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# AN IMPULSIVE STABILIZING CONTROL OF A NEW CHAOTIC SYSTEM

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**ABSTRACT.** In this paper, we design a novel impulsive control law to stabilize a new class of chaotic systems. Using a non-quadratic Lyapunov function candidate and a stability theorem in [10], we derive some algebraic sufficient conditions which ensure the asymptotical stability of the chaotic system under the impulsive control strategy. Finally, a numerical example is presented to illustrate the validity of our results.

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## 1. INTRODUCTION

Study of chaotic systems has been an active research area since the discovery of the Lorenz chaotic attractor in 1963 [3]. New chaotic attractors have been discovered, for example, those reported in [4], [5] and [6]. In particular, the family of chaotic systems obtained in [6] covers those reported in [4]–[6] as special cases. This new chaotic system is described by

$$\dot{x}(t) = Ax(t) + \phi(x) + \omega(t), \tag{1.1}$$

where

$$A = \begin{bmatrix} -\theta_1 & \theta_1 & 0 \\ f_1(\theta_1,\theta_2) & f_2(\theta_1,\theta_2) & 0 \\ 0 & 0 & -\theta_3 \end{bmatrix}, \phi(x) = \begin{bmatrix} 0 \\ -x_1x_3 \\ x_1x_2 \end{bmatrix},$$

$\omega(t) = \begin{bmatrix} \omega_1(t) \\ \omega_2(t) \\ \omega_3(t) \end{bmatrix}$, and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is the state vector. $\theta_1 > 0$, $\theta_2 > 0$ and $\theta_3 > 0$ are system parameters, $f_1 : R \times R \to R$ and $f_2 : R \times R \to R$ are linear functions, and $\omega(t)$ is a bounded exogenous disturbance. Consider system (1.1) with $\omega(t) = 0$, $f_1(\theta_1,\theta_2) = 0$ and $f_2(\theta_1,\theta_2) = \theta_2$. Then, the corresponding system reduces to the Lü

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chaotic system. System (1.1) with $\omega(t) = 0$, $f_1(\theta_1, \theta_2) = \theta_2 - \theta_1$ and $f_2(\theta_1, \theta_2) = \theta_2$ becomes the Chen chaotic system. The original Lorenz chaotic system can also be represented as system (1.1) with $\omega(t) = 0$, $f_1(\theta_1, \theta_2) = \theta_2$ and $f_2(\theta_1, \theta_2) = -1$.

For a practical control system, the design of a control law which stabilizes the controlled system is fundamentally important ([1] and [2]). In the literature, various design methods, such as those proposed in [6] and [11], have been proposed to control respective chaotic systems. The impulsive control method proposed in [7] has attracted a considerable attention because impulsive control laws have fast response time, low energy consumption, good robustness and resistance to disturbance. They have been used to stabilize different classes of chaotic systems in [8] and [9]. Finding a quadratic Lyapunov function is a common approach to derive sufficient conditions for stability and stabilization of chaotic systems. See, for example, [7] and [9]. Consequently, these sufficient conditions are in the forms of linear matrix inequalities or Riccati equations. They tend to be computationally expensive in practical applications.

In this paper, we derive some algebraic sufficient conditions, which ensure asymptotical stability of the new chaotic system (1.1) under an impulsive control law. These algebraic sufficient conditions are derived by using a non-quadratic Lyapunov function expressed in terms of $l_1$ norm of the state vector. The results are highly reliable, easy to use, and computationally inexpensive.

## 2. PRELIMINARIES

Let $\|\cdot\|$ denote the $l_1$ norm in $R^3$, and let $K$ denote the set of all continuous real valued functions $\zeta$ such that $\zeta$ are strictly increasing and $\zeta(0) = 0$. $C(R^3, R^3)$ denotes the set of all continuous functions defined on $R^3$ with values in $R^3$. $C(R_+ \times R^3, R^3)$ is defined similarly. Let $p : R_+ \to R_+$ be a function, which is continuous on $R_+$, except possibly at the time points in the sequence $\{\tau_k\}$, and is left-continuous and has right limit at $\tau_k$ for all $k$. Let $PC(R_+, R_+)$ denote the set of all such piecewise continuous functions $p$. For each $\rho > 0$, define $S_\rho = \left\{ x \in R^3 : \|x(t)\| = \sum_{i=1}^{3} |x_i| < \rho \right\}$. Let $V(\cdot,\cdot) : R_+ \times S_\rho \to R_+$ be a function which is continuous on $R_+ \times S_\rho$, except possibly at the time points in the sequence $\{\tau_k\}$, and satisfies the following two conditions:
1) For each $x \in S_\rho$, where $k = 1, 2, \ldots$, it holds that $\lim_{(t,y)\uparrow(\tau_k,x)} V(t, y) = V(\tau_k^-, x)$;
2) $V(t, x)$ is locally Lipschitz in $x$.
Let $V_0$ denote the set of all such functions $V(\cdot,\cdot)$.
For $(t, x) \in (\tau_{k-1}, \tau_k] \times R^3$, $k = 1, 2, \ldots$, define

$$D^+V(t, x) = \lim_{h\to 0} \sup \frac{1}{h} \left[ V(t + h, x + hf(t, x)) - V(t, x) \right].$$

## 3. IMPULSIVE STABILIZING CONTROL LAW

**Definition 3.1.** Let the sequence $\{\tau_k, u_k(x(\tau_k))\}$ be an impulsive control law for system (1.1). If system (1.1) is asymptotically stable under such a control law, then system (1.1) is said to be impulsively stabilizable and the corresponding control law is called an impulsive stabilizing control law.

**Assumption 3.2.** There exists a positive constant $h$ such that $\|\omega(t)\| = \sum_{i=1}^{3} |\omega_i(t)| \leq h\|x(t)\|$.

In this section, we shall devise a new impulsive stabilizing control law $\{\tau_k, u_k(x)\}$ for system (1.1), where $u_k(x) = [c_1x_1, c_2x_2, c_3x_3]^T$ with the superscript $T$ denoting transpose. Substituting such a control law into system (1.1), we obtain

$$\dot{x}(t) = Ax + \phi(x) + \omega(t), \ t \neq \tau_k, \tag{3.1a}$$

$$\Delta x(t) = Bx, \ t = \tau_k, \ k = 1, 2, \ldots, \tag{3.1b}$$

$$x\left(t_0^+\right) = x_0, \tag{3.1c}$$

where $B = diag(c_1, c_2, c_3)$.

The main result of the paper is presented in the following theorem.

**Theorem 3.3.** *Consider system* (3.1)*, i.e., system* (1.1) *under the impulsive control law* $\left\{\tau_k, u_k(x) = [c_1x_1, c_2x_2, c_3x_3]^T\right\}$*. Suppose that Assumption* 3.2 *is satisfied and that the following inequality holds,*

$$0 < (m_1 + m_2 + h)\varepsilon \leq -\ln \max_{1\leq i\leq 3} |1 + c_i|, \tag{3.2}$$

*where*
$m_1 = \max\{|f_1| - \theta_1, f_2 + \theta_1, -\theta_3\}, m_2 = \sup_{t\geq 0}|x_1(t)|,$ *and* $\varepsilon = \sup\{\tau_k - \tau_{k-1}\}$*. Then, the impulsive control law is an impulsive stabilizing control law for the chaotic system* (1.1)*, meaning that the controlled system* (3.1) *is asymptotically stable.*

*Proof.* Consider the following non-quadratic Lyapunov function candidate

$$V(t, x) = |x_1(t)| + |x_2(t)| + |x_3(t)|. \tag{3.3}$$

Clearly, it satisfies condition (i) of Lemma A.1 in Appendix A. At impulsive time points $t = \tau_k$, $k = 1, 2, \ldots$, we have

$$\begin{aligned} V(\tau_k, x(\tau_k) + u_k(x(\tau_k))) &= |1 + c_1||x_1(\tau_k)| + |1 + c_2||x_2(\tau_k)| + |1 + c_3||x_3(\tau_k)| \\ &\leq \max_{1\leq i\leq 3} |1 + c_i| (|x_1(\tau_k)| + |x_2(\tau_k)| + |x_3(\tau_k)|) \\ &= \max_{1\leq i\leq 3} |1 + c_i| V(\tau_k, x). \end{aligned} \tag{3.4}$$

Thus, condition (iii) of Lemma A.1 is satisfied with

$$g(s) = \max_{1\le i\le 3} |1+c_i|\, s. \tag{3.5}$$

Since $\|x+u_k(x)\| = \sum_{i=1}^{3} |1+c_i|\,|x_i|$, it is clear from (3.2) that $\|x+u_k(x)\| \le \sum_{i=1}^{3} |x_i|$, which shows that $x+u_k(x) \in S_\rho$ for all $k$. Thus, condition (ii) of Lemma A.1 is satisfied as well.

Now, by taking the upper Dini derivative of Lyapunov function (3.3) along the trajectory of system (3.1), it follows that when $t \ne \tau_k$, $k=1,2,\dots$,

$$\begin{aligned} D^+V(t,x(t)) = \sum_{i=1}^{3} sgn[x_i(t)]\dot{x}_i(t) &= sgn[x_1(t)]\left(-\theta_1 x_1(t) + \theta_1 x_2(t)\right) \\ &+ sgn[x_2(t)]\left(f_1 x_1(t) + f_2 x_2(t) - x_1(t)x_3(t)\right) \\ &+ sgn[x_3(t)]\left(-\theta_3 x_3(t) + x_1(t)x_2(t)\right) + \sum_{i=1}^{3} sgn[x_i(t)]\omega_i(t). \end{aligned} \tag{3.6}$$

Clearly,

$$sgn[x_1(t)]\left(-\theta_1 x_1(t) + \theta_1 x_2(t)\right) \le -\theta_1 |x_1(t)| + \theta_1 |x_2(t)|, \tag{3.7}$$

$$\begin{aligned} &sgn[x_2(t)]\left(f_1 x_1(t) + f_2 x_2(t) - x_1(t)x_3(t)\right) \\ &\quad \le |f_1|\,|x_1(t)| + f_2 |x_2(t)| - sgn[x_2(t)]x_1(t)x_3(t), \end{aligned} \tag{3.8}$$

and

$$sgn[x_3(t)]\left(-\theta_3 x_3(t) + x_1(t)x_2(t)\right) \le -\theta_3 |x_3(t)| + sgn[x_3]x_1(t)x_2(t). \tag{3.9}$$

Combining (3.7)–(3.9), we obtain

$$\begin{aligned} D^+V(t,x(t)) &\le (|f_1| - \theta_1)|x_1(t)| + (f_2+\theta_1)|x_2(t)| + (-\theta_3)|x_3(t)| \\ &\quad + |x_1(t)x_3(t)| + |x_1(t)x_2(t)| + h\sum_{i=1}^{3}|x_i(t)| \\ &\le \max\{|f_1|-\theta_1, f_2+\theta_1, -\theta_3\}\sum_{i=1}^{3}|x_i(t)| \\ &\quad + |x_1(t)|\left(|x_1(t)| + |x_2(t)| + |x_3(t)|\right) + hV(t,x(t)) \\ &= \max\{|f_1|-\theta_1, f_2+\theta_1, -\theta_3\}V(t,x(t)) + (|x_1(t)| + h)V(t,x(t)) \\ &\le (m_1+m_2+h)V(t,x(t)). \end{aligned} \tag{3.10}$$

Thus, condition (iv) of Lemma A.1 is satisfied with

$$p(s) = m_1 + m_2 + h, \tag{3.11}$$

and

$$\gamma(s) = s. \tag{3.12}$$

From (3.5) and (3.12), we have, for $\varepsilon < \infty$,

$$\int_{q}^{g_k(q)} \frac{ds}{\gamma(s)} = \ln(\max_{1\leq i\leq 3} |1+c_i|) \tag{3.13}$$

and from (3.11), we obtain

$$\int_{\tau_k}^{\tau_{k+1}} p\,(s)\,ds \leq (m_1 + m_2 + h)\,\varepsilon. \tag{3.14}$$

Combining (3.13) and (3.14), it follows that

$$\int_{q}^{g_k(q)} \frac{ds}{\gamma\,(s)} + \int_{\tau_k}^{\tau_{k+1}} p\,(s)\,ds \leq (m_1 + m_2 + h)\,\varepsilon + \ln\left(\max_{1\leq i\leq 3} |1+c_i|\right) \leq 0, \tag{3.15}$$

which shows that condition (v) of Lemma A.1 is satisfied. Now, we see that all the conditions of Lemma A.1 are satisfied. Therefore, by virtue of Lemma A.1, system (3.1) is asymptotically stable under the designed impulsive control law. This completes the proof. □

The following corollary follows from Theorem 1 for the case when $c_1 = c_2 = c_3 = c$.

**Corollary 3.4.** *Consider system* (3.1)*, i.e., system* (1.1) *under the impulsive control law* $\left\{\tau_k, u_k\,(x) = [cx_1, cx_2, cx_3]^T\right\}$*. Suppose that Assumption* 3.2 *is satisfied and that the following inequality holds,*

$$0 < (m_1 + m_2 + h)\,\varepsilon \leq -\ln|1+c|\,, \tag{3.16}$$

*where*

$$m_1 = \max\left\{|f_1| - \theta_1, f_2 + \theta_1, -\theta_3\right\}, m_2 = \sup_{t\geq 0} |x_1\,(t)|\,,$$

*and* $\varepsilon = \sup\left\{\tau_k - \tau_{k-1}\right\}$*. Then, the impulsive control law is an impulsive stabilizing control law for the chaotic system* (1.1)*.*

To illustrate the applicability of our results, we present a numerical example below.

**Example 3.5.** Consider the disturbed chaotic system (1.1) with the following specifications: $\theta_1 = 1,\ \theta_2 = 2,\ \theta_3 = 3,\ f_1\,(\theta_1, \theta_2) = \theta_2 - \theta_1,\ f_2\,(\theta_1, \theta_2) = \theta_2, \omega\,(t) = [0.2x_1\,(t)\,, -0.2x_2\,(t)\,, -0.2x_3\,(t)]^T,\ x\,(0) = [2, -3, 1]^T$. This system is Chen chaotic system with disturbance. The time sequence and phase charts for both the undisturbed and disturbed chaotic systems are shown in Figures 1–4. From Assumption 3.2, we can choose $h = 0.2$ and $m_1 = 3$. $m_2$, the upper bound of the absolute value of variable $x_1$, needs to be estimated. An approximate value of $m_2$ is obtained through simulations, giving $m_2 \approx 30.4089$. Let $\varepsilon = \tau_{k+1} - \tau_k = 0.01$, $k = 1, 2, \ldots,$ and $c_i = -0.35$,

$i = 1, 2, 3$. Then, it can be verified that $(m_1 + m_2 + h)\,\varepsilon + \ln|1 + c| = -0.0947 < 0$. By virtue of Corollary 3.4, this disturbed chaotic system is impulsively stabilizable under the impulsive control law $\left\{\tau_k, u_k(x) = [-0.35x_1, -0.35x_2, -0.35x_3]^T\right\}$. The time sequence and phase charts of the controlled Chen chaotic system are given in Figures 5 and 6, respectively. We observe that the state of the system reaches the equilibrium point after 0.4 second. From this example, it clearly indicates that the algebraic stability condition (3.16), and, in general, (3.2), are highly reliable and effective and it is easy to use.

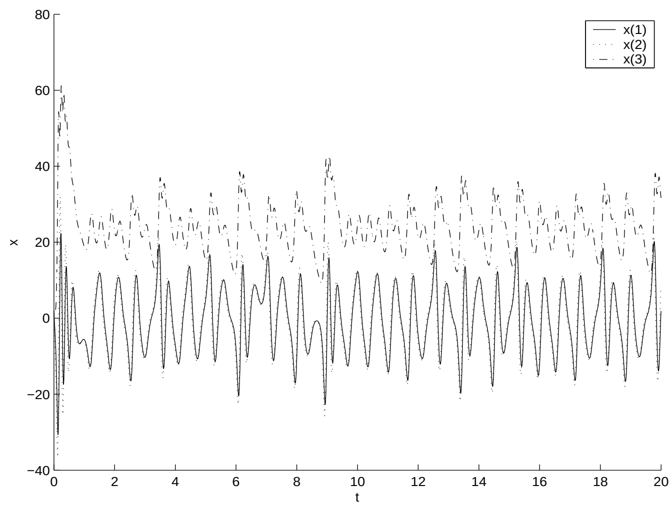


FIGURE 1. Time sequence chart for Chen chaotic system.

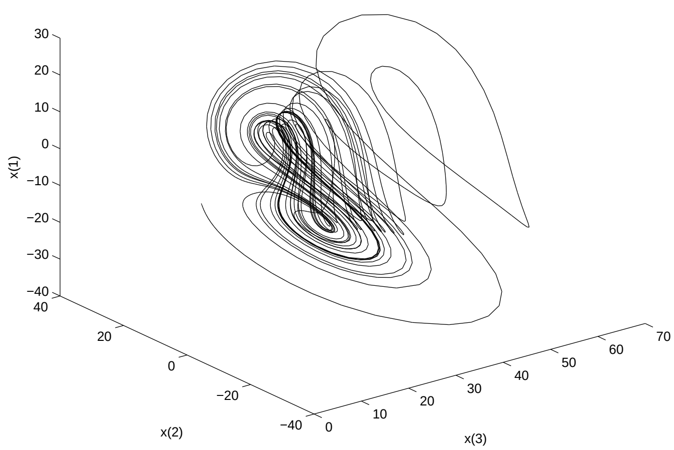


FIGURE 2. Phase chart for Chen chaotic system.

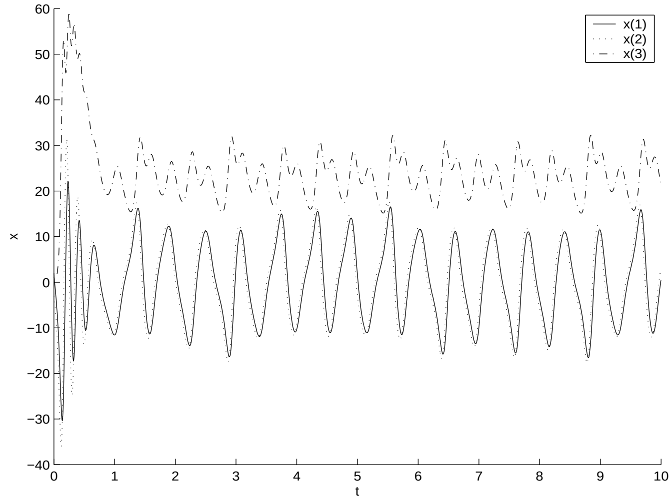


FIGURE 3. Time sequence chart for Chen chaotic system with disturbance.

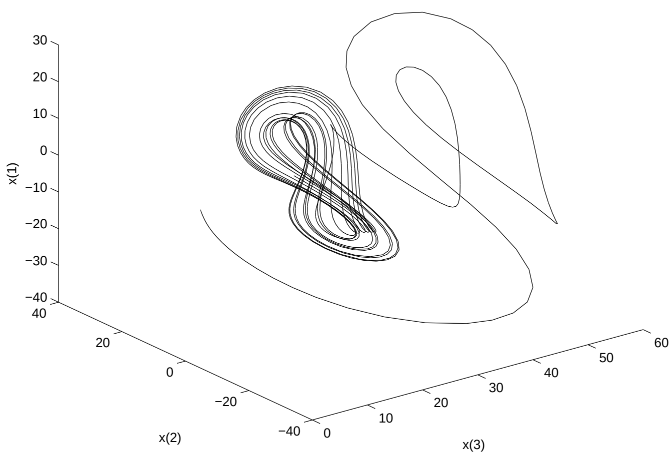


FIGURE 4. Phase chart for Chen chaotic system with disturbance.

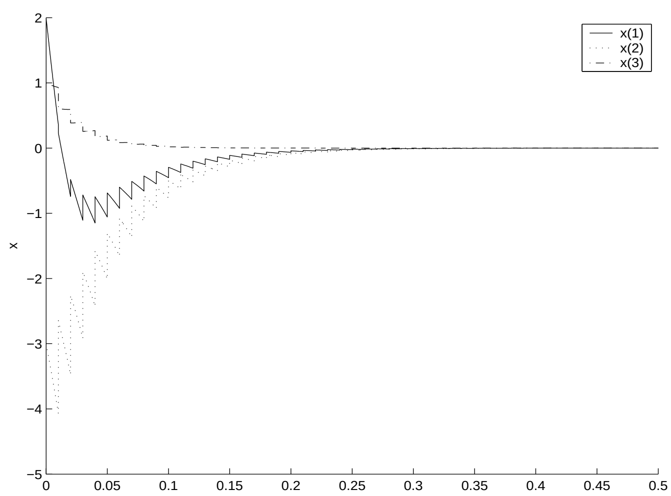


FIGURE 5. Time sequence chart for the controlled Chen chaotic system.

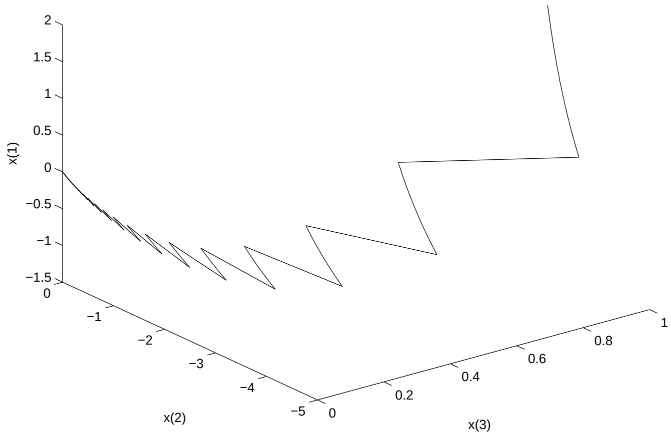


FIGURE 6. Phase chart for the controlled Chen chaotic system.

## 4. CONCLUSION

In this paper, we proposed a novel impulsive stabilizing control law for a new class of chaotic system. Based on Lyapunov stability theory and a non-quadratic Lyapunov function, we derive some algebraic sufficient conditions for asymptotical stability of the controlled chaotic systems. An example was solved using the results obtained, so as to illustrate its effectiveness and applicability.

## APPENDIX A

The main result of the paper is based on a result obtained in [10]. This result is recalled in this appendix for ease of references. Here, $K$, $C(\cdot,\cdot)$, $PC(\cdot,\cdot)$ are defined similarly as in Section 2. Consider the following impulsive controlled system with disturbance:

(A.1a) $$\dot{x}\left(t\right) = f\left(t, x\left(t\right)\right) + \omega\left(t\right), \, t \neq \tau_k,$$

(A.1b) $$y = \varphi\left(x\right), \; t \neq \tau_k$$

(A.1c) $$\Delta x\left(t\right) = u_k\left(y\right), \; t = \tau_k,$$

(A.1d) $$x\left(t_0^+\right) = x_0, \, k = 1, 2, \ldots,$$

where $x \in R^n$ is the state vector, $y \in R^m$ is the output vector, $\varphi \in C\left(R^n, R^m\right)$, $u_k \in C\left(R^m, R^n\right)$, $f \in C\left(R_+ \times R^n, R^n\right)$, $0 < \tau_1 < \cdots < \tau_k < \cdots$, with $\tau_k \to \infty$ as $k \to \infty$, $\Delta x\left(\tau_k\right) = x\left(\tau_k^+\right) - x\left(\tau_k^-\right) = x\left(\tau_k^+\right) - x\left(\tau_k\right)$, with $x\left(\tau_k^+\right) = \lim\limits_{t \downarrow \tau_k} x\left(t\right)$ and $x\left(\tau_k^-\right) = \lim\limits_{t \uparrow \tau_k} x\left(t\right)$.

**Lemma A.1** ([10]). *Assume that there exist* $\alpha, \beta, \gamma, g \in K$, $p \in PC(R_+, R_+)$, $V(t,x) \in V_0$ *and* $\sigma > 0$, *such that the following conditions are satisfied.*

*i)* $\beta(\|x\|) \le V(t,x) \le \alpha(\|x\|)$, $\forall (t,x) \in R_+ \times S_\rho$;

*ii) There exists a* $\rho_1 \in (0,\rho)$, *such that* $x \in S_{\rho_1}$, *i.e.*, $x + u_k(x) \in S_\rho$ *for all* $k$;

*iii)* $V(\tau_k, x + u_k(\varphi(x))) \le g(V(\tau_k, x))$, $k = 1, 2, \ldots,$;

*iv)* $D^+V(t,x) \le p(t)\gamma(V(t,x))$, *for* $t \ne \tau_k$ *and* $x \in S_\rho$;

*v)* $\int\limits_q^{g_k(q)} \frac{ds}{\gamma(s)} + \int\limits_{\tau_k}^{\tau_{k+1}} p(s)\,ds \le -l_k$ ,$\forall q \in (0,\sigma)$, *for all* $l_k \in (0,+\infty)$, $k = 1, 2, \ldots,$ *and*

$$\sum_{k=1}^{\infty} l_k = \infty.$$

*Then, system* (A.1) *is asymptotically stable.*

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