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# Repeated eigenstructure assignment in the computation of friends of output-nulling subspaces

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**Abstract**—This paper is concerned with the parameterisation of basis matrices and the simultaneous computation of friends of the output nulling subspaces  $\mathcal{V}^*$ ,  $\mathcal{V}_g^*$  and  $\mathcal{R}^*$  with the assignment of the corresponding inner and outer closed-loop free eigenstructure. Differently from the classical techniques presented in the literature so far on this topic, which are based on the standard pole assignment algorithms and are therefore applicable only in the non-defective case, the method presented in this paper can be applied in the case of closed-loop eigenvalues with arbitrary multiplicity.

## I. INTRODUCTION

In the last forty years, geometric control has played a fundamental role in the understanding of the structural properties of linear and non-linear dynamical systems and in the solution of several control and estimation problems, including disturbance decoupling, non-interacting control, fault detection, model matching and optimal control to name a few. The monographs [16], [2], [15], [3] provide surveys of the extensive literature in this area.

The subspaces that underpin the classic geometric theory of linear time-invariant (LTI) systems are the so-called output-nulling and input-containing subspaces. The most important output-nulling subspace is undoubtedly  $\mathcal{V}^*$ , which represents the set of initial states for which a control function exists that maintains the output function identically at zero; the second is  $\mathcal{R}^*$ , which represents the reachable subspace within  $\mathcal{V}^*$ , and can be interpreted as the set of initial states that are reachable from the origin of the state space by means of a control function that maintains the output function at zero. Finally, the subspace  $\mathcal{V}_g^*$  represents the set of initial states for which a control can be found that maintains the output at zero by means of state trajectories that converge to the origin. This latter subspace has played a central role in the solution of control problems with additional stability requirements. In the LTI case, these input functions can always be expressed as a static state feedback, by means of a feedback matrix usually referred to as a *friend* of the output-nulling subspace.<sup>1</sup>

The computation of friends of output nulling subspaces that assign the inner and outer assignable spectrum of the closed-loop has been considered by many authors and the texts [2] and [3] included publicly available MATLAB<sup>®</sup>

toolboxes. In the MATLAB<sup>®</sup> *GA* toolbox<sup>2</sup>, the `effesta.m` routine is used for computing the friends. Similarly, the SCB method of [4] was incorporated into the computation of the friends in the MATLAB<sup>®</sup> *Linsyskit* toolbox<sup>3</sup>; the `atea.m` routine is used for computing the friends, and is described in [6].

All the methods currently available in the literature are based on decompositions that reduce the problem to one where a feedback matrix  $F$  is sought that assigns all the eigenvalues of a closed-loop matrix, say  $\hat{A} + \hat{B}F$ , where the pair  $(\hat{A}, \hat{B})$  is reachable. Both the methods in the MATLAB<sup>®</sup> toolboxes *GA* and *Linsyskit* exploit the MATLAB<sup>®</sup> instruction `place.m` to this purpose, based on the algorithm of [5], which can only assign eigenvalues of  $\hat{A} + \hat{B}F$  with a multiplicity for each eigenvalue that must not exceed the rank of  $\hat{B}$ . This limitation of the routine `place.m` is thus inherited by the MATLAB<sup>®</sup> instructions of the toolboxes *GA* and *Linsyskit*, which can therefore compute the friend of the output-nulling subspace at hand only in the case of non-defective closed-loop.

A different approach for the computation of a basis matrix for  $\mathcal{R}^*$  and  $\mathcal{V}^*$  was proposed by Moore and Laub in [8], who presented an algorithm for the computation of  $\mathcal{R}^*$  and  $\mathcal{V}^*$  based on the computation of the null-spaces of the system Rosenbrock matrix pencil [12]. This procedure has the advantage of computing a basis matrix for  $\mathcal{R}^*$  (and  $\mathcal{V}^*$ ) and simultaneously delivering a corresponding friend  $F$  that assigns a certain inner closed-loop eigenstructure. The drawback was the number of restrictive assumptions that were made in that paper. These assumptions have been recently removed in [11] and [10]. Moreover, in these papers an additional generalisation of the procedure in [8] was proposed to the end of delivering a friend that also assigns the free outer eigenstructure of  $\mathcal{R}^*$  (or  $\mathcal{V}^*$ ). However, the most important aspect of the method presented in [8], which remained unexploited until very recent times, is the fact that the friend of  $\mathcal{R}^*$  (or  $\mathcal{V}^*$ ) that assigns the free inner and outer eigenstructure of the closed-loop with respect to  $\mathcal{R}^*$  is given in parameterised form. This fundamental aspect invites the formulation of optimisation problems aimed at exploiting the available freedom to deal with objectives such as minimum gain or improved robustness of the eigenstructure. The paper [11] is the first to propose a method for assigning friends that exploits this freedom.

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<sup>1</sup>This property does not necessarily hold outside the domain of finite dimensional LTI systems over a field.

<sup>2</sup>The geometric approach toolbox *GA* for MATLAB<sup>®</sup> is freely downloadable at [www3.deis.unibo.it/Staff/FullProf/GiovanniMarro/geometric.htm](http://www3.deis.unibo.it/Staff/FullProf/GiovanniMarro/geometric.htm).

<sup>3</sup>The Linear System Toolkit is available on request from the first author of [3]; see <http://vlab.ee.nus.edu.sg/~bmchen/>.

One of the restrictive assumptions of the method proposed in [8], which remains in the generalisations presented in [10] and [11], is the fact that the closed-loop eigenvalues to be assigned must be distinct. This paper addresses this issue: we generalise the method in [8] to also take into account the case of repeated closed-loop eigenvalues, with arbitrary multiplicity. This task is accomplished by introducing a new parameterisation of the basis matrices for  $\mathcal{R}^*$ ,  $\mathcal{V}^*$  and  $\mathcal{V}_g^*$ , which also provides a natural method for determining the associated friend which can place the assignable closed-loop eigenvalues to desired locations virtually without any assumptions on the location or on the multiplicity of such eigenvalues. Future research will consider the exploitation of this parameterisation of the friends of  $\mathcal{V}^*$ ,  $\mathcal{R}^*$  and  $\mathcal{V}_g^*$  to address problems of determining the friends with minimum Frobenius norm or improved robustness of the eigenstructure along the same lines of the non-defective case of [11].

**Notation.** Throughout this paper, the symbol  $0_q$  stands for the origin of the vector space  $\mathbb{R}^q$ . The image and the kernel of matrix  $A$  are denoted by  $\text{im } A$  and  $\text{ker } A$ , respectively. The Moore-Penrose pseudo-inverse of  $A$  is denoted by  $A^\dagger$ . Given a linear map  $A: \mathcal{X} \rightarrow \mathcal{Y}$  and a subspace  $\mathcal{S}$  of  $\mathcal{Y}$ , the symbol  $A^{-1}\mathcal{S}$  stands for the inverse image of  $\mathcal{S}$  with respect to the linear map  $A$ . If  $\mathcal{J} \subseteq \mathcal{X}$ , the restriction of the map  $A$  to  $\mathcal{J}$  is denoted by  $A|_{\mathcal{J}}$ . If  $\mathcal{X} = \mathcal{Y}$  and  $\mathcal{J}$  is  $A$ -invariant, the eigenvalues of  $A$  restricted to  $\mathcal{J}$  are denoted by  $\sigma(A|_{\mathcal{J}})$ . If  $\mathcal{J}_1$  and  $\mathcal{J}_2$  are  $A$ -invariant subspaces and  $\mathcal{J}_1 \subseteq \mathcal{J}_2$ , the mapping induced by  $A$  on the quotient space  $\mathcal{J}_2/\mathcal{J}_1$  is denoted by  $A|_{\mathcal{J}_2/\mathcal{J}_1}$ . The symbol  $\oplus$  stands for the direct sum of subspaces.

Given a map  $A: \mathcal{X} \rightarrow \mathcal{X}$  and a subspace  $\mathcal{B}$  of  $\mathcal{X}$ , we denote by  $\langle A, \mathcal{B} \rangle$  the smallest  $A$ -invariant subspace of  $\mathcal{X}$  containing  $\mathcal{B}$ . The symbol  $i$  stands for the imaginary unit, i.e.,  $i = \sqrt{-1}$ . The symbol  $\bar{\alpha}$  denotes the complex conjugate of  $\alpha \in \mathbb{C}$ . Given a matrix  $M$ , we denote by  $M_i$  its  $i$ -th row and by  $M^j$  its  $j$ -th column, respectively. The normal rank of a rational matrix  $M(\lambda)$  is defined as  $\text{normrank } M(\lambda) \triangleq \max_{\lambda \in \mathbb{C}} \text{rank } M(\lambda)$ .

## II. PRELIMINARIES

Consider an LTI system  $\Sigma$  modelled by

$$\Sigma: \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & x(0) = x_0, \\ y(t) = Cx(t) + Du(t), \end{cases} \quad (1)$$

where, for all  $t \geq 0$ ,  $x(t) \in \mathcal{X} = \mathbb{R}^n$  is the state,  $u(t) \in \mathcal{U} = \mathbb{R}^m$  is the control input,  $y(t) \in \mathcal{Y} = \mathbb{R}^p$  is the output, and  $A$ ,  $B$ ,  $C$  and  $D$  are appropriate dimensional constant real-valued matrices. Let the system  $\Sigma$  described by (1) be identified with the quadruple  $(A, B, C, D)$ . We assume with no loss of generality that all the columns of  $\begin{bmatrix} B \\ D \end{bmatrix}$  and all the rows of  $[C \ D]$  are linearly independent. We define the Rosenbrock system matrix pencil in the indeterminate  $\lambda \in \mathbb{C}$  as

$$P_\Sigma(\lambda) \triangleq \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix}, \quad (2)$$

[8]. The invariant zeros of  $\Sigma$  are identified with the values of  $\lambda \in \mathbb{C}$  for which the rank of  $P_\Sigma(\lambda)$  is strictly smaller than its normal rank. More precisely, the invariant zeros are the roots of the non-zero polynomials on the principal diagonal of the Smith form of  $P_\Sigma(\lambda)$ , see e.g. [1].

Given an invariant zero  $\lambda = z \in \mathbb{C}$ , the rank deficiency of  $P_\Sigma(\lambda)$  at the value  $\lambda = z$  is the geometric multiplicity of the invariant zero  $z$ , and is equal to the number of elementary divisors (invariant polynomials) of  $P_\Sigma(\lambda)$  associated with the complex frequency  $\lambda = z$ . The degree of the product of the elementary divisors of  $P_\Sigma(\lambda)$  corresponding to the invariant zero  $z$  is the algebraic multiplicity of  $z$ , see [7]. More explicitly, given the set of invariant zeros  $\mathcal{Z} = \{z_1, \dots, z_t\}$  of (2), if

$$\gamma_k(\lambda) = (\lambda - z_1)^{m_{1,k}} (\lambda - z_2)^{m_{2,k}} \dots (\lambda - z_t)^{m_{t,k}},$$

$k \in \{1, \dots, c\}$ , are the elementary divisors of  $P_\Sigma(\lambda)$  (ordered in such a way that  $m_{k,c} \geq m_{k,c-1} \geq \dots \geq m_{k,2} \geq m_{k,1}$  for any  $k \in \{1, \dots, t\}$ ), the geometric multiplicity of the invariant zero  $z_i$  equals the cardinality of the set  $\{m_{i,j} \neq 0 \mid j \in \{1, \dots, c\}\}$ , while the algebraic multiplicity of  $z_i$  is equal to  $\sum_{k=1}^c m_{i,k}$ . Finally, the invariant zero structure of  $\Sigma$  is given by  $\{m_{i,j} \mid i \in \{1, \dots, t\}, j \in \{1, \dots, c\}\}$ . Thus, the algebraic multiplicity of an invariant zero is not smaller than its geometric multiplicity. The set of invariant zeros of  $\Sigma$  is denoted with  $\mathcal{Z}$ , and the set of minimum-phase invariant zeros of  $\Sigma$  is denoted with  $\mathcal{Z}_g$ .

Given  $\lambda \in \mathbb{C}$ , we use the symbol  $N_\Sigma(\lambda)$  to denote a basis matrix for the null-space of  $P_\Sigma(\lambda)$ , and we denote by  $d(\lambda)$  the dimension of this null-space. Let  $d \triangleq n + m - \text{normrank } P_\Sigma(\lambda)$ . Clearly  $d(\lambda) = d$ , unless  $\lambda$  is an invariant zero of  $\Sigma$ , in which case  $d(\lambda) > d$ .

For any matrix  $M$  with  $n + m$  rows, we define  $\bar{\pi}\{M\}$  and  $\underline{\pi}\{M\}$  by taking the upper  $n$  and lower  $m$  rows of  $M$ , respectively.

**Geometric background.** Geometric objects extensively used in this paper are defined here. A controlled invariant subspace  $\mathcal{V}$  of the pair  $(A, B)$  is a subspace of  $\mathcal{X}$  satisfying  $A\mathcal{V} \subseteq \mathcal{V} + \text{im } B$ . An output-nulling subspace of  $\Sigma = (A, B, C, D)$  is a controlled invariant subspace  $\mathcal{V}$  of  $\Sigma$  which satisfies  $\begin{bmatrix} A \\ C \end{bmatrix} \mathcal{V} \subseteq (\mathcal{V} \oplus 0_p) + \text{im} \begin{bmatrix} B \\ D \end{bmatrix}$  or, equivalently, for which two matrices  $\Xi$  and  $\Omega$  exist such that  $\begin{bmatrix} A \\ C \end{bmatrix} V = \begin{bmatrix} V \\ 0 \end{bmatrix} \Xi + \begin{bmatrix} B \\ D \end{bmatrix} \Omega$ , where  $V$  is a basis matrix of  $\mathcal{V}$ .

These conditions are equivalent to the existence of a matrix  $F \in \mathbb{R}^{m \times n}$  such that  $(A + BF)\mathcal{V} \subseteq \mathcal{V} \subseteq \text{ker}(C + DF)$ . Any such matrix  $F$  is referred to as a *friend* of  $\mathcal{V}$ . The largest output-nulling subspace of  $\Sigma$  is denoted with  $\mathcal{V}^*$ , and represents the set of all initial states  $x_0$  of (1) for which a control  $u$  exists such that the corresponding output  $y$  is identically zero. Such input function can always be implemented as a static state feedback of the form  $u(t) = Fx(t)$  where  $F$  is a friend of  $\mathcal{V}^*$ . The so-called largest *reachability output-nulling subspace* on  $\mathcal{V}^*$ , here denoted with the symbol  $\mathcal{R}^*$ , is the smallest  $(A + BF)$ -invariant subspace of  $\mathcal{X}$  containing

the subspace  $\mathcal{V}^* \cap B \ker D$ , where  $F$  is a friend of  $\mathcal{V}^*$ . Loosely speaking, this subspace represents the states that are reachable from the origin on a state trajectory for which the output is zero, [15, Ch. 8], [9]. If  $F$  is a friend of  $\mathcal{V}^*$ , it is also a friend of  $\mathcal{R}^*$ . The spectrum  $\sigma(A + BF | \mathcal{R}^*)$  is assignable, whereas the spectrum  $\Gamma_{\text{in}} \triangleq \sigma(A + BF | \mathcal{V}^* / \mathcal{R}^*)$  is fixed, and its elements are the invariant zeros of  $\Sigma$ . Similarly, if we denote by  $\mathcal{R}_0$  the reachable subspace from the origin, i.e.,  $\mathcal{R}_0 = \langle A, \text{im} B \rangle = \text{im}[B \ AB \ \dots \ A^{n-1} B]$ , the spectrum  $\sigma(A + BF | \mathcal{V}^* + \mathcal{R}_0 / \mathcal{V}^*)$  is assignable, whereas the spectrum  $\Gamma_{\text{out}} \triangleq \sigma(A + BF | \mathcal{X} / \mathcal{V}^* + \mathcal{R}_0)$  is fixed. Finally,  $\mathcal{V}_g^*$  is the largest output-nulling subspace for which there exists a friend  $F$  such that  $(A + BF)\mathcal{V}_g^* \subseteq \mathcal{V}_g^* \subseteq \ker(C + DF)$  and  $\sigma(A + BF | \mathcal{V}_g^*) \subset \mathbb{C}_g$ , where  $\mathbb{C}_g$  denotes the left-half complex plane. Thus, there holds in general  $\mathcal{R}^* \subseteq \mathcal{V}_g^* \subseteq \mathcal{V}^*$ .

### III. THE NON-DEFECTIVE CASE

We now recall some results on the computation of basis matrices for  $\mathcal{R}^*$ ,  $\mathcal{V}^*$  and  $\mathcal{V}_g^*$  and the corresponding friends that assign the free closed-loop eigenstructure under the assumption that the closed-loop eigenvalues are distinct.

#### A. Computation of $\mathcal{R}^*$

Given a set of  $h$  self-conjugate complex numbers  $\mathcal{L} = \{\lambda_1, \dots, \lambda_h\}$  containing exactly  $s$  complex conjugate pairs, we say that  $\mathcal{L}$  is  $s$ -conformably ordered if the first  $2s$  values of  $\mathcal{L}$  are complex while the remaining are real, and for all odd  $k \leq 2s$  we have  $\lambda_{k+1} = \bar{\lambda}_k$ . For example, the sets  $\mathcal{L}_1 = \{1 + i, 1 - i, 3, -4\}$ ,  $\mathcal{L}_2 = \{10i, -10i, 2 + 2i, 2 - 2i, 7\}$  and  $\mathcal{L}_3 = \{3, -1\}$  are respectively 1-, 2- and 0-conformably ordered. We now recall the main result in [10], which provides a method to construct a basis for  $\mathcal{R}^*$  and *simultaneously* a friend  $F$  that assigns the **distinct** eigenstructure of the closed-loop restricted to  $\mathcal{R}^*$ .

*Theorem 3.1:* Let  $r = \dim \mathcal{R}^*$ . Let  $\mathcal{L} = \{\lambda_1, \dots, \lambda_r\}$  be  $s$ -conformably ordered and distinct, and such that  $\mathcal{L} \cap \mathcal{Z} = \emptyset$ . Let  $K \triangleq \text{diag}\{k_1, \dots, k_r\}$ , where  $k_i \in \mathbb{C}^d$  for each  $i \in \{1, \dots, 2s\}$ , and for all odd  $i \leq 2s$ , we have  $\bar{k}_i = k_{i+1}$ , whereas  $k_i \in \mathbb{R}^d$  for  $i \in \{2s+1, \dots, r\}$ . Let  $M_K$  be an  $(n+m) \times r$  complex matrix given by

$$M_K \triangleq [ N_{\Sigma}(\lambda_1) \mid N_{\Sigma}(\lambda_2) \mid \dots \mid N_{\Sigma}(\lambda_r) ] K \quad (3)$$

and let for all  $j \in \{1, \dots, r\}$

$$m_{K,j} \triangleq \begin{cases} \Re\{M_K^j\} & \text{if } j \leq 2s \text{ is odd} \\ \Im\{M_K^j\} & \text{if } j \leq 2s \text{ is even} \\ M_K^j & \text{if } j > 2s \end{cases} \quad (4)$$

Finally, let

$$X_K \triangleq \bar{\pi}\{[m_{K,1} \ m_{K,2} \ \dots \ m_{K,r}]\}, \quad (5)$$

$$Y_K \triangleq \underline{\pi}\{[m_{K,1} \ m_{K,2} \ \dots \ m_{K,r}]\}. \quad (6)$$

For almost every choice of the parameter matrix  $K = \text{diag}\{k_1, \dots, k_r\}$ , the rank of  $X_K$  is equal to  $r$ . Moreover, for all  $K$  such that  $\text{rank } X_K = r$ , there holds  $\mathcal{R}^* = \text{im } X_K$ , and the

set of all friends of  $\mathcal{R}^*$  such that  $\sigma(A + BF | \mathcal{R}^*) = \mathcal{L}$  is parameterised as

$$F_K = Y_K X_K^{-1}. \quad (7)$$

Theorem 3.1 contains a procedure for the construction of a friend  $F$  of the subspace  $\mathcal{R}^*$  that arbitrarily assigns all the eigenvalues of the closed-loop restricted to  $\mathcal{R}^*$ . However, we also know that the spectrum induced by the map  $A + BF$  on the quotient space  $\mathcal{R}_0 + \mathcal{R}^* / \mathcal{R}^* = \mathcal{R}_0 / \mathcal{R}^*$  is assignable using a friend  $F$ . In [10] and [11], a procedure is outlined to the end of generalising the statement of Theorem 3.1 to the case in which the free outer eigenstructure of  $A + BF$  with respect to  $\mathcal{R}^*$  is also assigned.

#### B. Computation of $\mathcal{V}^*$ and $\mathcal{V}_g^*$

We now recall another result in [10] which shows that it is always possible to parameterise all the friends that assign the internal and external eigenstructure of  $\mathcal{V}^*$  by means of a formula

$$F_K = Y_K X_K^{-1},$$

i.e., where this time  $X_K$  is square and invertible (for almost all choices of the parameter matrix  $K$ ). For the sake of simplicity of exposition, we assume that all the internal/external eigenvalues to be assigned, as well as all the invariant zeros and uncontrollable modes of the pair  $(A, B)$  are real and distinct.

*Theorem 3.2:* Let  $r = \dim \mathcal{R}^*$ ,  $v = \dim \mathcal{V}^*$  and  $q = \dim(\mathcal{V}^* + \mathcal{R}_0)$ . Let  $\mathcal{L}_{\text{in}} = \{\lambda_1, \dots, \lambda_r\}$  be real. Let  $\mathcal{Z} = \{z_{r+1}, \dots, z_v\}$  be the set of invariant zeros. Let  $\mathcal{L}_{\text{out}} = \{\mu_{v+1}, \dots, \mu_q\}$  be also real. Finally, let  $\mathcal{G} = \Gamma_{\text{out}} = \{\zeta_{q+1}, \dots, \zeta_n\}$ . We assume that  $\mathcal{L}_{\text{in}}$ ,  $\mathcal{Z}$ ,  $\mathcal{L}_{\text{out}}$  and  $\mathcal{G}$  are distinct, and that  $\mathcal{L}_{\text{in}} \cap \mathcal{Z} = \emptyset$ ,  $\mathcal{L}_{\text{in}} \cap \mathcal{G} = \emptyset$ ,  $\mathcal{L}_{\text{out}} \cap \mathcal{Z} = \emptyset$  and  $\mathcal{L}_{\text{out}} \cap \mathcal{G} = \emptyset$ . Define

$$M_K = [ N_{\Sigma}(\lambda_1) \ \dots \ N_{\Sigma}(\lambda_r) \mid N_{\Sigma}(z_{r+1}) \ \dots \ N_{\Sigma}(z_v) \mid S_{\Sigma}(\mu_{v+1}) \ \dots \ \dots \ S_{\Sigma}(\mu_q) \mid S_{\Sigma}(\zeta_{q+1}) \ \dots \ S_{\Sigma}(\zeta_n) ] K$$

where  $S_{\Sigma}(\mu)$  represents a basis matrix for  $\ker[A - \mu I_n \ B]$ , and where  $K = \text{diag}\{K_{\lambda}, K_z, K_{\mu}, K_{\zeta}\}$ , and

- $K_{\lambda} = \text{diag}\{k_1^{\lambda}, \dots, k_r^{\lambda}\}$ , with  $k_i^{\lambda} \in \mathbb{R}^d$ , and where  $d = \dim(\ker P_{\Sigma}(\lambda))$  when  $\lambda$  is not an invariant zero;
- $K_z = \text{diag}\{k_{r+1}^z, \dots, k_v^z\}$ , with  $k_i^z \in \mathbb{R}^{d_z}$ , and  $d_z = \dim(\ker P_{\Sigma}(z))$  when  $z \in \mathcal{Z}$ ;
- $K_{\mu} = \text{diag}\{k_{v+1}^{\mu}, \dots, k_q^{\mu}\}$ , with  $k_i^{\mu} \in \mathbb{R}^m$ , since  $m = \dim(\ker S_{\Sigma}(\mu))$  when  $\mu$  is not in  $\mathcal{G}$ ;
- $K_{\zeta} = \text{diag}\{k_{q+1}^{\zeta}, \dots, k_n^{\zeta}\}$ , with  $k_i^{\zeta} \in \mathbb{R}^{m_{\zeta}}$ , and where  $m_{\zeta} = \dim(\ker S_{\Sigma}(\zeta))$  when  $\zeta \in \Gamma_{\text{out}}$ .

Finally, define

$$X_K = \bar{\pi}\{M_K\} \in \mathbb{R}^{n \times n} \quad \text{and} \quad Y_K = \underline{\pi}\{M_K\} \in \mathbb{R}^{m \times n}. \quad (8)$$

For almost every choice of  $K$ , the matrix  $X_K$  is invertible, and the set of all friends of  $\mathcal{V}^*$  such that  $\sigma(A + BF | \mathcal{R}^*) = \mathcal{L}_{\text{in}}$ ,  $\sigma(A + BF | \mathcal{V}^* / \mathcal{R}^*) = \mathcal{Z}$  and  $\sigma(A + BF | (\mathcal{R}_0 + \mathcal{V}^*) / \mathcal{V}^*) = \mathcal{L}_{\text{out}}$  is parameterised in  $K$  as

$$F_K = Y_K X_K^{-1}, \quad (9)$$

where  $K$  is such that  $X_K$  is invertible. Moreover, for such  $K$  the first  $r$  columns of  $X_K$  are a basis for  $\mathcal{R}^*$ , the first  $v = r + t$

columns of  $X_K$  are a basis for  $\mathcal{V}^*$  and the first  $q$  are a basis for  $\mathcal{V}^* + \mathcal{R}_0$ .

The computation of a basis matrix for  $\mathcal{V}_g^*$  and the corresponding friend is obtained from Theorem 3.2, by replacing  $\mathcal{L}$  with  $\mathcal{L}_g$ , i.e., by only taking into account the minimum-phase invariant zeros.

#### IV. REPEATED EIGENVALUES AND INVARIANT ZEROS

In this section we develop a parametric formula for all friends of  $\mathcal{R}^*$ , such that the corresponding eigenstructure can have eigenvalues with any desired multiplicity, and any admissible Jordan form. We formulate the problem as follows. We let  $\mathcal{L} = \{\lambda_1, \dots, \lambda_\nu\}$  be  $s$ -conformably ordered and, for the sake of simplicity, distinct from the system invariant zeros. Let this desired eigenvalues have associated algebraic multiplicities  $\mathcal{M} = \{m_1, \dots, m_\nu\}$  satisfying  $m_1 + \dots + m_\nu = r$ ; if  $\lambda_{i+1} = \bar{\lambda}_i$ , then clearly  $m_{i+1} = m_i$ . We aim to obtain a real gain matrix  $F$  and a set of real vectors  $X$  such that

$$\begin{bmatrix} A+BF \\ C+DF \end{bmatrix} X = \begin{bmatrix} X \\ 0 \end{bmatrix} \Lambda, \quad (10)$$

where  $\text{im} X = \mathcal{R}^*$  and  $\Lambda$  is a real Jordan matrix in canonical form

$$\Lambda = \text{diag}\{J(\lambda_1), \dots, J(\lambda_\nu)\}, \quad (11)$$

where each  $J(\lambda_i)$  represents a real Jordan matrix for the eigenvalue  $\lambda_i$  of order  $m_i$ , and may be composed of up to  $g_i$  real mini-blocks, i.e.,

$$J(\lambda_i) = \text{diag}\{J_1(\lambda_i), \dots, J_{g_i}(\lambda_i)\}. \quad (12)$$

We use  $\mathcal{P} \triangleq \{p_{i,j} : 1 \leq i \leq \nu, 1 \leq j \leq g_i\}$  to denote orders of each Jordan mini-block  $J_j(\lambda_i)$ , and assume without loss of generality that for each  $i$ , they are in descending order  $p_{i,1} \geq p_{i,2} \geq \dots \geq p_{i,g_i}$ . If  $\mathcal{L}$ ,  $\mathcal{M}$  and  $\mathcal{P}$  satisfy the conditions of the Rosenbrock Theorem<sup>4</sup> [12], then we say that  $\mathcal{L}$ ,  $\mathcal{M}$  and  $\mathcal{P}$  define an *admissible Jordan structure* for  $\mathcal{R}^*$ . Given such a structure, we define a *parameter matrix*  $K = \text{diag}\{K_1, \dots, K_\nu\}$ , where, for each  $i \in \{1, \dots, 2s\}$ ,  $K_i \in \mathbb{C}^{d \times m_i}$ , for all odd  $i \leq 2s$ , we have  $\bar{K}_i = K_{i+1}$ ; and for  $i \in \{2s+1, \dots, \nu\}$ ,  $K_i \in \mathbb{R}^{d \times m_i}$ . Further, each  $K_i$  matrix is partitioned as

$$K_i = [K_{i,1}|K_{i,2}|\dots|K_{i,g_i}], \quad (13)$$

where each  $K_{i,j}$  is of dimension  $d \times p_{i,j}$ . Lastly we let

$$M_\Sigma(\lambda_i) \triangleq \begin{bmatrix} A - \lambda_i I_n & B \\ C & D \end{bmatrix}^\dagger \begin{bmatrix} I_n \\ 0_{p \times n} \end{bmatrix}. \quad (14)$$

The following theorem is the main result of this paper. It generalises the procedure of Theorem 3.1 to the end of computing the desired  $F$  in the case of repeated eigenvalues.

<sup>4</sup>In the case where the pair  $(A, B)$  of real matrices  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^{n \times m}$  is reachable, the eigenvalues of  $A + BF$ , along with their multiplicities, are freely assignable with a suitable real matrix  $F \in \mathbb{R}^{m \times n}$ , provided such eigenvalues are mirrored with respect to the real axis. However, the Jordan structure associated with such eigenvalues is not entirely free. The constraints on the Jordan structures that can be obtained in the closed-loop matrix are described in the celebrated Rosenbrock Theorem.

*Theorem 4.1:* Let  $\mathcal{L}$ ,  $\mathcal{M}$  and  $\mathcal{P}$  comprise an admissible Jordan structure for  $\mathcal{R}^*$ , and let  $K$  be a parameter matrix. For all odd  $i \in \{1, \dots, 2s\}$  and for each  $i \in \{2s+1, \dots, \nu\}$  and  $j \in \{1, \dots, g_i\}$ , build vector chains of length  $p_{i,k}$  as

$$s_{i,j,1} = N_\Sigma(\lambda_i) K_{i,j}^1 \quad (15)$$

$$s_{i,j,2} = M_\Sigma(\lambda_i) \bar{\pi}\{s_{i,j,1}\} + N_\Sigma(\lambda_i) K_{i,j}^2 \quad (16)$$

$\vdots$

$$s_{i,j,p_{i,j}} = M_\Sigma(\lambda_i) \bar{\pi}\{s_{i,j,p_{i,j}-1}\} + N_\Sigma(\lambda_i) K_{i,j}^{p_{i,j}} \quad (17)$$

From these column vectors and for such values of the index  $i$ , construct matrices

$$S_{i,j} = [s_{i,j,1}|s_{i,j,2}|\dots|s_{i,j,p_{i,j}}] \quad (18)$$

of dimension  $(n+m) \times p_{i,j}$ , and  $S_i = [S_{i,1}|S_{i,2}|\dots|S_{i,g_i}]$  of dimension  $(n+m) \times m_i$ , and finally

$$S = [S_1|S_2|\dots|S_\nu] \quad (19)$$

$$X_K = \Re\{e\{\bar{\pi}\{S\}\}\} \quad (20)$$

$$Y_K = \Re\{e\{\underline{\pi}\{S\}\}\}. \quad (21)$$

For almost every parameter matrix  $K$ , there holds  $\text{rank} X_K = r$ . For all  $K$  such that  $\text{rank} X_K = r$ , there holds  $\mathcal{R}^* = \text{im} X_K$ . Finally, the set of all friends of  $\mathcal{R}^*$  such that the Jordan structure of  $A + BF$  restricted to  $\mathcal{R}^*$  is described by  $\mathcal{L}$ ,  $\mathcal{M}$  and  $\mathcal{P}$  is parameterised in  $K$  as

$$F_K = Y_K X_K^\dagger. \quad (22)$$

where  $K$  is such that  $\text{rank} X_K = r$ .

**Proof:** For each  $i \in \{1, \dots, \nu\}$ , let  $K_i$  be an input parameter matrix as in (13), and for each  $j \in \{1, \dots, g_i\}$ , let  $S_{i,j}$  be constructed as in (18). We may partition  $S_{i,j}$  as

$$S_{i,j} = \begin{bmatrix} v'_{i,j,1} & v'_{i,j,2} & \dots & v'_{i,j,p_{i,j}} \\ w'_{i,j,1} & w'_{i,j,2} & \dots & w'_{i,j,p_{i,j}} \end{bmatrix}, \quad (23)$$

where the column vectors satisfy

$$\begin{bmatrix} A - \lambda_i I & B \\ C & D \end{bmatrix} \begin{bmatrix} v'_{i,j,1} \\ w'_{i,j,1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \vdots \\ \begin{bmatrix} A - \lambda_i I & B \\ C & D \end{bmatrix} \begin{bmatrix} v'_{i,j,l} \\ w'_{i,j,l} \end{bmatrix} = \begin{bmatrix} v'_{i,j,l-1} \\ 0 \end{bmatrix} \quad l = 2, \dots, p_{i,j}$$

Define matrices

$$V'_{i,j} = [v'_{i,j,1}|v'_{i,j,2}|\dots|v'_{i,j,p_{i,j}}],$$

$$W'_{i,j} = [w'_{i,j,1}|w'_{i,j,2}|\dots|w'_{i,j,p_{i,j}}],$$

and also  $V'_i = [V'_{i,1}|V'_{i,2}|\dots|V'_{i,g_i}]$  and  $W'_i = [W'_{i,1}|W'_{i,2}|\dots|W'_{i,g_i}]$ . Note that for all odd  $i \leq 2s$ ,  $\bar{K}_i = K_{i+1}$  implies that  $\bar{X}_i = X_{i+1}$ , and hence  $\bar{V}'_i = V'_{i+1}$  and  $\bar{W}'_i = W'_{i+1}$ . Also let

$$U_i = \frac{1}{2} \begin{bmatrix} I_{m_i} & -iI_{m_i} \\ I_{m_i} & iI_{m_i} \end{bmatrix} \quad (24)$$

Then for each odd  $i \leq 2s$ , we have  $[V'_i \ V'_{i+1}]U_i = [V_i \ V_{i+1}]$  and  $[W'_i \ W'_{i+1}]U_i = [W_i \ W_{i+1}]$ . Then,  $S$ ,  $X_K$  and  $Y_K$  in (19) may be written as

$$\begin{aligned} S &= [V'_1 \ V'_2 \ \dots \ V'_{2s} | V'_{2s+1} \ V'_{2s+2} \ \dots \ V'_v], \\ X_K &= [V_1 \ V_2 \ \dots \ V_{2s} | V_{2s+1} \ V_{2s+2} \ \dots \ V_v] \\ Y_K &= [W_1 \ W_2 \ \dots \ W_{2s} | W_{2s+1} \ W_{2s+2} \ \dots \ W_v]. \end{aligned}$$

Notice that, following the same argument of [11, Theorem 3.1], for almost all choices of  $K$  satisfying the conditions of the statement the rank of  $X_K$  equals the dimension of  $\mathcal{R}^*$ . For such a  $K$ , define  $F_K$  as in (22); we then have  $F_K V_i = W_i$  for all  $i \in \{1, \dots, v\}$ , and using (24) we obtain  $F_K [V'_i \ V'_{i+1}] = [W'_i \ W'_{i+1}]$  for odd  $i \in \{1, \dots, 2s\}$ . Hence (24-24) can be written for  $i \in \{2s+1, \dots, v\}$  as

$$\begin{bmatrix} A + BF_K \\ C + DF_K \end{bmatrix} V'_i = \begin{bmatrix} V'_i \\ 0 \end{bmatrix} J(\lambda_i) \quad (25)$$

while for odd  $i \in \{1, \dots, 2s\}$

$$\begin{bmatrix} A + BF_K \\ C + DF_K \end{bmatrix} \begin{bmatrix} V'_i & V'_{i+1} \end{bmatrix} = \begin{bmatrix} V'_i & V'_{i+1} \\ 0 & 0 \end{bmatrix} \text{diag}\{J(\lambda_i), J(\lambda_{i+1})\}. \quad (26)$$

Let  $\Lambda$  be given by (11). We then have

$$\begin{bmatrix} A + BF_K \\ C + DF_K \end{bmatrix} X_K = \begin{bmatrix} X_K \\ 0 \end{bmatrix} \Lambda, \quad (27)$$

where  $\Lambda$  is in the real Jordan canonical form described by  $\mathcal{L}$ ,  $\mathcal{M}$  and  $\mathcal{P}$ .

We now show that this parameterisation is exhaustive. Given  $\mathcal{L}$  and a friend  $F$  of  $\mathcal{R}^*$  such that  $(A + BF)\mathcal{R}^* \subseteq \mathcal{R}^* \subseteq \ker(C + DF)$  with  $\sigma(A + BF|_{\mathcal{R}^*}) = \mathcal{L}$ , we need to show that there exists  $K$  such that, building  $X_K$  and  $Y_K$  as in (20-21), there holds  $F = Y_K X_K^\dagger$ . First, notice that the set of friends  $F$  of  $\mathcal{R}^*$  such that  $\sigma(A + BF|_{\mathcal{R}^*}) = \mathcal{L}$  is parameterised as the solutions of the linear equation  $FR = -\Omega$ , where  $\Omega$  satisfies the linear equation  $\begin{bmatrix} A \\ C \end{bmatrix} R = \begin{bmatrix} R \\ 0 \end{bmatrix} \Lambda + \begin{bmatrix} B \\ D \end{bmatrix} \Omega$  with a certain  $\Lambda$  such that  $\sigma(\Lambda) = \mathcal{L}$  and where  $R$  is a basis matrix of  $\mathcal{R}^*$ . Let  $F$  be any of such friends of  $\mathcal{R}^*$ . The associated matrix  $\Lambda$  is such that  $\sigma(\Lambda) = \mathcal{L}$  satisfies  $\begin{bmatrix} A + BF \\ C + DF \end{bmatrix} R = \begin{bmatrix} R \\ 0 \end{bmatrix} \Lambda$ . Consider a change of coordinates  $T$  that brings  $\Lambda$  into the Jordan real canonical form. Let the blocks be ordered in such a way that the  $s$  complex conjugate pairs of eigenvalues are first. We can write

$$\begin{bmatrix} A + BF \\ C + DF \end{bmatrix} RT = \begin{bmatrix} R \\ 0 \end{bmatrix} T \underbrace{T^{-1} \Lambda T}_{\Lambda_J}, \quad (28)$$

and in (28) the matrix  $\Lambda_J$  can have Jordan mini-blocks of any order. In other words, (28) can be written as

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X \\ 0 \end{bmatrix} \text{diag}\{J_1(\lambda_1), \dots, J_{g_1}(\lambda_1), \\ J_1(\lambda_2), \dots, J_{g_2}(\lambda_2), \dots, J_1(\lambda_v), \dots, J_{g_v}(\lambda_v)\},$$

where  $\lambda_1, \dots, \lambda_v$  are the eigenvalues of  $A + BF$  restricted to  $\mathcal{R}^*$ ,  $g_i$  is the number of Jordan mini-blocks corresponding to the eigenvalue  $\lambda_i$  and the generic Jordan mini-block  $J_j(\lambda_i)$  is

of order  $p_{i,j}$ . Let us partition  $X$  and  $Y$  conformably with the corresponding Jordan mini-blocks that they multiply, i.e.,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_{1,1} & X_{1,2} & \dots & X_{v,g_v} \\ Y_{1,1} & Y_{1,2} & \dots & Y_{v,g_v} \end{bmatrix} \\ = \begin{bmatrix} X_{1,1} J_1(\lambda_1) & X_{1,2} J_2(\lambda_1) & \dots & X_{v,g_v} J_{g_v}(\lambda_v) \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$

Consider the generic term of this product

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_{i,j} \\ Y_{i,j} \end{bmatrix} = \begin{bmatrix} X_{i,j} \\ 0 \end{bmatrix} J_j(\lambda_i), \quad (29)$$

where  $J_j(\lambda_i)$  is the generic  $j$ -th Jordan mini-block relative to the eigenvalue  $\lambda_i$ . For the sake of simplicity, assume that its order  $p_{i,j}$  is denoted by  $t$ . First consider the case in which  $\lambda_i$  is real. Partitioning  $X_i = [v_{i,j,1} \ v_{i,j,2} \ \dots \ v_{i,j,t}]$  and  $Y_i = [w_{i,j,1} \ w_{i,j,2} \ \dots \ w_{i,j,t}]$ , (29) can be written as

$$\begin{bmatrix} A v_{i,j,1} + B w_{i,j,1} & A v_{i,j,2} + B w_{i,j,2} & \dots & A v_{i,j,t} + B w_{i,j,t} \\ C v_{i,j,1} + D w_{i,j,1} & C v_{i,j,2} + D w_{i,j,2} & \dots & C v_{i,j,t} + D w_{i,j,t} \end{bmatrix} \\ = \begin{bmatrix} v_{i,j,1} \lambda_i & v_{i,j,1} + \lambda_i v_{i,j,2} & \dots & v_{i,j,t-1} + \lambda_i v_{i,j,t} \\ 0 & 0 & \dots & 0 \end{bmatrix}. \quad (30)$$

Therefore,  $\begin{bmatrix} v_{i,j,1} \\ w_{i,j,1} \end{bmatrix} \in \ker \begin{bmatrix} A - \lambda_i I_n & B \\ C & D \end{bmatrix}$  implies that there exists  $K_{i,j}^1$  such that  $\begin{bmatrix} v_{i,j,1} \\ w_{i,j,1} \end{bmatrix} = N_{\Sigma}(\lambda_i) K_{i,j}^1$ . Moreover, from (30) we find that there exists  $K_{i,j}^2$  such that

$$\begin{bmatrix} v_{i,j,2} \\ w_{i,j,2} \end{bmatrix} = M_{\Sigma}(\lambda_i) \overline{\pi} \left\{ \begin{bmatrix} v_{i,j,1} \\ w_{i,j,1} \end{bmatrix} \right\} + N_{\Sigma}(\lambda_i) K_{i,j}^2.$$

Repeating this procedure for all  $k \in \{1, \dots, t\}$ , we find the parameters  $K_{i,j}^1, \dots, K_{i,j}^t$  which satisfy (15)-(17). This procedure can be carried out for all real Jordan mini-blocks. Consider now the case of a real mini-block associated with a complex conjugate eigenvalue  $\lambda_i = \sigma_i + i\omega_i$ . For the sake of argument assume that the Jordan mini-block has size 4 (so that  $\lambda_i$  and  $\bar{\lambda}_i$  have double multiplicity). Thus, (30) becomes

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} v_{i,j,1} + i v_{i,j,2} & v_{i,j,3} + i v_{i,j,4} \\ w_{i,j,1} + i w_{i,j,2} & w_{i,j,3} + i w_{i,j,4} \end{bmatrix} \\ = \begin{bmatrix} v_{i,j,1} + i v_{i,j,2} & v_{i,j,3} + i v_{i,j,4} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma_i + i\omega_i & 1 \\ 0 & \sigma_i + i\omega_i \end{bmatrix},$$

and the arguments above can be utilised after a re-labelling of the vectors.  $\blacksquare$

*Remark 4.1:* For every  $i \in \{1, \dots, v\}$ , there holds  $g_i \leq d$ . Indeed, if  $g_i > d$  (consider the case of real eigenvalues for the same of simplicity), then  $S = [s_{1,1,1} \ s_{1,2,1} \ \dots \ s_{1,g_i,1}]$ , where each  $s_{i,j,1}$  is a linear combination of the basis vectors of  $\ker P_{\Sigma}(\lambda_i)$ , whose dimension is  $d$ . This means that  $\text{rank } X_K < r$ , and therefore  $\mathcal{L}$ ,  $\mathcal{M}$  and  $\mathcal{P}$  are not an admissible Jordan structure. Thus,  $d$  also represents the largest multiplicity of each eigenvalue for which the corresponding Jordan structure can be made up by mini-blocks of unit size.

*Example 4.1:* Consider a quadruple  $(A, B, C, D)$  where

$$\begin{aligned} A &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & B &= \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 3 \end{bmatrix}, \\ C &= \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, & D &= \begin{bmatrix} 0 & 4 \end{bmatrix}. \end{aligned}$$

The only invariant zero of this system is  $z = 0$ . It is easy to verify that  $\mathcal{R}^*$  is spanned by the first two canonical basis vectors of  $\mathbb{R}^3$ . Hence,  $r = \dim \mathcal{R}^* = 2$ . Suppose we desire to assign the closed-loop eigenvalue  $-2$  with double multiplicity, i.e.,  $\mathcal{L} = \{-2\}$  and  $\mathcal{M} = \{2\}$ . Since the null-space of  $P_\Sigma(-2)$  is one-dimensional and spanned by  $[5 \ 4 \ 0 \mid -10 \ 0]^T$ , we need a single chain, i.e., we must have  $\mathcal{P} = \{2\}$ . Since  $g_1 = 1$ , in this case  $K = K_1 = K_{1,1}$  is  $d \times p_{i,j} = 1 \times 2$ . For example, let us take  $K = [1 \ 0]$ . Thus,  $s_{1,1,1} = [5 \ 4 \ 0 \mid -10 \ 0]^T$  and

$$s_{1,1,2} = M_\Sigma(-2) \bar{\pi}\{s_{1,1,1}\} + N_\Sigma(-2) K_{1,1}^2$$

$$= \begin{bmatrix} \frac{58}{141} & -\frac{4}{141} & 0 \\ -\frac{10}{141} & \frac{25}{141} & 0 \\ 0 & 0 & \frac{1}{2} \\ \frac{25}{141} & \frac{8}{141} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 5 \\ 4 \\ 0 \\ -10 \\ 0 \end{bmatrix} \cdot 0 = \frac{1}{141} \begin{bmatrix} 274 \\ 50 \\ 0 \\ 157 \\ 0 \end{bmatrix}$$

It follows that  $X_K = \begin{bmatrix} 5 & \frac{274}{141} \\ 4 & \frac{50}{141} \\ 0 & 0 \end{bmatrix}$  and  $Y_K = \begin{bmatrix} -10 & \frac{157}{141} \\ 0 & 0 \end{bmatrix}$ , which give  $F_K = Y_K X_K^\dagger = \begin{bmatrix} \frac{4}{3} & -\frac{25}{6} & 0 \\ 0 & 0 & 0 \end{bmatrix}$ . Since the rank of  $X_K$  is equal to 2, matrix  $F_K$  is a solution of  $F_K X_K = Y_K$ , and the Jordan form of the closed-loop matrix  $A + B F_K$  is indeed  $\text{diag} \left\{ \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}, 0 \right\}$ .

## V. CONCLUDING REMARKS

In this paper we have presented a generalisation of a result in [8] for the computation of a parameterisation of the friends of  $\mathcal{R}^*$  that assign the free inner and outer eigenstructure of the closed loop with no restrictions on the multiplicity of the eigenvalues to be assigned. The next step will be the exploitation of the parameterisation to the end of obtaining objectives such as minimum gain, improved robustness of the eigenstructure and improved departure from normality, along the same lines of [11] for the case of distinct eigenvalues.

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