \\ \hat{\theta} &= \tan^{-1}\frac{\bar{v}^* Q_z^{-1}\bar{x} - \bar{u}^* Q_z^{-1}\bar{y}}{\bar{u}^* Q_z^{-1}\bar{x} + \bar{v}^* Q_z^{-1}\bar{y}} \\ \hat{t}_x &= x_c - \hat{\lambda}\cos\hat{\theta}\, u_c - \hat{\lambda}\sin\hat{\theta}\, v_c \\ \hat{t}_y &= y_c - \hat{\lambda}\cos\hat{\theta}\, v_c + \hat{\lambda}\sin\hat{\theta}\, u_c \ , \end{aligned}\right. \tag{3.6}$$

where the weighted centred coordinates are defined by

$$\left\{\begin{aligned} &\bar{x} \triangleq P_c^{\perp}x \ , \quad \bar{y} \triangleq P_e^{\perp}y \ , \quad \bar{u} \triangleq P_e^{\perp}u \ , \quad \bar{v} \triangleq P_e^{\perp}v \\ &x_c \triangleq \frac{e^* Q_z^{-1}x}{e^* Q_z^{-1}e}, \quad y_c \triangleq \frac{e^* Q_z^{-1}y}{e^* Q_z^{-1}e} \ , \quad u_c \triangleq \frac{e^* Q_z^{-1}u}{e^* Q_z^{-1}e}, \quad v_c \triangleq \frac{e^* Q_z^{-1}v}{e^* Q_z^{-1}e} \\ &P_e^{\perp} \triangleq I_n - e\,(e^* Q_z^{-1}e)^{-1}\, e^* Q_z^{-1} \end{aligned}\right. \tag{3.7}$$

Note that if $Q_z = I_n$, solution (3.6) reduces to that of the well-known classical Helmert transformation (Helmert, 1893).

4. The 2D Symmetric Helmert transformation

We define the non-linear model of the *2D Symmetric Helmert transformation* with a rotational invariant covariance structure as :

$$E\left\{\begin{bmatrix} z \\ w \end{bmatrix}\right\} = \begin{bmatrix} S\otimes I_n & I_2\otimes e \\ I_{2n} & 0 \end{bmatrix}\begin{bmatrix} w' \\ t \end{bmatrix}, \quad \begin{bmatrix} I_2\otimes Q_z & 0 \\ 0 & I_2\otimes Q_w \end{bmatrix}. \tag{4.1}$$

We will assume that

$$Q_z = \sigma^2 Q_w \ , \quad \sigma \in R^+ . \tag{4.2}$$

Our problem is to solve for model (4.1) in a least-squares sense. We define

$$f(w', t, \lambda, \theta) \triangleq \left\| \begin{bmatrix} z \\ w \end{bmatrix} - \begin{bmatrix} S\otimes I_n & I_2\otimes e \\ I_{2n} & 0 \end{bmatrix}\begin{bmatrix} w' \\ t \end{bmatrix} \right\|^2 , \tag{4.3}$$

where

$$\|.\|^2 = (.)^* \begin{bmatrix} I_2 \otimes Q_z^{-1} & 0 \\ 0 & I_2 \otimes Q_w^{-1} \end{bmatrix} (.) \ . \tag{4.4}$$

In order to solve

$$\underset{w',t,\lambda,\theta}{min.} \ f(w',t,\lambda,\theta) \ , \tag{4.5}$$

we proceed in two steps. First we fix λ and θ, and solve for

$$\underset{w',t}{min.} \ f(w',t,\lambda,\theta) \ . \tag{4.6}$$

This is a *linear* least-squares problem. Its solution is denoted by $w'(\lambda,\theta)$, $t(\lambda,\theta)$. In the second step we solve for

$$\underset{\lambda,\theta}{min.} \ f_1(\lambda,\theta) = \underset{\lambda,\theta}{min.} \ f(w'(\lambda,\theta), t(\lambda,\theta), \lambda, \theta) \ . \tag{4.7}$$

Once we have found the solution $\hat{\lambda}$ and $\hat{\theta}$ of this *non-linear* least-squares problem, the complete solution of (4.5) is given by

$$\begin{cases} \hat{w}' = w(\hat{\lambda},\hat{\theta}) \\ \hat{t} = t(\hat{\lambda},\hat{\theta}) \\ \hat{\lambda} = \hat{\lambda} \\ \hat{\theta} = \hat{\theta} \end{cases} \tag{4.8}$$

Step 1 : (λ and θ fixed)

For λ and θ fixed we find from (4.2), (4.3), (4.4) and (4.6) that

$$\begin{bmatrix} w'(\lambda,\theta) \\ t(\lambda,\theta) \end{bmatrix} = \begin{bmatrix} (\lambda^2+\sigma^2) I_2 \otimes Q_z^{-1} & S^* \otimes Q_z^{-1} e \\ S \otimes e^* Q_z^{-1} & e^* Q_z^{-1} e I_2 \end{bmatrix}^{-1} \begin{bmatrix} S^* \otimes Q_z^{-1} & \sigma^2 I_2 \otimes Q_z^{-1} \\ I_2 \otimes e^* Q_z^{-1} & 0 \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} \tag{4.9}$$

$$= \begin{bmatrix} (\lambda^2+\sigma^2)^{-1} [I_2 \otimes Q_z + (\lambda/\sigma)^2 (e^* Q_z^{-1} e)^{-1} I_2 \otimes ee^*] & -\sigma^{-2} (e^* Q_z^{-1} e)^{-1} S^* \otimes e \\ -\sigma^{-2} (e^* Q_z^{-1} e)^{-1} S \otimes e^* & (\lambda^2+\sigma^2)\sigma^{-2} (e^* Q_z^{-1} e)^{-1} I_2 \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} S^* \otimes Q_z^{-1} & \sigma^2 I_2 \otimes Q_z^{-1} \\ I_2 \otimes e^* Q_z^{-1} & 0 \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix}$$

or

$$\begin{cases} w'(\lambda,\theta) = (\lambda^2+\sigma^2)^{-1}\,[S^*\otimes I_n)\,\bar{z} + \sigma^2\,\bar{w}] + w_c \otimes e \\ t\;\;(\lambda,\theta) = z_c - S\,w_c\;, \end{cases} \tag{4.10}$$

where the weighted centred coordinates are defined as

$$\begin{cases} z_c \otimes e \triangleq (I_2 \otimes P_e)\,z\;,\quad w_c \otimes e \triangleq (I_2 \otimes P_e)\,w \\ \bar{z} \triangleq z - z_c \otimes e\;,\quad \bar{w} \triangleq w - w_c \otimes e\;,\quad P_e \triangleq e\,(e^*\,Q_z^{-1} e)^{-1}\,e^*\,Q_z^{-1} \end{cases} \tag{4.11}$$

Step 2 :

Substitution of (4.10) into (4.3) gives

$$f_1(\lambda,\theta) = \left\| \begin{bmatrix} z \\ w \end{bmatrix} - \begin{bmatrix} S\otimes I_n & I_2 \otimes e \\ I_{2n} & 0 \end{bmatrix} \begin{bmatrix} w'(\lambda,\theta) \\ t\,(\lambda,\theta) \end{bmatrix} \right\|^2 \tag{4.12}$$

$$= \left\| \begin{bmatrix} \sigma^2\,(\lambda^2+\sigma^2)^{-1}\,[\bar{z} - (S\otimes I_n)\,\bar{w}] \\ -(\lambda^2+\sigma^2)^{-1}\,S^*\otimes I_n\,[\bar{z} - (S\otimes I_n)\,\bar{w}] \end{bmatrix} \right\|^2$$

or

$$f_1(\lambda,\theta) = \sigma^2\,(\lambda^2+\sigma^2)^{-1}\,\|\,\bar{z} - (S\otimes I_n)\,\bar{w}\,\|^2\;, \tag{4.13}$$

where

$$\|\,.\,\|^2 = (\,.\,)^*\,I_2 \otimes Q_z^{-1}(\,.\,)\;. \tag{4.14}$$

In order to find $\hat{\lambda}$ and $\hat{\theta}$ we need to minimize (4.13). Using the reparametrization

$$\lambda = \sigma\,\tan\varphi\;, \tag{4.15}$$

we can write (4.13) as

$$f_1(\sigma\,\tan\varphi\,,\theta) = \begin{bmatrix} \sin\phi\,\cos\theta \\ \sin\phi\,\sin\theta \\ \cos\phi \end{bmatrix}^* \begin{bmatrix} a & 0 & b \\ 0 & a & c \\ b & c & d \end{bmatrix} \begin{bmatrix} \sin\varphi\,\cos\theta \\ \sin\varphi\,\sin\theta \\ \cos\varphi \end{bmatrix}, \tag{4.16}$$

where

$$\begin{cases} a = \sigma^2 (\bar{u}^* Q_z^{-1} \bar{u} + \bar{v}^* Q_z^{-1} \bar{v}) \ , & d = \bar{x}^* Q_z^{-1} \bar{x} + \bar{y}^* Q_z^{-1} \bar{y} \\ b = -\sigma \, (\bar{u}^* Q_z^{-1} \bar{x} + \bar{v}^* Q_z^{-1} \bar{y}) \ , & c = -\sigma (\bar{v}^* Q_z^{-1} \bar{x} - \bar{u}^* Q_z^{-1} \bar{y}) \end{cases} \tag{4.17}$$

The minimization problem

$$\underset{\phi,\theta}{min.} \ f_1 \, (\sigma \, tan \, \phi \, , \theta) \ , \tag{4.18}$$

reduces to an ***eigenvalue problem*** :

$$\begin{vmatrix} a-\mu & 0 & b \\ 0 & a-\mu & c \\ b & c & d-\mu \end{vmatrix} = 0 \tag{4.19}$$

The three eigenvalues of (4.19) read :

$$\begin{cases} \mu & = a \\ \mu_{1,2} & = \dfrac{(a+d) \pm [(a+d)^2 + 4(b^2 + c^2 - ad)]^{\frac{1}{2}}}{2} \end{cases} \tag{4.20}$$

Hence, the smallest eigenvalue reads :

$$\mu_{min} = \frac{(a+d) - [(a-d)^2 + 4(b^2 + c^2)]^{\frac{1}{2}}}{2} \tag{4.21}$$

From the three equations

$$\begin{bmatrix} a-\mu_{min} & 0 & b \\ 0 & a-\mu_{min} & c \\ b & c & d-\mu_{min} \end{bmatrix} \begin{bmatrix} sin \ \hat{\phi} \, cos \, \hat{\theta} \\ sin \ \hat{\phi} \, cos \, \hat{\theta} \\ cos \, \hat{\phi} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} , \tag{4.22}$$

we find that

$$\begin{cases} \hat{\lambda} \, cos \, \hat{\theta} = \sigma \, tan \, \hat{\phi} \, cos \, \hat{\theta} = \dfrac{-\sigma b}{a - \mu_{min}} \\ \hat{\lambda} \, sin \, \hat{\theta} = \sigma \, tan \, \hat{\phi} \, sin \, \hat{\theta} = \dfrac{-\sigma c}{a - \mu_{min}} \end{cases} \tag{4.23}$$

From (4.10), (4.17), (4.21) and (4.23) follows therefore that the final least-squares solution of the ***non-linear*** 2D ***Symmetric Helmert transformation*** (4.1)–(4.2) is given by :

$$\hat{w}' = w_c \otimes e + (\sigma^2 + \hat{\lambda}^2)^{-1} [(\hat{S}^* \otimes I_n)\bar{z} + \sigma^2 \bar{w}] \ ,$$

$$\hat{t} = z_c - \hat{S} w_c \ , \quad \hat{S} = \hat{\lambda} \begin{bmatrix} \cos\hat{\theta} & \sin\hat{\theta} \\ -\sin\hat{\theta} & \cos\hat{\theta} \end{bmatrix} ,$$

$$\hat{\lambda}\cos\hat{\theta} = \frac{2\sigma^2 <\bar{z},\bar{w}>}{\sigma^2 \|\bar{w}\|^2 - \|\bar{z}\|^2 + [(\sigma^2 \|\bar{w}\|^2 - \|\bar{z}\|^2)^2 + 4\sigma^2 (<\bar{z},\bar{w}>^2 + <\bar{z},\bar{w}''>^2)]^{\frac{1}{2}}}$$

$$\hat{\lambda}\sin\hat{\theta} = \frac{2\sigma^2 <\bar{z},\bar{w}''>}{\sigma^2 \|\bar{w}\|^2 - \|\bar{z}\|^2 + [(\sigma^2 \|\bar{w}\|^2 - \|\bar{z}\|^2)^2 + 4\sigma^2 (<\bar{z},\bar{w}>^2 + <\bar{z},\bar{w}''>^2)]^{\frac{1}{2}}} \tag{4.24}$$

with the inner product

$$<.,.> = (.)^* I_2 \otimes Q_z^{-1} (.) \ ,$$

and $\quad \bar{w}'' = \begin{bmatrix} \bar{v} \\ -\bar{u} \end{bmatrix}$.

Note that this solution reduces to that of (3.6) if $\sigma^2 \to \infty$. Also note that the smallest eigenvalue μ_{min} of (4.21) is not unique if

$$\|\bar{z}\|^2 = \|\bar{w}\|^2 \ , \quad <\bar{z},\bar{w}> = 0 \ \text{and} \ <\bar{z},\bar{w}''> = 0$$

If this is the case the matrix of (4.16) reduces to a scaled unit matrix and the solution for $\hat{\lambda}$ and $\hat{\theta}$ becomes indeterminable. We shall disregard this exceptional case.

Furthermore if

$$<\bar{z},\bar{w}> = 0 \ \text{and} \ <\bar{z},\bar{w}''> = 0 \ ,$$

then $\mu_{min} = \sigma^2 \|\bar{w}\|^2$ if $\sigma^2 \|\bar{w}\|^2 < \|\bar{z}\|^2$ and $\mu_{min} = \|\bar{z}\|^2$ if $\|\bar{z}\|^2 < \sigma^2 \|\bar{w}\|^2$. From (4.22) follows then that $\hat{\theta}$ is indeterminable and $\cos\hat{\theta} = 0$ or $\sin\hat{\theta} = 0$. We shall also disregard this exceptional case.

Let λ be a scale parameter, i.e. positive. From (4.22) follows then that

$$\hat{\lambda}/\sigma = \tan\phi = \frac{(b^2 + c^2)^{\frac{1}{2}}}{a - \mu_{min}} = \frac{d - \mu_{min}}{(b^2 + c^2)^{\frac{1}{2}}} \tag{4.25}$$

Let us denote the scale estimators of the Helmert transformation (3.4) by $\hat{\lambda}_H$, of the corresponding transformation when interchanging the role of z and w by $\hat{\lambda}'_H$, of the Symmetric Helmert transformation (4.1) by $\hat{\lambda}_{SH}$ and of the corresponding transformation when interchanging the role of z and w by $\hat{\lambda}'_{SH}$. From (3.6), (4.17) and (4.25) follows then when $\sigma = 1$ that :

$$\begin{cases} \hat{\lambda}_H = \dfrac{(b^2+c^2)^{\frac{1}{2}}}{a} , & \hat{\lambda}'_H = \dfrac{(b^2+c^2)^{\frac{1}{2}}}{d} \\ \hat{\lambda}_{SH} = \dfrac{(b^2+c^2)^{\frac{1}{2}}}{a-\mu_{min}} = \dfrac{d-\mu_{min}}{(b^2+c^2)^{\frac{1}{2}}} , & \hat{\lambda}'_{SH} = \dfrac{(b^2+c^2)^{\frac{1}{2}}}{d-\mu_{min}} = \dfrac{a-\mu_{min}}{(b^2+c^2)^{\frac{1}{2}}} \end{cases} \tag{4.26}$$

This shows that for the 2D transformation :

$$\hat{\lambda}_H . \hat{\lambda}'_H \neq 1 \text{ , but } \hat{\lambda}_{SH} . \hat{\lambda}'_{SH} = 1 \tag{4.27}$$

From (4.26) also follows that :

$$\hat{\lambda}_{SH} = \hat{\lambda}_H . \frac{a}{a-\mu_{min}} \tag{4.28}$$

Hence, we see that in general the classical Helmert transformation systematically underestimates the scale.

The two scale estimates are identical if

$$b^2 + c^2 - ad = 0 \tag{4.29}$$

That is, when $\bar{z}$ is parallel to $\bar{w}$ or to $\bar{w}''$.

5. Concluding remarks

In this paper we discussed a particular class of non-linear least-squares problems for which a useful two-step procedure can be devised. Exact least-squares solutions are given for the 2D Helmert transformation and its non-linear symmetrical generalization. For the two dimensional case a rotational-invariant covariance structure was assumed. Solutions of the linearized versions and teststatistics were already given in (**Teunissen**, 1984). Our exact non-linear least-squares solutions make the computation of approximate values, linearization and iteration superfluous.

Although we had to make some simplifying assumptions in the covariance structure of the observational variates, it is felt that these assumptions are sufficiently general for many practical applications. When digitizing maps, the covariance matrix of the digitized coordinates can often even be simplified to a scaled unit matrix. The assumption of the rotational invariant covariance structure is also in many cases sufficient for geodetic networks. For instance, the Baarda-Alberda substitute matrix (see e.g. **Brouwer** et al., 1982 or **Teunissen**, 1984a) :

$$
\begin{array}{c|cccc}
 & x_i & x_j & y_i & y_j \\
\hline
x_i & d^2 & d^2 - d_{ij}^2 & 0 & 0 \\
x_j & d^2 - d_{ij}^2 & d^2 & 0 & 0 \\
y_i & 0 & 0 & d^2 & d^2 - d_{ij}^2 \\
y_j & 0 & 0 & d^2 - d_{ij}^2 & d^2
\end{array}
$$

d^2 : a parameter

d_{ij}^2 : a covariance function

is an example of a rotational-invariant covariance matrix. It describes the precision of many geodetic networks to a sufficient degree and can therefore be used in our formulae.

In a forthcoming contribution we will derive some local and global distributional properties of our non-linear least-squares estimators. The approach will make use of differential geometric methods of non-linear adjustment (Teunissen, 1984, 1985a, b).

For a discussion of the 3D Helmert transformation we refer to (Sansò, 1973), (Köchle, 1982) and (Krarup, 1985) and for the 3D Helmert transformation with its symmetrical generalization to (Teunissen, 1985).

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APPENDIX

Proof of :

$$\partial_z P_{A(z)} = (I - P_{A(z)})\,\partial_z A(z)\,A^-(z) + [Q_y^{-1}(I - P_{A(z)})\partial_z A(z)\,A^-(z)\,Q_y]^* . \quad \text{(A1)}$$

The orthogonal projector $P_{A(z)}$ and least-squares inverse $A^-(z)$ are given by

$$\begin{cases} \text{a)} \quad P_{A(z)} = A(z)\,N^{-1}(z)\,A^*(z)\,Q_y^{-1} \\ \text{b)} \quad A^-(z) = N^{-1}(z)\,A^*(z)\,Q_y^{-1} \quad , \end{cases} \quad \text{(A2)}$$

where

$$N(z) = A^*(z)\,Q_y^{-1}\,A(z) . \quad \text{(A3)}$$

From

$$\partial_z(N^{-1}(z)\,N(z)) = \partial_z I = 0 = \partial_z N^{-1}(z)\,N(z) + N^{-1}(z)\,\partial_z N(z)$$

follows that

$$\partial_z N^{-1}(z) = -N^{-1}(z)\,\partial_z N(z)\,N^{-1}(z) . \quad \text{(A4)}$$

From

$$\partial_z N(z) = \partial_z (A^*(z)\,Q_y^{-1}\,A(z))$$

follows that

$$\partial_z N(z) = \partial_z A^*(z)\,Q_y^{-1}\,A(z) + A^*(z)\,Q_y^{-1}\,\partial_z A(z) . \quad \text{(A5)}$$

From (A2) follows that

$$\partial_z P_{A(z)} = \partial_z A(z) N^{-1}(z) A^*(z) Q_y^{-1} + A(z) \partial_z N^{-1}(z) A^*(z) Q_y^{-1} \tag{A6}$$
$$+ A(z) N^{-1}(z) \partial_z A^*(z) Q_y^{-1} .$$

With (A4) and (A5) this gives :

$$\partial_z P_{A(z)} = \partial_z A(z) N^{-1}(z) A^*(z) Q_y^{-1} + A(z) [- N^{-1}(z) (\partial_z A^*(z) Q_y^{-1} A(z)$$
$$+ A^*(z) Q_y^{-1} \partial_z A(z)) N^{-1}(z)] A^*(z) Q_y^{-1} + A(z) N^{-1}(z) \partial_z A^*(z) Q_y^{-1}$$
$$= [I - A(z) N^{-1}(z) A^*(z) Q_y^{-1}] \partial_z A(z) N^{-1}(z) A^*(z) Q_y^{-1}$$
$$+ A(z) N^{-1}(z) \partial_z A^*(z) [I - Q_y^{-1} A(z) N^{-1}(z) A^*(z)] Q_y^{-1} .$$

or with (A2) :

$$\partial_z P_{A(z)} = (I - P_{A(z)}) \partial_z A(z) A^-(z) + Q_y A^{-*}(z) \partial_z A^*(z) (I - P_{A(z)})^* Q_y^{-1} , \tag{A7}$$

which is identical to (A1).

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