Journal of Geodesy (1998) 72: 107–117

Journal of Geodesy

The Ionosphere-weighted GPS baseline precision in canonical form

P. J. G. Teunissen

Delft Geodetic Computing Centre (LGR), Faculty of Geodesy, Delft University of Technology, Thijsseweg 11, 2629 JA Delft, The Netherlands; e-mail: lgr@geo.tudelft.nl

Received: 6 January 1997 / Accepted: 19 September 1997

Abstract. In this contribution we consider the precision of the floated and the fixed GPS baseline for the case of weighted ionosphere. Canonical forms of the baseline variance matrices are developed for different measurement scenarios. These forms make the relation between the various variance matrices transparent and thus present a simple way of studying their relative merits. It is also shown how these canonical forms give an intrinsic description of the gain in baseline precision which is experienced when the carrier-phase double-differenced ambiguities are treated as integers instead of as reals. The dependence of this gain on the various contributing factors, such as the decorrelation of the ionosphere, is also included.

Key words. GPS · Baseline precision · Gain-numbers · Ionosphere

1 Introduction

Many GPS positioning applications in surveying, navigation and geodesy make use of the integer nature of the double-differenced (DD) carrier-phase ambiguities. A survey of these applications can be found in textbooks such as Borre (1995), Hofmann-Wellenhof et al. (1994), Kleusberg and Teunissen (1996), Leick (1995), Parkinson et al. (1996) or Seeber (1993). Inclusion of the integer nature of the DD ambiguities enables one to make use of the carrier phases as if they were very precise pseudo-ranges. The topic of resolving the integer values of the DD ambiguities has therefore been a rich source of GPS research. It resulted in a variety of different methods and proposals for efficiently estimating the integer ambiguities, see e.g. Counselman and Gourevitch (1981), Hatch (1982), Remondi (1986), Blewitt (1989), Wübbena (1989), Frei (1991), Goad (1992), Teunissen (1993). The LAMBDA method for fast integer ambiguity estimation was introduced in the latter reference; results of the method can be found in Teunissen (1994), Teunissen et al. (1994, 1995) and de Jonge and Tiberius (1996).

Motivated by the purpose of integer ambiguity fixing, which in the positioning context is to improve upon the precision of the baseline, the goal of the present contribution is to give a qualitative description of the gain in baseline precision. To give a direct and transparent way of comparing the baseline precision before and after ambiguity fixing, we develop canonical forms of the various baseline variance matrices.

In Sect. 2 we introduce the single-baseline model on which our analysis is based. In it we treat the ionospheric delays as random variables, thus providing a way of inferring the impact of the ionospheric spatial decorrelation. In Sect. 3 we consider the precision of the floated and fixed baseline. In order to show how the various factors of the functional and stochastic model contribute, we make use of a stepwise approach. It commences with the geometry-free model, followed by the time-averaged model and concludes with the multi-epoch geometry-based model. In the first step it is primarily the weights of the observations which contribute; in the second step it is the receiver-satellite geometry that enters, and in the third step the change over time of this geometry provides the additional contribution.

The gain in baseline precision is the topic of Sect. 4. We make use of the gain-number concept and show how the gain-numbers depend, in terms of 'geometry', on the change of the receiver-satellite configuration, and in terms of 'time', on the observation time-span and the sampling rate. Through the gain-number concept we are in a position to develop canonical forms of the various baseline variance matrices. These forms reveal the intrinsic structure of the gain in baseline precision and allow one to infer the impact of using different measurement scenarios.

2 The single-baseline model

In this section we will introduce the single-baseline model that forms the basis of our study. When tracking satellite s at epoch i using two receivers u and v, the single-difference (SD) observation equations for L_1 single-frequency phase and code read (Hofmann-Wellenhof et al. 1994, Leick 1995; Kleusberg and Teunissen 1996):

$$\begin{aligned} \phi^s_{uv}(i) &= dt_{uv}(i) + \delta_{uv}(i) + \rho^s_{uv}(i) - \mu_1 I^s_{uv}(i) \\ &\quad + T^s_{uv}(i) + \delta m^s_{uv}(i) + \phi_{uv}(0) + \lambda_1 N^s_{uv} + n^s_{\phi,uv}(i) \\ p^s_{uv}(i) &= dt_{uv}(i) + d_{uv}(i) + \rho^s_{uv}(i) + \mu_1 I^s_{uv}(i) + T^s_{uv}(i) \\ &\quad + dm^s_{uv}(i) + n^s_{p,uv}(i) \end{aligned} \tag{1}$$

where $\phi^s_{uv}(i)$ is the SD phase observable expressed in units of range rather than cycles, $p^s_{uv}(i)$ is the SD code observable, $dt_{uv}(i)$ is the unknown relative receiver clock error, $\delta_{uv}(i)$ and $d_{uv}(i)$ are the carrier-phase and code equipment delays, $\rho^s_{uv}(i)$ is the unknown SD receiver-satellite range, $I^s_{uv}(i)$ and $T^s_{uv}(i)$ are the ionospheric and tropospheric delays, $\delta m^s_{uv}(i)$ and $dm^s_{uv}(i)$ are phase and code multipath terms, $\phi_{uv}(0)$ is the relative receiver non-zero initial phase offset, N^s_{uv} is the integer carrier-phase ambiguity, λ_1 is the wavelength of L_1 and $\mu_1 = \frac{\lambda_1}{\lambda_2}$ the wavelength ratio of L_1 and L_2, and $n^s_{\phi,uv}(i)$ and $n^s_{p,uv}(i)$ are the noises of phase and code, respectively.

In the following we will assume that multipath is absent and that the tropospheric delays can either be corrected for using an a priori model or that they are sufficiently small to be neglected. The unknown clock error, the equipment delays and the non-zero initial phase offset, can be eliminated by using double-differences instead of single-differences. When using satellites s and r, the L_1 DD observation equations follow as

$$\begin{aligned} \phi'^{rs}_{uv}(i) &= \phi^{rs}_{uv}(i) + \mu_1 I^{rs}_{uv}(i) = \rho^{rs}_{uv}(i) + \lambda_1 N^{rs}_{uv} \\ p'^{rs}_{uv}(i) &= p^{rs}_{uv}(i) - \mu_1 I^{rs}_{uv}(i) = \rho^{rs}_{uv}(i) \end{aligned} \tag{2}$$

where the noise terms have been left out for notational convenience. In case of dual-frequency data, a similar pair of equations is available for L_2. Thus for each pair of satellites, a total of four DD equations can be formed when dual-frequency phase and code data are used. In the following we will assume that a total of m satellites are tracked.

After a linearization of $\rho^{rs}_{uv}(i)$ with respect to the stationary baseline components of the two receivers, the $4(m-1)$ linear(ized) DD observation equations can conveniently be written in a compact vector-matrix form as

$$\begin{aligned} (I_2 \otimes D^T)\Phi'(i) &= (e_2 \otimes D^T A(i))b + (\Lambda \otimes I_{m-1})a \\ (I_2 \otimes D^T)P'(i) &= (e_2 \otimes D^T A(i))b \end{aligned} \tag{3}$$

with '$\otimes$' denoting the Kronecker product (Rao 1973) and where $i = 1, \ldots, k$ denotes the epoch number and k equals the total number of epochs. The unit matrix of order n will be denoted I_n and the n-vector having all 1's as its entries will be denoted as e_n.

The two $2m$-vectors $\Phi'(i)$ and $P'(i)$ contain the ionosphere-corrected SD phase and code observables on the two frequencies. The $(m-1) \times m$ matrix D^T is the DD matrix operator that transforms single differences into double differences. There are of course many different ways in which double differences can be formed. Each of the m satellites, for instance, can be taken as a reference satellite. This already gives m different ways of forming double differences. The DD transformation that takes the first satellite as reference reads $[-e_{m-1}, I_{m-1}]$, while the DD transformation that takes the last satellite as reference reads $[I_{m-1}, -e_{m-1}]$. Although the DD transformation itself is not unique, the adjustment results will be unique if proper care is taken of the correlation introduced by the DD process. We can therefore take any one of the admissible DD transformations to form DD observables.

The relative receiver-satellite geometry at epoch i is captured in the $m \times 3$ SD design matrix $A(i)$. The unknown parameters of the model are collected in the two vectors a and b. The $2(m-1)$-vector a contains the unknown *integer* DD ambiguities of the L_1 and L_2 phase data. The unknown increments of the three-dimensional baseline are collected in the 3-vector b. The known wavelengths of the two frequencies are collected in the diagonal matrix $\Lambda = \text{diag}(\lambda_1, \lambda_2)$.

The SD observables of the preceding system of observation equations are collected in the $2m$-vectors

$$\begin{aligned} \Phi'(i) &= \Phi(i) + \mu \otimes I(i) \\ P'(i) &= P(i) - \mu \otimes I(i) \end{aligned} \tag{4}$$

with $\Phi(i) = (\Phi_1(i)^T, \Phi_2(i)^T)^T$ and $P(i) = (P_1(i)^T, P_2(i)^T)^T$. The SD phase data of epoch i on L_1, respectively L_2 are collected in the m-vectors $\Phi_1(i)$ and $\Phi_2(i)$, and the SD code data of epoch i on L_1, respectively L_2, are collected in the m-vectors $P_1(i)$ and $P_2(i)$. The SD ionospheric delays of epoch i are collected in the m-vector $I(i)$. The 2-vector $\mu = (\mu_1, \mu_2)^T$ contains the wavelength ratios $\mu_1 = \frac{\lambda_1}{\lambda_2}$ and $\mu_2 = \frac{\lambda_2}{\lambda_1}$.

The variance matrix of the observables $(\Phi'(i)^T, P'(i)^T)^T$ is assumed to be given as

$$\begin{bmatrix} (Q_\phi + \sigma_I^2 \mu\mu^T) \otimes D^T D & -(\sigma_I^2 \mu\mu^T) \otimes D^T D \\ -(\sigma_I^2 \mu\mu^T) \otimes D^T D & (Q_p + \sigma_I^2 \mu\mu^T) \otimes D^T D \end{bmatrix} \tag{5}$$

Through the two variance matrices Q_ϕ and Q_p we allow the variances of the GPS observables on the two frequencies to differ. For some GPS receivers, these matrices can be assumed to be diagonal. For others however, this is not the case. Depending on how the measurement process is implemented in the GPS receivers, the observables may or may not be cross-correlated. In the presence of anti-spoofing (AS), for instance, some receivers use a hybrid technique to provide dual-frequency code measurements (Hofmann-Wellenhof et al. 1994). As a result, the code data become cross-correlated. For the main results obtained in this contribution, we therefore allow the presence of cross-correlation and assume the two variance matrices to be non-diagonal.

Note that the ionospheric delays are modelled as random variables. The use of an a priori weighted ionosphere has been discussed in, e.g., Wild and Beutler (1991), Schaer (1994) and Bock (1996). The sample values of the ionospheric delays can be taken from an externally provided ionospheric model, see e.g. Georgiadou (1994), Wild (1994), Wanninger (1995). In some applications it even suffices to take zero as sample value. The a priori uncertainty in the ionospheric delays is modelled through its variance σ_I^2. The value of σ_I^2 depends to a large extent on the interstation distance between the two receivers. Since the ionosphere decorrelates as a function of the interstation distance, σ_I^2 is at its maximum for baselines where the ionosphere is fully decorrelated, and it gets smaller the shorter the baseline becomes. For sufficiently short baselines it can be taken equal to zero. A proposal on how to describe σ_I^2 as function of the interstation distance can be found in Bock (1996).

The goal of the present contribution is to obtain a canonical description of the baseline variance matrices both before and after ambiguity fixing. These canonical forms will then reveal in a rather simple way the gain that is experienced in the baseline precision due to ambiguity fixing. At the same time they also allow one to study the various contributing factors of both the functional model and stochastic model. The least-squares solution of the baseline, when the ambiguities are treated as real-valued unknowns, is denoted $\hat{b}$. It is usually referred to as the *float* solution. The least-squares solution of the baseline, when the integer ambiguities are assumed known, is denoted as $\check{b}$. It is usually referred to as the *fixed* solution. When we denote the least-squares ambiguity vector as $\hat{a}$ and the corresponding *integer* least-squares ambiguity vector as $\check{a}$, the relation between the fixed and floated baseline solution can be expressed as (Teunissen 1995)

$$\check{b} = \hat{b} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1}(\hat{a} - \check{a}) \tag{6}$$

with $Q_{\hat{a}}$ being the variance matrix of $\hat{a}$ and $Q_{\hat{b}\hat{a}}$ being the covariance matrix between $\hat{b}$ and $\hat{a}$. This relation shows how the ambiguity residual vector $(\hat{a} - \check{a})$, being the difference between the real-valued least-squares ambiguity vector $\hat{a}$ and the integer-valued least-squares ambiguity vector $\check{a}$, is used to obtain the fixed-baseline solution $\check{b}$ from the float solution $\hat{b}$. If we assume the integer ambiguities to be known and non-stochastic, an application of the error propagation law gives

$$Q_{\check{b}} = Q_{\hat{b}} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} Q_{\hat{a}\hat{b}} \tag{7}$$

This shows that $Q_{\check{b}} < Q_{\hat{b}}$ and thus that the fixed baseline is of a better precision than its floated counterpart. This is in the present context also the whole purpose of ambiguity fixing. Once the ambiguities are fixed, the phase data start to act as if they were very precise code data, thus allowing one to solve for the baseline with a comparable high precision.

In order to construct the canonical forms of the baseline variance matrices, such that a deeper insight is obtained in the various contributing factors, our approach for solving the preceding model will be a step wise one. Note that the vector $r(i) = D^T A(i) b(i)$, with $b(i)$ being the baseline vector at epoch i, contains the DD ranges of epoch i. Depending on whether the observation equations given in Eq. (3) are parametrized in terms of $r(i)$, $b(i)$ or b, one can now discriminate between three different models. The receiver-satellite geometry as captured by the SD design matrix $A(i)$, gets eliminated if one chooses to parametrize the model in terms of the DD range vector $r(i)$. As a result one obtains the geometry-free model. It allows one to use the code data almost directly to determine the unknown integer ambiguities. The instantaneous receiver-satellite geometry enters when one uses, instead of a parametrization in terms of the DD ranges, the baseline parametrization $r(i) = D^T A(i) b(i)$. This is the model used in kinematic applications where instantaneous positioning is required. Finally, if the baseline can be assumed to be stationary, it is the parametrization $r(i) = D^T A(i) b$ which is used. And in this case it is the time-invariance of the baseline which allows the change over time of the receiver-satelite geometry to enter the model.

It will be clear that the 'strength' of the model increases when one changes the parametrization from $r(i)$ to $b(i)$ and then finally to b. In the first case it is primarily the weights of the observations that contribute to the precision of the solution. In the second case, one will have the additional contribution of the receiver-satellite geometry, provided there is satellite redundancy $(m > 4)$. And in the third case, again an important contributing factor enters, namely the change over time of the receiver-satellite geometry. One can therefore expect that each of these three contributing factors will be recognizable in the baseline solution of the model of Eq. (3).

3 The baseline precision before and after ambiguity fixing

In this section we will construct the baseline precision using the following three steps. First we will consider its counterpart in the geometry-free model, being the precision of the DD ranges. Then we will include the relative receiver-satellite geometry, for which the time-averaged geometry is taken as being representative. Then finally, the change over time of the receiver-satellite geometry is included.

3.1 The precision of the DD ranges

When we strip the model of Eq. (3) from the presence of the receiver-satellite geometry, we obtain the geometry-free model. Its two pairs of scalar DD observation equations for epoch i read

$$\begin{aligned} \phi'(i) &= (e_2 \otimes 1)\rho(i) + (\Lambda \otimes 1)N \\ p'(i) &= (e_2 \otimes 1)\rho(i) \end{aligned} \tag{8}$$

with the 2-vectors $\phi'(i)$ and $p'(i)$ having as their two entries the L_1 and corresponding L_2 component of $(I_2 \otimes D^T)\Phi'(i)$ and $(I_2 \otimes D^T)P'(i)$. The DD range of epoch i is denoted as $\rho(i)$ and the two DD ambiguities are collected in the integer vector N.

The geometry-free model is linear from the outset, and since the DD ranges $\rho(i)$ are not connected in time, it allows the two GPS receivers to be either stationary or moving. Due to the absence of the receiver-satellite geometry, the geometry-free model also allows the simplest approach to integer ambiguity estimation. Examples of such approaches can be found in, e.g., Hatch (1982), Euler and Goad (1990) and Teunissen (1996). For the purpose of this contribution, however, we are not so much interested in the integer ambiguity fixing process itself, but more in what it allows us to achieve in terms of the precision with which the DD ranges can be estimated. The following theorem gives an exact expression for the variance of the fixed DD range $\check{\rho}(\phi, p)$.

Theorem 1 (*single-epoch DD range precision*) The phase-and-code-based variance of the ambiguity-fixed, single-epoch least-squares estimator of the DD range, is given as

$$\sigma^2_{\check{\rho}}(\phi,p) = \frac{2}{e_2^T(Q_\phi^{-1}+Q_p^{-1})e_2} \cdot \frac{1+\sigma_I^2\mu^T(Q_\phi^{-1}+Q_p^{-1})\mu}{1+\sigma_I^2\left[\frac{(\mu^T d_2)^2}{d_2^T(Q_\phi^{-1}+Q_p^{-1})^{-1}d_2}+4\frac{(e_2^T Q_\phi^{-1}\mu)(\mu^T Q_p^{-1}e_2)}{e_2^T(Q_\phi^{-1}+Q_p^{-1})e_2}\right]} \quad (9)$$

with $e_2 = (1,1)^T$ and its orthogonal complement $d_2 = (1,-1)^T$.

Proof. See Appendix. □

Note, since Q_ϕ and Q_p are variance matrices of SD observables, the factor '2' in the preceding expression is due to the double differencing. Also note that we have added the argument (ϕ, p) to the variance of the DD range so as to show explicitly that it is based on using both phase and code data. The code-only variance $\sigma^2_{\check{\rho}}(p)$ and the phase-only variance $\sigma^2_{\check{\rho}}(\phi)$ follow by setting Q_ϕ and Q_p, respectively, equal to infinity.

To determine the impact of the ionospheric uncertainty on the range variance, we consider the two variance ratios $\sigma^2_{\check{\rho}}(\phi)/\sigma^2_{\check{\rho}|I}(\phi)$ and $\sigma^2_{\check{\rho}}(p)/\sigma^2_{\check{\rho}|I}(p)$, where the conditioning on the ionosphere implies that $\sigma_I^2 = 0$. The two variance ratios are both monotone increasing functions of σ_I^2. They attain their minimum of 1 when $\sigma_I^2 = 0$ and their maximum when $\sigma_I^2 = \infty$. For $Q_\phi = \sigma_\phi^2 I_2$ and $Q_p = \sigma_p^2 I_2$, this maximum equals

$$\max_{\sigma_I^2}\frac{\sigma^2_{\check{\rho}}(\phi)}{\sigma^2_{\check{\rho}|I}(\phi)} = \max_{\sigma_I^2}\frac{\sigma^2_{\check{\rho}}(p)}{\sigma^2_{\check{\rho}|I}(p)} = 2\frac{\mu_1^2+\mu_2^2}{(\mu_1-\mu_2)^2} \approx 17.7 \quad (10)$$

Note that this value also equals the factor by which the SD phase variance needs to be multiplied in order to get the single-epoch variance of the DD ionosphere-free linear phase combination.

3.2 Enter the receiver-satellite geometry

Following the geometry-free based, single-epoch DD range solution just given, we will now strengthen the model by including a single receiver-satellite configuration. As a reference configuration, we choose the time-averaged receiver-satellite geometry. This choice is motivated by the fact that the time-averaged geometry can be considered representative of the individual geometries as captured by the SD design matrices $A(i), i = 1,\ldots,k$. Moreover, the change over time of the receiver-satellite geometry, which will be considered in the next subsection, also has the time-averaged geometry as its reference. The DD observation equations of the time-averaged model are given as

$$\begin{aligned}(I_2\otimes D^T)\bar{\Phi}' &= (e_2\otimes D^T\bar{A})b + (\Lambda\otimes I_{m-1})a\\ (I_2\otimes D^T)\bar{P}' &= (e_2\otimes D^T\bar{A})b\end{aligned} \quad (11)$$

with $\bar{\Phi}' = \frac{1}{k}\sum_{i=1}^k \Phi'(i)$, $\bar{P}' = \frac{1}{k}\sum_{i=1}^k P'(i)$ and $\bar{A} = \frac{1}{k}\sum_{i=1}^k A(i)$. For $k = 1$, the model reduces to that used for instantaneous positioning. The following theorem gives the baseline variance matrices of the time-averaged model.

Theorem 2 (*time-averaged baseline precision*) The variance matrices of the time-averaged least-squares solutions of the floated, respectively fixed, baseline, are given as

$$\begin{aligned}&(i) \quad Q_{\hat{b}}(\bar{p}) = \sigma^2_{\check{\rho}}(p)[2k\bar{A}^T P\bar{A}]^{-1} = Q_{\hat{b}}(\bar{\phi},\bar{p})\\ &(ii) \quad Q_{\check{b}}(\bar{\phi}) = \frac{\sigma^2_{\check{\rho}}(\phi)}{\sigma^2_{\check{\rho}}(p)}Q_{\hat{b}}(\bar{p})\\ &(iii) \quad Q_{\check{b}}(\bar{\phi},\bar{p}) = \frac{\sigma^2_{\check{\rho}}(\phi,p)}{\sigma^2_{\check{\rho}}(p)}Q_{\hat{b}}(\bar{p})\end{aligned} \quad (12)$$

with the orthogonal projector $P = D(D^TD)^{-1}D^T = I_{m-1} - e_{m-1}(e_{m-1}^T e_{m-1})^{-1}e_{m-1}^T$.

Proof: See Appendix. □

Note that a phase-only, float-baseline solution does not exist. This is due to the presence of the ambiguities. Their presence makes it impossible to solve for the baseline when code data are absent and when only a single receiver-satellite geometry is used. Hence, in the float situation the phase data do not contribute and the solution depends entirely on the code data. Therefore $Q_{\hat{b}}(\bar{p}) = Q_{\hat{b}}(\bar{\phi},\bar{p})$.

Also note that all variance matrices of the theorem are scaled versions of each other. The scale factors are determined by the range variances of Theorem 1. Hence, the impact of, respectively, the observational weights, the cross-correlation and the weighting of the ionosphere, all come together in one scale factor. The correlation structure of the baseline variance matrices is not affected by it. It is determined by the receiver-satellite geometry through $\bar{A}$.

If we assume $Q_\phi = \sigma_\phi^2 I_2$ and $Q_p = \sigma_p^2 I_2$, then $\sigma_{\hat{\rho}}^2(p) = \sigma_p^2$ for $\sigma_I^2 = 0$ and $\sigma_{\hat{\rho}}^2(p) \approx 17.7\sigma_p^2$ for $\sigma_I^2 = \infty$. Thus the precision of the floated baseline is dominated by the relatively poor precision of the code data. Furthermore, the baseline standard deviations get enlarged by a factor of about 4.2 when one considers the ionospheric delays to be completely unknown and compares it to the short-baseline case for which $\sigma_I^2 = 0$ holds true.

To infer the improvement obtained when the ambiguities are fixed, we first compare the phase-only fixed baseline with the float solution, and therefore consider the variance ratio $\sigma_{\check{\rho}}^2(\phi)/\sigma_{\hat{\rho}}^2(p)$. For both the extreme cases $\sigma_I^2 = 0$ and $\sigma_I^2 = \infty$, it equals the phase-code variance ratio $\epsilon = \sigma_\phi^2/\sigma_p^2$. Since the phase data are so much more precise than the code data ($\epsilon \approx 10^{-4}$), this is a dramatic improvement over the float solution. And this is of course also what ambiguity fixing is all about; namely to be able to use the very precise phase data as if they were code data. The relative improvement in baseline precision is somewhat poorer when one considers the maximum of $\sigma_{\check{\rho}}^2(\phi)/\sigma_{\hat{\rho}}^2(p)$. It reads

$$\max_{\sigma_I^2} \frac{\sigma_{\check{\rho}}^2(\phi)}{\sigma_{\hat{\rho}}^2(p)} = \left[1 + \frac{(\alpha - 1)(1 - \epsilon)}{(\sqrt{\alpha\epsilon} + 1)^2}\right]\epsilon \approx 17.7\epsilon$$

with α given by Eq. (10). This maximum is reached for

$$\sigma_I^2 = \frac{\sigma_\phi \sigma_p}{\sqrt{\frac{1}{2}(\mu_1^2 + \mu_2^2)(\mu_1 - \mu_2)^2}} \approx 1.87\sigma_\phi\sigma_p$$

The gain in baseline precision, when code data are included as well in the fixed solution, is described by the variance ratio $\sigma_{\check{\rho}}^2(\phi, p)/\sigma_{\hat{\rho}}^2(p)$. Since the code data are so much less precise than the phase data, one cannot expect that the solution will be much better than the phase-only fixed solution. And indeed, we have

$$\frac{\sigma_{\check{\rho}}^2(\phi, p)}{\sigma_{\hat{\rho}}^2(p)} = \begin{cases} \frac{\epsilon}{1+\epsilon} \approx \epsilon \text{ for } \sigma_I^2 = 0 \\ \frac{\epsilon}{1+\epsilon}\left[1 + 4\frac{\epsilon}{(1+\epsilon)^2}\frac{(\mu_1+\mu_2)^2}{(\mu_1-\mu_2)^2}\right]^{-1} \approx \epsilon \text{ for } \sigma_I^2 = \infty \end{cases}$$

showing that it is still the very small phase-code variance ratio ϵ that dominates the large gain in baseline precision due to ambiguity fixing.

3.3 Enter the receiver-satellite geometry's change over time

Following the geometry-free-based solution and the single-satellite-configuration-based solution of the previous two subsections, we will now allow the receiver-satellite geometry to change over time as well. As the following theorem shows, each of the baseline variance matrices can now be written as an expansion of the variance matrices of Theorem 2.

Theorem 3 (*float- and fixed-baseline precision*) The float- and fixed-baseline variance matrices based on phase-only, code-only and phase and code data, are given as

$$(i) \quad Q_{\hat{b}}(\phi) = \sigma_{\check{\rho}}^2(\phi) \times \left[2\sum_{i=1}^{k}(A(i) - \bar{A})^T P(A(i) - \bar{A})\right]^{-1}$$

$$(ii) \quad Q_{\check{b}}(\phi) = \left[Q_{\check{b}}^{-1}(\bar{\phi}) + Q_{\hat{b}}^{-1}(\phi)\right]^{-1}$$

$$(iii) \quad Q_{\hat{b}}(p) = \left[Q_{\hat{b}}^{-1}(\bar{p}) + \frac{\sigma_{\check{\rho}}^2(\phi)}{\sigma_{\hat{\rho}}^2(p)} Q_{\hat{b}}^{-1}(\phi)\right]^{-1}$$

$$(iv) \quad Q_{\hat{b}}(\phi, p) = \left[Q_{\hat{b}}^{-1}(\bar{\phi}, \bar{p}) + \frac{\sigma_{\check{\rho}}^2(\phi)}{\sigma_{\check{\rho}}^2(\phi, p)} Q_{\hat{b}}^{-1}(\phi)\right]^{-1}$$

$$(v) \quad Q_{\check{b}}(\phi, p) = \left[Q_{\check{b}}^{-1}(\bar{\phi}, \bar{p}) + \frac{\sigma_{\check{\rho}}^2(\phi)}{\sigma_{\check{\rho}}^2(\phi, p)} Q_{\hat{b}}^{-1}(\phi)\right]^{-1} \quad (13)$$

Proof. See Appendix. □

In these expressions of the baseline variance matrices, we clearly recognize the following three contributing factors: the variance of the DD range, the time-averaged geometry and the change of the receiver-satellite geometry over time. In contrast with the variance matrices of Theorem 2, now not all of the variance matrices are scaled versions of each other. Instead, their inverse can be written as the inverse of their time-averaged counterpart plus a scaled version of the inverse of $Q_{\hat{b}}(\phi)$. It is in this last matrix where the change over time of the receiver-satellite geometry enters.

Note that the float, phase-only solution only exists when there is a change in the receiver-satellite geometry. Hence in the float case, no instantaneous phase-only solution exists. In the case of GPS, it is well known that the receiver-satellite geometry changes rather slowly over time due to the high-altitude orbits of the satellites. This implies for short observation time spans that $A(i) \approx \bar{A}$. This would make the matrix $\sum_{i=1}^{k}(A(i) - \bar{A})^T P(A(i) - \bar{A})$ near rank defect, thus posing potential problems in the inversion process. This does not automatically imply, however, that the baseline is poorly estimable. There is still the variance factor $\sigma_{\check{\rho}}^2(\phi)$, which is very small due to the very high precision of the phase data. We thus see here that data precision competes with the change in time of the satellite geometry. But of course, one can bring the matrix $\sum_{i=1}^{k}(A(i) - \bar{A})^T P(A(i) - \bar{A})$ arbitrarily close to a rank defect matrix by a sufficient shortening of the observation time-span, in which case the baseline will then indeed have a poor estimability.

It will be clear that the variance matrices of Theorem 3 can be ranked according to

$$Q_{\check{b}}(\phi, p) \leq Q_{\hat{b}}(\phi, p) \leq Q_{\hat{b}}(p) \text{ and } Q_{\check{b}}(\phi) \leq Q_{\hat{b}}(\phi)$$

How close these matrices are to one another or how much they differ, depends on the observational weights, on the uncertainty in the ionospheric delays and on the receiver-satellite geometry and its change over time. This also holds true when answering the question whether $Q_{\hat{b}}(p) \leq Q_{\hat{b}}(\phi)$ or $Q_{\hat{b}}(\phi) \leq Q_{\hat{b}}(p)$. The canonical decompositions of the baseline variance matrices that will be developed in the next section will make the relation among these matrices transparent, thus allowing one to study their relative merits.

4 The gain in baseline precision

The sole purpose of ambiguity fixing is to be able, via the inclusion of the integer constraints on the ambiguities, to improve upon the precision of the baseline. The impact of the integer constraints on the ambiguities therefore manifests itself in the change in baseline precision, when going from the 'float' situation to the 'fixed' situation. This gain in baseline precision is the topic of the present section. A qualitative description of it will be developed, based on the canonical decompositions of the baseline variance matrices. Through these decompositions the intrinsic structure of the baseline variance matrices is revealed.

4.1 Gain-numbers and the change in receiver-satellite geometry

In this subsection the following three issues are addressed. Following a definition of the gain in baseline precision due to ambiguity fixing, we show how the gain-numbers provide a measure for the change in receiver-satellite geometry and how they are related to the argument of time itself.

Gain-numbers measure the gain in baseline precision due to ambiguity fixing. The corresponding gain-vectors describe the direction in which the gain in baseline precision is experienced. Recall that $\hat{b}$ and $\check{b}$ are the least-squares estimates of the baseline, before and after ambiguity fixing, respectively. The corresponding least-squares estimates of the baseline component $f^T b$, then read: $\hat{\theta} = f^T \hat{b}$ and $\check{\theta} = f^T \check{b}$. Since the variance ratio $\sigma^2_{\hat{\theta}}/\sigma^2_{\check{\theta}}$ measures the improvement in precision when replacing the estimate $\hat{\theta}$ by $\check{\theta}$, the following definition for the gain-numbers and the gain-vectors is used.

Definition *Gain-numbers* are given as the ratio

$$\gamma(f) = \frac{f^T Q_{\hat{b}}(\phi) f}{f^T Q_{\check{b}}(\phi) f} \tag{14}$$

and the corresponding vectors $f \in R^3$ are called *gain-vectors*.

Note that we have defined the gain-numbers for the phase-only case. It will be clear that we could have equally well defined the gain-numbers for the case that phase *and* code data are used. In that case, the variance matrices in Eq. (14) would have to be replaced by $Q_{\hat{b}}(\phi, p)$ and $Q_{\check{b}}(\phi, p)$, respectively. The reason for using the definition as given, though, is that it results in gain-numbers that are *independent* of the observation weights used. Hence, the gain-number $\gamma(f)$ becomes solely dependent on the receiver-satellite geometry. This will therefore allow us in our further analysis clearly to separate the impact of the observation weights and the uncertainty in the ionospheric delays from the impact of the receiver-satellite geometry. Once the canonical decompositions of the baseline variance matrices are available, these different effects can then be taken together to study the gain in baseline precision for the case that both phase and code data are used. In addition, the definition allows us to make easy and economical use of some of the results of Teunissen (1997).

Note that, since the gain-numbers are defined as ratios, they are dimensionless. Moreover, the gain-numbers are invariant to a reparametrization of the baseline. Hence, one will obtain identical gain-numbers when using, for instance, a local {north, east, up}-frame or a global geocentric frame.

It will be intuitively clear that the gain-numbers must be larger than or equal to 1, since the whole purpose of ambiguity fixing is to improve upon the baseline precision. Hence the gain-numbers lie in the interval $\gamma(f) \in [1, \infty)$. It may happen that the ambiguity fixing fails to have an effect on the precision of *some* of the baseline components. In that case one or more of the gain-numbers is equal to 1. This occurs, for instance, when the time-averaged DD design matrix $D^T \bar{A}$ is not of full rank. Hence, in that case there exists a configuration defect in the time-averaged receiver-satellite geometry. Also note that then $Q_{\hat{b}}(\bar{p})$ and $Q_{\hat{b}}(\bar{\phi})$ will fail to exist.

The stationary values of the ratio in Eq. (14) correspond to the roots of the characteristic equation $| Q_{\hat{b}}(\phi) - \gamma Q_{\check{b}}(\phi) | = 0$. The roots of this characteristic equation, denoted as $\gamma_1 \leq \gamma_2 \leq \gamma_3$, will from now on simply be referred to as *the* gain-numbers.

It will be clear that in some way, the actual receiver-satellite configurations over the observation time-span must be instrumental in determining the values of the gain-numbers. It is therefore to be expected that the gain-numbers are closely linked to these receiver-satellite configurations. The following theorem, which gives a *geometric* interpretation of the gain-numbers, makes this dependency precise.

Theorem 4 (*gain-numbers and principal angles*) Let $R(U), R(V) \subset R^{mk}$ be the range spaces spanned by the columns of the two matrices

$$U = \begin{bmatrix} P(A(1) - \bar{A}) \\ \vdots \\ P(A(k) - \bar{A}) \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} PA(1) \\ \vdots \\ PA(k) \end{bmatrix}$$

Then

$$\gamma_i = \frac{1}{\cos^2 \theta_i}, \quad i = 1, 2, 3 \tag{15}$$

where θ_i, $i = 1, 2, 3$, are the *principal angles* between the subspaces $R(U)$ and $R(V)$, with $0 \leq \theta_1 \leq \theta_2 \leq \theta_3 \leq \pi/2$.

Proof. See Teunissen (1997). □

Note that the matrices U and V can be interpreted as being the design matrices of *before* and *after* ambiguity fixing. Matrix U corresponds then to the design matrix before ambiguity fixing, but after the ambiguity parameters have been eliminated, and matrix V to the corresponding design matrix after ambiguity fixing. The theorem therefore shows that the gain-numbers are a direct measure for the amount of *obliquity* between the range spaces of the two design matrices U and V.

The geometric interpretation of the gain-numbers can be completed by introducing the orthogonal complement of the range space of U, within the space spanned by the columns of U and V. Let $P_U^{\perp}$ be the orthogonal projector that projects onto the orthogonal complement of the range space of U. Then the range space of $P_U^{\perp}V$ is the required orthogonal complement. From $W = P_U^{\perp}V$ with $W = [(P\bar{A})^T, \ldots, (P\bar{A})^T]^T$, it follows that this orthogonal complement is given by the range space of matrix W. Note that this matrix corresponds to the design matrix of the time-averaged model.

It follows from Theorem 4 that if the principal angles θ_i are close to zero, then the gain-numbers are close to their minimum value of 1 and $R(V)$ is close to being orthogonal to $R(W)$. On the other hand, if the principal angles are large, then the experienced gain in baseline precision is large and $R(V)$ is close to being coincident with $R(W)$, in which case there is almost no difference between the model that includes the change over time of the receiver-satellite geometry and the one that excludes it. In the intermediate case, when the principal angles are close to $\pi/4$, then the gain-numbers are close to 2 and the two variance matrices $Q_{\hat{b}}(\phi)$ and $Q_{\hat{b}}(\bar{\phi})$ are almost identical.

Apart from this geometric point of view, we can also relate the gain-numbers to the argument of time itself. Here we will distinguish between the *sampling rate* and the *observation time-span*. In order to study the effect of the sampling rate, one should consider a constant observation time-span. Thus, in this case, k varies, while $(k-1)T$ stays constant, where T denotes the time-interval between two consecutive samples. However, in order to study the effect of the observation time-span, one should consider a constant sampling rate. Hence in this case, $(k-1)T$ varies, while k stays constant. With this in mind, the following theorem shows how 'time' affects the gain-numbers.

Theorem 5 (*gain, sampling rate and time-span*) Let $A = [a_1(t_c), \ldots, a_m(t_c)]^T$ and $\dot{A} = [\dot{a}_1(t_c), \ldots, \dot{a}_m(t_c)]^T$ with mid-point $t_c = \frac{1}{k}\sum_{i=1}^{k} t_i$ and $t_i = t_o + iT$, and let $v_i, i = 1,2,3$, be the eigenvalues in ascending order of $| A^T PA - v\dot{A}^T P\dot{A} | = 0$. Then to a first-order approximation

$$\gamma_i - 1 = \frac{12 v_i}{T^2(k^2-1)} \quad , \quad i = 1,2,3 \tag{16}$$

from which it follows that

$$\begin{cases} \gamma_i - 1 = \tan^2(\theta_i) \sim \frac{k-1}{k+1} \text{ (sampling rate)} \\ \gamma_i - 1 = \tan^2(\theta_i) \sim 1/[(k-1)T]^2 \text{ (time-span)} \end{cases}$$

Proof. See Teunissen (1997). □

The theorem shows that the sampling rate and the observation time-span have an opposite effect on the gain-numbers. The gain-numbers get larger as the sampling rate gets larger, but they get smaller as the observation time-span gets larger. In the latter case they approximately follow an *inverse-square law* in the observation time-span. The maximum factor by which the gain-numbers can be enlarged through an increase in the sampling rate equals 3. Numerical verification has shown that the actual time dependency of the gain-numbers shows an excellent agreement with the given approximations.

4.2 The canonical decomposition of the baseline precision

Now that we have seen how the gain-numbers γ_i, $i = 1,2,3$, measure the change of the receiver-satellite over time, we can use these results to develop canonical decompositions for the variance matrices of the baseline, before and after ambiguity fixing. The following theorem gives the canonical decompositions of each of the eight baseline variance matrices that we have met so far.

Theorem 6 (*canonical decomposition*) Let the gain-numbers and gain-vectors be collected in the two 3×3 matrices Γ and F as

$\Gamma = \text{diag}(\gamma_1, \gamma_2, \gamma_3)$ and $F = (f_1, f_2, f_3)$

Then F may be normalized such that for the *code-only* case

$$(i)\ Q_{\hat{b}}(p) = \sigma_{\rho}^2(p)[FF^T]^{-1}$$
$$(ii)\ Q_{\hat{b}}(\bar{p}) = \sigma_{\rho}^2(p)[F(I_3 - \Gamma^{-1})F^T]^{-1}$$

and for the *phase-only* case

$$(iii)\ Q_{\check{b}}(\phi) = \sigma_{\rho}^2(\phi)[FF^T]^{-1}$$
$$(iv)\ Q_{\check{b}}(\bar{\phi}) = \sigma_{\rho}^2(\phi)[F(I_3 - \Gamma^{-1})F^T]^{-1}$$
$$(v)\ Q_{\hat{b}}(\phi) = \sigma_{\rho}^2(\phi)[F\Gamma^{-1}F^T]^{-1}$$

and for the *phase and code* case

$$(vi)\ Q_{\check{b}}(\phi, p) = \sigma_{\rho}^2(\phi, p)[FF^T]^{-1}$$
$$(vii)\ Q_{\check{b}}(\bar{\phi}, \bar{p}) = \sigma_{\rho}^2(\phi, p)[F(I_3 - \Gamma^{-1})F^T]^{-1}$$
$$(viii)\ Q_{\hat{b}}(\phi, p) = \sigma_{\rho}^2(\phi, p) \left[F\left(\Gamma^{-1} + \frac{\sigma_{\rho}^2(\phi, p)}{\sigma_{\rho}^2(p)}(I_3 - \Gamma^{-1})\right)F^T\right]^{-1}$$

Proof: See Appendix. □

The importance of the preceding matrix decompositions is that they *simultaneously* diagonalize the eight variance matrices with respect to the same frame F. Hence, all eight variance matrices follow once F and Γ are known. Also the relation among these variance matrices is now directly clear, thus allowing one to study the respective gain matrices. In the following we will give some examples.

Pairwise comparisons of the various baseline variance matrices show how the respective gain matrices depend on either the change over time of the receiver-satellite geometry or on the precision of the DD ranges, or on both. For instance, when we compare the geometry-based variance matrices with their time-averaged counterparts, it follows that their ratios are governed by the gain-numbers only. In the case where there is almost no change over time of the receiver-satellite geometry, then $\Gamma^{-1} \approx 0$ and therefore

$$Q_{\hat{b}}(p) \approx Q_{\hat{b}}(\bar{p}), \; Q_{\check{b}}(\phi) \approx Q_{\check{b}}(\bar{\phi}), \; Q_{\check{b}}(\phi,p) \approx Q_{\check{b}}(\bar{\phi},\bar{p}) \tag{17}$$

Thus although the time-averaged results are known to be suboptimal, they are close to optimal when the gain-numbers are large.

When we compare the code-only solution with the phase-only, but fixed solution, we see that the gain in baseline precision is governed by the precision of the DD ranges and thus by the observational weights only. It follows that

$$Q_{\check{b}}(\phi) = \frac{\sigma^2_{\check{\rho}}(\phi)}{\sigma^2_{\hat{\rho}}(p)} Q_{\hat{b}}(p) \tag{18}$$

Since in both the extreme cases $\sigma_I^2 = 0$ and $\sigma_I^2 = \infty$, the variance ratio of the DD ranges equals the very small phase-code variance ratio $\epsilon = \sigma^2_\phi / \sigma^2_p$ when cross-correlation is absent, it follows that $Q_{\check{b}}(\phi) \ll Q_{\hat{b}}(p)$. This shows for short as well as for long baselines the relevance of being able to work with ambiguity fixed phase data instead of with code data.

Although the use of ambiguity fixed phase data results in a much more precise baseline than when using code data, this is not necessarily the case when phase data are used for which the ambiguities have not been resolved as integers. In this case, it is both the gain-numbers and the precision of the DD ranges which determine the gain in baseline precision. Instead of Eq. (18), we now have

$$Q_{\hat{b}}(\phi) = \frac{\sigma^2_{\check{\rho}}(\phi)}{\sigma^2_{\hat{\rho}}(p)} F^{-T} \Gamma F^T Q_{\hat{b}}(p) \tag{19}$$

This shows that the precision of the phase-only floated baseline is only better than the code-only solution when $\epsilon\gamma_i < 1$. In this case we see data precision competing with the change in receiver-satellite geometry. The inequality is satisfied when the change in geometry is sufficiently large (γ_i small) and/or when the precision of the code data is sufficiently poor with respect to the precision of the phase data (ϵ small).

The results of Theorem 6 can also be used to infer the impact of code data, both for the fixed solution as well as for the float solution. For the fixed solution we have

$$\begin{aligned} Q_{\check{b}}(\phi,p) &= \frac{\sigma^2_{\check{\rho}}(\phi,p)}{\sigma^2_{\check{\rho}}(\phi)} Q_{\check{b}}(\phi), \\ Q_{\check{b}}(\bar{\phi},\bar{p}) &= \frac{\sigma^2_{\check{\rho}}(\phi,p)}{\sigma^2_{\check{\rho}}(\phi)} Q_{\check{b}}(\bar{\phi}) \end{aligned} \tag{20}$$

Since $\sigma^2_{\check{\rho}}(\phi,p)/\sigma^2_{\check{\rho}}(\phi) = 1/(1+\epsilon) \approx 1$ when $\sigma_I^2 = 0$ and $\sigma^2_{\check{\rho}}(\phi,p)/\sigma^2_{\check{\rho}}(\phi) = 1/(1+\epsilon+4\frac{\epsilon}{1+\epsilon}\frac{(\mu_1+\mu_2)^2}{(\mu_1-\mu_2)^2}) \approx 1$ when $\sigma_I^2 = \infty$, it follows that the contribution of code data in the *fixed* solution is marginal, although it is a bit better for long baselines than for short ones. For the float solution we get a completely different picture, since now also the change in geometry counts. For the float solution we have

$$Q_{\hat{b}}(\phi,p) = \frac{\sigma^2_{\check{\rho}}(\phi,p)}{\sigma^2_{\check{\rho}}(\phi)} \times \left\{ F^{-T} \left[I_3 + \frac{\sigma^2_{\check{\rho}}(\phi,p)}{\sigma^2_{\hat{\rho}}(p)} (\Gamma - I_3) \right]^{-1} F^T \right\} Q_{\hat{b}}(\phi) \tag{21}$$

Compare this with Eq. (20) and note the additional term within the braces. This additional term vanishes in the case where all gain-numbers are at their minimum of 1, but this is unlikely to happen. Since $\sigma^2_{\check{\rho}}(\phi,p)/\sigma^2_{\hat{\rho}}(p) \approx \epsilon/(1+\epsilon)$, it follows that the contribution of the code data improves the larger $\epsilon\gamma_i$ gets. Thus either the code data need to be of sufficient precision and/or the change over time of the receiver-satellite geometry needs to be small. Only then will the inclusion of code data contribute significantly to the precision of the *floated* baseline.

The preceding are some examples of pairwise comparisons of baseline variance matrices. For inferring the impact of ambiguity fixing, however, the most relevant comparison is the one between the float solution $Q_{\hat{b}}(\phi,p)$ and the fixed solution $Q_{\check{b}}(\phi,p)$. To conclude, we have therefore summarized in the following corollary the gain in baseline precision which is obtained as a consequence of ambiguity fixing.

Corollary (*gain in baseline precision*) The gain in precision of the baseline component $f_i^T b$, obtained by means of integer ambiguity fixing, is given as

$$g_i = 1 + \frac{(\beta-1)(\gamma_i-1)}{\beta+\gamma_i+1}, \; i = 1,2,3 \tag{22}$$

with $\beta = \sigma^2_{\hat{\rho}}(p)/\sigma^2_{\check{\rho}}(\phi,p)$ and where the γ_i are the stationary values of $\gamma(f) = f^T Q_{\hat{b}}(\phi) f / f^T Q_{\check{b}}(\phi) f$, $f \in R^3$.

Proof: Follows directly from Theorem 6. □

This corollary shows how the combined effect of the change over time of the receiver-satellite geometry (γ_i) and of the observational weights including those of the ionospheric delays (β), contribute to the gain g_i. Note that the gain g_i is symmetric in β and γ_i, and that

$1 \leq g_i \leq \beta$ and $1 \leq g_i \leq \gamma_i$

This shows that there is no gain when either $\beta = 1$ or $\gamma_i = 1$. The first case corresponds to the situation where phase data are absent. Of course, due to the absence of the ambiguities, no gain is then possible. The second case occurs when the principal angles θ_i are all zero. This is not likely to happen, however, since it would require that the two range spaces $R(U)$ and $R(V)$ of Theorem 4 coincide.

The gain g_i reaches its two upperbounds β and γ_i for $\gamma_i = \infty$ and $\beta = \infty$, respectively. The second case corresponds to the situation where code data are absent. A change in the receiver-satellite geometry is then needed to be able to compute the float solution. In the first case, it is the change in the receiver-satellite geometry which is absent. Code data are then needed per se to be able to compute the float solution. This first case corresponds to the situation of instantaneous positioning.

If the ionospheric delays are assumed to be completely unknown ($\sigma_I^2 = \infty$) in case of instantaneous positioning, it follows in the absence of cross-correlation that the gain reduces to $g_i = (1 + \frac{1}{\epsilon})(1 + 4\frac{\epsilon}{(1+\epsilon)^2}\frac{(\mu_1+\mu_2)^2}{(\mu_1-\mu_2)^2})$. Although this gain is large when ϵ is small, instantaneous ambiguity fixing is not likely to be successful in this case. For instantaneous ambiguity validation to be successful when the ionospheric delays are assumed to be completely unknown, one requires namely very precise code data. Note that the more precise the code data become, the larger the phase-code variance ratio ϵ gets and thus the smaller the gain becomes. This stipulates the trade-off between, on the one hand, the gain in baseline precision, and on the other, one's ability to successfully validate the integer ambiguities. A similar effect can be seen if one considers the change over time of the receiver-satellite geometry instead of the precision of the code data. For instance, if the code data are not precise enough to successfully validate the integer ambiguities, one is forced to rely on the change in geometry. That is, one will then need smaller values for γ_i to be able to validate the integer ambiguities. But this also results in a smaller gain. We thus see, since successful ambiguity validation takes priority over a larger gain, that in the case of long baselines we are forced to be content with a smaller gain in baseline precision.

In case of short baselines ($\sigma_I^2 = 0$), the situation is much more favourable in terms of the gain that can be achieved. In this case, successful validation of the integer ambiguities is known to be feasible even in the absence of a change in receiver-satellite geometry. And since the gain specializes then to $g_i = 1 + \frac{1}{\epsilon}$, it follows that a very high gain is experienced due to the very small phase-code variance ratio.

5 Summary

In this contribution we studied the precision of the ionosphere-weighted GPS baseline before and after ambiguity fixing. The functional model used is of the single-baseline type and its observables are single-frequency or dual-frequency phase and code data. In the stochastic model, we admitted cross-correlation and allowed the variances of the observables to differ. We also included the uncertainty of the ionospheric delays in the stochastic model. Since the ionosphere decorrelates as function of the interstation distance, the a priori variance of the ionospheric delays is at its maximum for baselines where the ionosphere is fully decorrelated, and it becomes smaller the shorter the baseline becomes.

The approach followed for solving the single-baseline model was a stepwise one. First the geometry-free model was considered, then the time-averaged model and finally the multi-epoch geometry-based model. This stepwise approach was not only used because the result of each step is of relevance in its own right, but also because it shows how the various factors of both the functional model and stochastic model contribute to the overall solution. In the first step it is primarily the weights of the observations which contribute. In the second step, it is the receiver-satellite geometry that enters, and in the third step the change over time of this geometry that provides the additional contribution.

In order to give a direct and transparent way of comparing the properties of the baseline precision, we developed canonical forms of the various baseline variance matrices. These forms are based on the gain-number concept of Teunissen (1997). The gain-numbers $\gamma_i, i = 1, 2, 3$, lie in the interval $[1, \infty)$ and are an intrinsic measure for the change over time of the receiver-satellite geometry. They become infinite in case of instantaneous positioning, they are large for short observation time-spans and they get smaller as time progresses. The gain-numbers follow an inverse-square law in the observation time-span, and through an increase in the sampling rate, they can be enlarged by a factor of at most 3.

The various factors that contribute to the precision of both the float and the fixed baseline are easily recognized in the given analytical descriptions of the canonical forms. The precision of the phase and code data, the cross-correlation and the uncertainty in the ionospheric delays, all come together in a common scale factor that is described by the variance of the DD ranges. They do not affect the correlation structure of the baseline variance matrices. This correlation structure is determined by the receiver-satellite geometry and its change over time.

Since the given canonical forms diagonalize all the baseline variance matrices with respect to the same frame, they can also easily be compared. It allows one to infer the gain in baseline precision due to ambiguity fixing and to study its dependence on the use of different measurement scenarios (e.g. single-frequency versus dual-frequency data; phase-only versus phase and code data; instantaneous versus multi-epoch positioning; short versus longer baselines).

Appendix

In this appendix the proofs of the theorems are given.

Proof of Theorem 1 (*single-epoch DD range precision*) The variance matrix of $(\phi'(i)^T, p'(i)^T)^T$ is given by the 4×4 matrix

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \tag{23}$$

with

$$A = 2(Q_\phi + \sigma_I^2 \mu\mu^T)$$
$$B = -2\sigma_I^2 \mu\mu^T$$
$$C = 2(Q_p + \sigma_I^2 \mu\mu^T)$$

In order to obtain the ambiguity fixed range variance $\sigma^2_{\check{\rho}}(\phi, p)$, we need to invert the variance matrix, then take the sum of its entries and finally invert this sum again. The inverse of the variance matrix reads

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}^{-1} = \begin{bmatrix} N_A & N_B \\ N_B^T & N_C \end{bmatrix} \tag{24}$$

with

$$N_A = [A - BC^{-1}B^T]^{-1} = \tfrac{1}{2}Q_\phi^{-1} - \tfrac{1}{2}\sigma_I^2 \frac{Q_\phi^{-1}\mu\mu^T Q_\phi^{-1}}{1+\sigma_I^2\mu^T(Q_\phi^{-1}+Q_p^{-1})\mu}$$

$$N_C = [C - B^T A^{-1} B]^{-1} = \tfrac{1}{2}Q_p^{-1} - \tfrac{1}{2}\sigma_I^2 \frac{Q_p^{-1}\mu\mu^T Q_p^{-1}}{1+\sigma_I^2\mu^T(Q_\phi^{-1}+Q_p^{-1})\mu}$$

$$N_B = -A^{-1}B[C - B^T A^{-1} B]^{-1} = \tfrac{1}{2}\sigma_I^2 \frac{Q_\phi^{-1}\mu\mu^T Q_p^{-1}}{1+\sigma_I^2\mu^T(Q_\phi^{-1}+Q_p^{-1})\mu}$$

Taking the sum of the entries of the first two matrices, plus twice the sum of the entries of the third matrix then gives $1/\sigma^2_{\check{\rho}}(\phi, p)$, from which the result of the theorem follows.

Proof of Theorem 2 (*time-averaged baseline precision*) We will only consider the variance matrix $Q_{\check{b}}(\bar{\phi}, \bar{p})$. The other variance matrices follow from it by setting the appropriate observational weights equal to zero. The variance matrix of $(I_2 \otimes D^T)\bar{\Phi}'$ and $(I_2 \otimes D^T)\bar{P}'$ is given as

$$\begin{bmatrix} A \otimes \frac{1}{2k}(D^T D) & B \otimes \frac{1}{2k}(D^T D) \\ B^T \otimes \frac{1}{2k}(D^T D) & C \otimes \frac{1}{2k}(D^T D) \end{bmatrix}$$

with the matrices A, B and C as in the proof of Theorem 1. Its inverse is given as

$$\begin{bmatrix} N_A \otimes 2k(D^T D)^{-1} & N_B \otimes 2k(D^T D)^{-1} \\ N_B^T \otimes 2k(D^T D)^{-1} & N_C \otimes 2k(D^T D)^{-1} \end{bmatrix}$$

with the matrices N_A, N_B and N_C again as in the proof of Theorem 1. From this inverse and the observation equations of the model, it follows that

$$Q_{\check{b}}^{-1}(\bar{\phi}, \bar{p}) = e_2^T(N_A + 2N_B + N_C)e_2 \otimes 2k\bar{A}^T P \bar{A}$$

Taking the inverse and noting that $1/\sigma^2_{\check{\rho}}(\phi, p) = e_2^T(N_A + 2N_B + N_C)e_2$ then give the stated result. □

Proof of Theorem 3 (*float and fixed baseline precision*) We will only prove (*iv*) and (*v*) of the theorem. The remaining results can be obtained from it by setting the appropriate weights equal to zero. The normal matrix of the float solution is given as

$$\begin{bmatrix} N_{bb} & N_{ba} \\ N_{ab} & N_{aa} \end{bmatrix}$$

with

$$N_{bb} = e_2^T(N_A + 2N_B + N_C)e_2 \otimes 2\sum_{i=1}^{k} A(i)^T P A(i)$$
$$N_{ba} = e_2^T(N_A + N_B^T)\Lambda \otimes 2k\bar{A}^T D(D^T D)^{-1} = N_{ab}^T$$
$$N_{aa} = \Lambda^T N_A \Lambda \otimes 2k(D^T D)^{-1}$$

with the matrices N_A, N_B and N_C as in the proof of Theorem 1.
Case (iv): since $Q_{\hat{b}}^{-1}(\phi, p) = N_{bb} - N_{ba}N_{aa}^{-1}N_{ab}$, it follows that

$$Q_{\hat{b}}(\phi, p) = \left[\frac{2}{\sigma^2_{\hat{\rho}}(p)} k\bar{A}^T P \bar{A} + \frac{2}{\sigma^2_{\check{\rho}}(\phi, p)} \times \sum_{i=1}^{k}(A(i) - \bar{A})^T P (A(i) - \bar{A})\right]^{-1} \tag{25}$$

with the reciprocal range variances

$$\frac{1}{\sigma^2_{\hat{\rho}}(p)} = e_2^T C^{-1} e_2 = e_2^T(N_C - N_B^T N_A^{-1} N_B)e_2$$

and

$$\frac{1}{\sigma^2_{\check{\rho}}(\phi, p)} = e_2^T(N_A + 2N_B + N_C)e_2$$

Case (v): since $Q_{\check{b}}^{-1}(\phi, p) = N_{bb}$, it directly follows that $Q_{\check{b}}(\phi, p) = [\frac{2}{\sigma^2_{\check{\rho}}(\phi,p)}\sum_{i=1}^{k} A(i)^T P A(i)]^{-1}$, which can also be written as

$$Q_{\check{b}}(\phi, p) = \left[\frac{2}{\sigma^2_{\check{\rho}}(\phi, p)} k\bar{A}^T P \bar{A} + \frac{2}{\sigma^2_{\check{\rho}}(\phi, p)} \times \sum_{i=1}^{k}(A(i) - \bar{A})^T P (A(i) - \bar{A})\right]^{-1} \tag{26}$$

From Eqs. (25) and (26), the results of the theorem follow. □

Proof of Theorem 6 (*canonical decomposition*) Of the eight cases we only prove the three phase-only cases (*iii*), (*iv*) and (*v*). With the help of Theorems 2 and 3, the proofs of the other cases run along similar lines.

Case (iii): from their definition it follows that the gain-numbers and gain-vectors are related in matrix form as

$$Q_{\hat{b}}(\phi)F = Q_{\check{b}}(\phi)F\Gamma \tag{27}$$

Premultiplication with F^T gives

$$F^T Q_{\hat{b}}(\phi)F = F^T Q_{\check{b}}(\phi)F\Gamma \tag{28}$$

In order to prove (*iii*), it suffices to show that F can be chosen such that $F^T Q_{\check{b}}(\phi)F = \sigma^2_{\check{\rho}}(\phi)I_3$. It will be clear that the columns of F can always be normalized such that the diagonal terms of $F^T Q_{\check{b}}(\phi)F$ equal $\sigma^2_{\check{\rho}}(\phi)$. It therefore remains to be shown that the columns of F are $Q_{\check{b}}(\phi)$-orthogonal. It follows from the definition of the gain-numbers that

$$f_i^T Q_{\check{b}}(\phi)f_j = \begin{cases} \gamma_i f_i^T Q_{\check{b}}(\phi)f_j \\ \gamma_j f_i^T Q_{\check{b}}(\phi)f_j \end{cases}$$

This shows that $f_i^T Q_{\check{b}}(\phi)f_j = 0$ if $\gamma_i \neq \gamma_j$. Thus in this case the vectors f_i and f_j are $Q_{\check{b}}(\phi)$-orthogonal. In case $\gamma_i = \gamma_j$, any linear combination of f_i and f_j is again a gain-vector that corresponds with $\gamma_i = \gamma_j$. It is therefore always possible to find a linear combination of f_i and f_j that is $Q_{\check{b}}(\phi)$-orthogonal to either f_i or f_j. Hence, also for the case where the gain-numbers are identical, corresponding gain-vectors can be found that are $Q_{\check{b}}(\phi)$-orthogonal. This shows that F can be chosen such that $F^T Q_{\check{b}}(\phi)F = \sigma^2_{\check{\rho}}(\phi)I_3$ or $Q_{\check{b}}(\phi) = \sigma^2_{\check{\rho}}(\phi)[FF^T]^{-1}$ holds.

Case (iv): from Theorem 3, part (ii), it follows that

$$Q_{\check{b}}(\bar{\phi}) = [Q_{\check{b}}^{-1}(\phi) - Q_{\hat{b}}^{-1}(\phi)]^{-1}$$

Substitution of (*iii*) and (*v*) of Theorem 6 into this expression proves (*iv*).

Case (v): this result simply follows from $F^T Q_{\check{b}}(\phi)F = \sigma^2_{\check{\rho}}(\phi)I_3$ and Eq. (28). □

References

Blewitt G (1989) Carrier-phase ambiguity resolution for the Global Positioning System applied to geodetic baselines up to 2000 km. *J Geophys Res* 94: 10187–10203

Bock Y (1996) Medium distance GPS measurements. In: Kleusberg A, Teunissen PJG (eds) *GPS for geodesy*, lecture notes in earth sciences, vol 60. Springer, Berlin Heidelberg New York, chap 9

Borre K (1995) *GPS i landmaelingen*. Aalborg

Counselman CC, Gourevitch SA (1981) Miniature interferometer terminals for earth surveying: ambiguity and multipath with Global Positioning System. *IEEE Trans Geosci Remote Sensing* GE-19: 244–252

de Jonge PJ, Tiberius CCJM (1996) The LAMBDA method for integer ambiguity estimation: implementation aspects. *LGR Series* no. 12. Delft Geodetic Computing Centre, Delft

Euler HJ, Goad C (1990) On optimal filtering of GPS dual-frequency observations without using orbit information. *Bull Geod* 65: 130–143

Frei E (1991) *Rapid differential positioning with the Global Positioning System*. Geod Geophy Stud Switz 44

Georgiadou Y (1994) Modelling the ionosphere for an active control network of GPS stations. *LGR-Series* no 7. Delft Geodetic Computing Centre, Delft

Goad CC (1992) Robust techniques for determining GPS phase ambiguities. In: Proc *6th Geod Symp Satellite Positioning*, pp 245–254

Hatch R (1982) The synergism of GPS code and carrier-phase measurements. In: Proc *3rd Int Geod Symp Satellite Positioning*, vol 2. pp 1213–1231

Hofmann-Wellenhof B, Lichtenegger H, Collins J (1994) *Global positioning system: theory and practice*. 4th edn. Springer, Berlin Heidelberg New York

Kleusberg A, Teunissen PJG (eds) (1996) *GPS for geodesy*, lecture notes in earth sciences vol. 60. Springer, Berlin Heidelberg New York

Leick A (1995) *GPS satellite surveying*. 2nd edn. Wiley, New York

Parkinson B et al. (eds) (1996) *GPS: theory and applications*. vols 1 and 2. AIAA

Rao CR (1973) *Linear statistical inference and its applications*. 2nd edn, Wiley, New York

Remondi BW (1986) Performing cm-level surveys in seconds with GPS carrier phase: initial results. *J Navig* 3: 194–208

Schaer S (1994) Stochastische Ionosphärenmodellierung beim 'Rapid Static Positioning' mit GPS. Astronomisches Institut, Universität Berne

Seeber G (1993) *Satellite geodesy*. De Gruyter, Berlin

Teunissen PJG (1993) Least-squares estimation of the integer GPS ambiguities. *LGR Series*, no 6. Delft Geodetic Computing Centre, Delft

Teunissen PJG (1994) A new method for fast carrier-phase ambiguity estimation. In: *Proc IEEE position, location and navigation symposium PLANS'94*, pp 562–573

Teunissen PJG (1995) The least-squares ambiguity decorrelation adjustment: a method for fast GPS integer ambiguity estimation. *J Geod* 70: 65–82

Teunissen PJG (1996) An analytical study of ambiguity decorrelation using dual-frequency code and carrier phase. *J Geod* 70: 515–528

Teunissen PJG (1997) A canonical theory for short GPS baselines. Part I: the baseline precision. *J Geod* 71: 320–336. Part II: the ambiguity precision and correlation. *J Geod* 71: 389–401. Part III: the geometry of the ambiguity search space. *J Geod* 71: 486–501. Part IV: precision versus reliability. *J Geodesy*: in print.

Teunissen PJG, de Jonge, PJ Tiberius CCJM (1994) On the spectrum of the GPS DD ambiguities. In: *Proc ION GPS-94*. pp 115–124

Teunissen PJG, de Jonge PJ, Tiberius CCJM (1995) The LAMBDA method for fast GPS Surveying. In: *Proc Int Symp GPS technology Applications*. pp 203– 210

Wanninger L (1995) Improved ambiguity resolution by regional differential modelling of the ionosphere. In: *Proc ION GPS- 95*, pp 55–62

Wild U (1994) Ionosphere and geodetic satellite systems: permanent GPS tracking data for modelling and monitoring. *Geod Geophys Stud Switz* 48

Wild U, Beutler G (1991) Deterministische und stochastische Ionosphärenmodelle. *Z Vermessung, Photogr Kulturtech* 6: 298–302

Wübbena G (1989) The GPS adjustment software package – GEONAP – concepts and models. In: Proc *5th Geod Symp. Satellite Positioning*. pp 452–461