

# On the geometry of the continuous-time generalized algebraic Riccati equation arising in LQ optimal control\*

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**Abstract**—In this paper we analyze the properties of the set of solutions of the generalized continuous algebraic Riccati equation from a geometric perspective. In particular, we study the relationship existing between the solutions of the generalized Riccati equation and the output-nulling subspaces of the underlying system. This analysis reveals the presence of a subspace that plays an important role in the solution of the related optimal control problem.

## I. INTRODUCTION

The classic solution of the continuous-time regular infinite-horizon linear quadratic (LQ) problem is traditionally expressed in terms of a symmetric solution  $X$  of the Riccati equation

$$XA + A^T X - (S + XB)R^{-1}(S^T + B^T X) + Q = 0, \quad (1)$$

where  $A, Q \in \mathbb{R}^{n \times n}$ ,  $B, S \in \mathbb{R}^{n \times m}$ ,  $R \in \mathbb{R}^{m \times m}$  are such that the so-called *Popov matrix*  $\Pi$  satisfies

$$\Pi \stackrel{\text{def}}{=} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} = \Pi^T \geq 0, \quad (2)$$

and  $R$  is invertible. When  $R$  is allowed to be singular, the corresponding LQ problem is called *singular*, see e.g. [13], [20], [17], [16], [14]. In particular, in [13] and [20] it was proved that an optimal solution of the singular LQ problem exists for all initial conditions if the class of controls is extended to include distributions. In [15] the constrained generalized continuous algebraic Riccati equation was defined, in analogy with the discrete case, by replacing the inverse of the matrix  $R$  appearing in (1) with its Moore-Penrose pseudo-inverse. To our best knowledge, the recent papers [9], [10] were the first attempts to relate this equation to singular LQ optimal control problems. Indeed, in [9], [10] it was shown that the existence of symmetric solutions of the constrained generalized continuous-time Riccati equation is equivalent to the existence of impulse-free solutions of the associated singular LQ problem from any initial condition. This means, in particular, that an optimal control can always be expressed as a state-feedback. Now that the connection between the constrained generalized continuous-time algebraic Riccati equation and the singular LQ problem has been explained,

the important issue arises of analyzing the set of solutions of such equation and the relations of each of such solutions with the corresponding LQ control problem. Considerable effort has been devoted in providing a geometric characterization of the set of solutions of the discrete counterpart of this Riccati equation, see [18], [6], [7], [8]. A similar characterization for the continuous time generalized Riccati equation has never been considered.

Another reason for analyzing the geometric structure of the solutions of the generalized continuous-time Riccati equation is the fact that this equation is a particular case of a more general type of Riccati equation that arises in the literature that flourished in the past twenty years on stochastic optimal control, see e.g. [1], [2], [3], [11] and the references cited therein.

In this paper a geometric analysis is carried out on the structure of the symmetric solutions of the constrained generalized continuous-time algebraic Riccati equation. In particular, we will prove that the dynamics of the closed-loop system can be divided into a part that depends on the particular solution  $X$  considered, and one part which is independent of it. We also show that the latter dynamics, which is not necessarily stable, is confined to a reachability output nulling subspace, so that it does not contribute to the cost function. The spectrum associated with the reachable part of this dynamics can therefore be assigned without affecting the optimality of the cost. We show that the LQ optimal control problem may admit a stabilizing solution even in cases in which the generalized continuous-time Riccati equation does not admit a stabilizing solution. This new feature has no parallel in the regular LQ problems.

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## II. THE GENERALIZED RICCATI EQUATION AND LINEAR QUADRATIC OPTIMAL CONTROL

Consider the state differential equation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \in \mathbb{R}^n. \quad (3)$$

where  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^{n \times m}$ . The classic LQ problem can be stated as the problem of finding the control  $u(t)$ ,  $t \geq 0$ , that minimizes the quadratic performance index

$$J(x_0, u) = \int_0^\infty [x^T(t) \quad u^T(t)] \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt \quad (4)$$

subject to (3), where  $Q \in \mathbb{R}^{n \times n}$ ,  $S \in \mathbb{R}^{n \times m}$  and  $R \in \mathbb{R}^{m \times m}$  satisfy (2). We denote  $\Sigma \stackrel{\text{def}}{=} (A, B, \Pi)$ , which is sometimes referred to as *Popov triple*. When  $R$  is positive definite, the

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optimal control (when it exists) does not include distributions. If  $R$  is only positive semidefinite, the optimal solution can contain Dirac deltas and its derivatives. Important links exist between the existence of the solutions of the so-called *generalized continuous algebraic Riccati equation* GCARE( $\Sigma$ )

$$XA + A^T X - (S + XB)R^\dagger (S^T + B^T X) + Q = 0, \quad (5)$$

subject to the constraint

$$\ker R \subseteq \ker(S + XB), \quad (6)$$

and the non-impulsive optimal solutions of the infinite-horizon LQ problem, [9], [10]. The equation (5) along with the condition (6) is usually referred to as the *constrained generalized continuous algebraic Riccati equation*, and denoted by CGCARE( $\Sigma$ ).

The crucial difference between the discrete and the continuous time is that, whereas in the discrete time the existence of symmetric positive semidefinite solutions of the constrained generalized discrete algebraic Riccati equation is equivalent to the solvability of the infinite-horizon LQ problem, in the continuous time case this correspondence holds for the so-called *regular* solutions, i.e., the optimal controls of the LQ problem that do not contain distributions.

### III. PRELIMINARY RESULTS ON CGCARE( $\Sigma$ )

The purpose of this section is to provide a geometric characterisation for the set of solutions of the generalized continuous algebraic Riccati equation. Most of the results in the sequel hinge on the geometric concepts of output-nulling subspace and friend. These concepts are briefly recalled in Appendix A for the sake of completeness. We herein recall a standard linear algebra result that is used in the derivations of this paper.

*Lemma 3.1:* Consider  $\Pi = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} = \Pi^T \geq 0$ . Then,

- (i)  $\ker S \supseteq \ker R$ ;
- (ii)  $SR^\dagger R = S$ ;
- (iii)  $S(I_m - R^\dagger R) = 0$ ;
- (iv)  $Q - SR^\dagger S^T \geq 0$ .

In view of (i) in Lemma 3.1, (6) is equivalent to  $\ker R \subseteq \ker(XB)$ . The following notation is used throughout the paper. First, let  $G \stackrel{\text{def}}{=} I_m - R^\dagger R$  be the orthogonal projector that projects onto  $\ker R$ . Moreover, we consider a non-singular matrix  $T = [T_1 \mid T_2]$  where  $\text{im} T_1 = \text{im} R$  and  $\text{im} T_2 = \text{im} G$ , and we define  $B_1 \stackrel{\text{def}}{=} B T_1$  and  $B_2 \stackrel{\text{def}}{=} B T_2$ . Finally, to any  $X = X^T \in \mathbb{R}^{n \times n}$  we associate the following matrices

$$\begin{aligned} Q_X &\stackrel{\text{def}}{=} Q + A^T X + X A, \\ S_X &\stackrel{\text{def}}{=} S + X B, \\ K_X &\stackrel{\text{def}}{=} R^\dagger S_X^T, \\ A_X &\stackrel{\text{def}}{=} A - B K_X, \\ \Pi_X &\stackrel{\text{def}}{=} \begin{bmatrix} Q_X & S_X \\ S_X^T & R \end{bmatrix}. \end{aligned}$$

When  $X = X^T$  is a solution of CGDARE( $\Sigma$ ), then  $K_X$  is the corresponding gain matrix,  $A_X$  the associated closed-loop matrix.

*Remark 3.1:* A symmetric and positive semidefinite solution of the generalized discrete-time algebraic Riccati equation also solves the constrained generalized discrete-time algebraic Riccati equation, [6]. This fact does not hold in the continuous time, i.e., not all symmetric and positive semidefinite solutions of GCARE( $\Sigma$ ) are also solutions of CGCARE( $\Sigma$ ).

### IV. CHARACTERIZATION OF THE SOLUTIONS OF CGCARE

Since  $\Pi$  is assumed symmetric and positive semidefinite, we can consider a factorization of the form

$$\Pi = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} = \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix}, \quad (7)$$

where  $Q = C^T C$ ,  $S = C^T D$  and  $R = D^T D$ . Let us define

$$G(s) \stackrel{\text{def}}{=} C(sI_n - A)^{-1} B + D.$$

Let  $G^\sim(s) \stackrel{\text{def}}{=} G^T(-s)$ . The spectrum is defined as

$$\Phi(s) \stackrel{\text{def}}{=} G^\sim(s) G(s).$$

It is easy to see that

$$\Phi(s) = \begin{bmatrix} B^T(-sI_n - A^T)^{-1} & I_m \end{bmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} (sI_n - A)^{-1} B \\ I_m \end{bmatrix}.$$

We recall the following classical result.

*Lemma 4.1:* For any  $X = X^T \in \mathbb{R}^{n \times n}$ , there holds

$$\Phi(s) = \begin{bmatrix} B^T(-sI_n - A^T)^{-1} & I_m \end{bmatrix} \Pi_X \begin{bmatrix} (sI_n - A)^{-1} B \\ I_m \end{bmatrix}. \quad (8)$$

*Proof:* Let us define

$$L_X \stackrel{\text{def}}{=} \Pi_X - \Pi = \begin{bmatrix} A^T X + X A & X B \\ B^T X & 0 \end{bmatrix}.$$

It is enough to prove that

$$\begin{bmatrix} B^T(-sI_n - A^T)^{-1} & I_n \end{bmatrix} L_X \begin{bmatrix} (sI_n - A)^{-1} B \\ I_n \end{bmatrix} = 0.$$

Indeed

$$\begin{aligned} &\begin{bmatrix} B^T(-sI_n - A^T)^{-1} & I_n \end{bmatrix} L_X \begin{bmatrix} (sI_n - A)^{-1} B \\ I_n \end{bmatrix} \\ &= -B^T(sI_n + A^T)^{-1} [(sI_n + A^T)X - X(sI_n - A)](sI_n - A)^{-1} B \\ &\quad - B^T(sI_n + A^T)^{-1} X B + B^T X (sI_n - A)^{-1} B = 0. \end{aligned}$$

■

The following important result is the continuous-time counterpart of Theorem 3.1 in [6].

*Theorem 4.1:* Let  $X = X^T$  be a solution of CGCARE( $\Sigma$ ). Then,

- 1)  $W(s) \stackrel{\text{def}}{=} R^{\frac{1}{2}} R^\dagger S_X^T (sI_n - A)^{-1} B + R^{\frac{1}{2}}$  is a square spectral factor of  $\Phi(s)$ ;
- 2)  $\text{normrank} \Phi(s) = \text{rank} R$ .

Now we show that, given a solution  $X = X^T$  of GCARE( $\Sigma$ ):

(a)  $\ker X$  is an output-nulling subspace for the quadruple  $(A, B, C, D)$ , i.e.,

$$\begin{bmatrix} A \\ C \end{bmatrix} \ker X \subseteq (\ker X \oplus 0_p) + \text{im} \begin{bmatrix} B \\ D \end{bmatrix};$$

(b) the gain  $K_X$  is such that  $-K_X$  is a *friend* of  $\ker X$ , i.e.,

$$\begin{bmatrix} A - BK_X \\ C - DK_X \end{bmatrix} \ker X \subseteq \ker X \oplus 0_p.$$

In the case where  $X = X^T$  is the solution of GCARE( $\Sigma$ ) corresponding to the optimal cost, these properties are intuitive. Now we prove that the following stronger result holds.

**Theorem 4.2:** Let  $X = X^T$  be a solution of GCARE( $\Sigma$ ). Then,  $\ker X$  is an output-nulling subspace of the quadruple  $(A, B, C, D)$  and  $-K_X$  is a friend of  $\ker X$ .

**Proof:** Since  $X$  is a solution of GCARE( $\Sigma$ ), the Lyapunov equation

$$X A_X + A_X^T X + Q_{0X} = 0 \quad (9)$$

holds, where  $Q_{0X} \stackrel{\text{def}}{=} Q - SR^\dagger S^T + X B R^\dagger B^T X = C_X^T C_X \geq 0$ . Let  $\xi \in \ker X$ . If we multiply (9) to the left by  $\xi^T$  and to the right by  $\xi$ , we obtain  $C_X \xi = 0$ , which proves that  $\ker X \subseteq \ker C_X$ . We multiply the same equation to the right by  $\xi$ , and we obtain  $X A_X \xi = 0$ , which shows that  $\ker X$  is  $A_X$ -invariant. We have proved that  $\ker X$  is an  $A_X$ -invariant subspace contained in the null-space of  $C_X$ , and this shows that  $\ker X$  is an output-nulling subspace for  $(A, B, C, D)$  and  $-K_X = -R^\dagger S_X$  is an associated friend. ■

We will show that the reachable subspace  $\mathcal{R}_{\ker X}^*$  on the output-nulling subspace  $\ker X$ , coincides with the reachable subspace of the pair  $(A_X, B G_X)$ , that we denote by  $\mathcal{R}_{0,X}$ . Before we establish this fact, we first need to give some additional results on the solutions of CGCARE( $\Sigma$ ).

**Lemma 4.2:** Let  $X = X^T$  solve CGCARE( $\Sigma$ ) and let  $\mathcal{R}_{0,X}$  denote the reachable subspace associated with the pair  $(A_X, B G)$ , i.e.,

$$\mathcal{R}_{0,X} \stackrel{\text{def}}{=} \text{im} [B G \ A_X B G \ A_X^2 B G \ \dots \ A_X^{n-1} B G]. \quad (10)$$

Let  $C_X \stackrel{\text{def}}{=} C - D R^\dagger S_X^T$ . There holds

$$\mathcal{R}_{0,X} \subseteq \ker C_X. \quad (11)$$

The following result shows that the subspace  $\mathcal{R}_{0,X}$  is independent of the particular solution  $X = X^T$  of

CGCARE( $\Sigma$ ), and such is also the spectrum of the closed-loop matrix restricted to  $\mathcal{R}_{0,X}$ .

**Theorem 4.3:** Let  $X = X^T$  be a solution of CGCARE( $\Sigma$ ), and let  $\mathcal{R}_{0,X}$  be defined by (10). Then,

- $\mathcal{R}_{0,X}$  is independent of  $X$ ;
- $A_X|_{\mathcal{R}_{0,X}}$  is independent of  $X$ .

As a consequence of this result, if the spectrum of the closed-loop matrix restricted to  $\mathcal{R}_{0,X}$  contains unstable eigenvalues, no solutions of CGCARE( $\Sigma$ ) can stabilize the closed loop. Notice that this issue does not arise in the standard case because, when  $R$  is positive definite, matrix  $G$  is zero, and therefore  $\mathcal{R}_{0,X} = \{0\}$  for every symmetric solution  $X = X^T$  of CARE( $\Sigma$ ).

**Theorem 4.4:** Let  $X = X^T$  be a solution of CGCARE( $\Sigma$ ). Let  $\mathcal{R}_{\ker X}^*$  be the largest reachability subspace on  $\ker X$ . Then,  $\mathcal{R}_{\ker X}^* = \mathcal{R}_{0,X}$ .

**Proof:** Since  $\mathcal{R}_{0,X}$  is the reachable subspace of the pair  $(A_X, B G)$ , it is the smallest  $A_X$ -invariant subspace containing  $\text{im}(B G) = B \ker D$ . The reachability output-nulling subspace  $\mathcal{R}_{\ker X}^*$  on the output-nulling subspace  $\ker X$  is the smallest  $(A + B F)$ -invariant subspace containing  $\ker X \cap B \ker D$ , where  $F$  be an *arbitrary* friend of  $\ker X$ , see Appendix A. The subspace  $\mathcal{R}_{\ker X}^*$  does not depend on the choice of the friend  $F$ , [19, Theorem 7.18]. In view of Theorem 4.2,  $F = -K_X$  is a particular friend of  $\ker X$ . For this choice of  $F$ , we have  $A + B F = A - B K_X = A_X$ . Moreover,  $\ker X \cap B \ker D = B \ker D$ , because the inclusion  $\ker R \subseteq \ker(X B)$  implies  $\ker X \supseteq B \ker D$ . ■

## V. STABILIZATION

So far we have shown that the spectrum of the closed-loop matrix  $A_X$  restricted to the subspace  $\mathcal{R}_{0,X}$  is independent of the particular solution  $X = X^T$  of CGCARE( $\Sigma$ ) considered. This means that the corresponding eigenvalues are present in the closed-loop independently of the solution  $X = X^T$  of CGCARE( $\Sigma$ ) considered. Since we have shown that  $\mathcal{R}_{0,X} = \mathcal{R}_{\ker X}^*$ , it is always possible to find a matrix  $L$  that assigns all the eigenvalues of the map  $(A_X + B G L)$  restricted to  $\mathcal{R}_{\ker X}^*$ , by adding a further term  $B G L x(t)$  to the feedback control law. This operation does not change the value of the cost with respect to the one obtained by  $u(t) = -K_X x(t)$ , because this additional term only affects the part of the trajectory on  $\mathcal{R}_{\ker X}^*$  which is output-nulling. In doing so it may stabilize the closed-loop. Indeed, since  $\mathcal{R}_{0,X}$  is output-nulling with respect to the quadruple  $(A, B, C, D)$ , it is also output-nulling for the quadruple  $(A - B K_X, B, C - D K_X, D)$ , and two matrices  $\Xi$  and  $\Omega$  exist such that

$$\begin{bmatrix} A_X \\ C_X \end{bmatrix} R_{0,X} = \begin{bmatrix} R_{0,X} \\ 0 \end{bmatrix} \Xi + \begin{bmatrix} B \\ D \end{bmatrix} \Omega, \quad (12)$$

where  $R_{0,X}$  is a basis matrix of  $\mathcal{R}_{0,X}$ , see Appendix A. In order to find a feedback matrix which stabilizes the system, we solve (12) in  $\Xi$  and  $\Omega$ , so as to find  $L$  such that  $\begin{bmatrix} A_X + B L \\ C_X + D L \end{bmatrix} R_{0,X} = \begin{bmatrix} R_{0,X} \\ 0 \end{bmatrix} \Xi$ , where the eigenvalues of  $\Xi$  are the

eigenvalues of the map  $A_X + BL$  restricted to  $\mathcal{R}_{0,X}$ . We first compute the set of solutions of (12) in  $\Xi$  and  $\Omega$ , i.e.,

$$\begin{bmatrix} \Xi \\ \Omega \end{bmatrix} = \begin{bmatrix} \hat{\Xi} \\ \hat{\Omega} \end{bmatrix} + \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} K, \quad (13)$$

where

$$\begin{bmatrix} \hat{\Xi} \\ \hat{\Omega} \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} R_0 & B \\ 0 & D \end{bmatrix}^\dagger \begin{bmatrix} A_X \\ C_X \end{bmatrix} R_0$$

and  $\begin{bmatrix} H_1 \\ H_2 \end{bmatrix}$  is a basis matrix of  $\ker \begin{bmatrix} R_0 & B \\ 0 & D \end{bmatrix}$ . Since  $\mathcal{R}_{0,X}$  is an output-nulling reachability subspace, the pair  $(\hat{\Xi}, H_1)$  is completely reachable. This implies that a matrix  $K$  in (13) can always be found so that the eigenvalues of  $\Xi$  are freely assignable (provided they come in complex conjugate pairs). Hence, we use such  $K$  in (13) and then we compute  $L = -\Omega R_0^\dagger$ . This choice guarantees that only the eigenvalues of  $A_X$  restricted to  $\mathcal{R}_{0,X}$  get affected by the use of  $L$ .

*Example 5.1:* Consider the quadruple  $A = \begin{bmatrix} 1 & 1 & 1 \\ -3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ ,  $B = \begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$ ,  $Q = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ ,  $S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 0 \end{bmatrix}$  and  $R = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ . One can directly verify that the only two solutions of  $\text{CGCARE}(\Sigma)$  are  $X_0 = 0$  and  $X_1 = \text{diag}\{0, 0, 2\}$ . None of these two solutions is stabilizing. Indeed, the eigenvalues of the closed-loop matrix relative to  $X_0$  are  $\{1, 1 \pm i\sqrt{3}\}$ , while those of the one relative to  $X_1$  are  $\{-1, 1 \pm i\sqrt{3}\}$ . Thus,  $\text{CGCARE}(\Sigma)$  does not have a stabilizing solution. However, the infinite-horizon LQ problem admits an optimal solution. Consider for example  $X = X_0$ . We find  $\mathcal{R}_{0,X_0} = \langle A_{X_0}, BG \rangle = \text{im} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ . Since this subspace does not depend on the particular solution of  $\text{CGCARE}(\Sigma)$ , we have  $\mathcal{R}_{0,X_1} = \mathcal{R}_{0,X_0}$ . Since this subspace has dimension 2, we can stabilize at most 2 eigenvalues in the closed-loop matrix without altering the cost. On the other hand, since  $A_{X_0}$  has three unstable eigenvalues, we cannot obtain a stabilizing feedback from  $X = X_1$ . Using (13) we compute  $\hat{\Xi} = \begin{bmatrix} 1/5 & 1/5 \\ -3 & 1 \end{bmatrix}$  and  $\hat{\Omega} = \begin{bmatrix} 0 & 0 \\ 2/5 & 2/5 \end{bmatrix}$ . A basis for the null-space of  $\begin{bmatrix} R_{0,X} & B \\ 0 & D \end{bmatrix}$  is  $[H_1^T \mid H_2^T]^T = [-2 \ 0 \mid 0 \ 1]^T$ . We determine  $K$  so that the eigenvalues of  $\hat{\Xi} + H_1 K$  are equal to  $\{-2, -3\}$ . This yields  $K = [3.1 \ -1.9]$ , so that  $\Omega = \hat{\Omega} + H_2 K = \begin{bmatrix} 0 & 0 \\ 7/2 & -3/2 \end{bmatrix}$  and  $L = -\Omega R_0^\dagger = \begin{bmatrix} 0 & 0 & 0 \\ -7/2 & 3/2 & 0 \end{bmatrix}$ . Thus,  $\sigma(A_X + BL) = \{-1, -2, -3\}$ . It is easy to verify that the value of the cost does not change. This solution is optimal, and is also stabilizing. We found a stabilizing optimal control even if  $\text{CGCARE}(\Sigma)$  does not admit a stabilizing solution.

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## APPENDIX A

In this Appendix, we recall some concepts of classical geometric control theory that are used in this paper. More details can be found e.g. in [19]. Consider an LTI system described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), & x(0) &= x_0 \in \mathbb{R}^n \\ y(t) &= Cx(t) + Du(t), \end{aligned} \quad (14)$$

where  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ ,  $C \in \mathbb{R}^{p \times n}$  and  $D \in \mathbb{R}^{p \times m}$ . We recall that the *reachable subspace* is  $\mathcal{R}_0 = \text{im} [B \ AB \ \dots \ A^{n-1}B]$ , and coincides with the smallest  $A$ -invariant subspace of  $\mathbb{R}^n$  containing the image of  $B$ , i.e.  $\mathcal{R}_0 = \langle A, \text{im} B \rangle$ . An *output-nulling subspace*  $\mathcal{V}$  of (14) is a subspace of  $\mathbb{R}^n$  which satisfies the inclusion

$$\begin{bmatrix} A \\ C \end{bmatrix} \mathcal{V} \subseteq (\mathcal{V} \oplus \{0\}) + \text{im} \begin{bmatrix} B \\ D \end{bmatrix}, \quad (15)$$

which is equivalent to the existence of a matrix  $F \in \mathbb{R}^{m \times n}$  such that  $(A+BF)\mathcal{V} \subseteq \mathcal{V} \subseteq \ker(C+DF)$ . Any real matrix  $F$  satisfying these inclusions is referred to as a *friend* of  $\mathcal{V}$ . We denote by  $\mathfrak{F}(\mathcal{V})$  the set of friends of  $\mathcal{V}$ . We denote by  $\mathcal{V}^*$  the largest output-nulling subspace of (14), which represents the set of all initial states  $x_0$  of (14) for which a control input exists such that the corresponding output function is identically zero. Such an input function can always be implemented as a static state feedback of the form  $u(t) = Fx(t)$  where  $F \in \mathfrak{F}(\mathcal{V}^*)$ . Eq. (15) is equivalent to the existence of two matrices  $\Xi$  and  $\Omega$  such that

$$\begin{bmatrix} A \\ C \end{bmatrix} V = \begin{bmatrix} V \\ 0 \end{bmatrix} \Xi + \begin{bmatrix} B \\ D \end{bmatrix} \Omega, \quad (16)$$

where  $V$  is a basis matrix for the output-nulling subspace  $\mathcal{V}$ . The set of solutions of this equation is parameterised in  $K_1$  as

$$\begin{bmatrix} \Xi \\ \Omega \end{bmatrix} = \begin{bmatrix} V & B \\ 0 & D \end{bmatrix}^\dagger \begin{bmatrix} A \\ C \end{bmatrix} V + \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} K_1, \quad (17)$$

where  $\begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix}$  is a basis matrix for  $\ker \begin{bmatrix} V & B \\ 0 & D \end{bmatrix}$ . The set of friends  $F$  of  $\mathcal{V}$  are the solutions of the linear equation  $\Omega = -FV$ , where  $U$  is such that for a certain  $\Xi$  the equation (16) holds, and since  $V$  is full column-rank, the set of its solutions can be written as

$$F = -\Omega(V^\top V)^{-1}V^\top + K_2\Psi,$$

where  $\Psi$  is a full row-rank matrix such that  $\ker\Psi = \mathcal{V}$ , and  $K_2$  is arbitrary. Then, the set of friends of  $\mathcal{V}$  are parameterised in  $K_1$  and  $K_2$ , where  $K_1$  only affects the eigenstructure of the closed-loop restricted to  $\mathcal{V}$ , i.e.,  $\sigma(A+BF|_{\mathcal{V}})$ , whereas  $K_2$  only affects the eigenstructure  $\sigma(A+BF|_{\mathcal{X}/\mathcal{V}})$ . In other words, given a change of coordinate matrix  $T = [T_1 \ T_2]$  where  $T_1$  is a basis matrix for  $\mathcal{V}$ , we have  $T^{-1}(A+BF)T = \begin{bmatrix} A_{1,1}^F & A_{1,2}^F \\ 0 & A_{2,2}^F \end{bmatrix}$ , where  $A_{1,1}^F$  does not depend on  $K_2$  and  $A_{2,2}^F$  does not depend on  $K_1$ .

The so-called *output-nulling reachability subspace* on  $\mathcal{V}^*$ , herein denoted with  $\mathcal{R}^*$ , is the smallest  $(A+BF)$ -invariant subspace of  $\mathbb{R}^n$  containing the subspace  $\mathcal{V}^* \cap B \ker D$ , where  $F \in \mathfrak{F}(\mathcal{V}^*)$ , i.e.,  $\mathcal{R}^* = \langle A+BF, \mathcal{V}^* \cap B \ker D \rangle$  where  $F \in \mathfrak{F}(\mathcal{V}^*)$ . Let  $F \in \mathfrak{F}(\mathcal{V}^*)$ . The closed-loop spectrum can be partitioned as  $\sigma(A+BF) = \sigma(A+BF|_{\mathcal{V}^*}) \cup \sigma(A+BF|_{\mathcal{X}/\mathcal{V}^*})$ , where  $\sigma(A+BF|_{\mathcal{V}^*})$  is the spectrum of  $A+BF$  restricted to  $\mathcal{V}^*$  and  $\sigma(A+BF|_{\mathcal{X}/\mathcal{V}^*})$  is the spectrum of the mapping induced by  $A+BF$  on the quotient space  $\mathcal{X}/\mathcal{V}^*$ . The eigenvalues of  $A+BF$  restricted to  $\mathcal{V}^*$  can be further split into two disjoint sets: the eigenvalues of  $\sigma(A+BF|_{\mathcal{R}^*})$  are all freely assignable with a suitable choice of  $F$  in  $\mathfrak{F}(\mathcal{V}^*)$ . The eigenvalues in  $\sigma(A+BF|_{\mathcal{V}^*/\mathcal{R}^*})$  are fixed for all the choices of  $F$  in  $\mathfrak{F}(\mathcal{V}^*)$ .