The Importance of Using Deviations of the Vertical for the Reduction of Survey Data to a Geocentric Datum

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Abstract

This paper reviews the deviation of vertical and its use in the reduction of terrestrial survey data such as directions, azimuths, zenith angles and slope distances. The deviations of the vertical over Australia will change by an average of 6.8" due to the implementation of the Geocentric Datum of Australia. Therefore, for most applications, the deviation of the vertical may no longer be neglected in survey computations and adjustments. With the release of the AUSGeoid98, absolute deviations of the vertical at the geoid and with respect to the GRS80 ellipsoid are now available for these purposes. The improvements made when using deviations of the vertical are demonstrated for several worked examples. The exception is that the deviation of the vertical should not be applied when computing height differences from zenith angles and slope distances for use on the Australian Height Datum (AHD).

1. INTRODUCTION

Almost all terrestrial survey measurements, with the exception of spatial distances, are made with respect to the Earth's gravity vector. This is because a spirit bubble or electronic level sensor is usually used to level survey instruments and targets. Accordingly, the measurements are nominally oriented with respect to the level (equipotential) surfaces and plumblines of the Earth's gravity field, which undulate and are not parallel in a purely geometrical sense. This renders them impractical for survey computations and the representation of geographical positions. Therefore, account must be made for the orientation of a surveying instrument in the Earth's gravity field, so that the measurements are of practical use. This is achieved in practice through the reduction of survey measurements to a particular reference ellipsoid.

Historically, geodesists have introduced an ellipsoid that is a close fit to the geoid (the level surface that closely coincides with mean sea level) over the region to be surveyed and mapped. As the level surfaces and plumblines are orthogonal by definition, this is equivalent to closely aligning the ellipsoidal normals with the plumblines over the area of interest. This was the case for the Australian National Spheroid (ANS), which was oriented to give a best fit to the plumblines over Australia (Bomford, 1967). The result was that survey measurements made with respect to the gravity vector in Australia could be assumed to have been oriented with respect to the ANS, thereby simplifying most survey reductions and computations on the Australian Geodetic Datum (AGD). For most applications, the separation between the geoid and ANS and the angular differences between the plumbline and the ANS ellipsoidal normal could usually be neglected.

With the adoption of the Geocentric Datum of Australia or GDA94 (eg. Featherstone, 1996), these simplifying assumptions will not necessarily remain valid (Featherstone, 1997). This is because the geocentric GRS80 ellipsoid (Moritz, 1980), used with the GDA94, is a best fit to the level surfaces and plumblines of the Earth's gravity field on a global scale, and does not provide a best fit over Australia. Importantly, survey measurements made with respect to the gravity vector do not change with a change of datum (Heiskanen and Moritz, 1967), notwithstanding temporal variations due to geophysical phenomena. Terrestrial surveys conducted on a geocentric datum are more likely to require that the separation between the geoid and ellipsoid and the angular differences between the plumbline and the ellipsoidal normal are taken into account during the reduction of survey data. This also applies to the adjustment of existing terrestrial geodetic networks onto a geocentric datum, such as those undertaken by national and State/Territory mapping agencies to implement the GDA94.

This paper reviews two of the various definitions of the deviation of the vertical and illustrates the need for its inclusion, and the consequences of its neglect, in terrestrial surveying. Importantly, the change to a geocentric datum and ellipsoid from a regionally oriented datum and ellipsoid represents a substantial change in concept that has implications on the reduction and adjustment of terrestrial survey data. Since the deviation of the vertical in Australia will change by an average of 6.8” upon the implementation of the geocentric GRS80 ellipsoid, the corrections for its effect can no longer be ignored. Fortunately, however, the new national geoid model, AUSGeoid98 (Johnston and Featherstone, 1998), includes a model of the deviations of the vertical with respect to the GRS80 ellipsoid. This can be used to apply corrections to terrestrial survey data using the formulae summarised in this paper. Based on the worked examples in this paper, it is recommended that corrections for deviations of the vertical are routinely applied to all terrestrial surveys on the GDA94.

2. THE DEVIATION OF THE VERTICAL

2.1 Terminology

In this paper, the term ‘deviation of the vertical’ is used (eg. Fryer, 1971; Cooper, 1987; Bomford, 1980), whereas some readers may be more familiar with the synonymous term ‘deflection of the vertical’ (eg. Heiskanen and Moritz, 1967; Vanìček and Krakiwsky, 1986; Torge, 1991). The term ‘deviation of the vertical’ will be used here, since it is considered to more accurately describe what is being discussed, whereas the term deflection of the vertical could be misinterpreted as including the curvature of the plumbline (defined later). The plumbline is a field line of the Earth’s gravity field that is curved because it always orthogonal to the level surfaces, which themselves are curved.

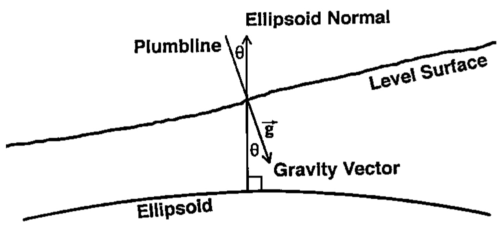

Figure 1. *The deviation of the vertical*

In Figure 1, the deviation of the vertical (θ) is the angular difference between the direction of the gravity vector (**g**), or plumbline at a point, and the ellipsoidal normal through the same point for a particular ellipsoid. Since the plumblines are orthogonal to the level surfaces, the deviation of the vertical effectively gives a measure of the gradient of the level surfaces (including the geoid) with respect to a particular ellipsoid. The deviation of the vertical is classified as absolute when it refers to a geocentric ellipsoid and relative when it refers to a local ellipsoid. Depending on the orientation - as well as the size and shape - of the ellipsoid used, the deviation of the vertical can reach 20” in lowland regions and up to 70” in regions of rugged terrain (Bomford, 1980). In Australia, the largest deviation of the vertical with respect to the GRS80 ellipsoid is 50.6” (cf. Table 1). Previously, the largest *measured* deviation of the vertical was approximately 30” with respect to the ANS (Fryer, 1971).

The deviation of the vertical, which is a vector quantity, is usually decomposed into two mutually perpendicular components: a north-south or meridional component (ξ), which is reckoned positive northward (i.e. the plumbline intersects the celestial sphere north of the ellipsoidal normal), and an east-west or prime vertical component (η), which is reckoned positive eastward (i.e. the plumbline intersects the celestial sphere east of the ellipsoidal normal). In other words, the deviation components are positive if the direction of the gravity vector points further south and further west than the corresponding ellipsoidal normal (Vanìček and Krakiwsky, 1986), or the level surface is rising to the south or west, respectively, with respect to the ellipsoid (Bomford, 1980). Taking Figure 1 as an example and assuming that left is west, right is east and θ = η, then the gravity vector points east, the plumbline intersects the celestial sphere in the west and the level surface rises to the east; therefore, η takes a negative value.

These two components reduce to the total deviation of the vertical (θ) in Figure 1 according to

$$\theta^2 = \xi^2 + \eta^2 \qquad (1)$$

The component of the deviation of the vertical can be resolved along any geodetic azimuth (α) by (Bomford, 1980; Vanìček and Krakiwsky, 1986)

$$\varepsilon = \xi \cos\alpha + \eta \sin\alpha \qquad (2)$$

which is most often used in the reduction of survey measurements (described later).

It is now important to distinguish the exact point at which the deviation of the vertical applies, since it varies depending upon position. The direction of the gravity vector and thus the orientation of a survey instrument or target, levelled using spirit bubbles or electronic level sensors, varies along the (curved) plumbline. The orientation of a survey instrument or target also varies from location to location because the plumblines curve by different amounts in different places. The level surfaces and plumblines are curved because of the mass distributions inside the Earth’s surface that generate the gravity field. Accordingly, the orientation of survey instruments and targets are always slightly different because of variations in the gravity field and must thus be reduced to a consistent orientation. It is acknowledged that there are several subtly different definitions of the deviation of the vertical (eg. Torge, 1991; Jekeli, 1999), but only two cases relevant to surveying in Australia will be considered here.

2.2 Deviation of the Vertical at the Geoid

The deviation of the vertical at the geoid (θ_G) is defined by Pizzetti (Torge, 1991) as the angular difference between the direction of the gravity vector and the ellipsoidal normal through the same point at the geoid. This can be an absolute quantity when using a geocentric ellipsoid (such as the newly adopted GRS80) or a relative quantity when using a locally oriented ellipsoid (such as the previous ANS). However, the deviation of the vertical at the geoid cannot be observed directly on land because of the presence of the Earth's topography. Therefore, deviations of the vertical that are observed at the Earth's surface (described later) have to be reduced to the geoid or *vice versa* by accounting for the curvature of the plumbline (also described later), which is notoriously problematic.

As an alternative, absolute deviations of the vertical with respect to a geocentric ellipsoid can be computed from gravity measurements using Vening-Meinesz's formula (eg. Heiskanen and Moritz, 1967; Vaníček and Krakiwsky, 1986). Nowadays, however, it is more convenient to estimate the deviation of the vertical at the geoid from the gradient of a gravimetric geoid model, which has been previously computed with respect to a geocentric ellipsoid. This determination of deviations of the vertical can be conceptualised as the reverse process of astro-geodetic levelling or astro-geodetic geoid determination (eg. Bomford, 1980; Heiskanen and Moritz, 1967), noting that it uses absolute deviations of the vertical. This use of a gravimetric geoid model is considered more convenient because many such models have already been computed for the transformation of GPS-derived ellipsoidal heights to orthometric heights.

The approach is as follows: given a regular grid of geoid-geocentric-ellipsoid separations, the meridional [north-south] (ξ_G) and prime vertical [east-west] (η_G) components of the absolute deviation of the vertical at the geoid can be estimated (eg. Torge, 1991) by

$$\xi_G = -\frac{\Delta N}{\rho\,\Delta\phi} \tag{3}$$

$$\eta_G = -\frac{\Delta N}{\nu\,\Delta\lambda\,\cos\phi} \tag{4}$$

where the subscript G is used to distinguish these components of the deviation of the vertical at the geoid (this notation which will be maintained throughout this paper), ρ is the radius of curvature of the ellipsoid in the meridian [north-south] at the point of interest, ν is the radius of curvature of the ellipsoid in the prime vertical [east-west] at the point of interest, ϕ is the geodetic latitude, and ΔN refers to the change in the gravimetric-geoid-geocentric-ellipsoid separation between grid nodes of latitude ($\Delta\phi$) and longitude ($\Delta\lambda$).

The two radii of curvature of the ellipsoid are computed from

$$\nu = \frac{a}{(1 - e^2 \sin^2\phi)^{1/2}} \tag{5}$$

$$\rho = \frac{a\,(1 - e^2)}{(1 - e^2 \sin^2\phi)^{3/2}} \tag{6}$$

where the first numerical eccentricity (e) and the flattening (f) of the ellipsoid are given by

$$e^2 = \frac{(a^2 - b^2)}{a^2} \tag{7}$$

$$f = \frac{(a - b)}{a} \tag{8}$$

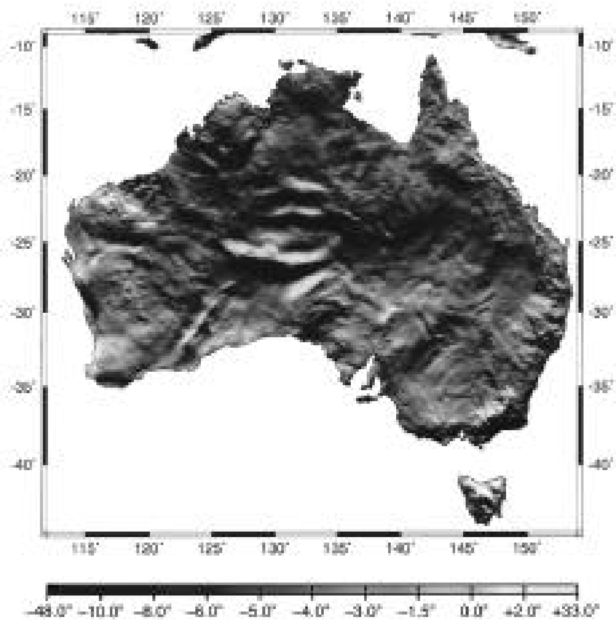

Figure 2. *Image of the magnitude of the north-south component of the deviation of the vertical at the geoid (ξ_G), computed from the AUSGeoid98 (units in arc seconds).*

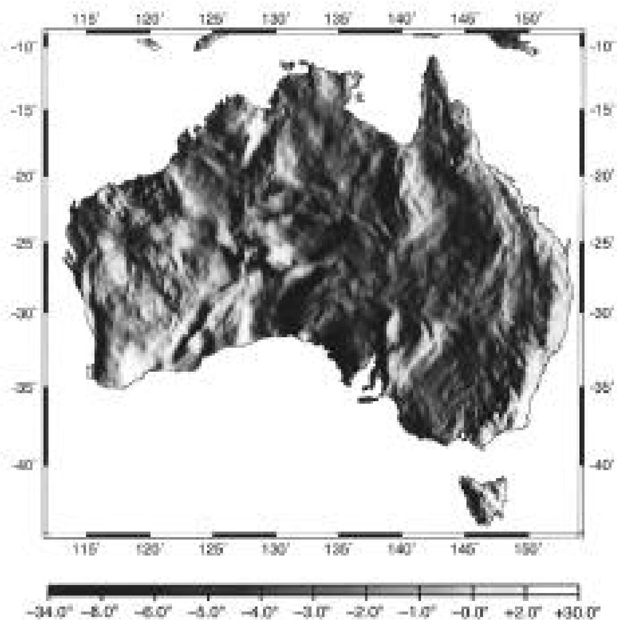

Figure 3. *Image of the magnitude of the east-west component of the deviation of the vertical at the geoid (η_G), computed from the AUSGeoid98 (units in arc seconds)*

and the semi-major and semi-minor axes of the ellipsoid are denoted by a and b, respectively. The GRS80 ellipsoid (Moritz, 1980) used for the GDA94 is based on the values a = 6,378,137 m and the flattening f = 1/298.257 222 101. For the GRS80 ellipsoid, the following values for e^2 and b apply:

$$b = a(1-f) \tag{9}$$

$$b = 6{,}356{,}752.314 \text{ m}$$

$$e^2 = 0.006\ 694\ 380\ 086$$

The numerical values for the GRS80 ellipsoid will be used throughout this paper.

The above determination of absolute deviations of the vertical from a gravimetric geoid model has already been applied to the AUSGeoid98 geoid model of Australia (Johnston and Featherstone, 1998). The east-west and north-south components of the deviation of the vertical at the geoid with respect to the GRS80 ellipsoid are available, together with Windows-based interpolation software, from the Australian Surveying and Land Information Group's world-wide web pages: http://www.auslig.gov.au/geodesy/ausgeoid/geoid.htm. These values can be bi-cubically interpolated in near-real-time over the Internet, also at the above address. Figures 2 and 3 show the north-south and east-west components of the deviation of the vertical over Australia, respectively, as derived from the AUSGeoid98 using equations (3) and (4). The images in Figures 2 and 3 only show the magnitude of the deviations of the vertical. This is because the scale of the maps and the 2' by 2' resolution of the AUSGeoid98 prevents the presentation of deviation vectors.

Table 1 shows the statistical properties of the deviations of the vertical for the 1,638,000 node points of AUSGeoid98 (10˚S < ϕ < 45˚S; 108˚E < λ < 160˚E). The maximum deviation is at 119.033˚ E and -10.067˚ S (Java Trench); on land, the largest value is 34.2" at 116.033˚E and 31.733˚S (near Perth).

Deviation	*max*	*min*	*mean*	*std*
north-south (ξ_G)	32.67	-48.11	-3.95	4.90
east-west (η_G)	28.84	-33.99	-2.98	4.09
total (θ_G)	50.60	0.00	6.81	4.33

Table 1. *Statistical properties of deviations of the vertical at the geoid, computed from the AUSGeoid98 (units in arc seconds)*

It is interesting to note that the mean value of the total deviation of the vertical (θ_G) in Table 1 agrees well with the value of 6" estimated by Featherstone (1997). Likewise, the mean values of the east-west and north-south deviation components in Table 1 agree reasonably well with the estimates of the orientation of the AGD made by Mather (1970).

Table 2 gives the geoid-GRS80-ellipsoid separations and the two components of the deviations of the vertical (at the geoid) as computed from AUSGeoid98 for the capital cities of Australia and for Alice Springs. The listed values refer to the GDA94 latitudes and longitudes shown. Brisbane has the smallest total deviation and Alice Springs the largest. All values are at least two times smaller than the maximum values listed in Table 1.

City	ϕ	λ	N (m)	ξ_G (")	η_G (")
Perth	–32˚00'	115˚53'	–32.163	+1.483	–16.743
Darwin	–12˚29'	131˚02'	+51.511	–5.342	–6.282
Alice Spr	–23˚48'	133˚53'	+14.276	–16.957	–5.847
Adelaide	–34˚48'	138˚37'	+0.215	–4.408	–8.974
Melbourne	-37˚50'	145˚10'	+5.309	–6.160	–4.670
Hobart	-42˚55'	147˚19'	–3.497	–5.754	+6.571
Canberra	-35˚14'	149˚09'	+19.836	–4.804	–6.245
Sydney	-33˚53'	151˚01'	+22.828	–6.241	+1.976
Brisbane	-27˚04'	152˚57'	+42.787	–5.149	+3.480

Table 2. *Components of the deviations of the vertical at the geoid (ξ_G, η_G) and the geoid-ellipsoid separation (N) computed from the AUSGeoid98, for each capital city in Australia and Alice Springs*

2.3 Deviation of the Vertical at the Earth's Surface

The deviation of the vertical at the surface of the Earth (θ_S) is defined by Helmert (Torge, 1991) as the angular difference between the direction of the gravity vector and the ellipsoidal normal through the same point at the Earth's surface. This can be an absolute quantity when using a geocentric ellipsoid (such as the newly adopted GRS80) or a relative quantity when using a locally oriented ellipsoid (such as the previous ANS). The deviation of the vertical at the surface of the Earth is of more direct use to surveyors than the deviation of the vertical at the geoid. This is because survey instruments and targets are levelled on or close to the Earth's surface and are thus affected by the deviation of the vertical at this point.

The deviation of the vertical at the Earth's surface can be computed simply by comparing astronomical and geodetic coordinates at the same point on the Earth's surface. The corresponding deviation in the meridian is the difference between astronomical latitude (Φ) and the geodetic latitude (ϕ) of the same point. Likewise, the deviation of the vertical in the prime vertical is the difference, scaled for meridional convergence, between astronomical longitude (Λ) and the geodetic longitude (λ) of the same point. These are given, respectively, by

$$\xi_S = \Phi - \phi \tag{10}$$

$$\eta_S = (\Lambda - \lambda)\cos\phi \tag{11}$$

where the subscript S is used to distinguish these components of the deviation of the vertical at the surface of the Earth. This notation will be used throughout the paper. In equations (10) and (11), it is assumed that the minor axis of the ellipsoid is parallel to the mean spin axis of the Earth's rotation (cf. Bomford, 1980), which is the case for the GDA and the AGD (Bomford, 1967).

The National Mapping Council (1986) and Fryer (1971) published, in diagram form, measured surface deviations of the vertical in relation to the previously used ANS. At present, there are no listings or figures available to the authors that give measured deviations of the vertical at the Earth's surface in relation to the GDA94 and the GRS80 ellipsoid. As stated, the deviations of the vertical at the points of measurement (usually at or close to the Earth's surface) should be used to reduce survey observations. Therefore, it would be very useful to compare the deviations of the vertical at the geoid as computed from AUSGeoid98 with measured values. Such an evaluation also requires the curvature of the plumbline (described next) with respect to the GRS80 ellipsoidal normal to be known. A direct comparison of measured values at the surface with computed values at the geoid would give an indication of the appropriateness of the deviations of the vertical from the AUSGeoid98 as approximations of the deviation of the vertical at the Earth's surface.

2.4 Curvature (Deflection) of the Plumbline

As stated, the deviation of the vertical changes with position along the curved plumbline. Therefore, the deviation of the vertical at the geoid (θ_G) does not necessarily equal that at the Earth's surface (θ_S) and *vice versa*. In order to relate these two quantities, the curvature (or, more accurately, the deflection caused by the curvature) of the plumbline between the geoid and Earth's surface ($\delta\theta_{GS}$) is required. This quantity cannot be observed directly because of the presence of the topography. However, it can be estimated using a model of the Earth's gravity field within the topographic masses (eg. Papp and Benedic, 2000) or by comparison of the deviations of the vertical at the geoid, derived from a geoid model or Vening-Meinesz's integral, and measured deviations of the vertical at the Earth's surface (see above). However, the latter approach is limited because the errors in the data are most probably of the same size as the curvature of the plumbline.

Alternatively, the curvature of the plumbline can be estimated using an approximate formula (Vaníček and Krakiwsky, 1986; Bomford, 1980). This is based on normal gravity and thus only affects the north-south deviation component ($\delta\xi_{GS}$).

$$\delta\xi_{GS} = \delta\theta_{GS} = 0.17'' \sin 2\phi \, H \qquad (12)$$

where the orthometric height H (in kilometres) is measured along the plumbline between the geoid and surface of the Earth. The evaluation of the actual curvature (deflection) of the plumbline presents a very difficult task because measurements cannot be made along the plumbline in the topography. A crude estimate of the curvature of the plumbline is $\delta\varepsilon_{GS} = 3.3''$ per (vertical) kilometre in rugged terrain (Vaníček and Krakiwsky, 1986). Given the relatively smooth terrain in Australia, the typical values in Australia are probably less than 1". However, this value may be too optimistic in areas where steep geoid gradients exist (cf. Figures 2 and 3). Bomford (1980) states that until the actual curvature of the plumbline is known, the above approximations are barely useful, and are thus ignored. Nevertheless, it is essential to acknowledge their existence.

3. THE USE OF DEVIATIONS OF THE VERTICAL IN SURVEY REDUCTIONS AND THE COMPUTATION OF HEIGHT DIFFERENCES

Historically, the most influential use of the deviation of the vertical led to the principle of isostasy, which is used to describe the broad geophysical structure of the Earth's crust. The deviations of the vertical, observed as part of the 1735-1744 Peruvian expedition to determine whether an oblate or prolate ellipsoid approximated the figure of the Earth, were shown by Bouguer to be smaller than expected. These and subsequent measurements formed the basis for the two models of isostatic compensation developed by Airy-Heiskanen and Pratt-Heyford. These models are analogous with Archimedes's principle, where the masses of mountains are buoyantly compensated either by a thickening of the crust (Airy-Heiskanen model) or a variation in the mass density of the crust (Pratt-Heyford model). However, these two simplified models do not always apply in practice because of the overriding geophysical and mechanical properties of the Earth's crust.

In terrestrial surveying, the deviation of the vertical has six primary uses:

1. transformation between astronomical coordinates and geodetic coordinates;
2. conversion between astronomic or gyro azimuths and geodetic azimuths;
3. reduction of measured horizontal directions (and angles) to the ellipsoid;
4. reduction of measured zenith angles to the ellipsoid;
5. reduction of slope electronic distance measurements (EDM) to the ellipsoid using zenith angles; and
6. determination of height differences from zenith angles and slope distances.

A reduction associated with item 5 is in the use of ellipsoidal heights for the reduction of EDM distances to the ellipsoid (cf. Featherstone, 1997), where large gradients in the geoid-ellipsoid separation (equivalent to the deviation of the vertical integrated over an EDM line) become important.

3.1 Transformation of Coordinates

The deviations of the vertical allow the transformation between astronomical (natural) coordinates (Φ, Λ), observed with respect to the gravity vector, and the desired geodetic coordinates (ϕ, λ) on the ellipsoid and *vice versa*. Rearranging equations (10) and (11), and adhering to the same approximations, gives this coordinate transformation as

$$\phi = \Phi - \xi_S \qquad (13)$$

$$\lambda = \Lambda - \frac{\eta_S}{\cos\phi} \qquad (14)$$

where the deviations of the vertical refer to the surface of the Earth, since this is the point at which astronomic coordinates are normally measured. If the deviations of the vertical at the geoid are used in equations (13) and (14), the limitation imposed by the curvature of the plumbline should be noted. Following the recommendation of Bomford (1980), the approximate formulae for the curvature of the plumbline are ignored because they may introduce more errors than they rectify. Therefore, it is important to state in the documentation that accompanies the survey data reduction if the deviations of the vertical at the geoid have been used and that curvature of the plumbline has been ignored.

Table 3 shows a worked example of the transformation of astronomic coordinates to geocentric geodetic coordinates using absolute deviations of the vertical at the geoid from AUSGeoid98. In this example, the deviation values were bi-cubically interpolated to the geocentric position in Table 3 using AUSLIG's internet-based facility. In practice, the use of the AUSGeoid98 to transform astronomical coordinates requires iteration, where the deviations of the vertical are interpolated to the astronomical position, equations (13) and (14) used to determine an approximate geocentric position, then this position is used to interpolate the appropriate deviations of the vertical. This iteration is necessary because the AUSGeoid98 grid nominally refers to a geocentric datum (cf. Featherstone, 1995).

Astro-geodetic Coordinates	Φ = 25° 56' 54.552" S
	Λ = 133° 12' 30.077" E
AUSGeoid98 Deviations of the Vertical	ξ_G = +2.312"
	η_G = –7.935"
GDA94 Geodetic Coordinates	ϕ = 25° 56' 56.864" S
	λ = 133° 12' 37.212" E

Table 3. *Sample transformation of astro-geodetic coordinates to geocentric (GDA) geodetic coordinates using the AUSGeoid98 deviations of the vertical at the geoid in equations (13) and (14)*

Note that the GDA94 coordinates in Table 3 will differ from the GDA94 coordinates as transformed from the astronomically determined coordinates using the seven-parameter, or other, datum transformation (AUSLIG, 1999). This is because the (assumed) astro-geodetic coordinates for the Johnston Memorial Cairn (Table 3) include an effect due to the orientation of the AGD (Bomford, 1967; Mather, 1970). There is also an effect due to the curvature of the plumbline over the ellipsoidal height of the station. These effects will not be discussed further since Table 3 is only to provide a numerical example of using equations (13) and (14).

3.2 Laplace's Equation for Azimuths

The deviations of the vertical at the Earth's surface are required to convert observed astronomic azimuths and observed gyro azimuths to geodetic azimuths. This is achieved rigorously using the formula (Vaníček and Krakiwsky, 1986)

$$\alpha = A - (\eta_S \tan\phi) - (\xi_S \sin\alpha - \eta_S \cos\alpha)\cot\zeta \qquad (15)$$

where α = geodetic azimuth of the measured line, clockwise through 360-degrees from north,
A = measured astronomic or gyro azimuth with respect to the gravity vector,
ζ = geodetic zenith angle between the observing and target stations (equation 18) – a measured zenith angle can also be used to a sufficient precision.

In relation to the accuracy with which the astronomic or gyro azimuth can be measured, the third term on the right-hand-side of equation (15) can be neglected, especially for zenith angles close to 90 degrees. Therefore, equation (15) reduces to the well-known Laplace correction

$$\alpha = A - (\eta_S \tan\phi) \qquad (16)$$

The Laplace correction is, thus, independent of the azimuth of a line. Historically, the most common use of equation (16) was at Laplace Stations, where astronomical azimuths (as well as astronomical latitudes and longitudes) were measured to constrain geodetic azimuths in terrestrial geodetic networks. The reduced geodetic azimuths provide the proper orientation of the geodetic network or traverse. For example, in the long traverses conducted in Australia during the establishment of the AGD, astronomical azimuths were used to control and decrease the propagation of errors in the traversed azimuths (Bomford, 1967).

Equation (16) can usually be neglected for solar determinations of astronomic azimuth, but not for stellar determinations of astronomic azimuth or gyro azimuths because of their increased precision. The Laplace correction is particularly relevant for precision gyro measurements in long tunnels, where neither the deviations of the vertical at the surface nor those at the geoid are necessarily applicable. Moreover, classical Laplace Stations cannot be established in a tunnel. For the planned 57 km rail base-tunnel under the Gotthard Pass in Switzerland, the effect of the deviation of the vertical on gyro measurements was investigated by Carosio and Ebneter (1998). They used an above-ground (open) traverse (ie. directions, zenith angles, slope distances, azimuths) over 8.6 km and an under-ground (open) traverse over 7.9 km. The under-ground traverse (in the service tunnel of the existing Gotthard road tunnel) included reciprocal precision gyro azimuth measurements on four legs.

The results of different adjustments of the under-ground traverse were compared with a (closed) precision traverse through the full length of the main tunnel, which was measured prior to the opening of the tunnel (Carosio and Ebneter, 1998). The different adjustments show that the computations *with* gyroscopic observations and *with* corrections for the deviations of the vertical produce the best results. Conversely, the computations *with* gyro observations but *without* corrections for the deviations of the vertical produce the worst results. The peak-to-peak variation of the η component of the deviations of the vertical is 7.5" and that of the ξ component 16.8" (Zanini *et al*., 1993). The maximum values were 7.6" and 11.1" for the η and ξ components, respectively. The same authors reported maximum coordinate differences (compared to the reference traverse) of 39 mm for the 7.9 km long traverse with gyro-azimuths and with Laplace corrections, and 137 mm for the traverse with gyro observations and without Laplace corrections. The magnitude of the Laplace corrections was greater than the accuracy of the azimuths (±2.3") measured with the Gyromat 2000 precision gyro-theodolite.

Table 4 shows a worked example of the calculation of a geodetic azimuth from a gyro azimuth using the deviation of the vertical at the Earth's surface in equation (16).

Gyro Azimuth from 2 to 3	A_{23}	=	306° 43' 28.2"
Deviation of the Vertical in Prime Vertical Direction	η_S	=	+7.27"
Geodetic Latitude	ϕ_2	=	46° 31' 30" N
Laplace Correction (*Equation 16*)		=	–7.67"
Geodetic Azimuth	α_{23}	=	306° 43' 20.5"

Table 4. *Sample computation of the conversion of a gyro azimuth to a geodetic azimuth using the deviation of the vertical at the Earth's surface in equation (16)*

3.3 Reduction of Measured Horizontal Directions and Angles to the Ellipsoid

Horizontal directions (and angles) have to be corrected for the deviation of the vertical at the Earth's surface when the instrument and the target are not at the same ellipsoidal height. This can be visualised as an error like that encountered for poorly levelled theodolites or total stations. Assuming that the skew normal correction has already been applied between stations (eg. Vaniček and Krakiwsky, 1986), the correction to a measured horizontal direction (Bomford, 1980; AUSLIG, 1999) is given by

$$d = D - (\xi_S \sin\alpha - \eta_S \cos\alpha)\cot\zeta \qquad (17)$$

where d = desired direction related to the ellipsoid,
D = measured direction with respect to the gravity vector at the Earth's surface,
ζ = geodetic zenith angle between the observing and target stations (equation 18) – a measured zenith angle can also be used to a sufficient precision,

It follows from equation (17), that the effect of the deviation of the vertical on horizontal directions is zero, if the observing and target stations are at the same height above the ellipsoid. The correction of horizontal angles follows from the difference of the corrections of the two directions concerned. The error caused by the neglect of this correction also propagates along a traverse, hence the need for regular Laplace Stations (cf. Bomford, 1967).

Table 5 illustrates the effect of neglecting deviations of the vertical on horizontal directions for a variety of geodetic azimuths and zenith angles from the GDA94 position in Table 3. The curvature of the plumbline is neglected since the AUSGeoid98 deviations of the vertical at the geoid are used in equation (17). The numerical examples in Table 5 show that the effect of the deviation of the vertical on a horizontal direction can be relatively large, especially for small zenith angles (ie. large height differences).

Measured Horizontal Direction (D)	*Geodetic Zenith Angle* (ζ)	*Geodetic Azimuth* (α)	*Reduced Horizontal Direction (d)*
45° 00' 00.00"	89°	45°	44° 59' 59.87"
45° 00' 00.00"	85°	45°	44° 59' 59.37"
45° 00' 00.00"	45°	45°	44° 59' 52.75"

Table 5. *Sample reductions of horizontal directions to the GRS80 ellipsoid for the GDA94 position in Table 3 using the AUSGeoid98 deviations of the vertical in Eq. (17)*

3.4 Reduction of Measured Zenith Angles to the Ellipsoid

Zenith angles also have to be corrected for the deviation of the vertical at the Earth's surface, for exactly the same reasons as horizontal directions and angles. Again, the skew normal corrections and the corrections for atmospheric refraction are assumed to have been applied. In the case of a single measured zenith angle, the component of the deviation of the vertical at the Earth's surface in the azimuth of the observation is required. Accordingly, equation (2) for the deviations of vertical at the surface of the Earth is applied to the observed zenith angle to yield the geodetic zenith angle with respect to the ellipsoidal normal (Vaniček and Krakiwsky, 1986)

$$\begin{aligned}\zeta &= z + (\xi_S \cos\alpha + \eta_S \sin\alpha) \\ &= z + \varepsilon_S\end{aligned} \qquad (18)$$

where ζ = geodetic zenith angle between the observing and target stations,
z = measured zenith angle between the observing and target stations, and
ε_S = deviation of the vertical at the Earth's surface in the geodetic azimuth of the observed line (equation 2).

In reciprocal trigonometric levelling, a large proportion of the effect of the deviation of the vertical may cancel on differencing. Nevertheless, equation (18) should still be applied to all observations of zenith angles, especially over long baselines, or in regions where the gravity field varies rapidly (cf. Figures 2 and 3 and Table 2). Since the deviations of the vertical at the geoid are readily available from the AUSGeoid98 and the reduction can be computed very easily, they should be included so as to reduce their effects on the survey results. It is important to point out that there may be cases where the curvature of the plumbline is sufficiently large to obscure any improvement offered by including the deviation of the vertical.

Table 6 illustrates the effect of neglecting the deviations of the vertical on zenith angles for a variety of azimuths measured from the GDA94 position in Table 3. Again, the curvature of the plumbline is neglected and the AUSGeoid98 deviations of the vertical at the geoid are used in equation (18). Accordingly, the effect of the curvature of the plumbline has also been ignored in these examples.

Measured Zenith Angle (z)	*Geodetic Azimuth* (α)	*Geodetic Zenith Angle* (ζ)
45° 00' 00.00"	0°	45° 00' 07.94"
85° 00' 00.00"	45°	85° 00' 07.25"
89° 00' 00.00"	90°	89° 00' 02.31"

Table 6. *Sample reductions of zenith angles to the GRS80 ellipsoid for the GDA position in Table 3 using AUSGeoid98 deviations of the vertical in equation (18)*

3.5 Reduction of Measured Distances to the Ellipsoid

Electronically measured distances (ie. EDM) can be reduced to distances on the ellipsoid by using a measured zenith angle and the ellipsoidal height at the instrument station, or by using known ellipsoidal heights at both ends of the line. The first method is usually used for short distances, where the ellipsoidal heights of the target stations are often not available. It is also the most widely used method nowadays. As it relies on measured zenith angles, it requires a knowledge of the deviation of the vertical at the instrument station (cf. equation 18). The second method was traditionally employed in connection with medium- to long-range distance measurements. Even though this second method does not require the deviations of the vertical, it will be reviewed briefly for the sake of completeness.

3.5.1 Reduction of Measured Distances to the Ellipsoid Using Zenith Angles

The geometry of this reduction is depicted in Figure 4, where

z_1, z_2 = measured zenith angles at stations P_1 and P_2
δ = refraction angle, assumed equal at P_1 and P_2
$\varepsilon_1, \varepsilon_2$ = deviations of vertical at the Earth's surface ε_S at P_1 and P_2 in the forward and reverse geodetic azimuths of the line
ζ_1, ζ_2 = geodetic zenith angles at P_1, P_2
β = angle between the wave-path normals through P_1 and P_2
R_α = radius of curvature of the ellipsoid in the geodetic azimuth of the line from P_1 to P_2
H_1, H_2 = orthometric heights (approximated by AHD heights in Australia) at P_1 and P_2
h_1, h_2 = GRS80 ellipsoidal heights at P_1 and P_2
N_1, N_2 = geoid-GRS80-ellipsoid separations at P_1 and P_2
d_1 = wave-path length
d_2 = wave-path chord length
d_4 = distance on the ellipsoid (assumed geodesic)
d_{MSL} = mean sea level distance

Figure 4 clearly assumes that the two (reciprocal) zenith angle measurements, as well as the distance measurement, are co-linear and that the trunnion axes heights (above the ground marks) of theodolites, EDM devices, reflectors, traversing targets, are thus the same for any one station. Additional corrections to the measured zenith angles and/or distances may apply, if these co-linearity conditions are not fulfilled. A detailed description of these corrections is beyond the scope of this paper, and the reader is referred to Rüeger (1996).

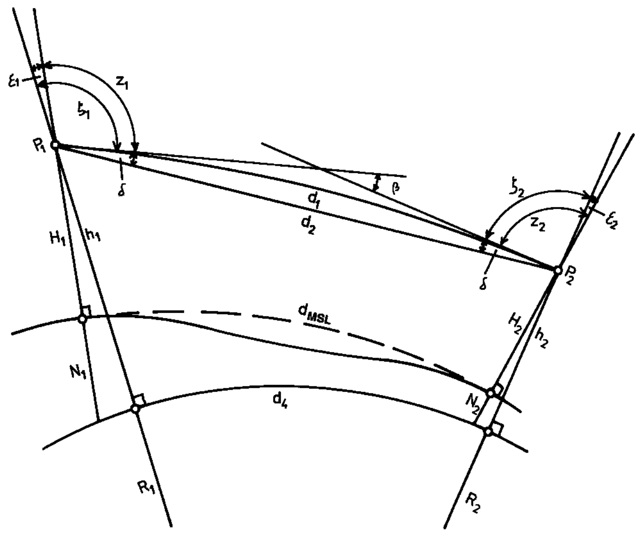

Figure 4. *Geometry of the reduction of slope distances to the ellipsoid using zenith angles and of the computation of height differences. It follows from equation (18) that ε_1 is a positive value and ε_2 a negative one.*

The GRS80 ellipsoidal heights at the end-points of the baseline refer to the common trunnion axes heights of all equipment used at the respective station. Therefore, for a measurement from P_1 to P_2, the values of h_1 and h_2 must include the height of the instrument (above the ground mark at P_1) and the height of reflector (above the ground mark at P_2), respectively. The heights of instrument and reflector are naturally measured along the plumbline and are thus strictly incompatible with the GRS80 ellipsoidal height required (see Section 2.4). Using the maximum deviation of the vertical relative to GRS80 of 50.6" (Table 1) and a height of instrument of 1.7 m, the maximum difference between a GRS80 ellipsoidal and orthometric height of instrument computes as only 50 nanometres. Accordingly, the difference between these two types of heights of instrument and heights of reflectors can be safely ignored in this and all other cases.

A rigorous reduction of the wave-path length (d_1) to the GRS80 ellipsoidal distance (d_4) is only possible if the GRS80 ellipsoidal heights (h), the deviations of the vertical at the Earth's surface (ε_S) and the radius of curvature (R_α) of the ellipsoid along the geodetic azimuth of the baseline are available (Figure 4).

- The GRS80 ellipsoidal height (h) can be computed from the orthometric height (H) by algebraic addition of the appropriate geoid-ellipsoid separation (N).

$$h = H + N \tag{19}$$

In Australia, the N values vary between approximately –40 m and approximately +70 m with respect to the GRS80 ellipsoid, and can be interpolated from the AUSGeoid98 (described earlier).

- The deviation of the vertical at the Earth's surface in the azimuth of the line (ε_S) can be computed from equation (2) if the surface components ξ_S and η_S from equations (10) and (11) are used. However, as stated, these are not routinely available, so the deviations of the vertical at the geoid from the AUSGeoid98 are used as an approximation.
- The radius of curvature of the GRS80 ellipsoid along the line from P_1 to P_2 can be computed from (eg. Cooper, 1987; Rüeger, 1996; AUSLIG 1999):

$$R_\alpha = \frac{\nu\rho}{\nu \cos^2\alpha + \rho \sin^2\alpha} \qquad (20)$$

where all quantities have been defined earlier.

The ellipsoidal distance (d_4) computes rigorously as (Rüeger, 1996)

$$d_4 = R_\alpha \arctan\left(\frac{d_2 \sin(z_1 + \varepsilon_1 + \delta)}{R_\alpha + h_1 + d_2 \cos(z_1 + \varepsilon_1 + \delta)}\right) \qquad (21)$$

where the argument of the arctan function is in radian measure and the other quantities have been defined earlier. Equation (21) can be simplified by assuming $d_2 \approx d_1$ and by expressing the refraction angle (δ) as a function of the coefficient of atmospheric refraction (k), as well as d_1 and R_α. For $k = 0.13$ and $R_\alpha = 6{,}371$ km, the difference between d_2 and d_1 (the first arc-to-chord correction) is only 0.02 mm for $d_1 = 10$ km and 0.47 mm for $d_1 = 30$ km. Therefore, the wave-path chord distance (d_2) may be safely replaced by the wave-path length (d_1) in all practical cases. Therefore, equation (21) reduces to (Rüeger, 1996)

$$d_4 = R_\alpha \arctan\left(\frac{d_1 \sin\left(z_1 + \varepsilon_1 + \frac{d_1 k}{2R_\alpha}\right)}{R_\alpha + h_1 + d_1 \cos\left(z_1 + \varepsilon_1 + \frac{d_1 k}{2R_\alpha}\right)}\right) \qquad (22)$$

where the argument of the arctan function is, again, in radian measure, as are the third terms of the arguments of the sine and cosine functions. It is essential to convert the values of z and ε into radians before solving equation (22).

As far as the reductions to the geocentric GRS80 ellipsoid are concerned, the propagation of any errors in R_α, N and ε into d_4 is of particular interest here. [The reader is referred to Rüeger (1996) for the error propagation in the terms d_1, z_1 and k.] The total differential of equation (22) with respect to the variables R_α, N and ε is (Rüeger, 1996)

$$\delta(d_4) = \left(\frac{d_4 (h_1 + \Delta h)}{R_\alpha^2}\right) \delta(R_\alpha) - \left(\frac{d_4}{R_\alpha}\right) \delta(N_1) + \left(\frac{\Delta h}{206265}\right) \delta(\varepsilon) \qquad (23)$$

where the terms $\delta(*)$ indicate the error in the quantities * in parentheses (defined earlier).

a) The Radius of Curvature of the Ellipsoid

For the practical reduction of measured slope distances to the ellipsoid, a sphere of radius R is always used between the terminals of a line. AUSLIG (1999) gives a diagram of the variation of R_α (equation (20)) with geodetic azimuth for the average Australian latitude of 26˚S (exactly). At this latitude, R_α varies by 34,559.159 m from ρ = 6,347,684.393 m (α = 0˚ or 180˚) to ν = 6,382,243.552 m (α = 90˚ or 270˚) for the GRS80 ellipsoid.

To simplify the calculations further, a geometric mean radius (R_M) is sometimes adopted for a certain area, which is computed for any point on the ellipsoid using

$$R_M = \sqrt{\nu\,\rho} \qquad (24)$$

where the values of ν and ρ are computed for the GDA94 geodetic latitude using equations (5) and (6). Using equation (24), the mean radius of the GRS80 ellipsoid at the average Australian latitude of 26˚S is R_M = 6,394,940.517. Table 7 lists values for ρ, ν and R_M on the GRS80 ellipsoid for the range of Australian geodetic latitudes at 5˚ intervals.

Latitude (degrees)	ρ *(north-south)* (m)	ν *(east-west)* (m)	R_M (m)
–10	6 337 358.120	6 378 780.843	6 358 035.748
–15	6 339 703.298	6 379 567.582	6 359 604.204
–20	6 342 888.481	6 380 635.807	6 361 734.215
–25	6 346 818.858	6 381 953.457	6 346 361.912
–30	6 383 480.917	6 367 408.777	6 367 408.777
–35	6 356 426.695	6 385 172.174	6 370 783.222
–40	6 361 815.826	6 386 976.165	6 374 383.582
–45	6 367 381.815	6 388 838.290	6 378 101.030

Table 7. *Variation of the radii of curvature of the GRS80 ellipsoid for selected latitudes across Australia*

Table 7 shows that, for the GRS80 ellipsoid, the meridional [north-south] radius of curvature changes by 30,024 m between latitudes of –10˚ and –45˚ and the prime vertical [east-west] radius of curvature changes by 10,057 m. Accordingly, the geodetic azimuth dependence of the radius of curvature (equation (20)) is more significant than the latitude dependence. It is important to note that the use of an average radius of curvature (ie. equation (24) instead of equation (20)) in equation (22) has the side effect that reduced north-south distances will have a different scale error to the reduced east-west distances. If used, this can introduce systematic distortions into terrestrial geodetic networks.

In terms of equation (22) it is now informative to investigate the magnitude of the errors introduced by selecting either a mean radius for a particular area or even a mean radius for the whole of Australia. For this purpose, only the first term on the right-hand-side of equation (23) need be considered. It can be expressed as a fraction

$$\frac{\delta(d_4)}{d_4} = \left(\frac{h_1 + \Delta h}{R_\alpha^2}\right) \delta(R_\alpha) \qquad (25)$$

which allows the presentation of the errors as parts per million (ppm) of the reduced ellipsoidal distance. As expected, equation (25) shows that the error caused by an erroneous radius of the ellipsoid is zero at a zero GRS80 ellipsoidal height.

The first case investigated uses a mean GRS80 ellipsoidal radius for a project area (equation (24)) as an approximation of the azimuth-dependent radius (equation (20)). Table 8 lists the relative error in the reduced distance (in ppm) caused by selecting a local mean radius for three extreme locations in Australia, namely Cape York (ϕ = 10˚S), Mt. Kosciusko (ϕ = 35˚S) and south-east Tasmania (ϕ = 45˚S). The upper part of Table 8 refers to extreme elevations at these latitudes; the lower part gives the values for an elevation of 500 m.

Latitude (degrees)	h (m)	$\delta(R_\alpha)$ (m)	$\delta(d_4) / d_4$ (ppm)
–10	1 000	20 711	0.51
–35	2 200	14 372	0.78
–45	1 500	10 728	0.40
–10	500	20 711	0.26
–35	500	14 372	0.18
–45	500	10 728	0.14

Table 8. *Maximum errors caused in reduced ellipsoidal distances (d_4) due to the use of an azimuth-independent mean GRS80 ellipsoidal radius (R_M) for a specific area*

To assess the feasibility of using a mean ellipsoidal radius R_M, the *rule of thumb* that each individual reduction error should be less than one third of the measuring or required precision is used. On this basis, Table 8 shows that surveyors can select a mean GRS80 ellipsoidal radius for a project area anywhere in Australia and at any elevation, where reduced distances do not have to be better than 2.5 ppm (ie. no better than 1 mm for maximum distances of 400 m). Surveyors working at elevations below 500 m can use a mean GRS80 ellipsoidal radius for a project area anywhere in Australia to an accuracy of 0.8 ppm in reduced distances (ie. for distances of less than 1.25 km being no better than 1 mm). For more accurate work, the correct and azimuth-dependent radius of curvature of a line, computed from equation (20), must be used.

The second case investigated uses a mean GRS80 ellipsoidal radius for the average latitude of Australia as an approximation of the azimuth-dependent radius. Table 9 shows the maximum errors (in ppm) incurred when using a mean radius of the GRS80 ellipsoid for the whole of Australia. This comparison is based on R_M = 6,364,940 m at ϕ = 26˚S (equation (24)) for GRS80. Again, the upper part of Table 9 refers to extreme elevations in Australia, and the lower part to elevations of 500 m.

Using the previous rule of thumb in Table 9, surveyors can select R_M = 6,364,940 m at any elevation where reduced distances do not have to be better than 3.3 ppm (ie. 1 mm for distances of less than 330 m). Surveyors working at elevations below 500 m can use the Australia-wide mean radius of curvature for an accuracy of 1.0 ppm in the reduced distances (ie. 1 mm for distances of less than 1 km). For more accurate work, the local mean radius (see Table 7) or, preferably, the correct and azimuth-dependent radius of curvature of the line to be reduced (equation (20)) must be used.

Latitude (degrees)	h (m)	$\delta(R_\alpha)$ (m)	$\delta(d_4) / d_4$ (ppm)
–10	1 000	27 582	0.68
–35	2 200	20 232	1.10
–45	1 500	23 898	0.88
–10	500	27 582	0.34
–35	500	20 232	0.25
–45	500	23 898	0.29

Table 9. *Maximum errors caused in reduced ellipsoidal distances (d_4) due to the use of a single, azimuth-independent mean GRS80 ellipsoidal radius for the whole of Australia (R_M = 6,364,940m at ϕ = 26˚S)*

b) The Geoid-Ellipsoid Separation

The second term on the right-hand-side of equation (23) is now used to assess the errors in the reduction of distances caused by the omission of, or errors in, the geoid-GRS80-ellipsoid separation. Expressing the error in a fractional form gives

$$\frac{\delta(d_4)}{d_4} = -\left(\frac{1}{R_\alpha}\right) \delta(N_1) \qquad (26)$$

Considering the previously stated mean radius for an Australian average latitude, equation (26) indicates that an error $\delta(N_1)$ in N_1 of 6.365 m generates a 1 ppm error $\delta(d_4)$ in the reduced ellipsoidal distance. Using the AUSGeoid98, the omission of N_1 creates errors of +6.3 ppm in the south-west of Western Australia (where $N \approx$ –40 m for GRS80) and errors of –11.0 ppm at Cape York (where $N \approx$ +70 m for GRS80). This shows that the geoid-GRS80-ellipsoid separation *cannot* be ignored unless a surveyor only requires these levels of accuracy in the reduced distances.

It is therefore essential that surveyors compute the required N values from the AUSGeoid98 for the reduction of distances to the GRS80 ellipsoid (cf. Featherstone, 1997). It is important to note that the N values obtained from the AUSGeoid98 model are not free from errors. Reliable estimates of the absolute errors in the AUSGeoid98 are difficult to determine and vary from location to location (described later). However, these errors are likely to be very much smaller than 6 m and, therefore, will only affect reduced distances by significantly less than 1 ppm. Since the reduced distances are weakly dependent on errors in the N values (equation (26)), it is often sufficient to compute an average value (N_M) for a small project area. However, in areas where there are steep geoid gradients (cf Figures 2 and 3 and Table 2), N values should be computed for each and every point.

c) The Deviation of the Vertical

The third term on the right-hand-side of equation (23) is used next to evaluate the effect of errors in, or even the complete omission of, the deviation of the vertical. Note that this error is a function of the ellipsoidal height difference (Δh). Table 10 lists the errors $\delta(d_4)$ in d_4 caused by some realistic errors in the deviations of the vertical. The absolute uncertainty of the AUSGeoid98-derived deviations of the vertical is difficult to assess, since direct comparisons with astro-geodetic surface deviation values have not yet been carried out. Instead, Fryer's (1971) estimate of ±2" for the accuracy of gravimetrically derived deviations of the vertical is taken as a guide. It is likely that the AUSGeoid98 deviations of the vertical are more accurate than this estimate because of the advances made in geoid modelling and improved data coverage and processing since 1971. Nevertheless, the following values for the errors $\delta(\varepsilon_1)$ are used.

1. 2" (the expected accuracy of deviations of the vertical at the geoid computed from the AUSGeoid98),
2. 7" (the average deviation of vertical at the geoid over Australia – Table 1), and
3. 50" (the maximum value of deviation of vertical at the geoid in Australia – Table 1).

The first value maps the unavoidable errors in the deviations of the vertical into the ellipsoidal height differences, which will be inherent to all reductions. Conversely, the latter two values give an indication of the errors in ellipsoidal height differences if the deviation of the vertical is ignored altogether.

Δh [m] =	10	50	100	200	500
$\delta(\varepsilon)$ = 2"	0.1	0.5	1.0	1.9	4.8
$\delta(\varepsilon)$ = 7"	0.3	1.7	3.4	6.8	17.0
$\delta(\varepsilon)$ = 50"	2.4	12.1	24.2	48.5	121.2

Table 10. *Errors $\delta(d_4)$ caused in the reduced ellipsoidal distances d_4 due to errors $\delta(\varepsilon)$ in the deviation of the vertical ε*

It follows from Table 10 that the deviations of the vertical cannot be ignored in most cases where the GRS80 ellipsoidal height difference is not zero. Clearly, the deviations of the vertical must be taken into account for most reductions of distances to the GRS80 ellipsoid, especially for large height differences. Therefore, it is essential that surveyors amend their reduction software accordingly. It is also suggested that surveyors interpolate the values of ξ_G and η_G from the AUSGeoid98 and use these in lieu of ξ_S and η_S in equation (2) to compute ε_S, which is then used in equation (22). The likely differences between the surface and the geoid values have been discussed in Section 2.4. Following this and given that the curvature of the plumbline is likely to be less than the mean deviation of the vertical across Australia (Table 1), the AUSGeoid98-derived deviations are certainly of importance.

d) A Sample Computation

Table 11 gives a numerical example for the reduction of distances to the GRS80 ellipsoid that is based on actual measurements. The components of the deviations of the vertical and the geoid-ellipsoid separation were obtained from the AUSGeoid98 on the basis of the latitudes and longitudes listed in Table 11. The azimuth (α) was derived from the (MGA94) plane bearing (θ), using the simplified relationship $\theta \approx \alpha + \gamma$ where $\gamma \approx -(\lambda - \lambda_0) \sin \phi$ and λ_0 is the MGA central meridian; the arc-to-chord corrections are ignored. For simplicity, the radius of curvature is computed with the latitudes of the starting points of the lines (as the deviations of the vertical must also be calculated for the instrument stations). Tables 8 and 9 indicate that this small simplification is of no consequence. For the sake of interest, the reductions of the forward and the backward measurements are shown. The small difference of 0.3 mm in the reduced distances is caused by the deviation of the actual coefficient of refraction from the assumed value (taken as 0.13 as the line crosses a deep valley) and the measuring uncertainty of the zenith angles. The ("sea-level") distances, that are obtained with a reduction that omits N and ε, are shown at the bottom of Table 11. Considering that the slope distance was measured with sub-ppm precision, the error of 3.3 mm in the "sea-level" distance is significant.

Parameter	*Unit*	*Distance* 4–>6	*Distance* 6–>4
Latitude GDA94	dec deg	-33.21874250	-33.22165528
Longitude GDA94	dec deg	151.1229361	151.1169625
Azimuth GDA94 ~	dec deg	239.879454	59.882807
N-S Radius ρ	m	6 354 580.7	6 354 583.7
E-W Radius ν	m	6 384 554.0	6 384 555.0
Radius in line R_α	m	6 376 979.4	6 376 983.4
N-S Dev of Vert ξ	"	-6.156	-6.106
E-W Dev of Vert η	"	-0.863	-0.848
Dev Vert in line ε	"	+3.836	–3.797
slope distance d_2	m	644.9391	644.9391
meas zenith angle z	dec deg	93.391933	86.612714
H (AHD)	m	173.4470	135.3171
N (AUSGeoid98)	m	25.334	25.322
h (ellipsoid)	m	198.7810	160.6391
H.I.	m	0.239	0.236
Coeff of Refract k		0.13	0.13
ellips distance d_4	m	643.7921	643.7918
Without N, ε, with R_m = 6 364 940 m :			
"sea level" distance	m	643.7953	643.7951

Table 11: *Sample reduction of a slope distance d_2 to the ellipsoid (d_4) using ellipsoidal zenith angles ζ and Eq. (22)*

3.5.2 Reduction of Measured Distances to the Ellipsoid Using Ellipsoidal Heights

As mentioned before, long distances were traditionally reduced to the ellipsoid using known ellipsoidal heights of both terminals, rather than with the height and the zenith angle at the instrument station (Section 3.5.1). A rigorous reduction of the wave-path length (d_1) to the ellipsoidal distance (d_4) is given by (Rüeger, 1996)

$$d_4 = 2\, R_\alpha \arcsin \sqrt{\frac{R_\alpha^2 \sin^2 \left(\frac{d_1\, k}{2\, R_\alpha}\right) - \frac{k^2}{4}\,(h_2 - h_1)^2}{k^2\, [R_\alpha + h_1]\, [R_\alpha + h_2]}} \qquad (27)$$

where all terms have been defined previously and the geometry of this reduction is depicted in Figure 4. The ellipsoidal heights in equation (27) again represent the trunnion axes elevations of the EDM instrument and the reflector at the terminals P_1 and P_2. As before, the GRS80 ellipsoidal heights must be calculated from the AHD (assumed orthometric) heights by algebraic addition of the appropriate geoid-ellipsoid separation (equation (19)).

Featherstone (1997) illustrates the need to include the geoid-GRS80-ellipsoid separations in this reduction of measured distances to the GRS80 ellipsoid. However, less emphasis was placed on the effect of the change in geoid heights over the baseline in question, which is directly related to the deviation of the vertical (see equations (3) and (4)). Both effects are revisited here, as is the effect of approximations of the radius of curvature of the GRS80 ellipsoid. [The reader is referred to Rüeger (1996) for the error propagation in this reduction with respect to d_1 and k.] Rüeger (1996, as amended) expresses the total differential of equation (27) with respect to the variables R_α, N (h) and ΔN (Δh) as

$$\begin{aligned}\delta(d_4) \;=\; & \left(\frac{d\, h_M}{R_\alpha{}^2}\right)\, \delta(R_\alpha) \;-\; \left(\frac{d^3}{12\, R_\alpha{}^3}\right)\, \delta(R_\alpha) \\ & + \left(\frac{k^2\, d^3}{12\, R_\alpha{}^3}\right)\, \delta(R_\alpha) \\ & - \left(\frac{\Delta h}{d}\right)\, \delta(\Delta \mathrm{h}) \;-\; \left(\frac{d}{R_\alpha}\right)\, \delta(h_M) \qquad (28)\end{aligned}$$

$$\begin{aligned}\text{where}\quad \Delta h \;&=\; h_2 - h_1 = H_2 - H_1 + N_2 - N_1 \\ &=\; \Delta H + \Delta N \\ h_M \;&=\; 0.5\,(h_1 + h_2) = 0.5(H_1 + H_2 + N_1 + N_2) \\ &=\; H_M + N_M\end{aligned}$$

Even though three out of five terms on the right-hand-side of equation (28) relate to the error in the radius $\delta(R_\alpha)$, only the first one is of significance. Omitting the second and third right-hand-terms in equation (28) and dividing by d gives an equation for the error in d_4 due to an error in R_α that is very similar to equation (25). However, rather than h_2, the value of h_M is required this time. The results shown in Tables 8 and 9 and the associated discussions in Section 3.5.1a also apply here.

The errors caused in the reduced ellipsoidal distance (d_4) by errors in the geoid-GRS80-ellipsoid separation can be obtained by assuming that the orthometric heights H_1 and H_2 are free from error and that, in consequence, $\delta(\Delta h) = \delta(\Delta N)$ and $\delta(h_M) = \delta(N_M)$. The error in d_4 caused by $\delta(N_M)$ can be developed from the last (fifth) term of equation (28) into the fractional form of equation (26). This accounts for the part of the error in N that is the same for N_1 and N_2. Absolute errors of the N values derived from the AUSGeoid98 can also be assessed this way (described later). As before, an error $\delta(N_M)$ in N_m of 6.365 m generates a 1 ppm error $\delta(d_4)$ in the reduced distance (cf. equation (26)).

The fourth term on the right-hand-side of equation (28) is more significant; it can be expressed as

$$\delta(d_4) \;=\; -\,\Delta h \left(\frac{\delta(\Delta N)}{d}\right) \qquad (29)$$

Equation (29) is now used to assess the effects of the precision of ΔN, or even the complete omission of ΔN. Note that the proportional error ($\delta(\Delta N)$ / d) is multiplied by Δh, the total ellipsoidal height difference. Therefore, Table 12 gives the errors in the reduced ellipsoidal distances (d_4) for some relevant geoid gradients ($\delta(\Delta N)$ / d) in Australia and for a range of GRS80 ellipsoidal height differences.

For values of ΔN derived from the AUSGeoid98, the fractional error ($\delta(\Delta N)$ / d) is crudely estimated to be typically less than 3 ppm, but this value is known to vary from place to place. Nevertheless, the following values for the errors in the geoid gradient ($\delta(\Delta N)$ / d) are used

1. 3 ppm (the expected overall accuracy of geoid gradients computed from the AUSGeoid98),
2. 30 ppm (the average gradient of the AUSGeoid98 with respect to the GRS80 ellipsoid, which is in a approximately north-easterly direction),
3. 100 ppm (the large geoid-GRS80-ellipsoid gradient across the Darling Fault near Perth (Friedlieb *et al*., 1997)), and
4. 115 ppm (the largest geoid-GRS80-ellipsoid gradient in central Australia, found on the geoid map shown in Featherstone (1997, Fig. 1(b), p. 47)).

The first value above maps the unavoidable errors in the computed geoid gradients into the ellipsoidal height differences, which will be inherent to all reductions. Conversely, the latter three values give an indication of the errors in ellipsoidal height differences if the geoid gradient is ignored altogether. This applies over all distances, and especially where the deviations of the vertical (ie. geoid gradients) are large. For instance, the geoid-ellipsoid separation changes by 1 m over approximately 8 km near Perth, Western Australia. It should be noted in this context that the largest total deviation of the vertical listed in Table 1 (50.6") is equivalent to an even larger geoid gradient of 245 ppm. So, for the worst possible case in Australia, the values listed for 115 ppm in Table 12 must be doubled.

Table 12 shows that the AUSGeoid98 provides ΔN values that are sufficiently accurate for 1 mm distance reductions provided that the ellipsoidal height difference is less than 100 m. However, totally ignoring the geoid gradient in equation (27) creates quite large errors for the more extreme geoid-GRS80-ellipsoid gradients in Australia. Even the average geoid gradients cannot be ignored if the height differences are larger than 10 m. However, Table 12 only reflects Australian conditions. In Alpine areas, for instance, the geoid slopes are not necessarily larger but the height differences covered by EDM measurements are. For example, Elmiger (1977) reports an error of 150 mm for a line of 2.5 km in Switzerland, because the geoid slope was omitted from the reduction.

Δh [m] =	10	50	100	200	500
$\delta(\Delta N)/d_4$ = 3 ppm	0.03	0.15	0.3	0.6	1.5
$\delta(\Delta N)/d_4$ = 30 ppm	0.3	1.5	3.0	6.0	15.0
$\delta(\Delta N)/d_4$ = 100 ppm	1.0	5.0	10.0	20.0	50.0
$\delta(\Delta N)/d_4$ = 115 ppm	1.2	5.8	11.5	23.0	57.5

Table 12. *Errors $\delta(d_4)$ (in mm) caused in reduced ellipsoidal distances (d_4) due to the omission of some* typical gradients $\delta(\Delta N)/d_4$ of the geoid-ellipsoid separation

Table 12 clearly shows that ignoring the change in the geoid-ellipsoid separation ΔN introduces a distortion whenever there is a large difference in geoid heights between the terminals of a baseline. Importantly, this is in addition to the distortion introduced by the omission of the mean geoid height (N_M) in the reduction of the slope distance to the ellipsoid (cf. Featherstone, 1997). Figure 4 shows the mean sea level distance d_{MSL} that is obtained if the geoid-GRS80-ellipsoid separations N_1 and N_2 are ignored. However, as described above and in Featherstone (1997), the sea level correction to d_{MSL} is no longer appropriate in the context of the GDA and the GRS80 ellipsoid. It is therefore recommended, that N values are computed for both end-points when distances are to be reduced using only the GRS80 ellipsoidal heights.

Parameter	*Unit*	*Distance* 4 –> 6
Latitude GDA94	dec deg	-33.21874250
Longitude GDA94	dec deg	151.1229361
Azimuth GDA94 ~	dec deg	239.879454
N-S Radius ρ	m	6 354 580.7
E-W Radius ν	m	6 384 554.0
Radius in line R_α	m	6 376 979.4
slope distance $d_1 = d_2$	m	644.9391
H_1 (AHD)	m	173.4470
N_1 (AUSGeoid98)	m	25.334
h_1 (ellipsoid)	m	198.7810
$H.I._1$	m	0.2359
H_2 (AHD)	m	135.3171
N_2 (AUSGeoid98)	m	25.322
h_2 (ellipsoid)	m	160.6391
$H.R._2$	m	0.2365
Coeff of Refract k		0.13
ellips distance d_4	m	643.7921
Without N, with R_M = 6 364 940 m :		
"sea level" distance	m	643.7954

Table 13. *Sample reduction of a slope distance to the ellipsoid using equation (27) and the ellipsoidal heights of both terminals*

Table 13 gives a numerical example (real data) for the reduction of a slope distance to the GRS80 ellipsoid with ellipsoidal heights at both end-points. The geoid-GRS80-ellipsoid separations N at both end-points were obtained using AUSLIG's internet-based interpolation of the AUSGeoid98. The reduced distance d_4 agrees very well with that obtained in Table 11. When reducing the data in Table 13 with an Australian average radius of curvature (R_M = 6,364,940 m), exactly the same result is obtained. For comparison, the result of the (erroneous) reduction to "sea-level" is shown at the bottom of Table 13.

3.6 Height Differences from Zenith Angles and the AHD

Typically, zenith angles are measured in connection with electronic distance measurements, especially when using a total station. In consequence, orthometric or ellipsoidal height differences can be computed from the same data. The question of the deviations of the vertical must be carefully considered in the context of using the computed height differences. Although Australia has adopted a geocentric horizontal datum, the Australian Height Datum (AHD) will be maintained as the vertical datum.

3.6.1 Height Differences for the AHD

Height differences, that are compatible with the AHD and spirit-levelled height differences, can be computed from measured slope distances and zenith angles using (Rüeger, 1996)

$$H_2 - H_1 = d_2 \cos z_1 + \frac{\left(1 - \frac{k}{\sin z_1}\right)}{2\,R_\alpha} \left(d_2 \sin z_1\right)^2 + h_{TH(1)} - h_{T(2)} \qquad (30)$$

where
$H_2 - H_1$ = orthometric height difference between P_1 and P_2
$h_{TH(1)}$ = height of instrument at P_1
$h_{T(2)}$ = height of target at P_2
z_1 = measured zenith angle at P_1.

All other quantities have been defined earlier (also see Figure 4).

Note that the measured zenith angle (z) must be used in equation (30), which is necessary since height differences related to the geoid, and not the ellipsoid, are required in this case. Therefore, the deviation of the vertical is not required in this reduction. Equation (30) also assumes co-linearity of the measured distance d_1 and the zenith angle z_1 (Figure 4). If this is not the case, additional corrections apply (Rüeger, 1996). Brunner (1973) gives a graph depicting the accuracy of equation (30). From this, the maximum error in ($H_2 - H_1$) is less than 0.1 mm for slope distances less than 2.5 km and ellipsoidal height differences less than 1000 m.

The only parameter in equation (30) that relates to the GRS80 ellipsoid is the radius of curvature along the line (R_α). Any error $\delta(R_\alpha)$ in this propagates as follows into the computed orthometric height difference ($H_2 - H_1$):

$$\delta(H_2 - H_1) = -\left(\left(1 - \frac{k}{\sin z_1}\right)\left(\frac{d_4^2}{2\,R_\alpha^{\,2}}\right)\right)\ \delta(R_\alpha) \qquad (31)$$

where the approximation $(d_2 \sin z_1) \approx d_4$ has been assumed. It is evident that the error in the orthometric height difference depends on k, d_4 and $\sin z_1$, but not on the measured height difference. Table 14 gives some values of the error in ΔH due to the maximum error $\delta(R_\alpha)$

= 27,582 m (Tables 8 and 9). The error in ΔH is always zero for k = +1 and z = 90˚.

	d_4 [m] =	100	500	1000	2000
k = 0	z = 90˚	0.0	–0.1	–0.3	–1.4
k = –1	z = 90˚	0.0	–0.2	–0.7	–2.7
k = –1	z = 45˚	0.0	–0.2	–0.8	–3.3
k = –3	z = 90˚	0.0	–0.3	–1.4	–5.4
k = –3	z = 45˚	0.0	–0.4	–1.8	–7.1

Table 14. *Errors $\delta(\Delta H)$ (in millimetres) caused in the orthometric height difference ΔH due to the use of an azimuth-independent and Australia-wide ellipsoidal mean radius (R_M = 6,364,940 m) rather than R_α*

The coefficients of refraction (k) used in Table 14 may seem large. However, in short-range EDM, the prevailing coefficient of refraction will vary considerably according to weather, time of day, season and (smallest) ground clearance. For these 'grazing' EDM rays, which are often encountered in short-range EDM, k varies between k = –3.0 (midday) and k = +3.0 (midnight).

Table 14 does not show the errors for positive coefficients of refraction, because they are smaller than those for the negative values. The errors listed in Table 14 can be considered insignificant since the errors in ΔH due to the uncertainty of the coefficient of refraction are, even in the best case of simultaneous reciprocal zenith angle measurements, at least six times larger (Rüeger, 1996) than those caused by the use of an azimuth-independent and Australia-wide ellipsoidal mean radius. Therefore, the use of an Australia-wide mean radius (R_M = 6,364,940 m) for the GRS80 ellipsoid in equation (30) is entirely justified. In this context, it should be noted that the computation of AHD height differences from zenith angles and slope distances is restricted to a few hundred metres, if height differences at centimetre level or better are required (Rüeger, 1996).

Table 15 gives a numerical example of the computation of AHD height differences from measured zenith angles using equation (30). Importantly, no deviations of the vertical and no geoid-ellipsoid separations are required in this case. Again, for convenience, the radii of curvature of the ellipsoid are computed with the latitude of the instrument stations (rather than the midpoint of the line). Since the actual value of the coefficient of refraction k is usually unknown, the height differences are computed with k = 0.0 for lines close to the ground and with k = 0.13 for lines that are several metres above the ground (Rüeger 1996). The AHD height differences were also computed with the mean radius for Australia. They differ by only 0.1 mm from the results shown in Table 15. The difference between the (absolute) forward and the return height differences is due to the uncertainty of the k value used as well as the measuring uncertainty of the zenith angles z.

For interest only, the GRS80 ellipsoidal height differences (see Section 3.6.2) have also been computed, using the geoid-GRS80 ellipsoid separations and the absolute deviations of the vertical shown in Table 11. The GRS80 ellipsoidal height differences are shown at the bottom of Table 15. Because of the geoid slope between Stations 4 and 6 (refer to the N values in Table 11), the ellipsoidal height differences differ by 12 mm.

Parameter	*Unit*	ΔH 4–>6	ΔH 6–>4
Latitude GDA94	dec deg	-33.21874250	-33.22165528
Longitude GDA94	dec deg	151.1229361	151.1169625
Azimuth GDA94 ~	dec deg	239.879454	59.882807
N-S Radius ρ	m	6 354 580.7	6 354 583.7
E-W Radius ν	m	6 384 554.0	6 384 555.0
Radius in line R_α	m	6 376 979.4	6 376 982.4
slope distance d_2	m	644.9391	644.9391
meas zenith angle z	dec deg	93.391933	86.612714
H (AHD)	m	173.4470	135.3171
H.I.	m	0.239	0.236
H.R.	m	0.236	0.241
Coeff of Refract k		0.13	0.13
$H_2 - H_1$ (AHD)	m	–38.1271	+38.1294
Ellipsoidal Δh:			
$h_2 - h_1$ (GRS80)	m	–38.1391	38.1413

Table 15. *Sample computation of height differences from zenith angles and slope distances using eqs. (30) and (32)*

3.6.2 Ellipsoidal Height Differences for GPS

Height differences, that are compatible with GRS80 ellipsoidal heights, can also be computed from measured slope distances and zenith angles provided that the deviation of the vertical with respect to GRS80 is known. However, the terrestrial observation and computation of GRS80 ellipsoidal height differences are not standard practice, since AHD differences are required for work in Australia. Instead, GRS80 ellipsoidal height differences are generally measured using GPS, principally because this technique is more accurate. The ellipsoidal heights obtained by GPS are then converted to AHD using the geoid-GRS80-ellipsoid separations. Therefore, the formula below is only given for completeness and for those rare occasions when a GRS80 ellipsoidal height difference has to be computed that is compatible with the ellipsoidal heights obtained by GPS.

Ellipsoidal height differences ($h_2 - h_1$) are obtained using (Rüeger, 1996)

$$h_2 - h_1 = d_2 \cos\zeta_1 + \frac{\left(1 - \dfrac{k}{\sin\zeta_1}\right)}{2\,R_\alpha}\left(d_2 \sin\zeta_1\right)^2 + h_{TH(1)} - h_{T(2)} \qquad (32)$$

where all quantities have been defined earlier (also see Figure 4). It is clearly not possible to compute GRS80 ellipsoidal height differences from measured zenith angles without a knowledge of ε_S. This is because the observed zenith angle (z) must first be converted to a geodetic zenith angle (ζ) using equation (18). As for equation (30), equation (32) assumes collinearity of the measured distance and the measured zenith angle. If this is not the case, additional corrections apply (Rüeger, 1996).

To be able to apply equation (32), the deviations of the vertical in the azimuth of the line need to be known. As mentioned before, the only practical way to achieve this is to compute these from the AUSGeoid98 and thus neglect the curvature of the plumbline. The effect of errors in R_α on Δh is the same as those on ΔH (Section 3.6.1), and can be estimated from Table 14 and equation (31).

Any error in the deviation of the vertical, or its complete omission, naturally affects the derived GRS80 ellipsoidal height difference. This is evaluated using the total differential of equation (32) with respect to ε, which is expressed as

$$\begin{aligned}\delta(h_2 - h_1) &\approx -\ (d_2 \sin \zeta_1)\left(\frac{\delta(\varepsilon_1)}{206265}\right) \\ &\approx -d_4 \left(\frac{\delta(\varepsilon_1)}{206265}\right) \qquad (33)\end{aligned}$$

where $(d_2 \sin \zeta_1) \approx d_4$ has now been assumed and $\delta(\varepsilon_1)$ is in seconds of arc. The secondary influence of Δh and k on $\delta(h_2 - h_1)$ is sufficiently small to be ignored. Table 16 summarises the error $\delta(h_2 - h_1)$ in the computed GRS80 ellipsoidal height difference due to an error $\delta(\varepsilon_1)$ in the deviation of the vertical. The values of $\delta(\varepsilon_1)$ are the same as those used in Table 10 and have been selected to show the error inherent to all reductions as well as the effect of neglecting the deviation of the vertical. Again note that the curvature of the plumbline has been ignored in all these examples.

d_4 [m] =	100	300	600	1000	2000
$\delta(\varepsilon_1)$ = 2"	–1.0	–2.9	–5.8	–9.7	–19.4
$\delta(\varepsilon_1)$ = 7"	–3.4	–10.2	–20.4	–33.9	–67.9
$\delta(\varepsilon_1)$ = 50"	–24.2	–72.7	–145.4	–242.4	–484.8

Table 16. *Errors $\delta(\Delta h)$ (in mm) caused in the ellipsoidal height difference (Δh) due to the error $\delta(\varepsilon_1)$ in the deviation of the vertical*

4. CONCLUDING REMARKS

This paper has become necessary to remind surveyors of the definition and use of the deviation of the vertical in terrestrial surveying. The need to seriously consider the effects of the deviation of the vertical has come about because of the introduction of the GDA94. It appears that the GDA94 was introduced primarily to make life easier for GPS users. As a consequence, life will be more difficult for terrestrial surveys on the GDA94. The GRS80 ellipsoid associated with this new datum is no longer a best fit to the level surfaces and plumblines of the Earth's gravity field over Australia. Therefore, the associated (absolute) deviations of the vertical generally become larger and more significant in survey data reductions.

Fortunately, absolute deviations of the vertical at the geoid are readily and freely available for the whole Australian continent as part of the AUSGeoid98 model. An alternative source of vertical deviations is to create a local geoid model and calculate the deviation components from this. As reported by Dymock et al. (1999) and Chen et al. (1999), this might be required for countries or regions where no accurate geoid information is available. However, due to the availability of AUSGeoid98, we believe that there is no need for surveyors to derive their own (local) geoid and deflection models for areas in Australia. Instead, all that surveyors need to know is the GDA94 latitude and longitude to compute the N, ξ_G and η_G values for the AUSGeoid98 at a particular point. The State and Territory survey organisations and AUSLIG (1999), on their respective web-sites, provide information on how to compute GDA latitudes and longitudes from coordinates known in the datum previously used in Australia.

We believe that it is appropriate for surveyors to routinely apply corrections for the deviations of the vertical and geoid-GRS80 ellipsoid separations to terrestrial survey data for all work on the new GDA94. Some possible simplifications have been discussed here as have the errors resulting from non-compliance. The only exception is that height differences from zenith angles and slope distances should normally be computed with zenith angles, that have not been corrected for the deviations of the vertical, since surveyors normally require height differences that are compatible with AHD heights.

In the context of this paper, it has been assumed that heights on the Australian Height Datum are free of error. As Morgan (1992), Featherstone (1998) and others have pointed out, this is clearly not the case; the Australian Height Datum of 1971 contains levelling errors and uncorrected sea-surface-topography effects on the tide gauge data. It is suggested that the AHD be readjusted, with new data included and all known errors of old data corrected. As noted earlier, we also suggest that the available measured deviations of the vertical (at the surface) be compared with the predicted (from AUSGeoid98) deviations of the vertical at the geoid.

5. REFERENCES

AUSLIG (1999) *Geocentric Datum of Australia –Technical Manual*, Version 2.0, http://www.anzlic.org.au/icsm/gdatm/gdatm.htm, Australian Surveying and Land Information Group, Canberra, August 1999, 62 pages.

Bomford, A. (1967) The Geodetic Adjustment of Australia, 1963-66, *Survey Review*, 144:52-71.

Bomford, G. (1980) *Geodesy* (fourth edition), Oxford University Press, Oxford.

Brunner, F. K. (1973) The Effect of the Deviation of the Vertical in Trigonometric Levelling with Steep Sights. *Österreichische Zeitschrift für Vermessungswesen und Photogrammetrie*, 61: 126-134 (in German)

Carosio, A., Ebneter, F. (1998) Geodetic Methods, Mathematical Models and Quality Management for Underground Surveying in the Swiss Alptransit Project. In: Proc., Commission 5, 21st International Congress (FIG'98), International Federation of Surveyors, 19-25 July 1998, Brighton, UK, 6: 407-426.

Chen, J. Y., G. Chang, Y. L. Lee, Z. L. Zhen. (1999) An Improved Local Geoid in the Mt. Everest Area. *Zeitschrift für Vermessungswesen (ZfV)*, 124(11): 362-363.

Cooper, M. A. R. (1987) *Control Surveys in Civil Engineering*, Blackwell, Oxford.

Dymock, P., J. Hannah and C. Pearson (1999) The Second Manapouri Tailrace Tunnel: the Integration of GPS and Conventional Surveying in an Area of Significant Deviation of the Vertical, *The New Zealand Surveyor*, 289: 13-18.

Elmiger, A. (1977) Three-Dimensional Adjustment of Terrestrial Geodetic Networks. In: Proc., *XV International Congress of Surveyors (FIG)*, Stockholm, 5: 251-257 (in German).

Featherstone, W. E. (1995) On the use of Australian geodetic datums in gravity field determination, *Geomatics Research Australasia*, 62: 17-36.

Featherstone, W. E. (1996) An Updated Explanation of the Geocentric Datum of Australia and its Effects upon Future Mapping, *The Australian Surveyor*, 41(2): 121-130.

Featherstone, W. E. (1997) The Importance of Including the Geoid in Terrestrial Survey Data Reduction to the Geocentric Datum of Australia, *The Australian Surveyor*, 42(3): 45-50.

Featherstone, W. E. (1998) Do we need a Gravimetric Geoid or a Model of the Australian Height Datum to Transform GPS Heights in Australia?, *The Australian Surveyor*, 43(4): 273-279.

Friedlieb, O. J., W. E. Featherstone and M. C. Dentith (1997) A WGS84-AHD Profile over the Darling Fault, Western Australia, *Geomatics Research Australasia*, 67: 17-32.

Fryer, J. G. (1971) *The Geoid in Australia -1971*, Technical Report 13, Division of National Mapping, Canberra.

Heiskanen, W. H. and H. Moritz (1967) *Physical Geodesy*, Freeman, San Francisco, USA.

Jekeli, C. (1999) An Analysis of Vertical Deflections Derived from High-Degree Spherical Harmonic Models, *Journal of Geodesy*, 73(1): 10-22.

Johnston, G. M. and W. E. Featherstone (1998) AUSGeoid98 Computation and Validation: Exposing the Hidden Dimension, *39th Australian Surveyors Congress*, Launceston, November, 105-116.

Mather, R. S. (1970) The Geocentric Orientation Vector for the Australian Geodetic Datum, *Geophysical Journal of the Royal Astronomical Society*, 22: 55-81.

Morgan, P. (1992) An Analysis of the Australian Height Datum – 1971. *The Australian Surveyor*, 37(1): 46-63.

Moritz, H. (1980) Geodetic Reference System 1980, *Bulletin Géodésique*, 54: 395-405.

National Mapping Council (1986) *Australian Geodetic Datum Technical Manual*, National Mapping Council of Australia, Canberra.

Papp, G. and R. Benedik (2000) Numerical modelling of gravitational field lines – the effect of mass attraction on horizontal coordinates, *Journal of Geodesy*, 73(12): 648-659.

Rüeger, J.M. (1996) *Electronic Distance Measurement– An Introduction* (fourth edition), Springer, Berlin-Heidelberg-New York.

Torge, W. (1991) *Geodesy* (second edition), de Gruyter, Berlin-Heidelberg-New York.

Vaniček, P and E. W. Krakiwsky (1986) Geodesy the Concepts (second edition), Elsevier, Amsterdam.

Zanini, M., Stengele, R., Plazibat, M. (1993) Kreiselazimute in Tunnelnetzen unter dem Einfluss des Erdschwerefeldes (Gyro-Azimuths in Tunnel Networks under the Influence of the Earth's Gravity Field), Report No. 214, Institute of Geodesy and Photogrammetry, Swiss Federal Institute of Technology, Zurich, 71 pages.