A geometric theory for 2-D systems including notions of stabilisability *

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Abstract

In this paper we consider the problem of internally and externally stabilising controlled invariant and output-nulling subspaces for two-dimensional (2-D) Fornasini-Marchesini models, via static feedback. A numerically tractable procedure for computing a stabilising feedback matrix is developed via linear matrix inequality techniques. This is subsequently applied to solve, for the first time, various 2-D disturbance decoupling problems subject to a closed-loop stability constraint.

1 Introduction

The notion of controlled invariance, introduced by Basile and Marro in [1], is central to the so-called geometric approach to linear control system analysis/synthesis with stationary state-space models, for systems that operate between signals defined over a one-dimensional (1-D) independent variable such as time. The most celebrated application of this concept is to the disturbance decoupling problem, solved for the first time in [1]. Disturbance decoupling with the additional requirement of internal stability was considered by Wonham and Morse in [24], via the introduction of (A, B) stabilisability subspaces. An improved solution to the same problem was subsequently suggested by Basile and Marro in [2], using the concept of self-bounded controlled invariance to avoid eigenspace computation; this permits the maximum number of eigenvalues of the closed-loop to be freely placed, as later shown by Malabre, Martínez-García, and Del-Muro-Cuéllar [18].

Over the same period of time, a significant stream of literature emerged regarding the modelling and analysis of two-dimensional (2-D) systems, which operate between signals defined over a 2-D

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independent variable (e.g. space and time). This includes the well-known Roesser [22] and Fornasini-Marchesini (FM) [8, 9] models, which are inherently related as shown in [10], for example. Many important results have been achieved in the attempt to develop a geometric theory for 2-D systems [5, 6, 13, 14]. In particular, a definition of controlled invariance was first proposed in [5] for FM models. This definition, even though less powerful than its 1-D counterpart, enjoys feedback properties that are very useful in synthesis problems. In [5], it is shown how to employ the notion of controlled invariance to solve 2-D decoupling problems with unmeasured and/or measured disturbances, but without stability constraints. The lack of guaranteed stability in the existing solutions of such problems poses the biggest limitation to the application of these techniques, particularly from the perspective of numerical implementation.

In part, the aim of this paper is to characterise a new notion of stabilisability for the invariant subspaces defined in [5]. More precisely, we investigate the problem of internally and externally stabilising a controlled invariant or output-nulling subspace, by means of a static feedback. The analysis yields, via established linear matrix inequality (LMI) based analysis techniques, a tractable procedure for computing a stabilising feedback matrix. Armed with these results, we present solutions to the aforementioned disturbance decoupling problems, subject to a closed-loop stability constraint. Finally, a full-information decoupling and model matching problem are also solved under a similar stability constraint.

Notation. Throughout, we denote by $\mathbb{Z}$ and $\mathbb{N}$ the integers, and positive integers including zero (i.e., natural numbers), respectively. The symbol $\mathbf{0}_n$ stands for the origin of the vector space $\mathbb{R}^n$. The image and the kernel of the linear map associated with multiplication by a matrix $M \in \mathbb{R}^{n\times m}$ are denoted by $\operatorname{im} M \subseteq \mathbb{R}^n$ and $\ker M \subseteq \mathbb{R}^m$, respectively. The $n \times m$ zero matrix is denoted by $0_{n\times m}$ and the $n \times n$ identity matrix is denoted by I_n. Given a matrix M, the symbols $M^\top$ and $M^\dagger$ denote the transpose and the Moore-Penrose pseudoinverse of M, respectively.

2 Invariant subspaces for autonomous FM models

In this section, we begin by considering the autonomous FM model

$$x_{i+1,j+1} = A_1 x_{i+1,j} + A_2 x_{i,j+1}, \tag{1}$$

where $A_1, A_2 \in \mathbb{R}^{n\times n}$ and the vector $x_{i,j} \in \mathbb{R}^n$ is called the *local state* at $(i, j) \in \mathbb{Z} \times \mathbb{Z}$. Defining, for each $k \in \mathbb{Z}$, the separation set

$$\mathbb{S}_k \triangleq \Big\{(i,j) \in \mathbb{Z}\times\mathbb{Z} \;\Big|\; i + j = k\Big\},$$

and the corresponding instance of the *global state*

$$\mathcal{X}_k \triangleq \Big\{x_{i,j} \;\Big|\; (i,j) \in \mathbb{S}_k\Big\},$$

it follows that $\mathcal{X}_k$ can be uniquely expressed in terms of $\mathcal{X}_{k-1}$ [8]. In particular, if we fix the values of $x_{i,j}$ on $\mathbb{S}_0$ (i.e., fix $\mathcal{X}_0$ as a boundary condition), equation (1) uniquely determines $\mathcal{X}_k$ for $k > 0$ (i.e.,

$x_{i,j}$ for $i+j>0$) .[1] Indeed, these are the boundary conditions usually associated with the FM model (1). In the sequel, given a subspace $\mathcal{W}\subseteq\mathbb{R}^n$, by a $\mathcal{W}$-valued boundary condition we intend $x_{i,j}\in\mathcal{W}$ for all $(i,j)\in\mathbb{S}_0$. Similarly, for each $k>0$, the global state $\mathcal{X}_k$ is said to be $\mathcal{W}$-valued when $x_{i,j}\in\mathcal{W}$ for all $(i,j)\in\mathbb{S}_k$.

Subspaces of $\mathbb{R}^n$ which are invariant under multiplication by A_1 and under multiplication by A_2, prove to be useful in analysing the dynamics of (1). In particular, given such a subspace $\mathcal{J}\subseteq\mathbb{R}^n$, it follows that for any $\mathcal{J}$-valued boundary condition, the global state $\mathcal{X}_k$ is $\mathcal{J}$-valued for all $k>0$. To see this, note from (1) that, by hypothesis and for $(i,j)\in\mathbb{S}_k$, whenever the elements $x_{i-1,j}$ and $x_{i,j-1}$ of the global state $\mathcal{X}_{k-1}$ are in $\mathcal{J}$, the elements $x_{i,j}=A_1x_{i,j-1}+A_2x_{i-1,j}$ of $\mathcal{X}_k$ also lie in $\mathcal{J}$. That is, the subspace $\mathcal{J}$ is invariant under the dynamics of the model (1). In what follows, the notion of (A_1,A_2)-invariance is discussed in more detail; it is shown how the dynamics of an autonomous FM model can be decomposed with respect to an (A_1,A_2)-invariant subspace, leading to definitions for internal and external stability, in preparation for the subsequent discussion of controlled-invariance for non-autonomous FM models in Section 3.

2.1 (A_1,A_2)-invariance

The theory expounded in this section parallels the one presented in [3, Section 3.2], for invariant subspaces of 1-D systems. Given the matrices $A_1,A_2\in\mathbb{R}^{n\times n}$ associated with an autonomous 2-D FM model of the form (1), a subspace $\mathcal{J}$ of $\mathbb{R}^n$ is said to be (A_1,A_2)-invariant if $\left[\begin{smallmatrix}A_1\\A_2\end{smallmatrix}\right]x\in\mathcal{J}\times\mathcal{J}$ for all $x\in\mathcal{J}$. It is standard to denote this kind of invariance property by means of the following inclusion

$$\begin{bmatrix}A_1\\A_2\end{bmatrix}\mathcal{J}\subseteq\mathcal{J}\times\mathcal{J},\tag{2}$$

where the left-hand side denotes the image of the subspace $\mathcal{J}$ under the linear map associated with multiplication by the matrix $\left[\begin{smallmatrix}A_1\\A_2\end{smallmatrix}\right]\in\mathbb{R}^{2n\times n}$. It follows that $\mathcal{J}$ is (A_1,A_2)-invariant if, and only if, $\mathcal{J}$ is both A_1-invariant and A_2-invariant in the usual 1-D sense. Moreover, we have the following result.

Lemma 2.1 *Let $\mathcal{J}$ be an r-dimensional subspace of $\mathbb{R}^n$ and let $J\in\mathbb{R}^{n\times r}$ be a basis matrix for $\mathcal{J}$; i.e.,* $\operatorname{im}J=\mathcal{J}$ *and* $\ker J=\mathbf{0}_n$. *The subspace $\mathcal{J}$ is (A_1,A_2)-invariant if, and only if, there exist two matrices $X_1,X_2\in\mathbb{R}^{r\times r}$ such that*

$$\begin{bmatrix}A_1\\A_2\end{bmatrix}J=\begin{bmatrix}J&0_{n\times r}\\0_{n\times r}&J\end{bmatrix}\begin{bmatrix}X_1\\X_2\end{bmatrix}.\tag{3}$$

Proof: The proof follows directly on noting that (3) is simply a matrix expression for the subspace inclusion (2). ■

[1] As shown in [9], other separation sets can be defined so that boundary conditions specified over them uniquely determine a local-state trajectory solution of (1) over a region of $\mathbb{Z}\times\mathbb{Z}$. An interesting and useful example is the separation set $\mathbb{S}_k\triangleq\Big\{(i,j)\in\{0\}\times[1,\infty)\cup[1,\infty)\times\{0\}\Big\}$, which with corresponding boundary conditions uniquely determines $x_{i,j}$ for $(i,j)\in\mathbb{N}\times\mathbb{N}$. Most of the considerations in this paper can be adapted to such separations sets.

Remark 2.1 Note that (3) can also be expressed as

$$\begin{bmatrix} A_1 & A_2 \end{bmatrix} \begin{bmatrix} J & 0_{n\times r} \\ 0_{n\times r} & J \end{bmatrix} = J \begin{bmatrix} X_1 & X_2 \end{bmatrix},$$

which is equivalent to the subspace inclusion $\begin{bmatrix} A_1 & A_2 \end{bmatrix} (\mathcal{J} \times \mathcal{J}) \subseteq \mathcal{J}$.

The following theorem is the 2-D counterpart of a well-known result [3, Theorem 3.2.1] concerning the decomposition of a 1-D system matrix A with respect to an invariant subspace.

Theorem 2.1 *The following are equivalent:*

(i) There exists an r-dimensional subspace $\mathcal{J} \subseteq \mathbb{R}^n$ that is (A_1, A_2)-invariant;

(ii) There exists a similarity transformation $T \in \mathbb{R}^{n\times n}$ such that

$$\hat{A}_1 \triangleq T^{-1} A_1 T = \begin{bmatrix} \hat{A}_{1,11} & \hat{A}_{1,12} \\ 0_{(n-r)\times r} & \hat{A}_{1,22} \end{bmatrix} \quad \textit{and} \quad \hat{A}_2 \triangleq T^{-1} A_2 T = \begin{bmatrix} \hat{A}_{2,11} & \hat{A}_{2,12} \\ 0_{(n-r)\times r} & \hat{A}_{2,22} \end{bmatrix}. \tag{4}$$

Proof: ***(i)*** $\Longrightarrow$ ***(ii)*** Let $J \in \mathbb{R}^{n\times r}$ be a basis matrix for $\mathcal{J}$. Then, by Lemma 2.1, two matrices $X_1, X_2 \in \mathbb{R}^{r\times r}$ exist such that (3) holds. Since J is of full column-rank, a non-singular matrix $T \in \mathbb{R}^{n\times n}$ exists such that $T^{-1} J = \left[\begin{smallmatrix} I_r \\ 0_{(n-r)\times r} \end{smallmatrix}\right]$. As such, with $\hat{A}_1 = T^{-1} A_1 T = \left[\begin{smallmatrix} \hat{A}_{1,11} & \hat{A}_{1,12} \\ \hat{A}_{1,21} & \hat{A}_{1,22} \end{smallmatrix}\right]$ and $\hat{A}_2 = T^{-1} A_2 T = \left[\begin{smallmatrix} \hat{A}_{2,11} & \hat{A}_{2,12} \\ \hat{A}_{2,21} & \hat{A}_{2,22} \end{smallmatrix}\right]$, it follows from (3) that

$$\begin{bmatrix} \hat{A}_{i,11} \\ \hat{A}_{i,21} \end{bmatrix} = \begin{bmatrix} \hat{A}_{i,11} & \hat{A}_{i,12} \\ \hat{A}_{i,21} & \hat{A}_{i,22} \end{bmatrix} \begin{bmatrix} I_r \\ 0_{(n-r)\times r} \end{bmatrix} = \hat{A}_i T^{-1} J = T^{-1} A_i J = T^{-1} J X_i = \begin{bmatrix} X_i \\ 0_{(n-r)\times r} \end{bmatrix}, \tag{5}$$

for $i = 1, 2$. That is, $\hat{A}_{1,21} = \hat{A}_{2,21} = 0$, as required in *(ii)*.

(ii) $\Longrightarrow$ ***(i)*** Let T be such that (4) holds. Then,

$$\hat{A}_1 \begin{bmatrix} I_r \\ 0_{(n-r)\times r} \end{bmatrix} = \begin{bmatrix} X_1 \\ 0_{(n-r)\times r} \end{bmatrix} \quad \text{and} \quad \hat{A}_2 \begin{bmatrix} I_r \\ 0_{(n-r)\times r} \end{bmatrix} = \begin{bmatrix} X_2 \\ 0_{(n-r)\times r} \end{bmatrix} \tag{6}$$

hold for $X_1 = \hat{A}_{1,11}$ and $X_2 = \hat{A}_{1,11}$. Pre-multiplying (6) by T yields

$$A_1 T \begin{bmatrix} I_r \\ 0_{(n-r)\times r} \end{bmatrix} = T \begin{bmatrix} I_r \\ 0_{(n-r)\times r} \end{bmatrix} X_1 \quad \text{and} \quad A_2 T \begin{bmatrix} I_r \\ 0_{(n-r)\times r} \end{bmatrix} = T \begin{bmatrix} I_r \\ 0_{(n-r)\times r} \end{bmatrix} X_2.$$

As such, $T \left[\begin{smallmatrix} I_r \\ 0_{(n-r)\times r} \end{smallmatrix}\right]$ is an r-dimensional (A_1, A_2)-invariant subspace, by Lemma 2.1. ∎

Remark 2.2 Consider the matrix $T = \begin{bmatrix} T_1 & T_2 \end{bmatrix} \in \mathbb{R}^{n\times n}$, where $T_1 \in \mathbb{R}^{n\times r}$ is a basis matrix for an r-dimensional (A_1, A_2)-invariant subspace $\mathcal{J}$ and $T_2 \in \mathbb{R}^{n\times(n-r)}$ is any matrix that makes T non-singular (such as, but not necessarily, a basis matrix for $\mathcal{J}^\perp$). Then in terms of the new coordinates $\left[\begin{smallmatrix} x'_{i,j} \\ x''_{i,j} \end{smallmatrix}\right] \triangleq T^{-1} x_{i,j}$, it follows from the proof of Theorem 2.1, that the model (1) is equivalent to

$$\begin{bmatrix} x'_{i+1,j+1} \\ x''_{i+1,j+1} \end{bmatrix} = \begin{bmatrix} \hat{A}_{1,11} & \hat{A}_{1,12} \\ 0 & \hat{A}_{1,22} \end{bmatrix} \begin{bmatrix} x'_{i+1,j} \\ x''_{i+1,j} \end{bmatrix} + \begin{bmatrix} \hat{A}_{2,11} & \hat{A}_{2,12} \\ 0 & \hat{A}_{2,22} \end{bmatrix} \begin{bmatrix} x'_{i,j+1} \\ x''_{i,j+1} \end{bmatrix}. \tag{7}$$

Note that any $\mathcal{J}$-valued boundary condition for the model (1) is such that the corresponding boundary conditions for the equivalent model (7) satisfy $x''_{i,j} = 0$ for all $(i,j) \in \mathbb{S}_0$. Moreover, in this case, it follows by the lower block part of (7) that $x''_{i,j} = 0$ for $(i,j) \in \mathbb{S}_k$ and $k > 0$. As such, in the original coordinates, the local state $x_{i,j} = T \begin{bmatrix} x'_{i,j} \\ x''_{i,j} \end{bmatrix} = T_1\, x'_{i,j}$, must lie in $\mathcal{J}$ for all $(i,j) \in \mathbb{S}_k$ and $k > 0$.

In light of Remark 2.2, it can be seen that $x'_{i,j}$, in the coordinates of the equivalent model (7), represents a projection of the local state vector onto an r-dimensional (A_1, A_2)-invariant subspace $\mathcal{J}$. On the other hand, the component $x''_{i,j}$ represents the canonical projection of the local state onto the $(n-r)$-dimensional quotient space $\mathbb{R}^n/\mathcal{J}$. Therefore, we refer to $x'_{i,j}$ as the *internal* (with respect to $\mathcal{J}$) component of the local state and to $x''_{i,j}$ as the *external* (with respect to $\mathcal{J}$) component of the local state. Similarly,

$$x'_{i+1,j+1} = \hat{A}_{1,11}\, x'_{i+1,j} + \hat{A}_{1,12}\, x''_{i+1,j} + \hat{A}_{2,11}\, x'_{i,j+1} + \hat{A}_{2,12}\, x''_{i,j+1}, \tag{8}$$

is said to govern the internal dynamics on $\mathcal{J}$ and

$$x''_{i+1,j+1} \quad \hat{A}_{1,22}\, x''_{i+1,j} + \hat{A}_{2,22}\, x''_{i,j+1}, \tag{9}$$

is said to govern the external dynamics of $\mathcal{J}$.

2.2 Internal and external stability of invariant subspaces

With $\|\mathcal{X}_k\| \triangleq \sup_{n\in\mathbb{Z}} \|x_{k-n,n}\|$, the system model (1) is said to be asymptotically stable if for any boundary condition satisfying $\|\mathcal{X}_0\| < \infty$, the corresponding sequence $\{\|\mathcal{X}_i\|\}_{i=0}^{\infty}$ converges to zero [8]. This is clearly invariant under coordinate transformation and with a slight abuse of nomenclature, the system matrix pair (A_1, A_2) is called asymptotically stable, in this case. It is well-known that the pair (A_1, A_2) is asymptotically stable if, and only if,

$$\det(I_n - A_1\, z_2 - A_2\, z_1) \neq 0 \qquad \forall\, (z_1, z_2) \in \mathfrak{P} \tag{10}$$

where $\mathfrak{P} = \Big\{(\zeta_1, \zeta_2) \in \mathbb{C} \times \mathbb{C} \;\Big|\; |\zeta_1| \leq 1 \text{ and } |\zeta_2| \leq 1\Big\}$ is the unit bidisc [8, Proposition 3]. Various, more computationally tractable, sufficient stability conditions have been proposed over the last two decades, expressed in terms of Lyapunov equations and/or spectral radius conditions of certain matrices, see e.g. [11, 12, 4]. In the very recent literature, new necessary and sufficient criteria have appeared for asymptotic stability in terms of conditions that can be checked in finite terms, see [25, 7]. For the sake of argument and clarity, however, the following simple sufficient condition for asymptotic stability, expressed in terms of an linear matrix inequality (LMI), will be used herein:

Lemma 2.2 ([12]) *The pair (A_1, A_2) is asymptotically stable if there exist two symmetric positive definite matrices P_1 and P_2 such that*

$$\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} - \begin{bmatrix} A_1^\top \\ A_2^\top \end{bmatrix} (P_1 + P_2) \begin{bmatrix} A_1 & A_2 \end{bmatrix} > 0. \tag{11}$$

The LMI condition in Lemma 2.2 is one of the most utilised for analysis and synthesis problems involving FM models. Here it forms the foundation of a procedure developed for computing the static feedback matrices that stabilise the internal and external dynamics of controlled invariant and output-nulling subspaces, which are defined shortly.

Our aim for the moment is to show that, as in the 1-D case, the stability of (1) can be studied in terms of two parts, with respect to a given (A_1, A_2)-invariant subspace $\mathcal{J}$. In particular, using the fact that the determinant of a block upper triangular matrix is the product of the determinants of the blocks on the diagonal, by (10) it follows that the equivalent model (7) is asymptotically stable if, and only if, the two matrix pairs $(\hat{A}_{1,11}, \hat{A}_{2,11})$ and $(\hat{A}_{1,22}, \hat{A}_{2,22})$ are each asymptotically stable. Moreover, when a $\mathcal{J}$-valued boundary condition is imposed (see Remark 2.2), so that for all $k \geq 0$ the global state $\mathcal{X}''_k$ associated with the external dynamics (9) satisfies $\|\mathcal{X}''_k\| = 0$ and the internal dynamics on $\mathcal{J}$ satisfy

$$x'_{i+1,j+1} = \hat{A}_{1,11}\, x'_{i+1,j} + \hat{A}_{2,11}\, x'_{i,j+1}, \tag{12}$$

if $(\hat{A}_{1,11}, \hat{A}_{2,11})$ alone is also asymptotically stable, then the global state $\mathcal{X}'_k$ associated with (12) satisfies $\|\mathcal{X}'_k\| \to 0$. Indeed, since in this case the global state $\mathcal{X}_k$ associated with (1) satisfies

$$\|\mathcal{X}_k\| \leq \bar{\sigma}(T_1)\|\mathcal{X}'_k\| + \bar{\sigma}(T_2)\|\mathcal{X}''_k\| = \bar{\sigma}(T_1)\|\mathcal{X}'_k\|,$$

where $\bar{\sigma}(\cdot)$ denotes maximum singular value and $T = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$ denotes the similarity transformation for the coordinate change used to obtain the equivalent model (7), we also have that $\|\mathcal{X}_k\| \to 0$.

Definition 2.1 *The (A_1, A_2)-invariant subspace $\mathcal{J}$ is said to be* internally stable *if the corresponding internal dynamics governed by (12) are asymptotically stable; i.e., the corresponding pair $(\hat{A}_{1,11}, \hat{A}_{2,11})$ is asymptotically stable.*

The following lemma, which follows directly from (5), will be useful in the sequel.

Lemma 2.3 *Let $\mathcal{J}$ be an r-dimensional (A_1, A_2)-invariant subspace, J be a basis matrix for $\mathcal{J}$, and $X_1, X_2 \in \mathbb{R}^{r\times r}$ be such that (3) holds. Then $\mathcal{J}$ is internally stable if, and only if, the pair (X_1, X_2) is asymptotically stable.*

Consider now a boundary condition that is not $\mathcal{J}$-valued, so that $\|\mathcal{X}''_0\| \neq 0$. It follows from (9) that $\|\mathcal{X}''_k\| \to 0$ if, and only if, the pair $(\hat{A}_{1,22}, \hat{A}_{2,22})$ is asymptotically stable, and in this case, the elements of the global state $\mathcal{X}_k$ associated with (1) approach the invariant subspace $\mathcal{J}$, as $k \to \infty$.

Definition 2.2 *The r-dimensional (A_1, A_2)-invariant subspace $\mathcal{J}$ is said to be* externally stable *if the corresponding external dynamics governed by (9) are asymptotically stable; i.e., the corresponding pair $(\hat{A}_{1,22}, \hat{A}_{2,22})$ is asymptotically stable.*

Finally, in view of the discussion above, note that the model (1) is asymptotically stable if, and only if, any (A_1, A_2)-invariant subspace is both internally and externally stable.

3 Controlled invariant subspaces for non-autonomous FM models

Consider the non-autonomous FM model

$$x_{i+1,j+1} = A_1\, x_{i+1,j} + A_2\, x_{i,j+1} + B_1\, u_{i+1,j} + B_2\, u_{i,j+1}, \tag{13}$$

where $A_1, A_2 \in \mathbb{R}^{n\times n}$, $B_1, B_2 \in \mathbb{R}^{n\times m}$ and, for $i,j \in \mathbb{Z}$, let $x_{i,j} \in \mathbb{R}^n$ and $u_{i,j} \in \mathbb{R}^m$ denote the local state and input, respectively. Given a $k > 0$, the instance $\mathcal{X}_k$ of the global state associated with the FM model (13), defined as before in the autonomous case, is uniquely determined given $\mathcal{X}_0$ and the inputs on $\bigcup_{i=0}^{k-1} \mathbb{S}_i \subset \mathbb{Z}\times\mathbb{Z}$. As such, the boundary conditions typically associated with (13) correspond to fixing the local state over $\mathbb{S}_0$.

Definition 3.1 ([5]) *The subspace $\mathcal{V} \subseteq \mathbb{R}^n$ is controlled invariant for (13) if it satisfies the subspace inclusion*

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \mathcal{V} \subseteq (\mathcal{V}\times\mathcal{V}) + \operatorname{im}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}. \tag{14}$$

A direct consequence of this definition is that the subspaces $\mathbf{0}_n$ and $\mathbb{R}^n$ are controlled invariant subspaces for (13). Moreover, if $\mathcal{V}$ is controlled invariant then it is both (A_1, B_1) and (A_2, B_2)-controlled invariant in the usual 1-D sense [1]. The converse, however, is not true in general, as observed in [14]. A controlled invariant subspace $\mathcal{V}$ implies the existence of a set of inputs $\{u_{i,j} \,|\, i+j \geq 0\}$ for which the corresponding local state solution of (13) lies in $\mathcal{V}$, for all $i+j > 0$ and any $\mathcal{V}$-valued boundary condition. While in the 1-D case the converse is true as well, with the above definition of controlled invariance for 2-D FM models, the subspace of minimal dimension which contains a given sequence satisfying (13) is not necessarily controlled invariant. Nonetheless, the definition enjoys good feedback properties, as shown for the first time in [5], and briefly recalled in Lemma 3.1.

Remark 3.1 It is worth mentioning that an alternative definition of controlled invariance was proposed in [14] for a different class of FM models described by

$$x_{i+1,j+1} = A_1\, x_{i+1,j} + A_2\, x_{i,j+1} + B\, u_{i,j}. \tag{15}$$

According to the definition used therein, controlled invariant subspaces are indeed loci of controlled local state, but they cannot be associated with local-state feedback control, since the structure of the model described by (15) is not preserved under local-state feedback.

Lemma 3.1 *Let $\mathcal{V}$ be an r-dimensional subspace of $\mathbb{R}^n$ and let $V \in \mathbb{R}^{n\times r}$ be a basis matrix for $\mathcal{V}$. The following are equivalent:*

i) The subspace $\mathcal{V}$ is controlled invariant for (13);

ii) There exist matrices $X \in \mathbb{R}^{2r\times r}$ and $\Omega \in \mathbb{R}^{m\times r}$ such that

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} V = \begin{bmatrix} V & 0 \\ 0 & V \end{bmatrix} X + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \Omega; \tag{16}$$

iii) There exists a matrix $F \in \mathbb{R}^{m\times n}$ such that $\mathcal{V}$ is $(A_1+B_1\,F, A_2+B_2\,F)$-invariant, i.e.,

$$\left[\begin{array}{c} A_1+B_1\,F \\ A_2+B_2\,F \end{array}\right] \mathcal{V} \subseteq \mathcal{V}\times\mathcal{V}; \tag{17}$$

iv) There exist matrices $F \in \mathbb{R}^{m\times n}$ and $X \in \mathbb{R}^{2\,r\times r}$ such that

$$\left[\begin{array}{c} A_1+B_1\,F \\ A_2+B_2\,F \end{array}\right] V = \left[\begin{array}{cc} V & 0 \\ 0 & V \end{array}\right] X. \tag{18}$$

Proof: The implication ***i)*** $\Longrightarrow$ ***ii)*** follows from Definition 3.1 on noting that (16) is simply a matrix representation of the subspace inclusion (14). To prove ***ii)*** $\Longrightarrow$ ***iii)*** it suffices to take $F=-\Omega\,(V^\top V)^{-1}V^\top$. It follows that $\Omega=-F\,V$, that can be replaced in (16) to get (17). The implication ***iii)*** $\Longrightarrow$ ***iv)*** follows directly from the fact that (18) is a matrix representation of the inclusion (17). Finally, the implication ***iv)*** $\Longrightarrow$ ***i)*** follows by re-writing (18) as

$$\left[\begin{array}{c} A_1 \\ A_2 \end{array}\right] V = \left[\begin{array}{cc} V & 0 \\ 0 & V \end{array}\right] X - \left[\begin{array}{c} B_1 \\ B_2 \end{array}\right] F\,V. \tag{19}$$

This completes the proof. ■

Remark 3.2 The pairs of matrices X and Ω which satisfy the linear equation (16) can be parameterised by

$$\left[\begin{array}{c} X \\ \Omega \end{array}\right] = W^\dagger \left[\begin{array}{c} A_1 \\ A_2 \end{array}\right] V + H\,K, \tag{20}$$

where $W \triangleq \left[\begin{smallmatrix} V & 0 & B_1 \\ 0 & V & B_2 \end{smallmatrix}\right]$, H is a basis matrix for $\ker W$ and K is an arbitrary matrix of suitable size.

Let F be such that (17) holds true. Applying a static local-state feedback $u_{i,j}=F\,x_{i,j}$ in (13) we find that

$$x_{i+1,j+1} = (A_1+B_1\,F)\,x_{i+1,j} + (A_2+B_2\,F)\,x_{i,j+1}. \tag{21}$$

Moreover, under such control action and given a $\mathcal{V}$-valued boundary condition, it follows as in the autonomous case discussed above, that the global state $\mathcal{X}_k$ is $\mathcal{V}$-valued for $k>0$. Given a controlled invariant subspace $\mathcal{J}$, the set of matrices F such that (17) holds is denoted by $\mathfrak{F}(\mathcal{V})$; when $F\in\mathfrak{F}(\mathcal{V})$ it is said to be a *friend* of the controlled invariant subspace $\mathcal{V}$. As in the 1-D case, and since $\mathcal{V}$ is $(A_1+B_1\,F, A_2+B_2\,F)$-invariant for all $F\in\mathfrak{F}(\mathcal{V})$, the definitions for internal and external stability of invariant subspaces introduced in Section 2.2 can be used to define notions of internal and external *stabilisability* with respect to a 2-D controlled invariant subspace.

Definition 3.2 *The controlled invariant subspace $\mathcal{V}$ is said to be* internally (resp. externally) stabilisable *if there exists an $F\in\mathfrak{F}(\mathcal{V})$ such that $\mathcal{V}$ is an internally (resp. externally) stable $(A_1+B_1\,F, A_2+B_2\,F)$-invariant subspace.*

To see how to choose a friend F of a controlled invariant subspace $\mathcal{V}$ to achieve internally (resp. externally) stability, a more explicit characterisation of the set $\mathfrak{F}(\mathcal{V})$ is required.

Lemma 3.2 *Let $\mathcal{V}$ be an r-dimensional controlled invariant subspace and let $V \in \mathbb{R}^{n\times r}$ be a basis matrix for $\mathcal{V}$. Each matrix $F \in \mathfrak{F}(\mathcal{V})$ is a solution of the linear equation $\Omega = -F\,V$, where $\Omega \in \mathbb{R}^{m\times r}$ is a solution of (16) for some $X \in \mathbb{R}^{2\,r\times r}$. In particular,*

$$\mathfrak{F}(\mathcal{V}) = \Big\{F = -\Omega\,(V^\top V)^{-1}V^\top + \Lambda \mid \Omega \text{ satisfies (16) for some } X \text{ and } \Lambda\,V = 0\Big\}. \tag{22}$$

Proof: The statement follows on noting that any $F \in \mathfrak{F}(\mathcal{V})$ satisfies (18) for some $X \in \mathbb{R}^{2\,r\times r}$. Hence, (18) can be written as (19). It follows that (16) is satisfied with this X and $\Omega = -F\,V$. To complete the proof, note that since V is full column-rank, all solutions of the linear equation $\Omega = -F\,V$ can be written as

$$F = F_\Omega + \Lambda, \tag{23}$$

where $F_\Omega = -\Omega\,(V^\top V)^{-1}V^\top$ and Λ is any matrix of suitable size such that $\Lambda V = 0$. ∎

Since all $F \in \mathfrak{F}(\mathcal{V})$ are such that $\mathcal{V}$ is $(A_1 + B_1\,F, A_2 + B_2\,F)$-invariant, it follows as discussed in Remark 2.2 that the similarity transformation $T = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$, with T_1 set to be a basis matrix for $\mathcal{V}$, is such that

$$T^{-1}(A_i + B_i\,F)\,T = \begin{bmatrix} \hat{G}_{i,11}(\Omega,\Lambda) & \hat{G}_{i,12}(\Omega,\Lambda) \\ 0 & \hat{G}_{i,22}(\Omega,\Lambda) \end{bmatrix} \qquad \text{for } i = 1, 2. \tag{24}$$

Equation (24) emphasises that for different values of Ω and Λ satisfying the conditions in (22), we obtain different matrices $\hat{G}_{i,*}(\Omega,\Lambda)$. Importantly, it is shown in Lemma 3.3 below that the matrices $\hat{G}_{1,11}(\Omega,\Lambda)$ and $\hat{G}_{2,11}(\Omega,\Lambda)$ do not depend on Λ, and similarly, the matrices $\hat{G}_{1,22}(\Omega,\Lambda)$ and $\hat{G}_{2,22}(\Omega,\Lambda)$ do not depend on Ω. In this way, the two matrices Ω and Λ can be chosen independently to build a friend of $\mathcal{V}$, so that the former does not affect $(\hat{G}_{1,22}, \hat{G}_{2,22})$ and the latter does not affect $(\hat{G}_{1,11}, \hat{G}_{2,11})$. In other words, when $\mathcal{V}$ is internally stabilisable, Ω can be chosen first so that F_Ω stabilises $(\hat{G}_{1,11}, \hat{G}_{2,11})$, and then Λ can be chosen to stabilise $(\hat{G}_{1,22}, \hat{G}_{2,22})$, if $\mathcal{V}$ is also externally stabilisable, without affecting the internal stabilisation achieved with F_Ω. These two independent stabilisation procedures are examined in the following sections.

Lemma 3.3 *The matrices $\hat{G}_{i,11}(\Omega,\Lambda)$ in (24) do not depend on Λ. The matrices $\hat{G}_{i,22}(\Omega,\Lambda)$ in (24) do not depend on Ω.*

Proof: First, we prove that the matrices $\hat{G}_{i,11}(\Omega,\Lambda)$ in (24) do not depend on Λ. Let $F_k = F_\Omega + \Lambda_k$ for $k = 1, 2$, where Λ_1 and Λ_2 are such that $\Lambda_1\,V = 0$ and $\Lambda_2\,V = 0$, and $F_\Omega = -\Omega\,(V^\top\,V)^{-1}\,V^\top$, where Ω is such that (16) holds for some X. Then, (24) can be written as

$$T^{-1}(A_i + B_i\,F_k)\,T = \begin{bmatrix} \hat{G}_{i,11}(\Omega,\Lambda_k) & \hat{G}_{i,12}(\Omega,\Lambda_k) \\ 0 & \hat{G}_{i,22}(\Omega,\Lambda_k) \end{bmatrix}. \tag{25}$$

Our aim is to show that $\hat{G}_{i,11}(\Omega,\Lambda_1) = \hat{G}_{i,11}(\Omega,\Lambda_2)$ for $i = 1,2$. From (25) we find

$$\begin{bmatrix} \hat{G}_{i,11}(\Omega,\Lambda_1) - \hat{G}_{i,11}(\Omega,\Lambda_2) & \hat{G}_{i,12}(\Omega,\Lambda_1) - \hat{G}_{i,12}(\Omega,\Lambda_2) \\ 0 & \hat{G}_{i,22}(\Omega,\Lambda_1) - \hat{G}_{i,22}(\Omega,\Lambda_2) \end{bmatrix}$$
$$= T^{-1}\,(A_i + B_i\,F_\Omega + B_i\,\Lambda_1)\,T - T^{-1}\,(A_i + B_i\,F_\Omega + B_i\,\Lambda_2)\,T$$
$$= T^{-1}\,B_i(\Lambda_1 - \Lambda_2)\begin{bmatrix} T_1 & T_2 \end{bmatrix} = \begin{bmatrix} 0 & T^{-1}\,B_i\,(\Lambda_1 - \Lambda_2)\,T_2 \end{bmatrix},$$

since $\Lambda_1\,T_1 = \Lambda_2\,T_1 = 0$. Thus, $\hat{G}_{i,11}(\Omega,\Lambda_1) - \hat{G}_{i,11}(\Omega,\Lambda_2) = 0$.

Now we show that the matrices $\hat{G}_{i,22}(\Omega,\Lambda)$ in (24) do not depend on Ω. To this end, let Ω_1 and Ω_2 be such that (16) holds for some X_1 and X_2, respectively. By difference,

$$\begin{bmatrix} V & 0 \\ 0 & V \end{bmatrix}(X_1 - X_2) + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}(\Omega_1 - \Omega_2) = 0. \tag{26}$$

With $F_k = -\Omega_k\,(V^\top V)^{-1}\,V^\top + \Lambda$, for $k = 1,2$, where Λ is any matrix such that $\Lambda\,V = 0$, it follows that (24) can be written as

$$T^{-1}(A_i + B_i\,F_k)\,T = \begin{bmatrix} \hat{G}_{i,11}(\Omega_k,\Lambda) & \hat{G}_{i,12}(\Omega_k,\Lambda) \\ 0 & \hat{G}_{i,22}(\Omega_k,\Lambda) \end{bmatrix}. \tag{27}$$

For the sake of conciseness, let $L_{i,*} \triangleq \hat{G}_{i,*}(\Omega_1,\Lambda) - \hat{G}_{i,*}(\Omega_2,\Lambda)$. Subtracting (27), with $k = 2$, from (27), with $k = 1$, gives

$$B_i\,(\Omega_2 - \Omega_1)\,(V^\top V)^{-1}V^\top \begin{bmatrix} T_1 & T_2 \end{bmatrix} = \begin{bmatrix} T_1 & T_2 \end{bmatrix}\begin{bmatrix} L_{i,11} & L_{i,12} \\ 0 & L_{i,22} \end{bmatrix},$$

which in particular, yields $B_i\,(\Omega_2 - \Omega_1)\,(V^\top V)^{-1}V^\top T_2 = T_1\,L_{i,12} + T_2\,L_{i,22}$. Since no generality is lost by assuming $T_1 = V$, we find that

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}(\Omega_2 - \Omega_1)(V^\top V)^{-1}V^\top T_2 = \begin{bmatrix} V\,L_{1,12} + T_2\,L_{1,22} \\ V\,L_{2,12} + T_2\,L_{2,22} \end{bmatrix}.$$

Then using (26) to obtain

$$\begin{bmatrix} V & 0 \\ 0 & V \end{bmatrix}(X_1 - X_2)(V^\top V)^{-1}V^\top T_2 = \begin{bmatrix} V & 0 \\ 0 & V \end{bmatrix}\begin{bmatrix} L_{1,12} \\ L_{2,12} \end{bmatrix} + \begin{bmatrix} T_2 & 0 \\ 0 & T_2 \end{bmatrix}\begin{bmatrix} L_{1,22} \\ L_{2,22} \end{bmatrix},$$

it follows that

$$\begin{bmatrix} T_2 & 0 \\ 0 & T_2 \end{bmatrix}\begin{bmatrix} L_{1,22} \\ L_{2,22} \end{bmatrix} = 0,$$

since V and T_2 have linearly independent columns. This in turns implies that $L_{1,22} = L_{2,22} = 0$ since T_2 has linearly independent columns. This means that $\hat{G}_{i,22}(\Omega_1,\Lambda) = \hat{G}_{i,22}(\Omega_2,\Lambda)$ for $i = 1,2$. ■

3.1 Internal stabilisation

By Lemma 2.3, finding a matrix F_Ω to internally stabilise $\mathcal{V}$ is equivalent to finding an F_Ω for which the solution $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ to (18) is such that the pair (X_1, X_2) is asymptotically stable. Since the only degree of freedom here lies in the choice of Ω, which in turn is given by (20), we find that

- when the nullspace of $W \triangleq \begin{bmatrix} V & 0 & B_1 \\ 0 & V & B_2 \end{bmatrix}$ is zero, i.e., when

$$(\mathcal{V} \times \mathcal{V}) \cap \operatorname{im} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \mathbf{0}_{2n}, \tag{28}$$

  there is only one solution to the linear equation (20), and this either achieves internal stabilisation or it does not.

- when W has non-trivial kernel, we can write (20) as

$$\begin{bmatrix} X_1 \\ X_2 \\ \Omega \end{bmatrix} = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} + \begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix} K, \tag{29}$$

  where $\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} \triangleq W^\dagger \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} V$, $\operatorname{im} \begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix} = \ker W$ and K is an arbitrary matrix of suitable size. The problem now considered is one of finding a K such that the pair (X_1, X_2) is asymptotically stable. If such a K exists, we can exploit it in order to compute Ω from (29) along with the corresponding asymptotically stable pair (X_1, X_2), to yield the required solution of (16). Moreover, with $F = -\Omega\,(V^\top V)^{-1} V^\top$, we find that (18) is also satisfied. This in turn implies that F stabilises $\mathcal{V}$ internally by Lemma 2.3.

The following result provides a computationally tractable sufficient condition for the internal stabilisability of a controlled invariant subspace.

Theorem 3.1 *The controlled invariant subspace $\mathcal{V}$ is internally stabilisable if there exist matrices $M = M^\top > 0$, $N = N^\top > 0$ and Q of suitable dimensions such that*

$$\begin{bmatrix} -M & 0 & N L_1^\top + Q^\top H_1^\top \\ 0 & -(N-M) & N L_2^\top + Q^\top H_2^\top \\ L_1 N + H_1 Q & L_2 N + H_2 Q & -N \end{bmatrix} < 0. \tag{30}$$

Given a (M, N, Q) in the convex set defined by (30), a matrix K such that (X_1, X_2) in (29) is asymptotically stable is given by $K = Q\,N^{-1}$.

Proof: The controlled invariant subspace $\mathcal{V}$ is internally stabilisable if, and only if, there exist symmetric positive definite matrices P_1 and P_2 such that (X_1, X_2) satisfies (11) in Lemma 2.2. Since

$X_i = L_i + H_i K$ $(i = 1, 2)$, this is equivalent to the existence of two symmetric and positive definite matrices Φ and Ψ such that

$$\begin{bmatrix} -\Phi & 0 & (L_1 + H_1 K)^\top \Psi \\ 0 & -(\Psi - \Phi) & (L_2 + H_2 K)^\top \Psi \\ \Psi(L_1 + H_1 K) & \Psi(L_2 + H_2 K) & -\Psi \end{bmatrix} < 0.$$

Pre- and post-multiplying this matrix inequality by $\mathrm{diag}\{\Psi^{-1}, \Psi^{-1}, \Psi^{-1}\}$ and defining $M = \Psi^{-1}\Phi\,\Psi^{-1}$, $N = \Psi^{-1}$, and $Q = K\,\Psi^{-1}$, yields (30). Finally, note that $K = Q\,N^{-1}$. ■

When (28) holds, the matrices H_i in (29) can be considered void. In this case, condition (30) in Theorem 3.1 reduces to the existence $M = M^\top > 0$ and $N = N^\top > 0$ satisfying the LMI

$$\begin{bmatrix} -M & 0 & N\,X_1^\top \\ 0 & -(N - M) & N\,X_2^\top \\ X_1\,N & X_2\,N & -N \end{bmatrix} < 0,$$

which is obviously another way of saying that the pair (X_1, X_2) satisfies the sufficient condition for stability (11). As mentioned above, in this case there is only one solution (X, Ω) of equation (20), so that there are no degrees of freedom in the choice of F_Ω. Indeed, $F_\Omega = -\Omega\,(V^\top V)^{-1} V^\top$ is uniquely determined in this case, and either the pair $(A_1 + B_1\,F_\Omega, A_2 + B_2\,F_\Omega)$ is asymptotically stable – and this happens if and only if (X_1, X_2) is asymptotically stable – or the controlled invariant $\mathcal{V}$ cannot be internally stabilised.

3.2 External stabilisation

Given a controlled invariant subspace $\mathcal{V}$ and a corresponding basis matrix V, let (X, Ω) be any solution of (20) and let $F_\Omega = -\Omega\,(V^\top V)^{-1} V^\top$ be a friend of $\mathcal{V}$ that is internally stabilising. We now consider the possibility of choosing a suitable Λ in order to stabilise $\mathcal{V}$ externally. Applying the static feedback control action $u_{i,j} = (F_\Omega + \Lambda)\,x_{i,j}$ in (13) yields

$$x_{i+1,j+1} = (G_{1,\Omega} + B_1\,\Lambda)\,x_{i+1,j} + (G_{2,\Omega} + B_2\,\Lambda)\,x_{i,j+1},$$

where $G_{i,\Omega} \triangleq A_i + B_i\,F_\Omega$. The problem can now be considered as one of finding Λ such that

$$\begin{cases} \text{The pair } (G_{1,\Omega} + B_1\,\Lambda, G_{2,\Omega} + B_2\,\Lambda) \quad \text{is asymptotically stable} \\ \Lambda\,V = 0 \end{cases}$$

Theorem 3.2 *Let $\mathcal{V}$ be a controlled-invariant subspace for (13), which is internally stabilised by the static feedback matrix F_Ω; i.e., $(G_{1,\Omega}, G_{2,\Omega})$ is internally stable with respect to $\mathcal{V}$. Then $\mathcal{V}$ is also externally stabilisable if there exist matrices $M = M^\top > 0$, $N = N^\top > 0$, $R = R^\top > 0$ and S of suitable dimensions such that*

$$\begin{bmatrix} -M & 0 & (G_{1,\Omega} + B_1\,S^\top Q^\top)^\top \\ 0 & -(N - M) & (G_{2,\Omega} + B_2\,S^\top Q^\top)^\top \\ G_{1,\Omega} + B_1\,S^\top Q^\top & G_{2,\Omega} + B_2\,S^\top Q^\top & -R \end{bmatrix} < 0 \qquad (31)$$

with

$$NR = I. \tag{32}$$

Proof: First note that the condition $\Lambda V = 0$ can also be written as $\mathrm{im}\,\Lambda^\top \subseteq \ker V^\top$. Then, consider a basis matrix Q of $\ker V^\top$, so that $\mathrm{im}\,\Lambda^\top \subseteq \mathrm{im}\,Q$. Then it follows that $\Lambda^\top = Q\,S$ for some matrix S so that $\Lambda = S^\top Q^\top$. Now by Lemma 2.2, the pair $(G_{1,\Omega} + B_1\,S^\top\,Q^\top, G_{2,\Omega} + B_2\,S^\top\,Q^\top)$ is asymptotically stable if there exist two symmetric positive definite matrices M and N and a matrix S of suitable dimension such that

$$\begin{bmatrix} -M & 0 & (G_{1,\Omega} + B_1\,S^\top\,Q^\top)^\top \\ 0 & -(N-M) & (G_{2,\Omega} + B_2\,S^\top\,Q^\top)^\top \\ G_{1,\Omega} + B_1\,S^\top\,Q^\top & G_{2,\Omega} + B_2\,S^\top\,Q^\top & -N^{-1} \end{bmatrix} < 0$$

which is equivalent to (31) when combined with (32). ■

The set defined by the inequality (31) with the constraint (32) is not convex. However, various established numerical techniques are available for finding feasible points. Here we consider the so-called *sequential linear programming matrix method* (SLPMM) developed in [15]. To this end, we first notice that condition (32) is satisfied if and only if $\mathrm{Trace}(NR) = n$ and

$$\begin{bmatrix} N & I \\ I & R \end{bmatrix} \geq 0. \tag{33}$$

The problem of finding (M, N, R, S) that satisfy (31-32) can then be tackled with the following algorithm.[2]

Algorithm 3.1 *(Leibfritz, 2001, [15])*

Step 1*: Check the existence of a pair (N, R) satisfying (31) and (33). If such pair exists, denote it with (N^0, R^0).*

Step 2*: Given (N^k, R^k), $k \geq 0$, obtain a solution (N, R) together with S, to the convex optimization problem*

$$\begin{aligned} \min \quad & \mathit{Trace}(N\,R^k + N^k R) \\ & \textit{subject to } (31), (33). \end{aligned}$$

Denote this solution with (N_T^k, R_T^k).

Step 3*: If*

$$\left| \mathit{Trace}(N_T^k R^k + N^k R_T^k) - 2 \cdot \mathit{Trace}(N^k R^k) \right| \leq \nu$$

then stop, where ν is a pre-defined sufficiently small positive scalar.

[2]This may not always yield a feasible point, even if the non-convex set defined by (31-32) is non-empty.

Step 4: *Compute* $\alpha \in [0,1]$ *by solving*

$$\min_{\alpha\in[0,1]} \mathit{Trace}\Big([N^k + \alpha(N_T^k - N^k)]\,[R^k + \alpha(R_T^k - R^k)] \Big).$$

Step 5: *Set* $N^{k+1} = (1-\alpha)N^k + \alpha N_T^k$ *and* $R^{k+1} = (1-\alpha)R^k + \alpha R_T^k$*, then go to Step 2.*

The optimisation problems described in Steps 2 and 4 are standard and easy to solve computationally, see e.g. the MATLAB® routines `mincx.m` and `fminbnd.m`, in the LMI and Optimization Toolboxes, respectively.

Example 3.1 Consider (13) with

$$A_1 = \begin{bmatrix} 0.05 & -0.3 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.1 & -3 \\ 0 & 0.8 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1.5 & -5 & 0 & 0 \\ -3.5 & 3 & 0 & -0.5 \\ 0 & 2.5 & 0.02 & 0 \\ 0 & 0 & 0 & 0.05 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -3 & 0 \\ 1 & -7 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & -1 \\ -5 & 0 \\ 0 & 0.5 \\ -3 & 0 \end{bmatrix}.$$

This system does not satisfy the sufficient condition (11) for stability. It is easily seen that the subspace

$$\mathcal{V} = \mathrm{im} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is controlled invariant. In this case $W = \left[\begin{smallmatrix} V & 0 & B_1 \\ 0 & V & B_2 \end{smallmatrix}\right]$ is singular and $H = \begin{bmatrix} 0 & 0 & 0.7 & 0.1 & -0.05 & 0 & 0 & 0.1 \end{bmatrix}^\top$ is a basis matrix of $\ker W$. Let $\left[\begin{smallmatrix} X \\ \Omega \end{smallmatrix}\right] = W^\dagger \left[\begin{smallmatrix} A_1 \\ A_2 \end{smallmatrix}\right] V$, so that

$$X_1 = \begin{bmatrix} 0.05 & 0 & 0 \\ -2.1 & 0.1 & 2.7 \\ -0.2356 & -0.0014 & -0.0044 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1.5663 & -0.0002 & 0.0137 \\ 0.0332 & 0.0199 & 0.0068 \\ 2.1 & 0 & 0.35 \end{bmatrix},$$

$$\Omega = \begin{bmatrix} 0.7 & 0 & 0.1 \\ 0.0663 & -0.0002 & 0.0137 \end{bmatrix}.$$

It is easy to check that the pair (X_1, X_2) does not satisfy condition (11) for stability. As such, by taking

$$F_\Omega = -\Omega\,(V^\top V)^{-1} V^\top = \begin{bmatrix} -0.7 & 0 & 0 & -0.1 \\ -0.0663 & 0 & -0.0002 & -0.0137 \end{bmatrix},$$

we find that the pair $(A_1 + B_1 F_\Omega, A_2 + B_2 F_\Omega)$ is not necessarily asymptotically stable. By changing coordinates according to the similarity transformation

$$T = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 2 \end{array}\right]$$

which is adapted to $\mathcal{V}$ in the sense that the first three columns span it, we find

$$T^{-1}(A_1 + B_1\, F_\Omega)\, T \;=\; \left[\begin{array}{ccc|c} 2.0171 & 2.8058 & -0.8900 & -4.5088 \\ -1.7342 & -2.8115 & 1.1800 & 3.6176 \\ -1.9671 & -2.8058 & 0.9400 & 4.2088 \\ \hline 0 & 0 & 0 & 0.1 \end{array}\right],$$

$$T^{-1}(A_2 + B_2\, F_\Omega)\, T \;=\; \left[\begin{array}{ccc|c} 0.5399 & -0.3098 & 0.8300 & 2.1590 \\ -1.0732 & 0.6463 & -1.7000 & -1.8317 \\ 1.0268 & 0.2963 & 0.7500 & -7.1317 \\ \hline 0 & 0 & 0 & 3 \end{array}\right].$$

These structures clearly display the $(A_1 + B_1\, F_\Omega, A_2 + B_2\, F_\Omega)$-invariance of $\mathcal{V}$. In order to find an F_Ω which internally stabilises the controlled invariant subspace $\mathcal{V}$, let us consider

$$\left[\begin{array}{c} X \\ \Omega \end{array}\right] = \left[\begin{array}{ccc} V & 0 & B_1 \\ 0 & V & B_2 \end{array}\right]^{\dagger} \left[\begin{array}{c} A_1 \\ A_2 \end{array}\right] V + HK,$$

where $H = \left[\begin{array}{cccccccc} 0 & 0 & 0.7 & 0.1 & -0.05 & 0 & 0 & 0.1 \end{array}\right]^{\top}$. In this case, the LMI (30) is feasible, which implies internal stabilisability of $\mathcal{V}$. One such feasible point yields $K = \left[\begin{array}{ccc} -12.5979 & 0.0018 & -0.1506 \end{array}\right]$. Using (20) we get

$$X_1 = \left[\begin{array}{ccc} 0.05 & 0 & 0 \\ -2.1 & 0.1 & 2.7 \\ -12.5539 & 0.0004 & -0.1517 \end{array}\right], \quad X_2 = \left[\begin{array}{ccc} -0.1934 & 0.0001 & -0.0074 \\ -0.8467 & 0.0200 & -0.0037 \\ 2.1 & 0 & 0.35 \end{array}\right],$$

and

$$\Omega = \left[\begin{array}{ccc} 0.7 & 0 & 0.1 \\ -1.6934 & 0.0001 & -0.0074 \end{array}\right].$$

Now the pair (X_1, X_2) is asymptotically stable, as it satisfies the stability condition (11). With this choice

$$F_\Omega = -\Omega\, (V^{\top} V)^{-1} V^{\top} = \left[\begin{array}{cccc} -0.7 & 0 & 0 & -0.1 \\ 1.6934 & 0 & 0.0001 & 0.0074 \end{array}\right].$$

Now

$$T^{-1}(A_1 + B_1\, F_\Omega)\, T \;=\; \left[\begin{array}{ccc|c} -10.3046 & 2.9513 & -13.3556 & -4.8034 \\ 22.9093 & -3.1027 & 26.1112 & 4.2068 \\ 10.3546 & -2.9513 & 13.4056 & 4.5034 \\ \hline 0 & 0 & 0 & 0.1 \end{array}\right],$$

$$T^{-1}(A_2 + B_2\, F_\Omega)\, T \;=\; \left[\begin{array}{ccc|c} 3.1803 & -0.3410 & 3.5012 & 2.2222 \\ -5.4738 & 0.6983 & -6.1520 & -1.9369 \\ -3.3738 & 0.3483 & -3.7020 & -7.2369 \\ \hline 0 & 0 & 0 & 3 \end{array}\right].$$

This shows that the pair $(0.1, 3)$ accounting for the external dynamics of $\mathcal{V}$ has not changed by modifying the feedback F_Ω to internally stabilise the controlled invariant subspace $\mathcal{V}$. Since the pair $(0.1, 3)$ is unstable, our goal now is to stabilise $\mathcal{V}$ externally, by means of a feedback matrix $F = F_\Omega + \Lambda$, where $\Lambda V = 0$. In this case, Algorithm 3.1 provides a feasible solution to the external stabilisation problem. By choosing $\nu = 10^{-4}$, after 16 iterations of Steps 1-3, the matrices N^k and R^k for which the condition in Step 3 is satisfied are found. Their values yield

$$N^k R^k = \begin{bmatrix} 1.000003 & 0.000008 & 0.000015 & 0.000101 \\ 0.000000 & 1.000001 & -0.000006 & 0.000035 \\ 0.000000 & -0.000000 & 1.000004 & 0.000000 \\ 0.000000 & 0.000000 & 0.000000 & 1.000003 \end{bmatrix}$$

and

$$Trace(N^k R^k) \simeq 4.000012,$$

and the corresponding solution is given by $S = \begin{bmatrix} -0.633 & 1.305 \end{bmatrix}$, so that $\Lambda = \begin{bmatrix} 0 & 0.6332 & 0 & 0 \\ 0 & -1.3051 & 0 & 0 \end{bmatrix}$ satisfies $\Lambda \mathcal{V} = \mathbf{0}_m$. It turns out that

$$F = F_\Omega + \Lambda = \begin{bmatrix} -0.7 & 0.6332 & 0 & -0.1 \\ 1.6934 & -1.3051 & 0.0001 & 0.0074 \end{bmatrix}$$

and

$$T^{-1}(A_1 + B_1 F)\,T = \left[\begin{array}{ccc|c} -10.3046 & 2.9513 & -13.3556 & 3.0662 \\ 22.9093 & -3.1027 & 26.1112 & -13.4320 \\ 10.3546 & -2.9513 & 13.4056 & -3.3662 \\ \hline 0 & 0 & 0 & 0.1 \end{array}\right],$$

$$T^{-1}(A_2 + B_2 F)\,T = \left[\begin{array}{ccc|c} 3.1803 & -0.3410 & 3.5012 & 4.6967 \\ -5.4738 & 0.6983 & -6.1520 & -7.5386 \\ -3.3738 & 0.3483 & -3.7020 & -8.4063 \\ \hline 0 & 0 & 0 & -0.1659 \end{array}\right].$$

Note that internal dynamics with respect to $\mathcal{V}$ has not changed by adding Λ to the static feedback; that is, the internal stabilisation previously performed has not been affected. On the other hand, $\mathcal{V}$ has been externally stabilised since the pair $(0.1, -0.1659)$ is now asymptotically stable.

4 Output-nulling controlled invariance

In this section we turn our attention to *output-nulling subspaces*. These are a particular type of controlled invariant subspaces for the FM model

$$\begin{aligned} x_{i+1,j+1} &= A_1\, x_{i+1,j} + A_2\, x_{i,j+1} + B_1\, u_{i+1,j} + B_2\, u_{i,j+1}, \\ y_{i,j} &= C\, x_{i,j} + D\, u_{i,j}, \end{aligned} \tag{34}$$

where $y_{i,j} \in \mathbb{R}^p$ is the output vector and the matrices C and D are of suitable dimensions.

The subspace $\mathcal{V} \subseteq \mathbb{R}^n$ is an output-nulling subspace for (34) if

$$\begin{bmatrix} A_1 \\ A_2 \\ C \end{bmatrix} \mathcal{V} \subseteq \Big(\mathcal{V} \times \mathcal{V} \times \mathbf{0}_p\Big) + \mathrm{im} \begin{bmatrix} B_1 \\ B_2 \\ D \end{bmatrix}. \tag{35}$$

An output-nulling subspace $\mathcal{V}$ is such that for any $\mathcal{V}$-valued boundary condition, there exists an input function such that the corresponding local state trajectory of (34) lies in $\mathcal{V}$ and the corresponding output is zero for all (i,j) such that $i+j \geq 0$. Such an input can always be expressed as a static state feedback. The following lemma summarizes the most important properties of output-nulling subspaces.

Lemma 4.1 *Let V be a basis matrix for an r-dimensional subspace $\mathcal{V} \subseteq \mathbb{R}^n$. The following statements are equivalent:*

(i) The subspace $\mathcal{V}$ is output-nulling for (34).

(ii) There exist two matrices $X \in \mathbb{R}^{2\,r \times r}$ and $\Omega \in \mathbb{R}^{m \times r}$ such that

$$\begin{bmatrix} A_1 \\ A_2 \\ C \end{bmatrix} V = \begin{bmatrix} V & 0 \\ 0 & V \\ 0 & 0 \end{bmatrix} X + \begin{bmatrix} B_1 \\ B_2 \\ D \end{bmatrix} \Omega. \tag{36}$$

(iii) There exists a matrix $F \in \mathbb{R}^{m \times n}$ such that

$$\begin{bmatrix} A_1 + B_1\,F \\ A_2 + B_2\,F \\ C + D\,F \end{bmatrix} \mathcal{V} \subseteq \Big(\mathcal{V} \times \mathcal{V} \times \mathbf{0}_p\Big). \tag{37}$$

Proof: The implication ***(i)*** $\Longrightarrow$ ***(ii)*** follows immediately from (35) on noting that (36) is simply a matrix representation of (35). In order to show ***(ii)*** $\Longrightarrow$ ***(iii)***, let $F = -\Omega\,(V^\top V)^{-1}\,V^\top$, so that $\Omega = -F\,V$ can be replaced in (35) to yield (37). The implication ***(iii)*** $\Longrightarrow$ ***(i)*** is immediate. ∎

The set of output-nulling controlled invariant subspaces of (34) is denoted with the symbol $\mathfrak{V}_0$. Given a $\mathcal{V} \in \mathfrak{V}_0$, any matrix F such that (37) holds is called an *output-nulling friend*. It is not difficult to see that, as in the 1-D case, the set $\mathfrak{V}_0$ is closed under subspace addition. Thus, the sum of all the output-nulling subspaces of (34) is the largest output-nulling subspace and this is denoted by $\mathcal{V}^\star$. The following algorithm enables computation of $\mathcal{V}^\star$ in finite terms, as the $(n-1)$-th term of a monotonically non-increasing sequence of subspaces. It is the 2-D counterpart of Algorithm 4.1.2 in [3].

Algorithm 4.1 *The sequence of subspaces $(\mathcal{V}_i)_{i\in\mathbb{N}}$ described by the recurrence*

$$\mathcal{V}_i = \begin{bmatrix} A_1 \\ A_2 \\ C \end{bmatrix}^{-1} \left((\mathcal{V}_{i-1} \times \mathcal{V}_{i-1} \times \mathbf{0}_p) + im \begin{bmatrix} B_1 \\ B_2 \\ D \end{bmatrix} \right), \quad \mathcal{V}_0 = \mathbb{R}^n,$$

is monotonically non-increasing. Moreover, there exists an integer $k \le n-1$ such that $\mathcal{V}_{k+1} = \mathcal{V}_k$. For such k the identity $\mathcal{V}^\star = \mathcal{V}_k$ holds.

Algorithm 4.1 is a generalisation of a corresponding result in [5, Proposition 2.7], to the case of 'non-strictly proper' systems. Due to the invariance property (37) of the set of all output-nulling friends associated with the elements of the output-nulling controlled invariant subspaces $\mathfrak{V}_0$ for (34), we can introduce the notions of internal stabilisability and external stabilisability for output-nulling subspaces: An output-nulling subspace $\mathcal{V} \in \mathfrak{V}_0$ is said to be *internally stabilisable* (resp. *externally stabilisable*) if there exists an output-nulling friend F such that $\mathcal{V}$ is an internally stable (resp. externally stable) $(A_1 + B_1 F, A_2 + B_2 F)$-invariant.

Given a $\mathcal{V}$-valued boundary condition for (34) with $\mathcal{V} \in \mathfrak{V}_0$, any control action $u_{i,j} = F\, x_{i,j}$ with F satisfying (37) – i.e., F is an output-nulling friend of $\mathcal{V}$ – is such that $x_{i,j} \in \mathcal{V}$ and $y_{i,j} = 0$ for all i, j such that $i + j \ge 0$. To see this, it suffices to substitute $u_{i,j} = F\, x_{i,j}$ in (34) to get

$$\begin{aligned} x_{i+1,j+1} &= (A_1 + B_1 F)\, x_{i+1,j} + (A_2 + B_2 F)\, x_{i,j+1} \\ y_{i,j} &= (C + D\, F)\, x_{i,j}, \end{aligned} \tag{38}$$

and to observe that when $x_{i+1,j}$ and $x_{i,j+1}$ belong to $\mathcal{V}$, so does $x_{i+1,j+1}$ because of (37). As a result, for any $\mathcal{V}$-valued boundary condition it is found that $x_{i,j} \in \mathcal{V}$ and $y_{i,j} = 0$ since $\mathcal{V} \subseteq \ker(C + D\, F)$. This shows that the control input required to maintain the output at zero and the local state on $\mathcal{V}$ can always be expressed as a static local state feeback. As such, all of the material developed in Section 3 for controlled invariant subspaces can be adapted straightforwardly to output-nulling subspaces with few modifications. Indeed, by replacing (16) with (36) and (28) with

$$(\mathcal{V} \times \mathcal{V}) \cap \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \ker D = \mathbf{0}_{2n},$$

the internal and external stabilisation of output-nulling subspaces via output-nulling static feedback can be carried out along the same lines as the internal and external stabilisation of arbitrary controlled invariant subspaces.

5 Disturbance decoupling problems

The theoretical and computational tools developed here for the stabilisation of controlled invariant and output-nulling subspaces can be used for the solution of the disturbance decoupling problem, which is a prototype problem for a large class of control and estimation problems amenable to geometric techniques, with the requirement that the closed-loop be asymptotically stable. Consider a FM model

$$\begin{aligned} x_{i+1,j+1} &= A_1\, x_{i+1,j} + A_2\, x_{i,j+1} + B_1\, u_{i+1,j} + B_2\, u_{i,j+1} + H_1\, w_{i+1,j} + H_2\, w_{i,j+1}, \\ y_{i,j} &= C\, x_{i,j} + D\, u_{i,j} + G\, w_{i,j}, \end{aligned} \tag{39}$$

where for all $i,j \in \mathbb{Z}$, $x_{i,j} \in \mathbb{R}^n$ is the local state, $u_{i,j} \in \mathbb{R}^m$ is the control input, $w_{i,j} \in \mathbb{R}^d$ is a disturbance to be decoupled from the output $y_{i,j} \in \mathbb{R}^p$, $A_k \in \mathbb{R}^{n\times n}$, $B_k \in \mathbb{R}^{n\times m}$, $H_k \in \mathbb{R}^{n\times d}$ for $k = 1, 2$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$ and $G \in \mathbb{R}^{p\times d}$. The corresponding 2-D counterpart of the disturbance decoupling problem (DDP) first considered in [1], was studied and solved for FM models by Conte and Perdon in [5] without requiring stability. Their approach consists of finding conditions which ensure that a static local state feedback input $u_{i,j} = F\, x_{i,j}$ exists such that the output function is not affected by the disturbance w. A sufficient condition for the solution of this problem is

$$\operatorname{im} \begin{bmatrix} H_1 \\ H_2 \\ G \end{bmatrix} \subseteq \mathcal{V}^\star \times \mathcal{V}^\star \times \mathbf{0}_p, \tag{40}$$

where $\mathcal{V}^\star$ is the largest output-nulling controlled invariant subspace of the *undisturbed* system (13). A necessary condition for (40) to be satisfied is that the feedthrough matrix G be zero, and this is equivalent to condition *(i)* of Proposition 3.1 in [5]. When condition (40) is satisfied, a feedback-state solution of this problem is given by any output-nulling friend F of $\mathcal{V}^\star$. The presence of the feedthrough matrices D and G appears to be more interesting in the second decoupling problem considered in [5]; i.e., the measurable signal decoupling problem (MSDP), in which the disturbance w is available for measurement. In this case, a decoupling control input can take advantage of the additional information provided by the direct measurement of the disturbance w, to take form $u_{i,j} = F\, x_{i,j} + S\, w_{i,j}$. A sufficient condition for the solution of the MSDP problem is characterised by the inclusion

$$\operatorname{im} \begin{bmatrix} H_1 \\ H_2 \\ G \end{bmatrix} \subseteq (\mathcal{V}^\star \times \mathcal{V}^\star \times \mathbf{0}_p) + \operatorname{im} \begin{bmatrix} B_1 \\ B_2 \\ D \end{bmatrix}, \tag{41}$$

which is the natural extension of condition *(ii)* of Proposition 3.1 in [5], to accommodate non-strictly proper systems. Notice that in this case G can be different from the zero matrix when (41) holds. Hence, this condition indeed encompasses condition *(ii)* of Proposition 3.1 in [5]. If the condition (41) holds true, there exist matrices Φ_1, Φ_2 and Ψ of suitable sizes such that

$$\begin{bmatrix} H_1 \\ H_2 \\ G \end{bmatrix} = \begin{bmatrix} V & 0 \\ 0 & V \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ D \end{bmatrix} \Psi, \tag{42}$$

where V is a basis matrix for $\mathcal{V}^\star$. Notice that the solutions Φ_1, Φ_2 and Ψ of the linear equation (42) are parameterised in the null-space of $\left[\begin{smallmatrix} V & 0 & B_1 \\ 0 & V & B_2 \\ 0 & 0 & D \end{smallmatrix}\right]$. If we take an output-nulling friend F of $\mathcal{V}^\star$ and

$S = -\Psi$, it is can be seen that the control input $u_{i,j} = F\, x_{i,j} + S\, w_{i,j}$ achieves exact decoupling. Indeed, by substituting this control input in (34) we obtain

$$\begin{aligned} x_{i+1,j+1} &= (A_1 + B_1 F)\, x_{i+1,j} + (A_2 + B_2 F)\, x_{i,j+1} + V\, \Phi_1\, w_{i+1,j} + V\, \Phi_2\, w_{i,j+1}, \\ y_{i,j} &= (C + DF)\, x_{i,j}, \end{aligned}$$

which is clearly disturbance decoupled, since given any $\mathcal{V}$-valued boundary condition over the separation set $\mathbb{S}_0$, we get $x_{i,j} \in \mathcal{V}$ and $y_{i,j} = 0$ for all i, j such that $i + j \geq 0$. The limitation of these sufficient conditions and of the corresponding solutions is that they are only *structural*, and they do not take into account stability requirements of the closed-loop. Hence, here we are concerned with the solution of the following two decoupling problems.

Problem 5.1 (DDP with stability) *Find $F \in \mathbb{R}^{m\times n}$ such that $u_{i,j} = F\, x_{i,j}$ decouples the disturbance w from the output y* and *such that the closed-loop pair $(A_1 + B_1\, F, A_2 + B_2\, F)$ is asymptotically stable.*

Problem 5.2 (MSDP with stability) *Find $F \in \mathbb{R}^{m\times n}$ and $S \in \mathbb{R}^{m\times d}$ such that $u_{i,j} = F\, x_{i,j} + S\, w_{i,j}$ decouples the disturbance w from the output y* and *such that the closed-loop pair $(A_1+B_1\, F, A_2+B_2\, F)$ is asymptotically stable.*

Theorem 5.1 *Problem 5.1 is solvable if*

(i) im $\begin{bmatrix} H_1 \\ H_2 \\ G \end{bmatrix} \subseteq \mathcal{V}^\star \times \mathcal{V}^\star \times \mathbf{0}_p$*;*

(ii) $\mathcal{V}^\star$ is internally and externally stabilisable.

When these conditions hold, any output-nulling friend F of $\mathcal{V}^\star$ which both internally and externally stabilises $\mathcal{V}^\star$ solves the problem. Similarly, Problem 5.2 is solvable if

(i) im $\begin{bmatrix} H_1 \\ H_2 \\ G \end{bmatrix} \subseteq (\mathcal{V}^\star \times \mathcal{V}^\star \times \mathbf{0}_p) + im \begin{bmatrix} B_1 \\ B_2 \\ D \end{bmatrix}$;

(ii) $\mathcal{V}^\star$ is internally and externally stabilisable.

When these conditions hold, any output-nulling friend F of $\mathcal{V}^\star$ that both internally and externally stabilises $\mathcal{V}^\star$, together with $S = -\Psi$, where Ψ satisfies (42) for some Phi_1 and Φ_2, is a solution to Problem 5.2.

Proof: This result follows by direct application of the results characterising the internal and external stabilisabilty of controlled invariant subspaces developed in preceding sections. ■

5.1 Full information decoupling

We now consider a different version of the measurable signal decoupling problem, in which a control action ensuring perfect decoupling is sought within the class of those generated by a *dynamic* feedforward compensator which exploits measurement of the disturbance w to be decoupled. We show that under condition (41), the *explicit* structure of a feedforward decoupling compensator $\Sigma_{\rm C}$ can be derived. In the next section, it is shown how to employ the solution of this problem to solve the so-called model matching problem.

We begin by presenting the formulation of the problem. First, let the global output $\mathcal{Y}_k$ on $\mathbb{S}_k$ associated with (39) be defined as

$$\mathcal{Y}_k \triangleq \Big\{ y_{i,j} \mid (i,j) \in \mathbb{S}_k \Big\}.$$

Problem 5.3 *Design a 2-D feedforward compensator $\Sigma_{\rm C}$ ruled by*

$$\begin{aligned} z_{i+1,j+1} &= K_1\, z_{i+1,j} + K_2\, z_{i,j+1} + L_1\, w_{i+1,j} + L_2\, w_{i,j+1}, \\ u_{i,j} &= M\, z_{i,j} + N\, w_{i,j} \end{aligned} \tag{43}$$

such that, for all admissible inputs w and all boundary conditions of Σ and $\Sigma_{\rm C}$, the sequence $\{\|\mathcal{Y}_i\|\}_{i=0}^{\infty}$ converges to zero.

Clearly, Problem 5.3 admits solutions only if a compensator $\Sigma_{\rm C}$ can be found so that the overall system is asymptotically stable. Since the decoupling scheme is feedforward, this is equivalent to requiring that Σ is asymptotically stable and that the compensator $\Sigma_{\rm C}$ is sought within the class of asymptotically stable 2-D systems ruled by (43). In the following theorem a sufficient solvability

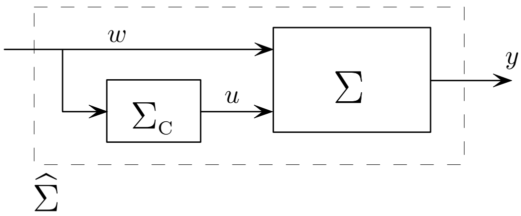

Figure 1: Block diagram of the feedforward compensation scheme.

condition for Problem 5.3 is presented, as well as the explicit structure of the feedforward compensator achieving perfect decoupling.

Theorem 5.2 *Let Σ be asymptotically stable. Problem 5.3 admits solutions if*

(i) $\mathrm{im} \begin{bmatrix} H_1 \\ H_2 \\ G \end{bmatrix} \subseteq (\mathcal{V}^\star \times \mathcal{V}^\star \times \mathbf{0}_p) + \mathrm{im} \begin{bmatrix} B_1 \\ B_2 \\ D \end{bmatrix}$;

(ii) $\mathcal{V}^\star$ *is internally stabilisable.*

If these conditions hold, a solution to Problem 5.3 is given as follows. If $\dim \mathcal{V}^\star > 0$, *let* Φ_1, Φ_2 *and* Ψ *be such that (42) holds, where V is a basis matrix of $\mathcal{V}^\star$. Let F be any output-nulling friend of $\mathcal{V}^\star$ that internally stabilises $\mathcal{V}^\star$, so that there exists an asymptotically stable pair (X_1, X_2) such that*

$$\begin{bmatrix} A_1 + B_1 F \\ A_2 + B_2 F \\ C + D F \end{bmatrix} V = \begin{bmatrix} V & 0 \\ 0 & V \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}. \tag{44}$$

The compensator $\Sigma_{\rm C}$ ruled by (43) with

$$(K_1, K_2, L_1, L_2, M, N) = (X_1, X_2, \Phi_1, \Phi_2, -\Omega, -\Psi) \qquad (45)$$

solves Problem 5.3. If $\mathcal{V}^\star = \mathbf{0}_n$, the compensator $\Sigma_{\rm C}$ solving Problem 5.3 reduces to the static unit $N = -\Psi$.

Proof: We first associate the sequence $s_{i,j}$ defined for all $i+j \geq 0$ with the following Laurent formal power series

$$\mathbf{s}(\lambda_h, \lambda_v) = \sum_{i+j \geq 0} s_{i,j}\, \lambda_h^i\, \lambda_v^j,$$

in the indeterminates λ_h and λ_v. Let $\mathbf{w}(\lambda_h, \lambda_v)$ and $\mathbf{y}(\lambda_h, \lambda_v)$ be the formal power series associated with the input and the output of the overall system $\widehat{\Sigma}$ depicted in Figure 1. Moreover, let

$$X_0(\lambda_h, \lambda_v) \triangleq \sum_{i \,\in\, \mathbb{Z}} \left[\begin{array}{c} x_{i,-i} \\ z_{i,-i} \end{array}\right] \lambda_h^i\, \lambda_v^{-i}.$$

Then, from (39) with (43) one gets

$$\begin{aligned} \mathbf{y}(\lambda_h, \lambda_v) \;=\; & \Big(\widehat{C}(I_n - \widehat{A}_1\lambda_v - \widehat{A}_2\lambda_h)^{-1}(\widehat{B}_1\lambda_v + \widehat{B}_2\lambda_h) + \widehat{D}\Big)\mathbf{w}(\lambda_h, \lambda_v) \\ & + \widehat{C}\,(I_n - \widehat{A}_1\lambda_v - \widehat{A}_2\lambda_h)^{-1} X_0(\lambda_h, \lambda_v) \end{aligned} \qquad (46)$$

where $\widehat{A}_i = \left[\begin{array}{cc} A_i & B_i M \\ 0 & K_i \end{array}\right]$ and $\widehat{B}_i = \left[\begin{array}{c} B_i N + H_i \\ L_i \end{array}\right]$ for $i = 1, 2$, $\widehat{C} = \left[\begin{array}{cc} C & DM \end{array}\right]$ and $\widehat{D} = DN + G$. Below it is shown that when $\dim \mathcal{V}^\star > 0$, the compensator given by (43) with (45) solves Problem 5.3. In other words, we show that with (45) the transfer matrix

$$G(\lambda_h, \lambda_v) \triangleq \widehat{C}(I_n - \widehat{A}_1\lambda_v - \widehat{A}_2\lambda_h)^{-1}(\widehat{B}_1\lambda_v + \widehat{B}_2\lambda_h) + \widehat{D}$$

in (46) is zero. In fact, if this is the case, the asymptotic stability of the overall system guarantees that for any given $X_0(\lambda_h, \lambda_v)$ the scalar sequence $\{\|\mathcal{Y}_i\|\}_{i=0}^{\infty}$ converges to zero. In order to show that $G(\lambda_h, \lambda_v)$ is zero, we consider zero boundary conditions for Σ and Σ_C – i.e., $x_{i,-i}$ and $z_{i,-i}$ are zero for all $i \in \mathbb{Z}$ – and prove that any $\mathbf{w}(\lambda_h, \lambda_v)$ leads to $\mathbf{y}(\lambda_h, \lambda_v)$ being the zero polynomial. To this end, it is first shown that the identities $x_{i,j} = V\, z_{i,j}$ and $y_{i,j} = 0$ hold for all $i+j \geq 0$. By substitution of the control action (39) we get

$$\begin{aligned} x_{i+1,j+1} &= A_1\, x_{i+1,j} + A_2\, x_{i,j+1} + B_1\, F\, V\, z_{i+1,j} + B_2\, F\, V\, z_{i,j+1} \\ &\quad + (H_1 - B_1\, \Psi)\, w_{i+1,j} + (H_2 - B_2\, \Psi)\, w_{i,j+1}, \\ y_{i,j} &= C\, x_{i,j} + D\, F\, V\, z_{i,j} + (G - D\, \Psi)\, w_{i,j}. \end{aligned}$$

Then by taking (42) into account we find

$$\begin{aligned} x_{i+1,j+1} \;&=\; A_1\, x_{i+1,j} + A_2\, x_{i,j+1} + B_1\, F\, V\, z_{i+1,j} + B_2\, F\, V\, z_{i,j+1} + V\, \Phi_1\, w_{i+1,j} + V\, \Phi_2\, w_{i,j+1}, \\ y_{i,j} \;&=\; C\, x_{i,j} + D\, F\, V\, z_{i,j}. \end{aligned}$$

Now, if for any $(i,j) \in \mathbb{S}_0$ the identity $x_{i,j} = V\, z_{i,j}$ holds, in view of (44) it follows that

$$\begin{aligned} x_{i+1,j+1} &= (A_1 + B_1\, F)\, V\, z_{i+1,j} + (A_2 + B_2\, F)\, V\, z_{i,j+1} + V\, \Phi_1\, w_{i+1,j} + V\, \Phi_2\, w_{i,j+1} \\ &= V\, X_1\, z_{i+1,j} + V\, X_2\, z_{i,j+1} + V\, \Phi_1\, w_{i+1,j} + V\, \Phi_2\, w_{i,j+1} = V\, z_{i+1,j+1}, \\ y_{i,j} &= (C + D\, F)\, V\, z_{i,j} \end{aligned}$$

As a result, for all (i,j) such that $i+j \geq 0$ we get $x_{i,j} = V\, z_{i,j}$ and $y_{i,j} = 0$ for any input sequence w. It follows that $G(\lambda_h, \lambda_v) = 0$. Moreover, since the overall system is asymptotically stable, if the boundary conditions of Σ and $\Sigma_{\rm C}$ are not at zero, the norm of the global state $\|\mathcal{X}_k\|$, and hence of the output $\|\mathcal{Y}_k\|$, converges to zero as k goes to infinity, so that $\Sigma_{\rm C}$ solves Problem 5.3. ■

Example 5.1 Let (13) be defined over $\mathbb{N} \times \mathbb{N}$ with

$$A_1 = \begin{bmatrix} 0 & 1.2 & -1.6 \\ 0 & 0.03 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.01 & 0 & 0.02 \\ 0 & 0 & 0 \\ -0.3 & 0.6 & 0.04 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 7 & -9 \\ 0 & 0 \\ 0 & -9 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 3 \\ 4 & 0 \\ 5 & 0 \end{bmatrix},$$

$$H_1 = \begin{bmatrix} -3.5 \\ 0 \\ -1 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0.1 \\ -1.6 \\ -1.3 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 9 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} -6 & 0 \\ 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 2.4 \\ 0 \end{bmatrix}.$$

The associated boundary conditions are random assignments of the local state over the region $\big(\{0\} \times [1,\infty)\big) \cup \big([1,\infty) \times \{0\}\big)$. The boundary conditions on each component of the local state are depicted in Figure 2. This system is asymptotically stable, since it satisfies the stability condition (11). By

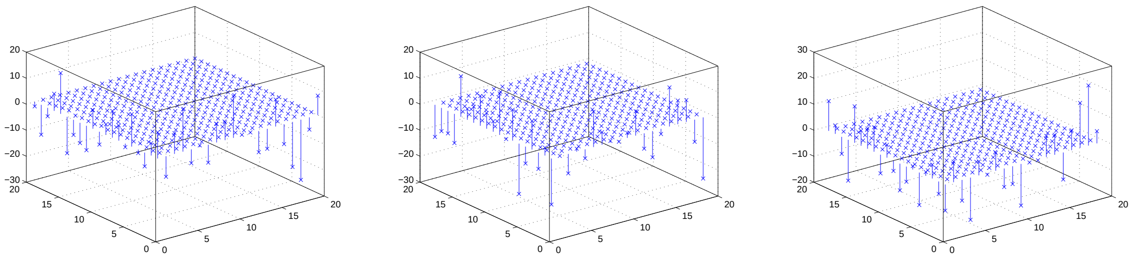

Figure 2: Boundary conditions.

using Algorithm 4.1 it is also readily seen that

$$\mathcal{V}^\star = \operatorname{im} V, \qquad \text{where} \quad V = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

In this case $W = \begin{bmatrix} V & 0 & B_1 \\ 0 & V & B_2 \\ 0 & 0 & D \end{bmatrix}$ is singular and $H = \begin{bmatrix} 9 & 9 & -3 & 0 & 0 & 1 \end{bmatrix}^\top$ is a basis matrix for its kernel. It is easy to check that the pair (X_1, X_2) computed from $\begin{bmatrix} X_1 \\ X_2 \\ \Omega \end{bmatrix} = W^\dagger \begin{bmatrix} A_1 \\ A_2 \\ C \end{bmatrix} V$ does not satisfy condition (11) for stability. As such, by taking $F_\Omega = -\Omega\,(V^\top V)^{-1} V^\top$ we find that the pair $(A_1 + B_1\,F_\Omega, A_2 + B_2\,F_\Omega)$ does not satisfy (11). In order to find an F_Ω which internally stabilises the controlled invariant subspace $\mathcal{V}$, let us condider

$$\begin{bmatrix} X \\ \Omega \end{bmatrix} = \begin{bmatrix} V & 0 & B_1 \\ 0 & V & B_2 \\ 0 & 0 & D \end{bmatrix}^\dagger \begin{bmatrix} A_1 \\ A_2 \\ C \end{bmatrix} V + HK \tag{47}$$

where $H = \begin{bmatrix} 9 & 9 & -3 & 0 & 0 & 1 \end{bmatrix}^\top$. In this case, the LMI (30) is feasible, which implies internal stabilisability of $\mathcal{V}^\star$. One such feasible point yields $K = \begin{bmatrix} 0.0003 & -0.0436 \end{bmatrix}$. By using this value of K in (47) we find

$$X_1 = \begin{bmatrix} 0.0018 & -0.8733 \\ 0.0018 & 0.7267 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0.0094 & -0.2222 \\ -0.3 & 0.04 \end{bmatrix}, \quad \Omega = \begin{bmatrix} 0 & 0 \\ 0.0002 & 0.0807 \end{bmatrix}.$$

Now the pair (X_1, X_2) is asymptotically stable, as it satisfies the stability condition (11). With this choice we find

$$F_\Omega = -\Omega\,(V^\top V)^{-1} V^\top = \begin{bmatrix} 0 & 0 & 0 \\ -0.0002 & 0 & -0.0807 \end{bmatrix}.$$

Note from

$$T^{-1}(A_1 + B_1\,F_\Omega)\,T = \left[\begin{array}{cc|c} 0.0018 & -0.8733 & -1.2282 \\ 0.0018 & 0.7267 & -0.3982 \\ \hline 0 & 0 & 0.03 \end{array}\right]$$

and

$$T^{-1}(A_2 + B_2\,F_\Omega)\,T = \left[\begin{array}{cc|c} 0.0094 & -0.2222 & 0.0094 \\ -0.3 & 0.04 & -0.9 \\ \hline 0 & 0 & 3 \end{array}\right],$$

that the pair $(0.03, 3)$ accounting for the external dynamics of $\mathcal{V}^\star$ has not changed by selecting the feedback F_Ω in order to stabilise the controlled invariant subspace $\mathcal{V}^\star$ internally.

Now since $\mathcal{V}^\star$ is internally stabilisable and the structural condition $\operatorname{im}\begin{bmatrix} H_1 \\ H_2 \\ G \end{bmatrix} \subseteq (\mathcal{V}^\star \times \mathcal{V}^\star \times \mathbf{0}_p) + \operatorname{im}\begin{bmatrix} B_1 \\ B_2 \\ D \end{bmatrix}$ is satisfied, the result of Theorem 5.2 can be applied. It is found that

$$\begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Psi \end{bmatrix} = \begin{bmatrix} V & 0 & B_1 \\ 0 & V & B_2 \\ 0 & 0 & D \end{bmatrix}^\dagger \begin{bmatrix} H_1 \\ H_2 \\ G \end{bmatrix} = \begin{bmatrix} 0.1791 \\ -0.1209 \\ 0.207 \\ 0.7 \\ -0.4 \\ 0.0977 \end{bmatrix}$$

is a solution of (42). As such, the compensator Σ_{C} ruled by

$$K_1 = \begin{bmatrix} 0.0018 & -0.8733 \\ 0.0018 & 0.7267 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0.0094 & -0.2222 \\ -0.3 & 0.04 \end{bmatrix}, \quad L_1 = \begin{bmatrix} 0.1791 \\ -0.1209 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0.207 \\ 0.7 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & 0 \\ 0.0002 & 0.0807 \end{bmatrix}, \quad N = \begin{bmatrix} -0.4 \\ 0.0977 \end{bmatrix}$$

solves the full information problem. Let the overall system be subject to the randomly generated input depicted in Figure 3.

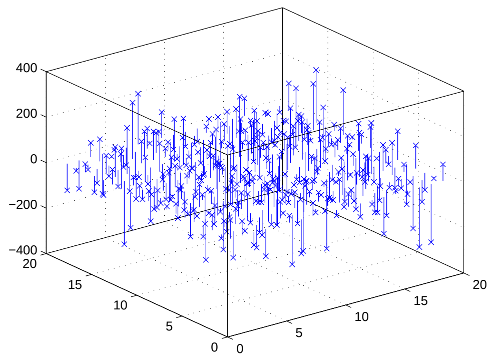

Figure 3: Disturbance w in the bounded frame $[0, 20] \times [0, 20]$.

The asymptotic stability of the overall system guarantees that the two outputs go to zero as the double index (i, j) moves away from the axes, as shown in Figure 4. In order to see that as the index

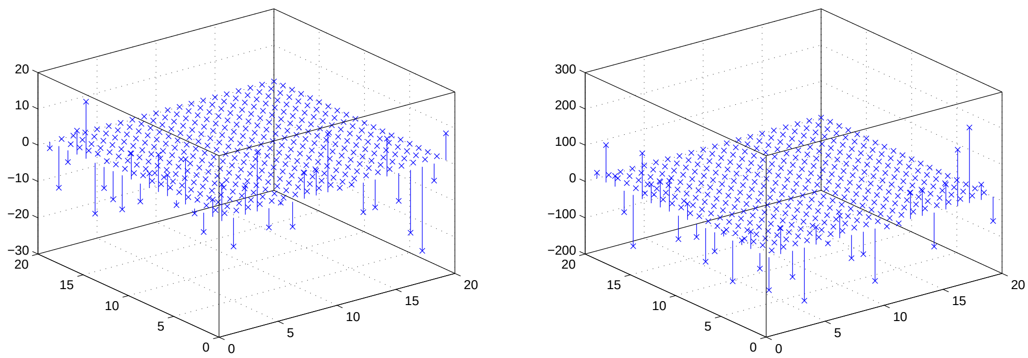

Figure 4: Controlled outputs y^1 and y^2 in the bounded frame $[0, 20] \times [0, 20]$.

(i, j) moves away from the axis the two controlled outputs $y^1_{i,j}$ and $y^2_{i,j}$ decrease in an exponential fashion, Figure 5 shows the base 10 logarithms of the two outputs $|y^1_{i,i}|$ and $|y^2_{i,i}|$ against the variable i in the interval $[0, 20]$.

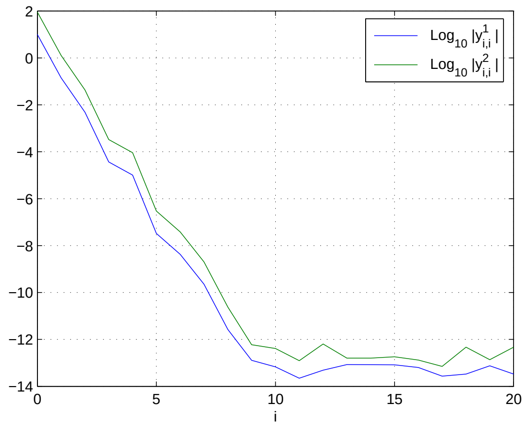

Figure 5: Logarithms of $|y^1_{i,i}|$ and $|y^2_{i,i}|$ for $i \in [0, 20]$.

5.2 Model Matching with Stability

The material presented in the previous sections is now exploited for the solution of another well-known and deeply investigated (both in a 1-D and in a 2-D setting) control problem: The so-called *model matching* problem. Different approaches have been proposed for the solution of this problem in the two-dimensional framework, see e.g. [21, 23] and [5], where the model matching problem is solved via polynomial and geometric approaches, respectively. In this paper, we propose a different perspective for the solution of this problem, where stability is also taken into account. In particular, we show how the solution of the full information decoupling can be employed to tackle the model matching problem, following a well-known procedure for 1-D systems [20, 16, 17, 19]. Given a system Σ along with a model $\Sigma_{\rm M}$ governed respectively by (34) and

$$
\begin{aligned}
x^{\rm M}_{i+1,j+1} &= A^{\rm M}_1\, x^{\rm M}_{i+1,j} + A^{\rm M}_2\, x^{\rm M}_{i,j+1} + B^{\rm M}_1\, r_{i+1,j} + B^{\rm M}_2\, r_{i,j+1},\\
y^{\rm M}_{i,j} &= C^{\rm M}\, x^{\rm M}_{i,j} + D^{\rm M}\, r_{i,j},
\end{aligned}
$$

and having the same output spaces, the exact model matching consists of finding a compensator $\Sigma_{\rm C}$ ruled by (43) such that the input/output behaviour of the series connection between Σ and $\Sigma_{\rm C}$ equals that of the given model $\Sigma_{\rm M}$. In other words, if we denote by e the difference between the output of the original system Σ and that of the model $\Sigma_{\rm M}$, see Figure 6, and $\mathcal{E}_k \triangleq \{e_{i,j} \,|\, (i,j) \in \mathbb{S}_k\}$, the aim is to determine the inner structure of the compensator $\Sigma_{\rm C}$ connected in series of the plant Σ such that the sequence $\{\|\mathcal{E}_i\|\}_{i=0}^{\infty}$ converges to zero for all reference input functions r and for all boundary conditions for Σ and $\Sigma_{\rm M}$. As shown in Figure 6, the model matching problem can be easily turned into a full information decoupling problem, where now $\bar{\Sigma}$ is the system with input $\left[\begin{smallmatrix} u \\ r \end{smallmatrix}\right]$, output e and local state $\left[\begin{smallmatrix} x \\ x^{\rm M} \end{smallmatrix}\right]$; this system is completely characterised by the collection of matrices

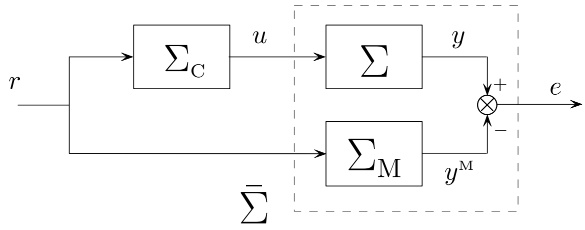

Figure 6: Block diagram for model matching.

$(\bar{A}_1, \bar{A}_2, [\,\bar{B}_1 \;\; \bar{H}_1\,], [\,\bar{B}_2 \;\; \bar{H}_2\,], \bar{C}, [\,\bar{D} \;\; \bar{G}\,])$, where

$$\bar{A}_k = \begin{bmatrix} A_k & 0 \\ 0 & A_k^{\rm M} \end{bmatrix}, \quad \bar{B}_k = \begin{bmatrix} B_k \\ 0 \end{bmatrix}, \quad \bar{H}_k = \begin{bmatrix} 0 \\ B_k^{\rm M} \end{bmatrix}, \quad k \in \{1,2\},$$
$$\bar{C} = \begin{bmatrix} C & -C^{\rm M} \end{bmatrix}, \quad \bar{D} = D, \quad \bar{G} = -D^{\rm M}.$$

The reference input r can be thought of as a signal to be decoupled from the output e by means of a feedforward compensator $\Sigma_{\rm C}$. Hence, the model matching problem with stability can be solved if, given the largest output-nulling $\bar{\mathcal{V}}^\star$ of the system $(\bar{A}_1, \bar{A}_2, \bar{B}_1, \bar{B}_2, \bar{C}, \bar{D})$, the following two conditions hold:

(i) $\operatorname{im} \begin{bmatrix} \bar{H}_1 \\ \bar{H}_2 \\ \bar{G} \end{bmatrix} \subseteq (\bar{\mathcal{V}}^\star \times \bar{\mathcal{V}}^\star \times \mathbf{0}_p) + \operatorname{im} \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \\ \bar{D} \end{bmatrix}$;

(ii) $\bar{\mathcal{V}}^\star$ is internally stabilisable.

In this case, a compensator $\Sigma_{\rm C}$ solving the model matching problem can be devised from Theorem 5.2 with the due substitutions.

Concluding remarks

The problem of internally and externally stabilising, via static feedback, controlled invariant and output-nulling subspaces for 2-D systems is here considered for the first time. The main results permit various standard compensator synthesis problems (e.g. disturbance decoupling with and without full information) to be solved subject to a closed-loop stability constraint, via geometric techniques. By contrast, existing geometric treatments of such problems omit the stability requirement and only focus on achieving controlled and output-nulling invariance. Being able to handle a stability constraint is important, particularly from the perspective of numerical implementation of the compensator schemes. In fact, even when the signals are only of interest over a bounded index set $[0, N] \times [0, M]$, say, numerical problems arise for large N or M if the controlled (i.e., the closed-loop under static feedback) system is unstable. The techniques developed here can be adapted to characterisations of stability other than Lemma 2.2, which is used here for the sake of argument and clarity in view of its simple form.

It is also worth noting that due to the equivalence between FM and Roesser latent variable models, clearly a geometric setting similar to the one developed in this paper can be established within the context of Roesser models, as well.

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