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Research Article

Positive Solutions for $(n-1,1)$-Type Singular Fractional Differential System with Coupled Integral Boundary Conditions

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We study the positive solutions of the $(n-1,1)$-type fractional differential system with coupled integral boundary conditions. The conditions for the existence of positive solutions to the system are established. In addition, we derive explicit formulae for the estimation of the positive solutions and obtain the unique positive solution when certain additional conditions hold. An example is then given to demonstrate the validity of our main results.

1. Introduction

This paper is motivated by the boundary value problem

$$\begin{gathered}
D_{0^+}^{5/2}u(t)+\frac{(\sin t+\cos t)\sqrt{u}}{\sqrt[3]{t(1-t)v}}=0,\\
D_{0^+}^{5/2}v(t)+\frac{\sqrt[3]{v}}{e^t\sqrt{t(1-t)u}}=0,\quad 0<t<1,\\
u(0)=u'(0)=0,\\
u(1)=\frac{1}{2}\int_0^1 v(t)\,dt,\\
v(0)=v'(0)=0,\\
v(1)=\int_0^1 u(t)\,dt^2,
\end{gathered}\tag{1}$$

which arises in a variety of disciplinary areas such as mechanics, chemical physics, mathematical biology, flows, fluid electrical networks, and viscoelasticity (see [1–6] and the references cited therein). In problem (1), the nonlinearity $f(t,x,y)$ may be singular at $t=0,1$ and $y=0$; $g(t,x,y)$ may be singular at $t=0,1$ and $x=0$.

Research on fractional order integrodifferential operators dates back to the end of the 19th century, when Riemann and Liouville introduced the first definition of the fractional derivative. However, this field of study started to become attractive to engineers only in the late 1960s, when fractional derivative description of some real systems was observed. It was found that fractional operators are nonlocal and are more suitable for constructing models possessing memory effect in a long time period, and hence fractional differential equations possess many advantages.

In this paper, we consider the existence of positive solutions for a nonlinear singular fractional differential system with coupled boundary conditions:

$$\begin{gathered}
D_{0^+}^{\alpha_1}u(t)+f(t,u(t),v(t))=0,\\
D_{0^+}^{\alpha_2}v(t)+g(t,u(t),v(t))=0,\quad 0<t<1,\\
u(0)=u'(0)=\cdots=u^{(n-2)}=0,\\
u(1)=\mu_1\int_0^1 v(s)\,dA_1(s),
\end{gathered}$$

$$v(0) = v'(0) = \cdots = v^{(n-2)} = 0,$$
$$v(1) = \mu_2 \int_0^1 u(s)\, dA_2(s), \tag{2}$$

where $n-1 < \alpha_i \le n, n \ge 2$, and $D_{0^+}^{\alpha_i}$ is the standard Riemann-Liouville derivative. $\mu_i > 0$, A_i is right continuous on $[0,1)$, left continuous at $t=1$, and nondecreasing on $[0,1]$, $A_i(0) = 0$, and $\int_0^1 x(s)dA_i(s)$ denotes the Riemann-Stieltjes integrals of x with respect to A_i $(i = 1,2)$. $f : (0,1) \times [0,+\infty) \times (0,+\infty) \to [0,+\infty)$ and $g : (0,1) \times (0,+\infty) \times [0,+\infty) \to [0,+\infty)$ are two continuous functions, and $f(t,x,y)$ may be singular at $t = 0,1$ and $y = 0$, while $g(t,x,y)$ may be singular at $t = 0,1$ and $x = 0$.

Coupled boundary value problem arises naturally in the research of Sturm-Liouville problems, reaction-diffusion equations, mathematical biology, and so on. In recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives with coupled boundary conditions, as shown by [7–16] and the references therein. By using the nonlinear alternative of the Leray and Schauder theorem and the Krasnoselskii fixed point theorem in a cone, Bai and Fang in [17] obtained some results of existence of positive solutions by considering the singular coupled system of nonlinear fractional differential equations:

$$\begin{gathered} D^s u + f(t,v) = 0, \\ D^p v + g(t,u) = 0, \quad 0 < t < 1, \end{gathered} \tag{3}$$

where $0 < s, p < 1$, D^s, D^p are two standard Riemann-Liouville fractional derivative, and $f, g : (0,1] \times [0,+\infty) \to [0,+\infty)$ are two given continuous functions and are singular at $t = 0$.

Wang et al. [18] study the following system of nonlinear fractional differential equations:

$$\begin{gathered} D_{0^+}^{\alpha} u(t) + f(t, v(t)) = 0, \\ D_{0^+}^{\beta} v(t) + g(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = v(0) = 0, \\ u(1) = a u(\xi), \\ v(1) = b v(\xi), \end{gathered} \tag{4}$$

where $1 < \alpha, \beta < 2$, $0 \le a, b < 1$, $0 < \xi < 1$, $f, g : [0,1] \times [0,+\infty) \to [0,+\infty)$ are continuous functions, and $D_{0^+}^{\alpha}$, $D_{0^+}^{\beta}$ are two standard Riemann-Liouville fractional derivatives. By using the Banach fixed point theorem and the nonlinear alternative of Leray-Schauder type, the existence and uniqueness of a positive solution are obtained.

In this paper, we consider the existence and uniqueness of positive solutions for the singular system (2). The work presented in this paper has the following new features. Firstly, until now, coupled integral boundary value problems for fractional differential system as system (2) have seldom been considered when $f(t,x,y)$ may be singular at $t = 0,1$ and $y = 0$, and $g(t,x,y)$ may be singular at $t = 0,1$ and $x = 0$. Also $\int_0^1 x(s)dA_i(s)$ denotes the Riemann-Stieltjes integral, and thus system (2) includes the multipoint problems and integral problems as special cases. Secondly, by using the well-known fixed point theorem due to Guo-Krasnoselskii, we not only obtain the existence of positive solutions for system (2), but also obtain the uniqueness of system (2).

A vector (u, v) is said to be a positive solution of system (2) if and only if (u, v) satisfies (2) and $u(t) > 0$, $v(t) > 0$ for any $t \in (0, 1]$.

2. Preliminaries and Lemmas

In what follows, we present some necessary definitions about fractional calculus theory.

Definition 1 (see [2, 19]). Let $\alpha > 0$ and let u be piecewise continuous on $(0,+\infty)$ and integrable on any finite subinterval of $[0,+\infty)$. Then, for $t > 0$, we call

$$I_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds \tag{5}$$

the Riemann-Liouville fractional integral of u of order α.

Definition 2 (see [2, 19]). The Riemann-Liouville fractional derivative of order $\alpha > 0$, $n-1 \le \alpha < n$, $n \in \mathbb{N}$, is defined as

$$D_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} u(s)\, ds, \tag{6}$$

where $\mathbb{N}$ denotes the natural number set and the function $u(t)$ is n times continuously differentiable on $[0,+\infty)$.

Lemma 3 (see [2]). *Let $\alpha > 0$; then,*

$$\begin{aligned} & I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) \\ & \quad = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \end{aligned} \tag{7}$$

where $c_1, c_2, \ldots, c_n \in (-\infty, +\infty)$ and n is the smallest integer greater than or equal to α.

Lemma 4. *Let $h_i \in C(0,1) \cap L^1(0,1)$ $(i = 1,2)$, and the following condition $(\mathbf{H}_0)$ holds:*

$(\mathbf{H}_0)$

$$\begin{gathered} k_1 = \int_0^1 t^{\alpha_2 - 1} dA_1(t) > 0, \\ k_2 = \int_0^1 t^{\alpha_1 - 1} dA_2(t) > 0, \\ 1 - \mu_1 \mu_2 k_1 k_2 > 0. \end{gathered} \tag{8}$$

Then the system subjected to the coupled boundary conditions

$$\begin{gathered} D_{0^+}^{\alpha_1} u(t) + h_1(t) = 0, \\ D_{0^+}^{\alpha_2} v(t) + h_2(t) = 0, \\ 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)} = 0, \\ u(1) = \mu_1 \int_0^1 v(s)\, dA_1(s), \\ v(0) = v'(0) = \cdots = v^{(n-2)} = 0, \\ v(1) = \mu_2 \int_0^1 u(s)\, dA_2(s) \end{gathered} \tag{9}$$

has an integral representation

$$\begin{aligned} u(t) &= \int_0^1 K_1(t,s) h_1(s)\, ds + \int_0^1 H_1(t,s) h_2(s)\, ds, \\ v(t) &= \int_0^1 K_2(t,s) h_2(s)\, ds + \int_0^1 H_2(t,s) h_1(s)\, ds, \end{aligned} \tag{10}$$

where

$$\begin{aligned} K_1(t,s) &= \frac{\mu_1\mu_2 k_1 t^{\alpha_1-1}}{1-\mu_1\mu_2k_1k_2} \int_0^1 G_1(t,s)\, dA_2(t) + G_1(t,s), \\ H_1(t,s) &= \frac{\mu_1 t^{\alpha_1-1}}{1-\mu_1\mu_2k_1k_2} \int_0^1 G_2(t,s)\, dA_1(t), \\ K_2(t,s) &= \frac{\mu_2\mu_1 k_2 t^{\alpha_2-1}}{1-\mu_1\mu_2k_1k_2} \int_0^1 G_2(t,s)\, dA_1(t) + G_2(t,s), \\ H_2(t,s) &= \frac{\mu_2 t^{\alpha_2-1}}{1-\mu_1\mu_2k_1k_2} \int_0^1 G_1(t,s)\, dA_2(t), \end{aligned} \tag{11}$$

$$G_i(t,s) = \frac{1}{\Gamma(\alpha_i)} \begin{cases} [t(1-s)]^{\alpha_i-1} \\ \quad -(t-s)^{\alpha_i-1}, & 0 \le s \le t \le 1, \quad i = 1,2. \\ [t(1-s)]^{\alpha_i-1}, & 0 \le t \le s \le 1, \end{cases} \tag{12}$$

Proof. By Lemma 3, the system (9) is equivalent to the following integral equations system:

$$u(t) = u(1) t^{\alpha_1-1} + \int_0^1 G_1(t,s) h_1(s)\, ds, \tag{13}$$

$$v(t) = v(1) t^{\alpha_2-1} + \int_0^1 G_2(t,s) h_2(s)\, ds. \tag{14}$$

Integrating (13) and (14) with respect to $dA_2(t)$ and $dA_1(t)$, respectively, we have

$$\begin{aligned} &\int_0^1 u(t)\, dA_2(t) \\ &\quad = u(1) \int_0^1 t^{\alpha_1-1} dA_2(t) + \iint_0^1 G_1(t,s) h_1(s)\, ds\, dA_2(t), \\ &\int_0^1 v(t)\, dA_1(t) \\ &\quad = v(1) \int_0^1 t^{\alpha_2-1} dA_1(t) + \iint_0^1 G_2(t,s) h_2(s)\, ds\, dA_1(t), \end{aligned} \tag{15}$$

which yield

$$\begin{aligned} \frac{1}{\mu_1} u(1) - k_1 v(1) &= \iint_0^1 G_2(t,s) h_2(s)\, ds\, dA_1(t), \\ -k_2 u(1) + \frac{1}{\mu_2} v(1) &= \iint_0^1 G_1(t,s) h_1(s)\, ds\, dA_2(t). \end{aligned} \tag{16}$$

It follows from

$$\begin{vmatrix} \frac{1}{\mu_1} & -k_1 \\ -k_2 & \frac{1}{\mu_2} \end{vmatrix} = \frac{1-\mu_1\mu_2k_1k_2}{\mu_1\mu_2} \neq 0 \tag{17}$$

that the system of (16) has a unique solution, which can be represented as

$$\begin{aligned} u(1) &= \frac{\mu_1}{1-\mu_1\mu_2k_1k_2} \\ &\quad \cdot \left(\iint_0^1 G_2(t,s) h_2(s)\, ds\, dA_1(t) \right. \\ &\quad \left. + \mu_2 k_1 \iint_0^1 G_1(t,s) h_1(s)\, ds\, dA_2(t) \right), \\ v(1) &= \frac{\mu_2}{1-\mu_1\mu_2k_1k_2} \\ &\quad \cdot \left(\iint_0^1 G_1(t,s) h_1(s)\, ds\, dA_2(t) \right. \\ &\quad \left. + \mu_1 k_2 \iint_0^1 G_2(t,s) h_2(s)\, ds\, dA_1(t) \right). \end{aligned} \tag{18}$$

Substituting (18) into (13) and (14), we have

$$
\begin{aligned}
u(t) &= \frac{\mu_1 t^{\alpha_1-1}}{1-\mu_1\mu_2 k_1 k_2} \\
&\quad \cdot \left(\iint_0^1 G_2(t,s) h_2(s)\, ds\, dA_1(t) \right. \\
&\quad \left. + \mu_2 k_1 \iint_0^1 G_1(t,s) h_1(s)\, ds\, dA_2(t) \right) \\
&\quad + \int_0^1 G_1(t,s) h_1(s)\, ds \\
&= \int_0^1 K_1(t,s) h_1(s)\, ds + \int_0^1 H_1(t,s) h_2(s)\, ds, \\
v(t) &= \frac{\mu_2 t^{\alpha_2-1}}{1-\mu_1\mu_2 k_1 k_2} \\
&\quad \cdot \left(\iint_0^1 G_1(t,s) h_1(s)\, ds\, dA_2(t) \right. \\
&\quad \left. + \mu_1 k_2 \iint_0^1 G_2(t,s) h_2(s)\, ds\, dA_1(t) \right) \\
&\quad + \int_0^1 G_2(t,s) h_2(s)\, ds \\
&= \int_0^1 K_2(t,s) h_2(s)\, ds + \int_0^1 H_2(t,s) h_1(s)\, ds.
\end{aligned}
\tag{19}
$$

So (10) holds. This completes the proof of the lemma. □

Lemma 5. *For $t, s \in [0,1]$, the functions $K_i(t,s)$ and $H_i(t,s)$ ($i = 1, 2$) defined by (11) possess the following properties:*

$$
\begin{aligned}
K_1(t,s), H_2(t,s) &\le \rho s (1-s)^{\alpha_1-1}, \\
K_2(t,s), H_1(t,s) &\le \rho s (1-s)^{\alpha_2-1},
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
K_1(t,s), H_1(t,s) &\le \rho t^{\alpha_1-1}, \\
K_2(t,s), H_2(t,s) &\le \rho t^{\alpha_2-1},
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
K_1(t,s) &\ge \varrho t^{\alpha_1-1} s (1-s)^{\alpha_1-1}, \\
H_2(t,s) &\ge \varrho t^{\alpha_2-1} s (1-s)^{\alpha_1-1},
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
K_2(t,s) &\ge \varrho t^{\alpha_2-1} s (1-s)^{\alpha_2-1}, \\
H_1(t,s) &\ge \varrho t^{\alpha_1-1} s (1-s)^{\alpha_2-1},
\end{aligned}
\tag{23}
$$

where

$$
\begin{aligned}
\rho = \max \Bigg\{ & \frac{1}{\Gamma(\alpha_1-1)} \left(\frac{\mu_1\mu_2 k_1}{1-\mu_1\mu_2 k_1 k_2} \int_0^1 dA_2(t) + 1 \right), \\
& \frac{\mu_1}{\Gamma(\alpha_2-1)(1-\mu_1\mu_2 k_1 k_2)} \int_0^1 A_1(t), \\
& \frac{1}{\Gamma(\alpha_2-1)} \left(\frac{\mu_2\mu_1 k_2}{1-\mu_1\mu_2 k_1 k_2} \int_0^1 dA_1(t) + 1 \right), \\
& \frac{\mu_2}{\Gamma(\alpha_1-1)(1-\mu_1\mu_2 k_1 k_2)} \int_0^1 dA_2(t) \Bigg\}, \\
\varrho = \min \Bigg\{ & \frac{\mu_1\mu_2 k_1}{\Gamma(\alpha_1)(1-\mu_1\mu_2 k_1 k_2)} \int_0^1 (1-t) t^{\alpha_1-1} dA_2(t), \\
& \frac{\mu_1}{\Gamma(\alpha_2)(1-\mu_1\mu_2 k_1 k_2)} \int_0^1 (1-t) t^{\alpha_2-1} dA_1(t), \\
& \frac{\mu_2\mu_1 k_2}{\Gamma(\alpha_2)(1-\mu_1\mu_2 k_1 k_2)} \int_0^1 (1-t) t^{\alpha_2-1} dA_1(t), \\
& \frac{\mu_2}{\Gamma(\alpha_1)(1-\mu_1\mu_2 k_1 k_2)} \int_0^1 (1-t) t^{\alpha_1-1} dA_2(t) \Bigg\}.
\end{aligned}
\tag{24}
$$

Proof. By [20], for any $t, s \in [0,1]$, we have

$$
\begin{aligned}
& \frac{(1-t) t^{\alpha_i-1} s (1-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} \\
& \quad \le G_i(t,s) \le \frac{s(1-s)^{\alpha_i-1}}{\Gamma(\alpha_i-1)}, \quad i = 1, 2.
\end{aligned}
\tag{25}
$$

It follows from (11) and (25) that

$$
\begin{aligned}
& K_1(t,s) \\
& = \frac{\mu_1\mu_2 k_1 t^{\alpha_1-1}}{1-\mu_1\mu_2 k_1 k_2} \int_0^1 G_1(t,s)\, dA_2(t) + G_1(t,s) \\
& \le \frac{\mu_1\mu_2 k_1 t^{\alpha_1-1}}{1-\mu_1\mu_2 k_1 k_2} \int_0^1 \frac{s(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1-1)} dA_2(t) + \frac{s(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1-1)} \\
& \le \frac{1}{\Gamma(\alpha_1-1)} \left(\frac{\mu_1\mu_2 k_1 t^{\alpha_1-1}}{1-\mu_1\mu_2 k_1 k_2} \int_0^1 dA_2(t) + 1 \right) s(1-s)^{\alpha_1-1} \\
& \le \frac{1}{\Gamma(\alpha_1-1)} \left(\frac{\mu_1\mu_2 k_1}{1-\mu_1\mu_2 k_1 k_2} \int_0^1 dA_2(t) + 1 \right) s(1-s)^{\alpha_1-1} \\
& \le \rho s (1-s)^{\alpha_1-1},
\end{aligned}
$$

$$\begin{aligned}
&H_2(t,s)\\
&= \frac{\mu_2 t^{\alpha_2-1}}{1-\mu_1\mu_2k_1k_2}\int_0^1 G_1(t,s)\,dA_2(t)\\
&\le \frac{s(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1-1)}\frac{\mu_2 t^{\alpha_2-1}}{1-\mu_1\mu_2k_1k_2}\int_0^1 dA_2(t)\\
&\le \left(\frac{\mu_2}{\Gamma(\alpha_1-1)(1-\mu_1\mu_2k_1k_2)}\int_0^1 dA_2(t)\right)s(1-s)^{\alpha_1-1}\\
&\le \rho s(1-s)^{\alpha_1-1}.
\end{aligned} \tag{26}$$

As for the proof of (26), we have

$$K_2(t,s), H_1(t,s) \le \rho s(1-s)^{\alpha_2-1}, \quad t,s \in [0,1]; \tag{27}$$

that is, (20) holds.

By [20], for any $t, s \in [0,1]$, we have

$$\begin{aligned}
&\frac{(1-t)t^{\alpha_i-1}s(1-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}\\
&\quad \le G_i(t,s) \le \frac{t^{\alpha_i-1}(1-t)}{\Gamma(\alpha_i-1)}, \quad i=1,2.
\end{aligned} \tag{28}$$

So, by (11) and (28), we have

$$\begin{aligned}
&K_1(t,s)\\
&= \frac{\mu_1\mu_2k_1 t^{\alpha_1-1}}{1-\mu_1\mu_2k_1k_2}\int_0^1 G_1(t,s)\,dA_2(t) + G_1(t,s)\\
&\le \frac{\mu_1\mu_2k_1 t^{\alpha_1-1}}{1-\mu_1\mu_2k_1k_2}\int_0^1 \frac{t^{\alpha_1-1}(1-t)}{\Gamma(\alpha_1-1)}dA_2(t)\\
&\quad + \frac{t^{\alpha_1-1}(1-t)}{\Gamma(\alpha_1-1)}\\
&\le \frac{\mu_1\mu_2k_1 t^{\alpha_1-1}}{1-\mu_1\mu_2k_1k_2}\int_0^1 \frac{1}{\Gamma(\alpha_1-1)}dA_2(t) + \frac{t^{\alpha_1-1}}{\Gamma(\alpha_1-1)}\\
&\le \frac{1}{\Gamma(\alpha_1-1)}\left(\frac{\mu_1\mu_2k_1}{1-\mu_1\mu_2k_1k_2}\int_0^1 dA_2(t)+1\right)t^{\alpha_1-1}\\
&\le \rho t^{\alpha_1-1},\\
&H_2(t,s)\\
&= \frac{\mu_2 t^{\alpha_2-1}}{1-\mu_1\mu_2k_1k_2}\int_0^1 G_1(t,s)\,dA_2(t)\\
&\le \frac{\mu_2 t^{\alpha_2-1}}{1-\mu_1\mu_2k_1k_2}\int_0^1 \frac{t^{\alpha_1-1}(1-t)}{\Gamma(\alpha_1-1)}dA_2(t)\\
&\le \frac{\mu_2 t^{\alpha_2-1}}{1-\mu_1\mu_2k_1k_2}\int_0^1 \frac{1}{\Gamma(\alpha_1-1)}dA_2(t)\\
&\le \rho t^{\alpha_2-1}.
\end{aligned} \tag{29}$$

Proceeding as for the proof of (29), we have

$$\begin{aligned}
&K_2(t,s) \le \rho t^{\alpha_2-1},\\
&H_1(t,s) \le \rho t^{\alpha_1-1},\\
&t \in [0,1];
\end{aligned} \tag{30}$$

thus (21) holds.

On the other hand, it follows from (11) and (25) that

$$\begin{aligned}
&K_1(t,s)\\
&= \frac{\mu_1\mu_2k_1 t^{\alpha_1-1}}{1-\mu_1\mu_2k_1k_2}\int_0^1 G_1(t,s)\,dA_2(t) + G_1(t,s)\\
&\ge \frac{\mu_1\mu_2k_1 t^{\alpha_1-1}}{1-\mu_1\mu_2k_1k_2}\int_0^1 \frac{(1-t)t^{\alpha_1-1}s(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}dA_2(t)\\
&\ge \left(\frac{\mu_1\mu_2k_1}{\Gamma(\alpha_1)(1-\mu_1\mu_2k_1k_2)}\int_0^1 (1-t)t^{\alpha_1-1}dA_2(t)\right)\\
&\quad \cdot t^{\alpha_1-1}s(1-s)^{\alpha_1-1}\\
&\ge \varrho t^{\alpha_1-1}s(1-s)^{\alpha_1-1},\\
&H_2(t,s)\\
&= \frac{\mu_2 t^{\alpha_2-1}}{1-\mu_1\mu_2k_1k_2}\int_0^1 G_1(t,s)\,dA_2(t)\\
&\ge \frac{\mu_2 t^{\alpha_2-1}}{1-\mu_1\mu_2k_1k_2}\int_0^1 \frac{(1-t)t^{\alpha_1-1}s(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}dA_2(t)\\
&\ge \left(\frac{\mu_2}{\Gamma(\alpha_1)(1-\mu_1\mu_2k_1k_2)}\int_0^1 (1-t)t^{\alpha_1-1}dA_2(t)\right)\\
&\quad \cdot t^{\alpha_2-1}s(1-s)^{\alpha_1-1}\\
&\ge \varrho t^{\alpha_2-1}s(1-s)^{\alpha_1-1},
\end{aligned} \tag{31}$$

which implies that (22) holds. Similarly, we also have

$$\begin{aligned}
&K_2(t,s) \ge \varrho t^{\alpha_2-1}s(1-s)^{\alpha_2-1},\\
&H_1(t,s) \ge \varrho t^{\alpha_1-1}s(1-s)^{\alpha_2-1},\\
&t \in [0,1].
\end{aligned} \tag{32}$$

This completes the proof of the lemma. □

From Lemma 5, we have the following conclusion.

Remark 6. For $t, \tau, s \in [0,1]$, we have

$$\begin{gathered} K_i(t,s) \geq \omega t^{\alpha_i - 1} K_i(\tau, s), \\ H_i(t,s) \geq \omega t^{\alpha_i - 1} H_i(\tau, s), \\ i = 1, 2, \\ K_1(t,s) \geq \omega t^{\alpha_1 - 1} H_2(\tau, s), \\ H_2(t,s) \geq \omega t^{\alpha_2 - 1} K_1(\tau, s), \\ K_2(t,s) \geq \omega t^{\alpha_2 - 1} H_1(\tau, s), \\ H_1(t,s) \geq \omega t^{\alpha_1 - 1} K_2(\tau, s), \end{gathered} \tag{33}$$

where $\omega = \varrho/\rho$, $0 < \omega < 1$.

Throughout this paper, we will work in the space $X = C[0,1] \times C[0,1]$, which is a Banach space if it is endowed with the norm

$$\begin{gathered} \|(u,v)\| = \max\{\|u\|, \|v\|\}, \\ \|u\| = \max_{0 \leq t \leq 1} |u(t)|, \\ \|v\| = \max_{0 \leq t \leq 1} |v(t)|. \end{gathered} \tag{34}$$

Let

$$\begin{aligned} K = \Big\{ &(u,v) \in X : u(t) \geq \omega t^{\alpha_1 - 1} \|u, v\|, \\ &v(t) \geq \omega t^{\alpha_2 - 1} \|u, v\|, \ t \in [0,1] \Big\}; \end{aligned} \tag{35}$$

then K is a cone in X. For $0 < r < R$, denote

$$\begin{gathered} K_{[r,R]} = \{(u,v) \in K : r \leq \|u,v\| \leq R\}, \\ K_r = \{(u,v) \in K : \|u,v\| < r\}. \end{gathered} \tag{36}$$

In what follows, we list some conditions to be used later:

($\mathbf{H}_1$) $f : (0,1) \times [0,+\infty) \times (0,+\infty) \to [0,+\infty)$ is continuous, $f(t,x,y)$ is nondecreasing in x and nonincreasing in y, and there exist $\lambda_1, \delta_1 \in (0,1)$ such that

$$\begin{gathered} c^{\lambda_1} f(t,x,y) \leq f(t,cx,y), \quad x, y > 0, \ c \in (0,1), \\ f(t,x,cy) \leq c^{-\delta_1} f(t,x,y), \quad x, y > 0, \ c \in (0,1); \end{gathered} \tag{37}$$

$g : (0,1) \times (0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is continuous, $g(t,x,y)$ is nonincreasing in x and nondecreasing in y, and there exist $\lambda_2, \delta_2 \in (0,1)$ such that

$$\begin{gathered} c^{\lambda_2} g(t,x,y) \leq g(t,x,cy), \quad x, y > 0, \ c \in (0,1), \\ g(t,cx,y) \leq c^{-\delta_2} g(t,x,y), \quad x, y > 0, \ c \in (0,1); \end{gathered} \tag{38}$$

($\mathbf{H}_2$)

$$\begin{gathered} \int_0^1 s(1-s)^{\alpha_1 - 1} f\left(s, 1, s^{\alpha_2 - 1}\right) ds < +\infty, \\ \int_0^1 s(1-s)^{\alpha_2 - 1} g\left(s, s^{\alpha_1 - 1}, 1\right) ds < +\infty. \end{gathered} \tag{39}$$

Remark 7. By ($\mathbf{H}_1$), we have

$$\begin{gathered} f\left(s, s^{\alpha_2 - 1}, 1\right) \leq f\left(s, 1, s^{\alpha_2 - 1}\right), \\ g\left(s, 1, s^{\alpha_1 - 1}\right) \leq g\left(s, s^{\alpha_1 - 1}, 1\right). \end{gathered} \tag{40}$$

This together with ($\mathbf{H}_2$) yields

$$\begin{aligned} &\int_0^1 s(1-s)^{\alpha_1 - 1} f\left(s, s^{\alpha_2 - 1}, 1\right) ds \\ &\quad \leq \int_0^1 s(1-s)^{\alpha_1 - 1} f\left(s, 1, s^{\alpha_2 - 1}\right) ds < +\infty, \\ &\int_0^1 s(1-s)^{\alpha_2 - 1} g\left(s, 1, s^{\alpha_1 - 1}\right) ds \\ &\quad \leq \int_0^1 s(1-s)^{\alpha_1 - 1} g\left(s, s^{\alpha_1 - 1}, 1\right) ds < +\infty. \end{aligned} \tag{41}$$

From the above assumptions ($\mathbf{H}_0$)–($\mathbf{H}_2$), for any $(u,v) \in K \backslash \{(0,0)\}$, we define an integral operator $T : K \backslash \{(0,0)\} \to X$ by

$$\begin{aligned} &T(u,v)(t) \\ &\quad = \left(T_1(u,v)(t), T_2(u,v)(t)\right), \quad 0 \leq t \leq 1, \end{aligned} \tag{42}$$

where

$$\begin{aligned} &T_1(u,v)(t) \\ &\quad = \int_0^1 K_1(t,s) f(s, u(s), v(s)) ds \\ &\qquad + \int_0^1 H_1(t,s) g(s, u(s), v(s)) ds, \\ &T_2(u,v)(t) \\ &\quad = \int_0^1 K_2(t,s) g(s, u(s), v(s)) ds \\ &\qquad + \int_0^1 H_2(t,s) f(s, u(s), v(s)) ds. \end{aligned} \tag{43}$$

Now we claim that T is well defined for $(u,v) \in K \setminus \{(0,0)\}$. In fact, for any $(u,v) \in K \setminus \{(0,0)\}$, we have

$$\begin{gathered} \omega t^{\alpha_1 - 1} \|(u,v)\| \leq u(t) \leq \|(u,v)\|, \\ \omega t^{\alpha_2 - 1} \|(u,v)\| \leq v(t) \leq \|(u,v)\|, \\ t \in [0,1]. \end{gathered} \tag{44}$$

Let c be a positive number such that $\|(u,v)\|/c < 1$, $c > 1$. From $(\mathbf{H}_1)$ and (44), we have

$$\begin{aligned}
& f(t,u(t),v(t)) \\
&\quad \le f\left(t,c,\omega t^{\alpha_2-1}\|(u,v)\|\right) \le c^{\lambda_1} f\left(t,1,\frac{\omega\|(u,v)\|}{c}t^{\alpha_2-1}\right) \\
&\quad \le c^{\lambda_1+\delta_1}(\omega\|(u,v)\|)^{-\delta_1} f\left(t,1,t^{\alpha_2-1}\right), \\
& g(t,u(t),v(t)) \\
&\quad \le g\left(t,\omega t^{\alpha_1-1}\|(u,v)\|,c\right) \le c^{\lambda_2} g\left(t,\frac{\omega\|(u,v)\|}{c}t^{\alpha_1-1},1\right) \\
&\quad \le c^{\lambda_2+\delta_2}(\omega\|(u,v)\|)^{-\delta_2} g\left(t,t^{\alpha_1-1},1\right).
\end{aligned} \tag{45}$$

Hence, for any $t \in [0,1]$, by Lemma 5 and (45), we have

$$\begin{aligned}
& T_1(u,v)(t) \\
& = \int_0^1 K_1(t,s) f(s,u(s),v(s))\,ds \\
& \quad + \int_0^1 H_1(t,s) g(s,u(s),v(s))\,ds \\
& \le \rho\left(c^{\lambda_1+\delta_1}(\omega\|(u,v)\|)^{-\delta_1}\int_0^1 s(1-s)^{\alpha_1-1} f\left(s,1,s^{\alpha_2-1}\right)ds\right. \\
& \qquad + c^{\lambda_2+\delta_2}(\omega\|(u,v)\|)^{-\delta_2} \\
& \qquad \left.\cdot\int_0^1 s(1-s)^{\alpha_2-1} g\left(s,s^{\alpha_1-1},1\right)ds\right) \\
& < +\infty.
\end{aligned} \tag{46}$$

Similarly, for any $t \in [0,1]$, we have

$$\begin{aligned}
& T_2(u,v)(t) \\
& \le \rho\left(c^{\lambda_1+\delta_1}(\omega\|(u,v)\|)^{-\delta_1}\int_0^1 s(1-s)^{\alpha_1-1} f\left(s,1,s^{\alpha_2-1}\right)ds\right. \\
& \qquad + c^{\lambda_2+\delta_2}(\omega\|(u,v)\|)^{-\delta_2} \\
& \qquad \left.\cdot\int_0^1 s(1-s)^{\alpha_2-1} g\left(s,s^{\alpha_1-1},1\right)ds\right) \\
& < +\infty.
\end{aligned} \tag{47}$$

Together with the continuity of $K_i(t,s)$ and $H_i(t,s)$ $(i = 1,2)$, it is easy to see that $T_i(u,v) \in C[0,1]$, for $(u,v) \in K \setminus \{(0,0)\}$. Therefore $T : K \setminus \{(0,0)\} \to X$ is well defined.

Obviously, (u,v) is a positive solution of system (2) if and only if (u,v) is a fixed point of T in $K \setminus \{(0,0)\}$.

Lemma 8. *Assume that* $(\mathbf{H}_0)$–$(\mathbf{H}_2)$ *hold. Then* $T : K_{[r_1,r_2]} \to K$ *is a completely continuous operator.*

Proof. First, we show $T\left(K_{[r_1,r_2]}\right) \subseteq K$.

For any $(u,v) \in K_{[r_1,r_2]}$, $0 \le t,\tau \le 1$, by Remark 6, we have

$$\begin{aligned}
& T_1(u,v)(t) \\
& \quad = \int_0^1 K_1(t,s) f(s,u(s),v(s))\,ds \\
& \qquad + \int_0^1 H_1(t,s) g(s,u(s),v(s))\,ds \\
& \quad \ge \int_0^1 \omega t^{\alpha_1-1} K_1(\tau,s) f(s,u(s),v(s))\,ds \\
& \qquad + \int_0^1 \omega t^{\alpha_1-1} H_1(\tau,s) g(s,u(s),v(s))\,ds \\
& \quad \ge \omega t^{\alpha_1-1}\left(\int_0^1 K_1(\tau,s) f(s,u(s),v(s))\,ds\right. \\
& \qquad \left. + \int_0^1 H_1(\tau,s) g(s,u(s),v(s))\,ds\right) \\
& \quad \ge \omega t^{\alpha_1-1} T_1(u,v)(\tau), \\
& T_1(u,v)(t) \\
& \quad = \int_0^1 K_1(t,s) f(s,u(s),v(s))\,ds \\
& \qquad + \int_0^1 H_1(t,s) g(s,u(s),v(s))\,ds \\
& \quad \ge \int_0^1 \omega t^{\alpha_1-1} H_2(\tau,s) f(s,u(s),v(s))\,ds \\
& \qquad + \int_0^1 \omega t^{\alpha_1-1} K_2(\tau,s) g(s,u(s),v(s))\,ds \\
& \quad \ge \omega t^{\alpha_1-1}\left(\int_0^1 H_2(\tau,s) f(s,u(s),v(s))\,ds\right. \\
& \qquad \left. + \int_0^1 K_2(\tau,s) g(s,u(s),v(s))\,ds\right) \\
& \quad \ge \omega t^{\alpha_1-1} T_2(u,v)(\tau).
\end{aligned} \tag{48}$$

Then, we have

$$\begin{aligned}
T_1(u,v)(t) &\ge \omega t^{\alpha_1-1}\left\|T_1(u,v)\right\|, \\
T_1(u,v)(t) &\ge \omega t^{\alpha_1-1}\left\|T_2(u,v)\right\|;
\end{aligned} \tag{49}$$

that is,

$$T_1(u,v)(t) \ge \omega t^{\alpha_1-1}\left\|\left(T_1(u,v),T_2(u,v)\right)\right\|. \tag{50}$$

In the same way as (48), we can prove that

$$T_2(u,v)(t) \ge \omega t^{\alpha_2-1}\left\|\left(T_1(u,v),T_2(u,v)\right)\right\|. \tag{51}$$

Therefore, we have $T\left(K_{[r_1,r_2]}\right) \subseteq K$.

Next, we show $T : K_{[r_1,r_2]} \to K$ is continuous.

Let $(u_n, v_n), (u, v) \in K_{[r_1,r_2]}$, such that $\|(u_n, v_n) - (u, v)\| \to 0$ $(n \to +\infty)$. Obviously, $r_1 \le \|(u,v)\|, \|(u_n, v_n)\| \le r_2$ for all n; choose c such that $\|(u,v)\|/c < 1$, $\|(u_n, v_n)\|/c < 1$, $c > 1$. So, by Lemma 5, $(\mathbf{H}_1)$, and (43), we have

$$\begin{aligned}
&\left|T_1(u_n, v_n)(t) - T_1(u,v)(t)\right| \\
&= \Bigg|\left(\int_0^1 K_1(t,s) f(s, u_n(s), v_n(s))\,ds + \int_0^1 H_1(t,s) g(s, u_n(s), v_n(s))\,ds\right) \\
&\quad - \left(\int_0^1 K_1(t,s) f(s, u(s), v(s))\,ds + \int_0^1 H_1(t,s) g(s, u(s), v(s))\,ds\right)\Bigg| \\
&\le 2\rho \int_0^1 s(1-s)^{\alpha_1-1} \cdot \left|f(s, u_n(s), v_n(s)) + f(s, u(s), v(s))\right| ds \\
&\quad + 2\rho \int_0^1 s(1-s)^{\alpha_2-1} \cdot \left|g(s, u_n(s), v_n(s)) + g(s, u(s), v(s))\right| ds \\
&\le \rho\Bigg(c^{\lambda_1+\delta_1}\left((\omega\|(u,v)\|)^{-\delta_1} + (\omega\|(u_n,v_n)\|)^{-\delta_1}\right) \cdot \int_0^1 s(1-s)^{\alpha_1-1} f\left(s, 1, s^{\alpha_2-1}\right) ds \\
&\quad + c^{\lambda_2+\delta_2}\left((\omega\|(u,v)\|)^{-\delta_2} + (\omega\|(u_n,v_n)\|)^{-\delta_2}\right) \cdot \int_0^1 s(1-s)^{\alpha_2-1} g\left(s, s^{\alpha_1-1}, 1\right) ds\Bigg) \\
&\le \rho\Bigg(2c^{\lambda_1+\delta_1}(\omega r_2)^{-\delta_1} \int_0^1 s(1-s)^{\alpha_1-1} f\left(s, 1, s^{\alpha_2-1}\right) ds \\
&\quad + 2c^{\lambda_2+\delta_2}(\omega r_2)^{-\delta_2} \int_0^1 s(1-s)^{\alpha_2-1} g\left(s, s^{\alpha_1-1}, 1\right) ds\Bigg) \\
&< +\infty, \quad t \in [0,1].
\end{aligned} \tag{52}$$

By (52), for any $\varepsilon > 0$, we can find a sufficiently large natural number $m > 0$, for all n, such that

$$\begin{aligned}
&\rho \int_{H(m)} s(1-s)^{\alpha_1-1} \cdot \left|f(s, u_n(s), v_n(s)) + f(s, u(s), v(s))\right| ds \\
&\quad + \rho \int_{H(m)} s(1-s)^{\alpha_2-1} \cdot \left|g(s, u_n(s), v_n(s)) + g(s, u(s), v(s))\right| ds < \frac{\varepsilon}{2},
\end{aligned} \tag{53}$$

where $H(m) = [0, 1/m] \cup [1 - 1/m, 1]$. On the other hand, for each $(\overline{u}, \overline{v}) \in K_{[r_1,r_2]}$ and $t \in [1/m, 1-(1/m)]$, we have

$$\begin{aligned}
&\omega\varpi r_1 \le \overline{u}(t), \\
&\overline{v}(t) \le r_2,
\end{aligned} \tag{54}$$

where $\varpi = \min\left\{t^{\alpha_i - 1} : t \in [1/m, 1-(1/m)],\ i = 1, 2\right\}$. Since $f(t,x,y)$ and $g(t,x,y)$ are uniformly continuous in $[1/m, 1-(1/m)] \times [\omega\varpi r_1, r_2] \times [\omega\varpi r_1, r_2]$, we have

$$\begin{aligned}
&\lim_{n\to+\infty} \left|f(s, u_n(s), v_n(s)) - f(s, u(s), v(s))\right| = 0, \\
&\lim_{n\to+\infty} \left|g(s, u_n(s), v_n(s)) - g(s, u(s), v(s))\right| = 0
\end{aligned} \tag{55}$$

hold uniformly on $s \in [1/m, 1-(1/m)]$. Then the Lebesgue dominated convergence theorem yields that

$$\begin{aligned}
&\rho \int_{1/m}^{1-(1/m)} s(1-s)^{\alpha_1-1} \cdot \left|f(s, u_n(s), v_n(s)) - f(s, u(s), v(s))\right| ds \longrightarrow 0, \quad \text{as } n \longrightarrow +\infty, \\
&\rho \int_{1/m}^{1-(1/m)} s(1-s)^{\alpha_2-1} \cdot \left|g(s, u_n(s), v_n(s)) - g(s, u(s), v(s))\right| ds \longrightarrow 0, \quad \text{as } n \longrightarrow +\infty.
\end{aligned} \tag{56}$$

So, for the above $\varepsilon > 0$, there exists a sufficiently large natural number N_0 such that when $n > N_0$, we have

$$\begin{aligned}
&\rho \int_{1/m}^{1-(1/m)} s(1-s)^{\alpha_1-1} \cdot \left|f(s, u_n(s), v_n(s)) - f(s, u(s), v(s))\right| ds < \frac{\varepsilon}{4}, \\
&\rho \int_{1/m}^{1-(1/m)} s(1-s)^{\alpha_2-1} \cdot \left|g(s, u_n(s), v_n(s)) - g(s, u(s), v(s))\right| ds < \frac{\varepsilon}{4}.
\end{aligned} \tag{57}$$

It follows from (53) and (57) that

$$\begin{aligned}
&\left\|T_1(u_n,v_n)-T_1(u,v)\right\| \\
&\le \int_0^1 K_1(t,s)\left|f(s,u_n(s),v_n(s)) - f(s,u(s),v(s))\right| ds \\
&\quad + \int_0^1 H_1(t,s)\left|g(s,u_n(s),v_n(s)) - g(s,u(s),v(s))\right| ds \\
&\le \rho\int_{1/m}^{1-(1/m)} s(1-s)^{\alpha_1-1}\left|f(s,u_n(s),v_n(s)) - f(s,u(s),v(s))\right| ds \\
&\quad + \rho\int_{1/m}^{1-(1/m)} s(1-s)^{\alpha_2-1}\left|g(s,u_n(s),v_n(s)) - g(s,u(s),v(s))\right| ds \\
&\quad + \rho\int_{H(m)} s(1-s)^{\alpha_1-1}\left|f(s,u_n(s),v_n(s)) + f(s,u(s),v(s))\right| ds \\
&\quad + \rho\int_{H(m)} s(1-s)^{\alpha_2-1}\left|g(s,u_n(s),v_n(s)) + g(s,u(s),v(s))\right| ds \\
&< \varepsilon, \quad n > N_0.
\end{aligned} \tag{58}$$

This implies that the operator $T_1 : K_{[r_1,r_2]} \to C[0,1]$ is continuous. Similarly, we can prove $T_2 : K_{[r_1,r_2]} \to C[0,1]$ is continuous. So $T : K_{[r_1,r_2]} \to K$ is continuous.

Finally, we show $T : K_{[r_1,r_2]} \to K$ is compact.

Let $D \subset K_{[r_1,r_2]}$ be any bounded set; then, for any $(u,v) \in D$, we have $r_1 \le \|(u,v)\| \le r_2$. Choose c, such that $\|(u,v)\|/c < 1$, $c > 1$. By (45), for any $(u,v) \in D$, $t \in [0,1]$, we have

$$\begin{aligned}
&T_i(u,v)(t) \\
&\le \rho\left(\int_0^1 s(1-s)^{\alpha_1-1} f(s,u(s),v(s))\,ds + \int_0^1 s(1-s)^{\alpha_2-1} g(s,u(s),v(s))\,ds\right) \\
&\le \rho\left(c^{\lambda_1+\delta_1}(\omega r_1)^{-\delta_1}\int_0^1 s(1-s)^{\alpha_1-1} f\left(s,1,s^{\alpha_2-1}\right) ds + c^{\lambda_2+\delta_2}(\omega r_1)^{-\delta_2}\int_0^1 s(1-s)^{\alpha_2-1} g\left(s,s^{\alpha_1-1},1\right) ds\right) \\
&< +\infty, \quad i = 1,2.
\end{aligned} \tag{59}$$

So $T(D)$ is bounded in X.

In what follows, we show that $T_i(D)$ is equicontinuous. In fact, by (59), for any $\varepsilon > 0$, there exists a sufficiently large natural number m_0, for all $(u,v) \in D$, such that

$$\rho\left(\int_{H(m_0)} s(1-s)^{\alpha_1-1} f(s,u(s),v(s))\,ds + \int_{H(m_0)} s(1-s)^{\alpha_2-1} g(s,u(s),v(s))\,ds\right) < \frac{\varepsilon}{4}. \tag{60}$$

Let

$$\begin{aligned}
M_0 &= \max\left\{f(t,x,y) : \frac{1}{m_0} \le t \le 1-\frac{1}{m_0},\ \omega\varpi_0 r_1 \le x \le r_2,\ \omega\varpi_0 r_1 \le y \le r_2\right\}, \\
M_0' &= \max\left\{g(t,x,y) : \frac{1}{m_0} \le t \le 1-\frac{1}{m_0},\ \omega\varpi_0 r_1 \le x \le r_2,\ \omega\varpi_0 r_1 \le y \le r_2\right\},
\end{aligned} \tag{61}$$

where $\varpi_0 = \min\left\{t^{\alpha_i-1} : t \in \left[1/m_0, 1-(1/m_0)\right],\ i = 1,2\right\} > 0$.

By the uniform continuity of $K_1(t,s)$, $H_1(t,s)$ on $[0,1]\times[0,1]$, for the above $\varepsilon > 0$, there exists $\delta_0 > 0$ such that, for any $t_1, t_2 \in [0,1]$, $|t_1 - t_2| < \delta_0$, $s \in \left[1/m_0, 1-(1/m_0)\right]$, we have

$$\begin{aligned}
\left|K_1(t_1,s) - K_1(t_2,s)\right| &< \frac{\varepsilon}{4}\left(M_0\left(1-\frac{2}{m_0}\right)\right)^{-1}, \\
\left|H_1(t_1,s) - H_1(t_2,s)\right| &< \frac{\varepsilon}{4}\left(M_0'\left(1-\frac{2}{m_0}\right)\right)^{-1}.
\end{aligned} \tag{62}$$

Thus, when $t_1, t_2 \in [0,1]$, $|t_1 - t_2| < \delta_0$, $s \in \left[1/l_0, (l_0-1)/l_0\right]$, for any $(u,v) \in D$, we get

$$\begin{aligned}
&\left|T_1(u,v)(t_1) - T_1(u,v)(t_2)\right| \\
&= \left|\int_0^1 K_1(t_1,s) f(s,u(s),v(s))\,ds + \int_0^1 H_1(t_1,s) g(s,u(s),v(s))\,ds \right. \\
&\quad \left. - \left(\int_0^1 K_1(t_2,s) f(s,u(s),v(s))\,ds + \int_0^1 H_1(t_2,s) g(s,u(s),v(s))\,ds\right)\right|
\end{aligned}$$

$$\leq \int_{1/m_0}^{1-(1/m_0)} \left|K_1(t_1,s) - K_1(t_2,s)\right| \cdot \left|f(s,u(s),v(s))\right| ds + \int_{1/m_0}^{1-(1/m_0)} \left|H_1(t_1,s) - H_1(t_2,s)\right| \cdot \left|g(s,u(s),v(s))\right| ds + \int_{H(m_0)} \left|K_1(t_1,s) - K_1(t_2,s)\right| \cdot f(s,u(s),v(s))\, ds + \int_{H(m_0)} \left|H_1(t_1,s) - H_1(t_2,s)\right| \cdot g(s,u(s),v(s))\, ds$$
$$\leq \frac{\varepsilon}{2} + 2\rho \int_{H(m_0)} s(1-s)^{\alpha_1-1} \cdot f(s,u(s),v(s))\, ds + 2\rho \int_{H(m_0)} s(1-s)^{\alpha_2-1} \cdot g(s,u(s),v(s))\, ds < \varepsilon. \quad (63)$$

This means that $T_1(D)$ is equicontinuous. By the Arzela-Ascoli theorem, $T_1(D)$ is a relatively compact set. In the same way, we can show that $T_2(D)$ is a relatively compact set. So $T : K_{[r_1,r_2]} \to K$ is compact.

From the above discussion, together with the fact that $T : K_{[r_1,r_2]} \to K$ is continuous, we get that $T : K_{[r_1,r_2]} \to K$ is completely continuous. This completes the proof of the lemma. □

In order to obtain the existence of positive solutions of system (2), we will use the following cone compression and expansion fixed point theorem.

Lemma 9 (see [21]). *Let P be a positive cone in a Banach space E, Ω_1 and Ω_2 are bounded open sets in E, $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and $A : P \cap \overline{\Omega}_2 \setminus \Omega_1 \to P$ is a completely continuous operator. If the following conditions are satisfied:*

$$\|Ax\| \leq \|x\|, \quad \forall x \in P \cap \partial\Omega_1, \qquad \|Ax\| \geq \|x\|, \quad \forall x \in P \cap \partial\Omega_2, \quad (64)$$

or

$$\|Ax\| \geq \|x\|, \quad \forall x \in P \cap \partial\Omega_1, \qquad \|Ax\| \leq \|x\|, \quad \forall x \in P \cap \partial\Omega_2, \quad (65)$$

then A has at least one fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Main Results

Theorem 10. *Assume that $(\mathbf{H}_0)$–$(\mathbf{H}_2)$ hold; then system (2) has at least one positive solution (u^*, v^*), and there exists a real number $0 < l < 1$ satisfying*

$$l t^{\overline{\alpha}-1} \leq u^*(t) \leq l^{-1} t^{\alpha^*-1}, \qquad l t^{\overline{\alpha}-1} \leq v^*(t) \leq l^{-1} t^{\alpha^*-1}, \qquad t \in [0,1], \quad (66)$$

where $\overline{\alpha} = \max\{\alpha_1, \alpha_2\}$, $\alpha^* = \min\{\alpha_1, \alpha_2\}$.

Proof. First, we show that system (2) has at least one positive solution.

Choose r, R, such that

$$0 < r \leq \min\left\{\left(\left(\frac{1}{3}\right)^{\alpha_1-1} \varrho\omega^{\lambda_1} \cdot \int_0^1 s(1-s)^{\alpha_1-1} f\left(s, s^{\alpha_1-1}, 1\right) ds\right)^{1/(1-\lambda_1)}, \frac{1}{2}\right\},$$
$$R \geq \max\left\{\left(\rho \int_0^1 s(1-s)^{\alpha_1-1} f\left(s, 1, s^{\alpha_2-1}\right) ds + \rho \int_0^1 s(1-s)^{\alpha_2-1} \cdot g\left(s, s^{\alpha_1-1}, 1\right) ds\right)^{1/(1-\max\{\lambda_1,\lambda_2\})}, \frac{1}{\omega}, 2\right\}. \quad (67)$$

For any $(u, v) \in \partial K_r$, we have

$$r\omega t^{\alpha_1-1} \leq u(t) \leq r, \qquad r\omega t^{\alpha_2-1} \leq v(t) \leq r, \qquad t \in [0,1]. \quad (68)$$

By Lemma 5, Remark 7, and $(\mathbf{H}_1)$, for any $(u, v) \in \partial K_r$, we get

$$T_1(u,v)(t) = \int_0^1 K_1(t,s) f(s,u(s),v(s))\, ds + \int_0^1 H_1(t,s) g(s,u(s),v(s))\, ds \geq \int_0^1 K_1(t,s) f(s,u(s),v(s))\, ds \geq \varrho \int_0^1 t^{\alpha_1-1} s(1-s)^{\alpha_1-1} f\left(s, r\omega s^{\alpha_1-1}, r\right) ds$$

$$\begin{aligned}
&\geq \varrho \int_0^1 t^{\alpha_1-1} s(1-s)^{\alpha_1-1} f\left(s, r\omega s^{\alpha_1-1}, 1\right) ds \\
&\geq \varrho t^{\alpha_1-1} r^{\lambda_1} \omega^{\lambda_1} \int_0^1 s(1-s)^{\alpha_1-1} f\left(s, s^{\alpha_1-1}, 1\right) ds \\
&\geq \left(\frac{1}{3}\right)^{\alpha_1-1} \varrho r^{\lambda_1} \omega^{\lambda_1} \int_0^1 s(1-s)^{\alpha_1-1} f\left(s, s^{\alpha_1-1}, 1\right) ds \\
&\geq r = \|(u,v)\|, \quad t \in \left[\frac{1}{3}, \frac{2}{3}\right].
\end{aligned} \tag{69}$$

This guarantees

$$\|T(u,v)\| \geq \|(u,v)\|, \quad (u,v) \in \partial K_r. \tag{70}$$

On the other hand, for any $(u,v) \in \partial K_R$, we have

$$\begin{gathered}
R\omega t^{\alpha_1-1} \leq u(t) \leq R, \\
R\omega t^{\alpha_2-1} \leq v(t) \leq R, \\
t \in [0,1].
\end{gathered} \tag{71}$$

By Lemma 5, $(\mathbf{H}_1)$, and $(\mathbf{H}_2)$, for any $(u,v) \in \partial K_R$, $t \in [0,1]$, we get

$$\begin{aligned}
&T_1(u,v)(t) \\
&\quad = \int_0^1 K_1(t,s) f(s, u(s), v(s)) ds \\
&\quad\quad + \int_0^1 H_1(t,s) g(s, u(s), v(s)) ds \\
&\quad \leq \rho \int_0^1 s(1-s)^{\alpha_1-1} f\left(s, R, R\omega s^{\alpha_2-1}\right) ds \\
&\quad\quad + \rho \int_0^1 s(1-s)^{\alpha_2-1} g\left(s, R\omega s^{\alpha_1-1}, R\right) ds \\
&\quad \leq \rho \int_0^1 s(1-s)^{\alpha_1-1} f\left(s, R, s^{\alpha_2-1}\right) ds \\
&\quad\quad + \rho \int_0^1 s(1-s)^{\alpha_2-1} g\left(s, s^{\alpha_1-1}, R\right) ds \\
&\quad \leq \rho R^{\lambda_1} \int_0^1 s(1-s)^{\alpha_1-1} f\left(s, 1, s^{\alpha_2-1}\right) ds \\
&\quad\quad + \rho R^{\lambda_2} \int_0^1 s(1-s)^{\alpha_2-1} g\left(s, s^{\alpha_1-1}, 1\right) ds \\
&\quad \leq \rho R^{\max\{\lambda_1,\lambda_2\}} \left(\int_0^1 s(1-s)^{\alpha_1-1} f\left(s, 1, s^{\alpha_2-1}\right) ds\right. \\
&\quad\quad \left. + \int_0^1 s(1-s)^{\alpha_2-1} g\left(s, s^{\alpha_1-1}, 1\right) ds\right) \\
&\quad \leq R = \|(u,v)\|.
\end{aligned} \tag{72}$$

In the same way as (72), we have

$$T_2(u,v)(t) \leq R = \|(u,v)\|, \quad (u,v) \in \partial K_R. \tag{73}$$

So,

$$\|T(u,v)\| \leq \|(u,v)\|, \quad (u,v) \in \partial K_R. \tag{74}$$

It follows from (70), (74), and Lemmas 8 and 9 that T has a fixed point (u^*, v^*) with $0 < r \leq \|(u^*, v^*)\| \leq R$. It is easy to see that (u^*, v^*) is a positive solution of system (2).

Next, we show there exists a real number $0 < l < 1$ such that the positive solution (u^*, v^*) in system (2) satisfies

$$\begin{gathered}
l t^{\overline{\alpha}-1} \leq u^*(t) \leq l^{-1} t^{\alpha^*-1}, \\
l t^{\overline{\alpha}-1} \leq v^*(t) \leq l^{-1} t^{\alpha^*-1}, \\
t \in [0,1],
\end{gathered} \tag{75}$$

where $\overline{\alpha} = \max\{\alpha_1, \alpha_2\}$, $\alpha^* = \min\{\alpha_1, \alpha_2\}$.

From Lemma 8, we know $(u^*, v^*) \in K \setminus \{(0,0)\}$. So, we have

$$\begin{gathered}
\omega t^{\alpha_1-1} \|(u^*, v^*)\| \leq u^*(t) \leq \|(u^*, v^*)\|, \\
\omega t^{\alpha_2-1} \|(u^*, v^*)\| \leq v^*(t) \leq \|(u^*, v^*)\|, \\
t \in [0,1].
\end{gathered} \tag{76}$$

Choose c, such that $\|(u^*, v^*)\|/c < 1$, $c > 1/\omega$; by Lemma 5 and $(\mathbf{H}_1)$, for $t \in [0,1]$, we have

$$\begin{aligned}
u^*(t) &= \int_0^1 K_1(t,s) f(s, u^*(s), v^*(s)) ds \\
&\quad + \int_0^1 H_1(t,s) g(s, u^*(s), v^*(s)) ds \\
&\leq \int_0^1 \rho t^{\alpha_1-1} f\left(s, c, \omega s^{\alpha_2-1} \|(u^*, v^*)\|\right) ds \\
&\quad + \int_0^1 \rho t^{\alpha_1-1} g\left(s, \omega t^{\alpha_1-1} \|(u^*, v^*)\|, c\right) ds \\
&\leq \rho t^{\alpha_1-1} \int_0^1 f\left(s, c, \frac{\omega \|(u^*, v^*)\|}{c} s^{\alpha_2-1}\right) ds \\
&\quad + \rho t^{\alpha_1-1} \int_0^1 g\left(s, \frac{\omega \|(u^*, v^*)\|}{c} s^{\alpha_1-1}, c\right) ds \\
&\leq c^{\lambda_1+\delta_1} \left(\omega \|(u^*, v^*)\|\right)^{-\delta_1} \rho t^{\alpha_1-1} \int_0^1 f\left(s, 1, s^{\alpha_2-1}\right) ds \\
&\quad + c^{\lambda_2+\delta_2} \left(\omega \|(u^*, v^*)\|\right)^{-\delta_2} \rho t^{\alpha_1-1} \int_0^1 g\left(s, s^{\alpha_1-1}, 1\right) ds \\
&\leq c^{\lambda_1+\delta_1} (\omega R)^{-\delta_1} \rho t^{\alpha_1-1} \int_0^1 f\left(s, 1, s^{\alpha_2-1}\right) ds \\
&\quad + c^{\lambda_2+\delta_2} (\omega R)^{-\delta_2} \rho t^{\alpha_1-1} \int_0^1 g\left(s, s^{\alpha_1-1}, 1\right) ds \\
&\leq M t^{\alpha^*-1},
\end{aligned} \tag{77}$$

where

$$M = c^{\lambda_1+\delta_1} (\omega R)^{-\delta_1} \rho \int_0^1 f\left(s, 1, s^{\alpha_2-1}\right) ds + c^{\lambda_2+\delta_2} (\omega R)^{-\delta_2} \rho \int_0^1 g\left(s, s^{\alpha_1-1}, 1\right) ds. \quad (78)$$

In the same way as (77), we also have $v(t) \le Mt^{\alpha^*-1}, t \in [0,1]$. Choose

$$l = \min\left\{\omega r, \frac{1}{M}, \frac{1}{2}\right\}; \quad (79)$$

then we have

$$lt^{\overline{\alpha}-1} \le u^*(t) \le l^{-1} t^{\alpha^*-1}, \quad lt^{\overline{\alpha}-1} \le v^*(t) \le l^{-1} t^{\alpha^*-1}, \quad t \in [0,1]. \quad (80)$$

This completes the proof of Theorem 10. □

Theorem 11. *Assume that* $(\mathbf{H}_0)$–$(\mathbf{H}_2)$ *hold. If* $\lambda_1 + \delta_1 < 1$ *and* $\lambda_2 + \delta_2 < 1$, *then system (2) has a unique positive solution for* $\alpha_1 = \alpha_2$.

Proof. Assume that system (2) has two different positive solutions (u_1, v_1) and (u_2, v_2). Denote $\alpha = \alpha_1 = \alpha_2$; by Theorem 10, there exists $0 < l_1 < 1$, $0 < l_2 < 1$, such that

$$l_1 t^{\alpha-1} \le u_1(t) \le \frac{1}{l_1} t^{\alpha-1}, \quad l_1 t^{\alpha-1} \le v_1(t) \le \frac{1}{l_1} t^{\alpha-1}, \quad t \in [0,1], \quad l_2 t^{\alpha-1} \le u_2(t) \le \frac{1}{l_2} t^{\alpha-1}, \quad l_2 t^{\alpha-1} \le v_2(t) \le \frac{1}{l_2} t^{\alpha-1}, \quad t \in [0,1]. \quad (81)$$

Thus, we have

$$l_1 l_2 u_2(t) \le u_1(t) \le \frac{1}{l_1 l_2} u_2(t), \quad l_1 l_2 v_2(t) \le v_1(t) \le \frac{1}{l_1 l_2} v_2(t), \quad t \in [0,1]. \quad (82)$$

Obviously, $l_1 l_2 \ne 1$. Let

$$L = \sup\left\{l : lu_2(t) \le u_1(t) \le l^{-1} u_2(t), \; lv_2(t) \le v_1(t) \le l^{-1} v_2(t), \; t \in [0,1]\right\}. \quad (83)$$

It is easy to see that $0 < l_1 l_2 \le L < 1$, and

$$Lu_2(t) \le u_1(t) \le \frac{1}{L} u_2(t), \quad lv_2(t) \le v_1(t) \le \frac{1}{L} v_2(t), \quad t \in [0,1]. \quad (84)$$

By $(\mathbf{H}_1)$, we have

$$\begin{aligned} f(t, u_1(t), v_1(t)) &\ge f\left(t, Lu_2(t), \frac{1}{L} v_2(t)\right) \ge L^{\lambda_1+\delta_1} f(t, u_2(t), v_2(t)) \\ &\ge L^{\sigma} f(t, u_2(t), v_2(t)), \\ g(t, u_1(t), v_1(t)) &\ge g\left(t, Lu_2(t), \frac{1}{L} v_2(t)\right) \ge L^{\lambda_2+\delta_2} g(t, u_2(t), v_2(t)) \\ &\ge L^{\sigma} g(t, u_2(t), v_2(t)), \end{aligned} \quad (85)$$

where $\sigma = \max\{\lambda_1 + \delta_1, \lambda_2 + \delta_2\} < 1$. Consider the following:

$$\begin{aligned} f(t, u_2(t), v_2(t)) &\ge f\left(t, Lu_1(t), \frac{1}{L} v_1(t)\right) \ge L^{\lambda_1+\delta_1} f(t, u_1(t), v_1(t)) \\ &\ge L^{\sigma} f(t, u_1(t), v_1(t)), \\ g(t, u_2(t), v_2(t)) &\ge g\left(t, Lu_1(t), \frac{1}{L} v_1(t)\right) \ge L^{\lambda_2+\delta_2} g(t, u_1(t), v_1(t)) \\ &\ge L^{\sigma} g(t, u_1(t), v_1(t)). \end{aligned} \quad (86)$$

From (85), for $t \in [0,1]$, we have

$$\begin{aligned} u_1(t) &= T_1(u_1, v_1)(t) = \int_0^1 K_1(t,s) f(s, u_1(s), v_1(s)) ds + \int_0^1 H_1(t,s) g(s, u_1(s), v_1(s)) ds \\ &\ge L^{\sigma} \int_0^1 K_1(t,s) f(s, u_2(s), v_2(s)) ds + L^{\sigma} \int_0^1 H_1(t,s) g(s, u_2(s), v_2(s)) ds \\ &\ge L^{\sigma} T_1(u_2, v_2)(t) = L^{\sigma} u_2(t), \end{aligned} \quad (87)$$

$$\begin{aligned}
&u_2(t)\\
&= T_1(u_2, v_2)(t) = \int_0^1 K_1(t,s) f(s, u_2(s), v_2(s)) ds\\
&\quad + \int_0^1 H_1(t,s) g(s, u_2(s), v_2(s)) ds\\
&\geq L^\sigma \int_0^1 K_1(t,s) f(s, u_1(s), v_1(s)) ds\\
&\quad + L^\sigma \int_0^1 H_1(t,s) g(s, u_1(s), v_1(s)) ds\\
&\geq L^\sigma T_1(u_1, v_1)(t) = L^\sigma u_1(t).
\end{aligned} \tag{88}$$

Similarly, from (86), we also have

$$\begin{aligned}
v_1(t) &\geq L^\sigma v_2(t),\\
v_2(t) &\geq L^\sigma v_1(t),\\
t &\in [0,1].
\end{aligned} \tag{89}$$

Therefore, we obtain

$$\begin{gathered}
L^\sigma u_2(t) \leq u_1(t) \leq \frac{1}{L^\sigma} u_2(t),\\
L^\sigma v_2(t) \leq v_1(t) \leq \frac{1}{L^\sigma} v_2(t),\\
t \in [0,1].
\end{gathered} \tag{90}$$

Notice that $0 < L < 1$, $0 < \sigma < 1$; we have $L^\sigma > L$, so it is a contradiction with the maximality of L. Therefore, system (2) has a unique positive solution (u^*, v^*). This completes the proof of Theorem 11. □

Remark 12. By the proof of Theorems 10 and 11, we obtained the positive solutions of system (2) suppose $(\mathbf{H}_0)-(\mathbf{H}_2)$ hold; and the uniqueness of the solution to the system is established provided that system (2) satisfies the additional conditions ($\lambda_1 + \delta_1 < 1$ and $\lambda_2 + \delta_2 < 1$) in theorem.

4. Example

Now we consider the existence and uniqueness of positive solutions for the fractional differential system (1). Obviously,

$$\begin{gathered}
\alpha_1 = \alpha_2 = \frac{5}{2},\\
\mu_1 = \frac{1}{2},\\
\mu_2 = 1,\\
A_1(t) = t,\\
A_2(t) = t^{1/2}.
\end{gathered} \tag{91}$$

We also have

$$\begin{aligned}
k_1 &= \int_0^1 t^{\alpha_2 - 1} dA_1(t) = \int_0^1 t^{3/2} dt = \frac{2}{5} > 0,\\
k_2 &= \int_0^1 t^{\alpha_1 - 1} dA_2(t) = \int_0^1 t^{3/2} dt^{1/2}\\
&= \frac{1}{2}\int_0^1 t dt = \frac{1}{4} > 0,\\
1 - \mu_1\mu_2 k_1 k_2 &= \frac{19}{20} > 0.
\end{aligned} \tag{92}$$

So, the condition $(\mathbf{H}_0)$ holds. For

$$\begin{aligned}
f(t,x,y) &= \frac{(\sin t + \cos t)\sqrt{x}}{\sqrt[3]{t(1-t)y}},\\
g(t,x,y) &= \frac{\sqrt[3]{y}}{e^t\sqrt{t(1-t)x}},
\end{aligned} \tag{93}$$

it is easy to see that $f : (0,1) \times [0,+\infty) \times (0,+\infty) \to (0,+\infty)$ is continuous, $f(t,x,y)$ is nondecreasing in x and nonincreasing in y, $g : (0,1) \times (0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is continuous, and $g(t,x,y)$ is nonincreasing in x and nondecreasing in y. Take

$$\begin{gathered}
\lambda_1 = \frac{11}{20},\\
\delta_1 = \frac{2}{5},\\
\lambda_2 = \frac{3}{5},\\
\delta_2 = \frac{1}{5}.
\end{gathered} \tag{94}$$

Then, we know that the condition $(\mathbf{H}_1)$ holds. As

$$\begin{aligned}
&\int_0^1 s(1-s)^{\alpha_1 - 1} f(s, 1, s^{\alpha_2 - 1}) ds\\
&\quad \leq 2\int_0^1 s^{1/2}(1-s) ds = \frac{8}{15} < +\infty,\\
&\int_0^1 s(1-s)^{\alpha_2 - 1} g(s, s^{\alpha_1 - 1}, 1) ds\\
&\quad \leq \int_0^1 s^{-1/4}(1-s) ds = \frac{16}{21} < +\infty,
\end{aligned} \tag{95}$$

the condition $(\mathbf{H}_2)$ also holds.

Therefore, by Theorem 10, we get that system (1) has at least one positive solution $(u^*, v^*) \in K_{[r_1, r_2]}$. For

$$\begin{gathered}
\alpha_1 = \alpha_2,\\
\lambda_1 + \delta_1 = \frac{11}{20} + \frac{2}{5} = \frac{19}{20} < 1,\\
\lambda_2 + \delta_2 = \frac{3}{5} + \frac{1}{5} = \frac{4}{5} < 1,
\end{gathered} \tag{96}$$

by Theorem 11, we get that (u^*, v^*) is the unique positive solution of system (1).

Remark 13. The example not only implies that $f(t, x, y)$ can be singular at $t = 0, 1$, $y = 0$, and $g(t, x, y)$ can be singular at $t = 0, 1$, $x = 0$, but also indicates that there is a large number of functions that satisfy the conditions of the theorems which we discuss in this paper. Also, the conditions in our theorems are easy to check.

5. Conclusions

In this paper, the $(n-1, 1)$-type singular fractional differential system with coupled boundary conditions has been investigated. Based on the well-known Guo-Krasnoselskii fixed point theorem, the existence of solutions for the $(n-1, 1)$-type fractional differential system with coupled integral boundary conditions is presented; also the uniqueness of positive solution is established when certain additional conditions are satisfied. The example given demonstrates the effectiveness and feasibility of our results.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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