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On the Minimum-Energy Problem for Positive Discrete Time Linear Systems

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Abstract

The nonnegativity of controls in positive discrete-time linear systems (PDLS) usually give rise to complementarity conditions in the first-order Karush-Kuhn-Tucker optimality conditions – this complicates the analytical solution and usually leads to numerical solutions. At the same time the appeal and the advantages of analytic solutions are well appreciated. In this paper the minimum energy problem for PDLS with fixed terminal state is reduced to a problem with a free final state and analytic solution to the latter is obtained. The relationship between the two problems is studied in full detail..

Key words: positive linear systems, discrete-time systems, minimum energy problem, optimality conditions
MSC2000: 93C55, 49N10, 90C46, 90C35

1. Introduction

The minimum-energy problem for time invariant linear systems is a classical problem in control theory. It has elegant analytic solutions if no restrictions are imposed on the state and control variables [8]. Positive discrete-time linear systems (PDLS) are defined on cones and not on linear spaces since the control and the trajectory are non-negative [6, 10]. The non-negativity of control in such systems gives rise to complementarity conditions in the first-order Karush-Kuhn-Tucker (KKT) optimality conditions [3], which complicates the analytic solution and usually leads to numerical solutions. At the same time the appeal and the advantages of analytic solutions are well appreciated.

Kaczorek [6] has obtained analytical solution to the minimum-energy problem for PDLS with fixed final state under some assumptions, among which the assumption of reachability of the system and the assumption of zero initial state seem to be quite restrictive. The relation of the minimum-energy problem with the reachable sets that is the geometry of the problem is not studied in [6] either. In the very recent work [11], the authors, on the basis of analysis of geometry of the problem, obtain a more general result for PDLS with any non-negative pair of fixed terminal (initial and final) states and scalar controls without the assumption of reachability of the system.

Related work for continuous-time systems with non-negative controls is published in [5, 7, 9] but the positivity of the system is not exploited in these papers. Positivity is an intrinsic property of positive systems and in many cases it helps to simplify the analysis and the results.

In this paper the minimum-energy problem for PDLS with any non-negative pair of fixed terminal (initial and final) states is reduced to a problem with free final state and an analytic solution to the latter is developed using the dynamic programming approach [1].The relationship between the two problems is analysed. A numerical example is also provided.

2. Problem formulation and preliminaries

The *minimum-energy problem* for scalar positive discrete-time linear systems (PDLS) *with fixed final state* is formulated as follows [6, 11]

Minimize
$$J = \frac{1}{2}\sum_{t=0}^{T-1} u^2(t) \tag{1}$$

Subject to

$$x(t+1) = ax(t) + bu(t),\ t = 0,\dots,T-1 \tag{2}$$

$$a,b \geq 0 \quad u(t) \in \mathrm{R}_+ \tag{3}$$

where $x(t)$ is the state of the system at time $t = 0, 1, 2, \dots, T$, $u(t) \in \mathrm{R}_+$ is the control sequence, the symbol R_+ denotes the set of all non-negative real numbers, T is a finite-time horizon, and the initial and final state are given by

$$x(0) = x_0 \geq 0 \text{ and } x(T) = x_T \geq 0. \tag{4}$$

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The state variables $x(t)$, $t = 0,\dots,T$, are, clearly, non-negative for any non-negative initial state $x_0 \geq 0$, and any (non-negative) control sequence (3). So, the minimum-energy problem consists in finding a (non-negative) control sequence $\{u(t) \geq 0, t = 0,\dots,T-1\}$ and the corresponding trajectory $\{x(t), t = 0,1,\dots,T\}$ that satisfy (2) – (4) and minimize the "energy" (1) of the input signal. Note that the dynamics of the positive discrete-time linear systems is described by the difference equation (2) and (3).

Under the natural assumption that $x_T \in R_T(x_0)$, where $R_T(x_0)$ denotes the T – steps reachable set [2, 10, 11], the optimal control sequence that minimizes the cost function (1) in the minimum-energy problem (1) – (4) with fixed final state is given by [11]

$$u^*(t) = \begin{cases} \dfrac{a^{T-(t+1)}\left(x_T - a^{T-t}x^*(t)\right)}{b\sum\limits_{i=t}^{T-1} a^{2(T-(i+1))}}, & b > 0 \\ 0, & b = 0 \end{cases}, \quad t = 0, 1, 2, \dots, T-1, \tag{5}$$

where $x^*(t)$ is the corresponding optimal trajectory, and the optimal value of the cost function (1) is

$$J_0^* = \begin{cases} \dfrac{1}{2}\dfrac{\left(x_T - a^T x_0\right)^2}{b^2\sum\limits_{i=0}^{T-1} a^{2(T-(i+1))}}, & b > 0 \\ 0, & b = 0 \end{cases}. \tag{6}$$

By relaxing the boundary condition $x(T) = x_T$, problem (1) – (4) can be reduced to the following *minimum-energy problem with free final state*

Minimize
$$J = \frac{1}{2}[(x_T - x(T))^2 + \sum_{t=0}^{T-1} u^2(t)] \tag{7}$$

Subject to

$$x(t+1) = ax(t) + bu(t), \; t = 0,\dots,T-1 \tag{8}$$

$$a, b \geq 0 \quad u(t) \in \mathrm{R}_+ \tag{9}$$

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$$x(0) = x_0 \geq 0 \tag{10}$$

The new term in the cost function (1) reflects how close to the given terminal state the terminal state in the reduced problem is. The solution to the minimum-energy problem with free terminal state then gives a control sequence that minimises the energy (1) of the input and at the same time the corresponding (to that control sequence) trajectory ends at a point, which is in a closed proximity of the targeted terminal point x_T.. In other words, the optimum control sequence resolves the "trade-off" between minimizing the energy of the input signal and the deviation from the given terminal point x_T.

In the next section we obtain an analytic solution to the relaxed minimum-energy problem (7) – (10) and provide some comparative analysis with the generic problem (1) – (4).

3. Main results

Theorem 1. *Let on the optimal trajectory* $a^{T-t}x^*(t) \leq x_T$, $t = 0,\dots,T-1$. *Then the optimal control sequence that minimizes the cost function (7) in the minimum-energy problem with free final state (7) – (10) is given by*

$$u^*(t) = \frac{a^{T-(t+1)}b(x_T - a^{T-t}x^*(t))}{1 + b^2\sum_{i=t}^{T-1} a^{2(T-(i+1))}}, \qquad t = 0,\dots,T-1, \tag{11}$$

where $x^*(t)$ *is the corresponding optimal trajectory, and the optimal value of the cost function (1) is*

$$J_0^* = \frac{1}{2}\frac{(x_T - a^T x_0)^2}{1 + b^2\sum_{i=0}^{T-1} a^{2(T-(i+1))}}. \tag{12}$$

Proof.

To find an analytic solution to the optimal control problem (7) – (10) we use the dynamic programming approach [1].

The Bellman equation can be written as

$$J_t(x) = \min_{u\geq 0}\left\{\frac{1}{2}u^2 + J_{t+1}(x)\right\},\ u = u(t), x = x(t), t = 0,..,T-1$$

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with

$$J_T(x) = \frac{1}{2}\left(x_T - x(T)\right)^2$$

We try for $t = T-1$. We have:

$$J_{T-1}(x) = \min_{u \geq 0}\left\{\frac{1}{2}u^2 + \frac{1}{2}\left(x_T - (ax+bu)\right)^2\right\},$$

where $x = x(T-1)$ and $u = u((T-1)$ is to be determined by the initial condition (10). A differentiation of the above expression with respect to u leads to

$$u^*(T-1) = \frac{b(x_T - ax)}{1+b^2}.$$

where to satisfy $u(t) \in \mathbb{R}_+$ we impose the condition $ax(T-1) \leq x_T$, and, therefore,

$$J_{T-1}(x) = \frac{1}{2}\frac{(x_T - ax)^2}{1+b^2}.$$

Similarly, for $t = T-2$ we have:

$$u^*(T-2) = \frac{ab\left(x_T - a^2x\right)}{1+b^2+a^2b^2} \text{ with } a^2x(T-1) \leq x_T,$$

and, respectively,

$$J_{T-2}(x) = \frac{1}{2}\frac{\left(x_T - a^2x\right)^2}{1+b^2+a^2b^2}.$$

We form, now, the induction hypothesis:

$$J_t(x) = \frac{1}{2}\frac{\left(x_T - a^{T-t}x\right)^2}{1+b^2\sum_{i=t}^{T-1}a^{2(T-(i+1))}} \qquad (13)$$

and

$$u^*(t) = \frac{a^{T-(t+1)}b\left(x_T - a^{T-t}x\right)}{1+b^2\sum_{i=t}^{T-1}a^{2(T-(i-1))}},\ ax(t) \leq x_T. \qquad (14)$$

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Let the expression (13) and (14) be true for $t = k + 1$, that is

$$J_{k+1}(x) = \frac{1}{2}\frac{\left(x_T - a^{T-(k+1)}x\right)^2}{1 + b^2 \sum_{i=k+1}^{T-1} a^{2(T-i-1)}} \tag{15}$$

and, respectively,

$$u^*(k+1) = \frac{a^{T-(k+2)}b\left(x_T - a^{T-(k+1)}x\right)}{1 + b^2 \sum_{i=k+1}^{T-1} a^{2(T-(i+1))}},\ a^{k+1}x(k+1) \le x_T. \tag{16}$$

We prove that (13) and (14) hold also for $t = k$ namely

$$J_k(x) = \frac{1}{2}\frac{\left(x_T - a^{T-k}x\right)^2}{1 + b^2 \sum_{i=k}^{T-1} a^{2(T-i-1)}}$$

and, respectively,

$$u^*(k) = \frac{a^{T-(k+1)}b\left(x_T - a^{T-k}x\right)}{1 + b^2 \sum_{i=k}^{T-1} a^{2(T-(i+1))}},\ a^k x(k) \le x_T.$$

For $t = k$ the Bellman equation is specified as

$$J_k(x) = \min_{u\ge 0}\left\{\frac{1}{2}u^2 + J_{k+1}(x)\right\}. \tag{17}$$

A substitution of the state equation (2) (or (8)) in (15) yields

$$J_k(x) = \min_{u\ge 0}\left\{\frac{1}{2}u^2 + \frac{1}{2}\frac{\left(x_T - a^{T-k-1}(ax+bu)\right)^2}{1 + b^2 \sum_{i=k+1}^{T-1} a^{2(T-i-1)}}\right\}, \tag{18}$$

where $u = u(k)$ and the state $x(k) = x$ is a parameter that is to be specified by the initial condition $x(0) = x_0$. The differentiation of the expression $\frac{1}{2}u^2 + \frac{1}{2}\frac{\left(x_T - a^{T-k-1}(ax+bu)\right)^2}{1 + b^2 \sum_{i=k+1}^{T-1} a^{2(T-i-1)}}$ with respect to u results in

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$$u^*(k) = \frac{a^{T-(k+1)} b\left(x_T - a^{T-k} x\right)}{1 + b^2 \sum_{i=k}^{T-1} a^{2(T-(i+1))}}, \quad (19)$$

where the condition $a^{T-k} x \le x_T$ is imposed in order to satisfy $u(t) \in \mathbb{R}_+$. Furthermore, the substitution of (19) in (18) leads to

$$J_k^*(x) = \frac{1}{2} \frac{\left(x_T - a^{T-k} x\right)^2}{1 + b^2 \sum_{i=k}^{T-1} a^{2(T-i-1)}}.$$

So, the assumptions (13) and (14) hold for $t = k$, and, therefore, they are true by induction for any k. For $t = 0$ we have

$$u^*(0) = \frac{a^{T-1} b\left(x_T - a^T x\right)}{1 + b^2 \sum_{i=0}^{T-1} a^{2(T-(i+1))}}.$$

This concludes the proof.

□

Remark

Assume that $x^*(T) = x_T$. Then, from (11) we have

$$u^*(T-1) = \frac{b(x_T - a x^*(T-1))}{1 + b^2}.$$

Substituting the above expression for $u^*(T-1)$ and taking into account $x^*(T) = x_T$ we obtain that $x_T = a x^*(T-1)$ and, hence,

$$x^*(T-1) = \frac{x_T}{a},$$

which implies that the optimal control

$$u^*(T-1) = 0.$$

Continue the process for $t = T-2$ and $t = T-3$, and form the induction hypothesis for $k = t$

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$$x^*(T-t) = \frac{x_T}{a^t}, \tag{20}$$

and

$$u^*(T-t) = 0. \tag{21}$$

Now, assume that the expressions (20) and (21) are true for $k = t$ and show that they hold for $k = t + 1$ that is

$$x^*(T-(t+1)) = x_T / a^{t+1}, \tag{22}$$

and

$$u^*\left(T-\left(t+1\right)\right) = 0.$$

For $k = t + 1$, the expression (14) for the optimal control becomes

$$u^*\left(T-\left(t+1\right)\right) = \frac{a^t b\left(x_T - a^{t+1}x\right)}{1 + b^2 \sum_{i=T-(t+1)}^{T-1} a^{2(T-(i+1))}}, \tag{23}$$

where $x = x(T-(t+1))$. Then, a substitution of (22) and (23) in the state equation (8) yields

$$\frac{x_T}{a^t} = ax + \frac{a^t b^2\left(x_T - a^{t+1}x\right)}{1 + b^2 \sum_{i=T-(t+1)}^{T-1} a^{2(T-(i+1))}},$$

which results in

$$x(T-(t+1)) = x_T / a^{t+1}, \tag{24}$$

so that (20) is true. Taking into account (24) it is not difficult to see that the optimal control (23) becomes $u^*\left(T-\left(t+1\right)\right) = 0$. The hypothesis is thus proved and this concludes the proof. The optimal control sequence is

$u^*(t) = 0$ for $t = 0, 1, 2, \ldots, T-1$,

the corresponding optimal trajectory

$$\{x_0, ax_0, a^2x_0, \ldots, a^T x_0\},$$

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and the optimal cost function

$$J_0^*(x_0) = 0.$$

□

4. Example

To illustrate the approach adopted in this paper we consider the following simple minimum-energy problem with fixed final state.

Minimize
$$J = \frac{1}{2}\sum_{t=0}^{T-1} u^2(t) \tag{25}$$

Subject to
$$x(t+1) = x(t) + u(t),\ t = 0,\dots,3 \tag{26}$$
$$u(t) \in R_+ \tag{27}$$
$$x(0) = x_0 = 1 \text{ and } x(4) = x_4 = 5. \tag{28}$$

The 4-steps reachable set (see [11]) for the PDLS (26) – (27) is $R_4(1) = [1,\infty)$ and the final state $x_4 = 5$ is an interior point of $R_4(1)$. Note also that the system is reachable and stable but not asymptotically [2, 10, 11].

Using expressions (5), (6) and the state equation (2) we obtain the optimal control sequence

$$u^*(0) = 1, u^*(1) = 1, u^*(2) = 1, u^*(3) = 1,$$

the corresponding optimal trajectory

$$x_0 = 1, x^*(1) = 2, x^*(2) = 3, x^*(3) = 4, x^*(4) = x_4 = 5,$$

and the optimal cost function

$$J_0^* = 2.$$

By relaxing the boundary condition $x_4 = 5$ we reduce the problem (25) – (28) to the following minimum-energy problem with free final state

Minimize
$$J = \frac{1}{2}[(5 - x(4))^2 + \sum_{t=0}^{T-1} u^2(t)] \tag{29}$$

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Subject to

$$x(t+1) = x(t) + u(t), \; t = 0,\ldots,3 \tag{30}$$

$$u(t) \in \mathrm{R}_+ \tag{31}$$

$$x(0) = x_0 = 1 \tag{32}$$

To determine the optimal control sequence we use expression (11). For $t = 0$ we have

$$u^*(0) = \frac{4}{5},$$

and using the state equation (30) we find the next state $x^*(1) = \frac{9}{5}$. Consequently, we obtain

$$u^*(1) = \frac{4}{5} \text{ and } x^*(2) = \frac{13}{5},$$

$$u^*(2) = \frac{4}{5} \text{ and } x^*(3) = \frac{17}{5},$$

and, finally,

$$u^*(3) = \frac{4}{5} \text{ and } x^*(4) = \frac{21}{5}.$$

We see that the condition $x^{*}(t) \le x_f$ is satisfied on the optimal trajectory. So, the optimal control sequence is $u^* = \{\frac{4}{5}, \frac{4}{5}, \frac{4}{5}, \frac{4}{5}\}$, the corresponding optimal trajectory $\{x^*(t)\} = \{1, \frac{9}{5}, \frac{13}{5}, \frac{17}{5}, \frac{21}{5}\}$, and the optimal cost function $J_0^* = 1.6$.

The above results tell us that by relaxing the minimum-energy problem (25) - (28) with fixed final state to the minimum-energy problem (29) - (32) with free final state we decrease the energy of the input (control) from 2 to $\frac{32}{25}$ at the expense of not reaching the final state – the deviation from the desired final state 5 is $\frac{4}{5}$.

5. Concluding remarks

In this paper the minimum energy problem for positive discrete-time linear systems with fixed final state is reduced to a minimum energy problem with free final state by including in the cost function a term that reflects the deviation of the final state in the reduced problem from the targeted final state. Using the dynamic

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programming approach an analytic solution of the reduced minimum energy problem with free final state is obtained and analysed. It is shown that the relaxation of the problem leads to a decrease of the "consumed" energy of the input but at the expense of not reaching the desired final state. Such a "trade-off" might be quite appealing in a number of real-life problems.

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