

H_∞ OPTIMAL STABILIZATION OF A CLASS OF UNCERTAIN IMPULSIVE SYSTEMS: AN LMI APPROACH

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ABSTRACT. This paper studies H_∞ optimal control problems for a class of impulsive dynamical systems with norm-bounded time-varying uncertainty. By using a linear matrix inequality approach, some sufficient conditions are established to ensure both internally asymptotical stability and H_∞ optimal performance of the impulsive closed-loop system. Moreover, based on the stability criteria, a linear time-invariant stabilizing control law is designed. Finally, a numerical example is presented to illustrate the effectiveness of our results.

1. Introduction. In the recent years, impulsive dynamical systems have been attracting much attention in the applied mathematics and control community because the problems involving impulsive effects are not only theoretically challenging, but also of practical significance. An impulsive system is a special class of hybrid dynamical systems, which contain many abrupt changes of states at some certain time instants. Many examples can be found in finance, engineering, and other disciplines [1], [2]. Systems with not only uncertain disturbances but also abrupt state changes at certain time instants may not be well modelled by employing purely continuous or purely discrete systems. However, an uncertain impulsive system can be regarded as a natural framework for mathematical modelling of such dynamical systems. Significant research attention has been focussed on establishing new techniques and methods for treating this class of special dynamical systems, and as a result there have been rapid developments in the control theory of this class of systems over the past two decades. Fundamental theory problems of impulsive dynamical systems with or without uncertainties have been intensively studied in recent literatures, see [2], [3] and [8] and references therein. Moreover, stability and stabilization results of impulsive systems involving switching phenomena and

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time lag effects have also been established in [4]-[6], [9]. However, to the best of our knowledge, few papers dealing with H_∞ optimal control on impulsive systems, except for the work by Guan et al. [10] have been reported. The corresponding theory and results for uncertain impulsive systems have not been fully developed and need to be investigated further. In this paper, we shall study the robust H_∞ optimal control problem of uncertain impulsive systems. We address the problem of designing a feedback control law such the uncertain closed-loop system is asymptotically stable, and the H_∞ norm bound constraints on disturbance attenuation for all admissible uncertainties are satisfied. By employing a linear matrix inequality approach, sufficient conditions for existence of such a control law are presented and an associated linear state feedback control law is constructed by solving a certain linear matrix inequality.

The rest of this paper is organized as follows. In section 2, we introduce the class of uncertain impulsive dynamical systems which will be considered in this paper, along with the associated definitions. In section 3, H_∞ asymptotic stability criteria for this class of uncertain linear impulsive systems are established. Based on the stability results obtained, the appropriate time-invariant feedback control law is designed. In section 4, an illustrative example is given to demonstrate our main results. Finally, concluding remarks are presented in Section 5.

2. Problem Statement. Let $R_+ = [0, +\infty)$ and R_n denote the n -dimensional Euclidean space where $\|\cdot\|$ denotes the Euclidean norm. Let I be the identity matrix. The matrix $\Omega > (\geq, <, \leq) 0$ means that Ω is a symmetric positive definite (positive-semidefinite, negative definite, negative-semidefinite) matrix. Let $\lambda_{max}(\cdot)$ (respectively, $\lambda_{min}(\cdot)$) be the maximum (respectively, minimum) eigenvalue of the matrix .

Usually a linear dynamical system can be described by

$$\dot{x}(t) = A(t)x(t) + C(t)u(t). \quad (1)$$

When the system (1) is time-invariant and involves uncertain disturbances and proportional impulsive effects, it becomes

$$\begin{cases} \dot{x}(t) = Ax(t) + Bw(t) + Cu(t) & t \neq t_k \\ \Delta x(t) = d_k x(t) & t = t_k \\ z(t) = Ex(t) \\ x(t) = 0, & t = t_0 = 0 \end{cases}, \quad (2)$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control input, $w(t) \in R^p$ is the disturbance input with limited energy, i.e. $w(t) \in L_2[0, \infty)$, $z(t) \in R^q$ is the controlled output, $A \in R^{n \times n}$, $B \in R^{n \times p}$, $C \in R^{n \times m}$, $E \in R^{q \times n}$ are constant real matrices that describe the known nominal system, d_k is a real number, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^-) = \lim_{h \rightarrow 0^+} x(t_k - h)$, $x(t_k^+) = \lim_{h \rightarrow 0^+} x(t_k + h)$, t_k is an impulsive jumping point, $k = 1, 2, \dots$, $t_0 < t_1 < t_2 < \dots < t_k < \dots < t_\infty$, $t_k \rightarrow \infty$ as $k \rightarrow \infty$, $\lim_{h \rightarrow 0^+} x(t_k - h) = x(t_k^-) = x(t_k)$, it means that the solution of impulsive system (2) is left continuous.

In the subsequent discussion, we need the following stability concepts for the impulsive dynamical system (2).

Definition 2.1. System (2) is said to be asymptotically stable if for any $\epsilon > 0$, there exists a $\delta = \delta(\epsilon)$, such that if $\|x_0\| \leq \delta$, then the following conditions hold:

$$\|x(t)\| < \epsilon, \text{ for every } t \geq 0 \text{ and } \lim_{t \rightarrow \infty} x(t) = 0.$$

Definition 2.2. The H_∞ optimal control problem is to find a controller which achieves the H_∞ optimal performance, i.e., $\|z(t)\|_2 < \gamma \|w(t)\|_2$, $\gamma > 0$.

The objective of this paper is to design an associated linear state feedback law

$$u(t) = Fx(t), \quad F \in R^{m \times n}, \quad (3)$$

such that the impulsive closed-loop system

$$\begin{cases} \dot{x}(t) = (A + CF)x(t) + Bw(t) & t \neq t_k \\ \Delta x(t) = d_k x(t) & t = t_k \\ z(t) = Ex(t) \\ x(t) = 0, & t = t_0 = 0 \end{cases} \quad (4)$$

is internally asymptotically stable with a given H_∞ optimal performance for all admissible uncertainties.

Lemma 2.3. [4] Given a matrix $G \in R^{p \times q}$ such that $G^T G \leq I$ then

$$2x^T G y \leq x^T x + y^T y \quad (5)$$

for all $x \in R^p$ and $y \in R^q$. In the case $G = I$, (5) reduces to $2x^T y \leq x^T x + y^T y$.

Lemma 2.4. [4] Given a positive definite matrix $P \in R^{n \times n}$ and a symmetric matrix $Q \in R^{n \times n}$ then

$$\lambda_{\min}(P^{-1}Q)x(t)^T P x(t) \leq x(t)^T Q x(t) \quad (6)$$

for all $x(t) \in R^n$.

Definition 2.5. System (4) is called to be internally asymptotically stable if the impulsive closed-loop system (4) is asymptotically stable when $w(t) = 0$.

3. Main Results. In this section, we shall establish asymptotical stability criteria for an uncertain linear impulsive system by a linear matrix inequality approach. Moreover, by using the stability results obtained, we design the time-invariant feedback control law.

Theorem 3.1. For a given $\gamma > 0$, $\prod_{i=1}^k (1+d_i)^2 < 1$, $k \in N$, the impulsive closed-loop system (4) is internally asymptotically stable and H_∞ optimal performance is guaranteed for all admissible uncertainties if there exists a positive definite symmetric matrix $P \in R^{n \times n}$ which satisfies the following linear matrix inequality,

$$\begin{bmatrix} (A + CF)^T P + P(A + CF) & \gamma^{-1} P B & E^T \\ \gamma^{-1} B^T P & -I & 0 \\ E & 0 & -I \end{bmatrix} < 0. \quad (7)$$

Proof. Firstly, we consider the system's H_∞ optimal performance. When $t \in (t_k, t_{k+1}]$, under the conditions of the theorem, choose the candidate Lyapunov functional

$$V(t) = x(t)^T P x(t). \quad (8)$$

It follows that the Lyapunov derivative corresponding to the closed-loop system (4) is given by

$$\begin{aligned}
\dot{V}(t) &= \dot{x}(t)^T P x(t) + x(t)^T P \dot{x}(t) \\
&= (Ax(t) + Bw(t) + Cu(t))^T P x(t) + x(t)^T P (Ax(t) + Bw(t) + Cu(t)) \\
&= x(t)^T A^T P x(t) + x(t)^T P A x(t) + u(t)^T C^T P x(t) + x(t)^T P C u(t) \\
&\quad + w(t)^T B^T P x(t) + x(t)^T P B w(t) \\
&= x(t)^T A^T P x(t) + x(t)^T P A x(t) + u(t)^T C^T P x(t) + x(t)^T P C u(t) \\
&\quad + 2x(t)^T P B w(t),
\end{aligned}$$

where $u(t) = Fx(t)$.

From Lemma 2.3, we have that

$$\begin{aligned}
2x(t)^T P B w(t) &= 2(x(t)^T P B \gamma^{-1})(\gamma w(t)) \\
&\leq \gamma^{-2} x(t)^T P B B^T P x(t) + \gamma^2 w(t)^T w(t).
\end{aligned}$$

Applying Schur complement theorem [7], the linear matrix inequality (7) is equivalent to

$$(A + CF)^T P + P(A + CF) + \gamma^{-2} P B B^T P + E^T E < 0.$$

Then, the derivative of $V(t)$ will yield

$$\begin{aligned}
\dot{V}(t) &\leq x(t)^T ((A + CF)^T P + P(A + CF) + \gamma^{-2} P B B^T P) x(t) \\
&\quad + \gamma^2 w(t)^T w(t) \\
&< -x(t)^T E^T E x(t) + \gamma^2 w(t)^T w(t) \\
&= -\|E x(t)\|^2 + \gamma^2 \|w(t)\|^2,
\end{aligned}$$

i.e.,

$$\dot{V}(t) < -\|z(t)\|^2 + \gamma^2 \|w(t)\|^2. \quad (9)$$

It can be rewritten as

$$\|z(t)\|^2 < -\dot{V}(t) + \gamma^2 \|w(t)\|^2.$$

Integrating from 0 to τ on both sides, we obtain

$$\int_0^\tau \|z(t)\|^2 dt < -\int_0^\tau \dot{V}(t) dt + \gamma^2 \int_0^\tau \|w(t)\|^2 dt, \quad \tau \in (t_k, t_{k+1}]. \quad (10)$$

From (8), we have $V(0) = 0$, $V(\tau) > 0$, $\tau \in (t_k, t_{k+1}]$.

When $\prod_{i=1}^k (1 + d_i)^2 < 1$, $k = 1, 2, \dots$, it follows that

$$\begin{aligned}
\int_0^\tau \dot{V}(t) dt &= \int_0^{t_1} \dot{V}(t) dt + \int_{t_1}^{t_2} \dot{V}(t) dt + \dots + \int_{t_{k-1}}^{t_k} \dot{V}(t) dt + \int_{t_k}^\tau \dot{V}(t) dt \\
&= V(t_1^-) - V(0) + V(t_2^-) - V(t_1^+) + \dots \\
&\quad + V(t_k^-) - V(t_{k-1}^+) + V(\tau) - V(t_k^-) \\
&= \sum_{i=1}^k [1 - (1 + d_k)^2] V(t_i) + V(\tau) - V(0) \\
&> 0.
\end{aligned}$$

Then, $\int_0^\tau \|z(t)\|^2 dt < \gamma^2 \int_0^\tau \|w(t)\|^2 dt$, $\tau \in (t_k, t_{k+1}]$. Hence, when $\tau \rightarrow \infty$, we obtain

$$\left(\int_0^\infty \|z(t)\|^2 dt \right)^{1/2} < \gamma \left(\int_0^\infty \|w(t)\|^2 dt \right)^{1/2},$$

which is equivalent to

$$\|z(t)\|_2 < \gamma \|w(t)\|_2$$

by applying Parseval's theorem and the time-domain expression of the L_2 -norm.

Therefore, H_∞ optimal performance of the impulsive closed-loop system (4) is satisfied.

Next, we shall derive sufficient conditions to ensure internally asymptotical stability of the impulsive system (4). When $w(t) = 0$, then it follows from (9) that

$$\dot{V}(t) < -\|z(t)\| = -x(t)^T E^T E x(t) \quad (11)$$

By Lemma 2.4, we get that

$$x(t)^T E^T E x(t) \geq \lambda_{\min}(P^{-1} E^T E) x(t)^T P x(t).$$

Then, from (11) we have

$$\dot{V}(t) + \eta V(t) < 0, \quad t \in (t_k, t_{k+1}],$$

where $\eta = \lambda_{\min}(P^{-1} E^T E) > 0$. From (4) and (8), we derive

$$\begin{aligned} V(t_k^+) &= x(t_k^+)^T P x(t_k^+) \\ &= (1 + d_k)^2 x(t_k)^T P x(t_k) \\ &= (1 + d_k)^2 V(t_k) \end{aligned}$$

Then, it follows that

$$\begin{aligned} V(t) &< V(t_k^+) \exp(-\eta(t - t_k)) \\ &= (1 + d_k)^2 V(t_k) \exp(-\eta(t - t_k)), \quad t \in (t_k, t_{k+1}]. \end{aligned}$$

When $t \in (t_0, t_1]$,

$$\begin{aligned} V(t) &< V(t_0) \exp(-\eta(t - t_0)) \\ V(t_1) &< V(t_0) \exp(-\eta(t_1 - t_0)). \end{aligned}$$

When $t \in (t_1, t_2]$,

$$\begin{aligned} V(t) &< V(t_1^+) \exp(-\eta(t - t_1)) \\ &= (1 + d_1)^2 V(t_1) \exp(-\eta(t - t_1)) \\ &< (1 + d_1)^2 V(t_0) \exp(-\eta(t - t_0)) \end{aligned}$$

Hence, when $t \in (t_k, t_{k+1}]$,

$$\begin{aligned} V(t) &< V(t_k^+) \exp(-\eta(t - t_k)) \\ &= (1 + d_k)^2 V(t_k) \exp(-\eta(t - t_k)) \\ &< \prod_{i=1}^k (1 + d_i)^2 V(t_0) \exp(-\eta(t - t_0)) \\ &< V(t_0) \exp(-\eta(t - t_0)). \end{aligned}$$

Thus, the impulsive closed-loop system (4) is asymptotically stable when $w(t) = 0$. Therefore, the impulsive closed-loop system (4) is internally asymptotically stable and satisfies H_∞ optimal performance. This completes the proof. \square

Theorem 3.2. *For a given $\gamma > 0$ and $\prod_{i=1}^k (1 + d_i)^2 < 1$, $k \in N$, the impulsive closed-loop system (4) is internally asymptotically stable and H_∞ optimal performance is guaranteed for all admissible uncertainties if there exist a constant $\varepsilon > 0$ and a positive definite symmetric matrix $P \in R^{n \times n}$ which satisfies the following linear matrix inequality*

$$\begin{bmatrix} A^T P + P A + P(\gamma^{-2} B B^T - \varepsilon^{-2} C C^T) P & E^T \\ E & -I \end{bmatrix} < 0. \quad (12)$$

Moreover, a suitable feedback control law is given by

$$u(t) = F x(t), \quad F = -\frac{1}{2\varepsilon^2} C^T P. \quad (13)$$

Proof. Let $u(t) = Fx(t)$, $F = -\frac{1}{2\varepsilon^2}C^T P$, $\varepsilon > 0$. Substituting $F = -\frac{1}{2\varepsilon^2}C^T P$ into (7) and applying theorem 3.1, we can obtain

$$\begin{bmatrix} A^T P + PA + P(\gamma^{-2}BB^T - \varepsilon^{-2}CC^T)P & E^T \\ E & -I \end{bmatrix} < 0. \quad (14)$$

This means that if (14) holds, the feedback control law $u(t) = Fx(t)$, $F = -\frac{1}{2\varepsilon^2}C^T P$ guarantees both internally asymptotical stability and H_∞ optimal performance of the impulsive closed-loop system (4).

Thus, the proof is complete. \square

4. A Numerical Example. In this section, we shall give a numerical example to demonstrate the applicability of the proposed approach.

Consider an uncertain impulsive system with following specification:

$$A = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0.5 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad d_k = -0.5, \quad E = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

Then, system (2) becomes

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 & 1 \\ 1 & 0.5 \end{bmatrix} w(t) & t \neq t_k, \\ \Delta x(t) = -0.5x, & t = t_k, \\ z(t) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} x(t). \end{cases}$$

It is easily check that $\prod_{i=1}^k (1 + d_i)^2 < 1$ holds.

Choose $\gamma = \varepsilon = 1$, by solving (12) under the MATLAB environment with the LMI Toolbox, we obtain the positive definite symmetric matrix

$$P = \begin{bmatrix} 1.4604 & -0.2898 \\ -0.2898 & 1.9320 \end{bmatrix}.$$

Hence, the required state feedback control law is

$$u(t) = Fx(t), \quad F = \begin{bmatrix} 0.1449 & -0.9660 \\ -0.7302 & 0.1449 \end{bmatrix},$$

which internally asymptotically stabilizes the impulsive systems with time varying uncertainty (2) and guarantees $\|z\|_2 < \gamma \|x\|_2$.

5. Conclusion. We have developed a state feedback robust H_∞ optimal control technique for a class of impulsive dynamical systems with time varying uncertainty. Based on a positive definite solution of a linear matrix inequality, the proposed robust H_∞ static state feedback control law guarantees both internally asymptotical stability and robust H_∞ optimal performance for a class of impulsive systems with time varying uncertainty. An illustrative example has been given to demonstrate the applicability of the proposed approach.

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