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A canonical theory for short GPS baselines. Part II: the ambiguity precision and correlation

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Abstract. The present contribution is the second of four parts. It considers the precision and correlation of the least-squares estimators of the carrier phase ambiguities. It is shown how the precision and correlation of the double-differenced ambiguities as well as of the widelane ambiguities are effected by the observation weights, by the number of satellites tracked, by the number of observation epochs used, and by the change over time of the relative receiver-satellite geometry. Also the ability of the widelane transformation to decorrelate and to improve the precision is investigated.

1 Introduction

In this contribution, we continue our canonical analysis of the three single GPS baseline models: the geometry-based model, the time-averaged model, and the geometry-free model. For the *geometry-based* model, the linearized set of double-differenced (DD) observation equations reads as

$$\begin{aligned} D^T\phi_j(i) &= D^T A_i b + \lambda_j a_j \ , \\ D^T p_j(i) &= D^T A_i b \ , \end{aligned} \tag{1}$$

with $j = 1, 2$ and where $i = 1, \ldots, k$ denotes the epoch number and k equals the total number of epochs; ϕ_1, ϕ_2, p_1, and p_2 are the m-vectors containing the (observed minus computed) metric *single-differenced* (SD) phase and code observables on L_1 and L_2; D^T is the $(m-1) \times m$ DD matrix operator; A_i is the $m \times 3$ SD design matrix that captures the relative receiver-satellite geometry at epoch i; b is the 3-vector that contains the unknown increments of the three-dimensional baseline; λ_1 and λ_2 are the wavelengths of L_1 and L_2; and a_1 and a_2 are the two $(m-1)$-vectors that contain the unknown *integer* DD ambiguities. Time correlation is assumed to be absent and the time-invariant weight matrix (inverse variance matrix) at epoch i is assumed to be given as the block diagonal matrix

$$Q^{-1} = \text{diag}(\alpha_1, \alpha_2, \beta_1, \beta_2) \otimes (D^T D)^{-1} \ , \tag{2}$$

where '$\otimes$' denotes the Kronecker product. The scalars α_1, α_2, β_1, and β_2 are the weights of the L_1 and L_2 phase and code observables.

The *time-averaged* model follows from taking the time-average of the vectorial observation equations of Eq.(1). The *geometry-free* model follows from the geometry-based model if we disregard the presence of the receiver-satellite geometry. Hence, it follows if we replace $A_i b$ in Eq.(1) by the SD range vector r_i.

In *Part I* Teunissen (1996b) we focussed our attention on the first set of unknown parameters in the single-baseline model, the baseline vector, and studied the gain in baseline precision due to ambiguity fixing. The gain was defined as the *variance ratio* of baseline components before and after ambiguity fixing

$$\gamma(f) = \frac{f^T Q_{\hat{b}}(\phi) f}{f^T Q_{\check{b}}(\phi) f} \quad \text{with} \quad f \in R^3 \ , \tag{3}$$

with $Q_{\hat{b}}(\phi)$ and $Q_{\check{b}}(\phi)$ being the variance matrices of the 'floated' and 'fixed' baseline based on phase data only. The stationary values of the variance ratio Eq.(3) were referred to as the *gain numbers*.

In the present contribution we turn our attention to the second set of unknown parameters, the ambiguity vector, and study the precision and correlation of its least-squares estimator. As it will be shown, it is again the gain numbers which allow us to reveal the intrinsic properties of ambiguity precision and correlation.

In Sect. 2 we give the least-squares estimators of the ambiguities together with their variance-covariance matrices. It shows how the ambiguity estimators for the three different single-baseline models can be ranked precision-wise. In Sect. 3 we consider the average precision. This is done for the DD ranges, for the DD ambiguities, and for the widelane ambiguities. Special attention is given to the dependency on the choice of reference satellite. An easy-to-compute relation is established between the average precision of the DD ambiguities and the average gain in baseline precision due to ambiguity fixing. In Sect. 4 we study the precision of the widelane ambiguities in relation to the precision of the DD ambiguities. It is shown that it is *not* guaranteed that the widelane ambiguities are of a better precision than their DD counterparts. For each of the three different single-baseline models, the conditions are derived which state when and to what extent the precision of the widelane ambiguities is better than that of the DD ambiguities.

The ambiguity correlation is taken up in Sects. 5 and 6. In the former we study the correlation between the L_1 and the L_2 ambiguities and the correlation between the widelane ambiguities and the L_2 ambiguities. It is shown how the correlation depends on the change in the receiver-satellite geometry and on the observation weights used. The correlation is in particular large when the phase-code variance ratio is small and the gain numbers are large. In Sect. 6 we study the L_1 and the L_2 ambiguities in pairs and we formulate the necessary and sufficient conditions that need to be fulfilled in order to be able to decorrelate the ambiguities.

In *Part I* it was shown how the results that hold true for the time-averaged model or for the geometry-free model can be obtained from the results of the geometry-based model. For instance, the results for the time-averaged model follow from the results of the geometry-based model by simply letting the gain numbers tend to infinity. The known relations between the three models will therefore become useful in the present contribution as well, since it will enable us to present proofs of some of our results in a compact manner.

2 The least-squares ambiguities

In this section the least-squares estimates of the DD ambiguities will be given. This will be done for the geometry-based model, the time-averaged model, and the geometry-free model. The results for the latter two models follow rather straightforwardly from the least-squares ambiguity solution of the geometry-based model. The estimates and corresponding variance-covariance matrices are given in the following theorem.

Theorem 1 (*Least-squares DD ambiguities*)

The least-squares ambiguity estimates and their variance-covariance matrices of the *geometry-based*, the *time-averaged*, and the *geometry-free* model, are given as

$$\begin{aligned} \hat{a}_1 &= \frac{1}{\lambda_1} D^T[\bar{\phi}_1 - r] \ , \\ \hat{a}_2 &= \frac{1}{\lambda_2} D^T[\bar{\phi}_2 - r] \ , \end{aligned} \tag{4}$$

and

$$\begin{aligned} Q_{\hat{a}_1} &= \frac{1}{\lambda_1^2} D^T\left[\frac{1}{\alpha_1 k} I_m + Q_r\right] D \ , \\ Q_{\hat{a}_2} &= \frac{1}{\lambda_2^2} D^T\left[\frac{1}{\alpha_2 k} I_m + Q_r\right] D \ , \\ Q_{\hat{a}_1\hat{a}_2} &= \frac{1}{\lambda_1\lambda_2} D^T[Q_r]D \ , \end{aligned} \tag{5}$$

with the least-squares estimates of the time average of the DD range vector given as

$$D^T r = \begin{cases} \hat{r}(\phi, p) = D^T \bar{A}\,\hat{b}(\phi, p) & \text{(geometry-based)} \\ \bar{r}(\bar{p}) = D^T \bar{A}\,\hat{b}(\bar{p}) & \text{(time-averaged)} \\ \tilde{r}(\bar{p}) = D^T \bar{p} & \text{(geometry-free)} \end{cases} \tag{6}$$

where $\bar{\phi}_j = \frac{1}{k}\sum_{i=1}^{k} \phi_j(i)$, $\bar{p}_j = \frac{1}{k}\sum_{i=1}^{k} p_j(i)$, $j = 1, 2$, $\bar{A} = \frac{1}{k}\sum_{i=1}^{k} A_i$ and $\bar{p} = (\beta_1\bar{p}_1 + \beta_2\bar{p}_2)/(\beta_1 + \beta_2)$.

Proof: see Appendix. □

This result shows that the least-squares estimates of the DD ambiguities can be given a rather straightforward interpretation. They are simply the double-difference version of the difference between a biased and an unbiased range. The biased range is given by the time average of the phase data and the unbiased range is given by the least-squares estimate of the time average of the receiver-satellite ranges. These two types of range will generally differ greatly in precision. The biased range is highly precise, due to the high precision of the phase data. The unbiased range, however, is generally of a rather poor precision, in particular when short observation time-spans are used.

For the three models the least-squares ambiguity estimates only differ in their use of the range estimate. Hence, a comparison of their precision can be based on a comparison of the precision of the three types of range estimator. Since $Q_{\hat{b}}(\phi, p) \leq Q_{\hat{b}}(\bar{p})$, we have $Q_{\hat{r}} \leq Q_{\bar{r}}$. We also have $Q_{\bar{r}} \leq Q_{\tilde{r}}$. This can be seen as follows. Since $Q_{\bar{r}} = \frac{1}{(\beta_1+\beta_2)k} D^T\bar{A}(\bar{A}^T P\bar{A})^{-1}\bar{A}^T D$ and $Q_{\tilde{r}} = \frac{1}{(\beta_1+\beta_2)k} D^T D$, we have to show that $D^T\bar{A}(\bar{A}^T P\bar{A})^{-1}\bar{A}^T D \leq D^T D$. But this follows since $PD = D$ and $P\bar{A}(\bar{A}^T P\bar{A})^{-1}\bar{A}^T P$ is a projector having eigenvalues equal to 1 or zero. Thus precision-wise, the three types of range estimator can be ranked according to

$$Q_{\hat{r}} \leq Q_{\bar{r}} \leq Q_{\tilde{r}} \ . \tag{7}$$

The same ranking then holds of course also for the corresponding ambiguities. The first equality is satisfied when $A_i = \bar{A}$, for all i, and the second equality is satisfied when $D^T\bar{A}$ is invertible.

3 Average precision

In this section we will use the results of Theorem 1 to show how the average precision is related to the average gain in baseline precision. The ambiguities, being of a double-differenced nature, are like all DD estimators very much dependent on the arbitrary choice of reference satellite. This implies that a simple averaging of the ambiguity precision would still leave us with an unwanted dependency on the arbitrary choice of reference satellite. That is, the trace of the ambiguity variance-covariance matrix is *not* invariant for changes in the choice of reference satellite. This implies that a different averaging operation has to be performed. The idea we will use is one of a double averaging. First we average over the $(m-1)$ double-difference variances and then we average over the m different reference satellites that can be chosen. The first average is still dependent on the choice of reference satellite, but with the second average, the final result will be independent of it. Our definition of the average variance of a DD estimator is therefore given as follows.

Definition (*Average DD precision*)

Let σ_{is}^2 be the variance of the ith component of a DD $(m-1)$-estimator, $i \in \{1,\ldots,m\}/\{s\}$, having satellite $s \in \{1,\ldots,m\}$ as reference. Then the average variance of the DD estimator is defined as

$$\bar{\sigma}^2 = \frac{1}{m}\sum_{s=1}^{m}\frac{1}{m-1}\sum_{i=1, i\neq s}^{m}\sigma_{is}^2 \ . \tag{8}$$

Based on this definition, the following theorem shows how the average variance is computed from the entries of the DD variance-covariance matrix.

Theorem 2 (*Average DD precision*)

Let Q be the $m \times m$ variance matrix of a SD estimator and let $D^T Q D$ be the $(m-1)\times(m-1)$ variance matrix of its DD counterpart. The average variance of the DD estimator then reads

$$\begin{aligned}\bar{\sigma}^2 &= \frac{2}{m-1}[\mathrm{trace} QP] \\ &= \frac{2}{m-1}[\mathrm{trace} Q - \frac{1}{m}e_m^T Q e_m] \\ &= \frac{2}{m-1}[\mathrm{trace} D^T Q D - \frac{1}{m}e_{m-1}^T D^T Q D e_{m-1}] \ ,\end{aligned} \tag{9}$$

with the orthogonal projector $P = D(D^T D)^{-1} D^T$ and where the m-vector e_m and the $(m-1)$-vector e_{m-1} have entries which all are equal to 1.

Proof: see Appendix. □

The first two equations of the theorem show that the average variance as defined is indeed independent of the choice of reference satellite. The last equation shows how the average variance can be computed from the entries of a DD variance-covariance matrix. The computation is very straightforward. The term within the square brackets of Eq.(9) equals $(m-1)/m$ times the trace of the variance matrix, minus $2/m$ times the sum of the entries of the upper triangular part (excluding the diagonal) of the variance matrix.

We are now in a position to apply the theorem. First we will consider the average variance of the DD range estimator and then the average variance of the ambiguities. The following theorem shows how the average variance of the DD range estimator depends on the receiver-satellite geometry.

Theorem 3 (*Average precision of DD range*)

The average variances of the least-squares estimators of the time-averaged DD range, $\bar{\sigma}_{\hat{r}}^2$, $\bar{\sigma}_{\bar{r}}^2$ and $\bar{\sigma}_{\check{r}}^2$, of, respectively, the *geometry-based*, the *time-averaged*, and the *geometry-free* model, are given as

$$\begin{aligned}&\text{(i)} \quad \bar{\sigma}_{\hat{r}}^2 = \frac{6(\bar{\delta}-1)\epsilon}{(\beta_1+\beta_2)k(m-1)} \quad (m \geq 4) \ , \\ &\text{(ii)} \quad \bar{\sigma}_{\bar{r}}^2 = \frac{6}{(\beta_1+\beta_2)k(m-1)} \quad (m \geq 4) \ , \\ &\text{(iii)} \quad \bar{\sigma}_{\check{r}}^2 = \frac{2}{(\beta_1+\beta_2)k} \quad (m \geq 2) \ ,\end{aligned} \tag{10}$$

with the *average gain* in baseline precision

$$\bar{\delta} = \frac{1}{3}\sum_{i=1}^{3}\frac{(1+\epsilon)\gamma_i}{1+\epsilon\gamma_i} \tag{11}$$

and the weight ratio $\epsilon = (\beta_1+\beta_2)/(\alpha_1+\alpha_2)$.

Proof: see Appendix. □

This result clearly shows how the average precision of the DD range estimator depends on the observation weights of phase and code $(\alpha_1, \alpha_2, \beta_1, \beta_2)$, the number of epochs used (k), the number of satellites tracked (m), and the change in receiver-satellite geometry (γ_i). The condition $m \geq 4$ is due to the presence of the three-dimensional baseline in the geometry-based model and the time-averaged model. For the geometry-free model we have the condition $m \geq 2$, since in this case, due to the absence of the baseline, two satellites already suffice to be able to estimate a DD range.

Since $\epsilon(\gamma_i - 1)/(\epsilon\gamma_i + 1) \leq 1$ and $3/(m-1) \leq 1$ for $m \geq 4$, it is clear that the precision of the DD range based on the geometry-free model is poorer than that based on the time-averaged model, which in turn is poorer than that of the geometry-based model. The three average variances are related as

$$\bar{\sigma}_{\check{r}}^2 \ \xrightarrow{\frac{3}{m-1}} \ \bar{\sigma}_{\bar{r}}^2 \ \xrightarrow{(\bar{\delta}-1)\epsilon} \ \bar{\sigma}_{\hat{r}}^2 \ . \tag{12}$$

The variance $\bar{\sigma}_{\hat{r}}^2$ is a monotone increasing function of the gain-numbers and reaches its maximum $\bar{\sigma}_{\bar{r}}^2$ when $\gamma_i = \infty$. The variance $\bar{\sigma}_{\bar{r}}^2$ gets larger as m gets smaller and reaches its maximum $\bar{\sigma}_{\check{r}}^2$ when $m = 4$.

Note that $\bar{\sigma}^2_{\hat{r}}$ depends on the receiver-satellite geometry, whereas $\bar{\sigma}^2_{\bar{r}}$ does not. This is also what one would expect. But note that also $\bar{\sigma}^2_{\bar{r}}$ is independent of the receiver-satellite geometry. At a first instance, this is not what one would expect, since the time-averaged model depends on $\bar{A}$ and thus on the receiver-satellite geometry. The explanation for the absence of the dependence on the receiver-satellite geometry is as follows. The variance matrix of the range estimator $\bar{r}(\bar{p})$ is based on a projector. This projector depends on $\bar{A}$, but its trace does not. This implies that in the averaging process the explicit dependency on the receiver-satellite geometry vanishes. What remains is the factor $3/(m-1)$, with the 3 due to the dimension of the baseline and the $(m-1)$ due to the averaging.

We now turn to the average precision of the ambiguities. Apart from the time-averaged phases, the ambiguity estimates are constructed from the least-squares estimators of the DD ranges. Hence, we can use the above results to obtain the average ambiguity variances. The results are stated in the following theorem; we have included the average precision of the *widelane* ambiguities as well.

Theorem 4 (*Average precision of ambiguities*)

The average variances of the L_1, the L_2, and the widelane (L_w) ambiguities are given for the *geometry-based*, the *time-averaged*, and the *geometry-free* model as

$$\begin{array}{lll} \text{(i)} & \bar{\sigma}^2_{\hat{a}} = c_1\left[1 + c_2\frac{3(\bar{\delta}-1)}{m-1}\right] & (m \geq 4) \;, \\ \text{(ii)} & \bar{\sigma}^2_{\bar{a}} = c_1\left[1 + c_2\frac{3}{(m-1)\epsilon}\right] & (m \geq 4) \;, \\ \text{(iii)} & \bar{\sigma}^2_{\check{a}} = c_1\left[1 + c_2\frac{1}{\epsilon}\right] & (m \geq 2) \;, \end{array} \tag{13}$$

with

$$\begin{array}{lll} L_1\!: & c_1 = \dfrac{2}{\alpha_1\lambda_1^2 k} & c_2 = \dfrac{\alpha_1}{\alpha_1+\alpha_2} \;, \\ L_2\!: & c_1 = \dfrac{2}{\alpha_2\lambda_2^2 k} & c_2 = \dfrac{\alpha_2}{\alpha_1+\alpha_2} \;, \\ L_w\!: & c_1 = 2\dfrac{\alpha_1\lambda_1^2+\alpha_2\lambda_2^2}{\alpha_1\alpha_2\lambda_1^2\lambda_2^2 k} & c_2 = \dfrac{\alpha_1\alpha_2(\lambda_2-\lambda_1)^2}{(\alpha_1+\alpha_2)(\alpha_1\lambda_1^2+\alpha_2\lambda_2^2)} \;. \end{array}$$

Proof: see Appendix. □

As with the DD ranges, the precision of the ambiguities is poorest when based on the geometry-free model and best when based on the geometry-based model. Note that the gains in precision when switching between the three models, are somewhat larger for the DD ranges than they are for the ambiguities. But since ϵ is small these differences are negligible. Hence for all practical purposes, Eq.(12) holds for the ambiguities as well.

Note that since ϵ is very small and $(\bar{\delta}-1)$ large for short time-spans, the second terms within the square brackets of Eq.(13) will be much larger than 1. This shows that the precision of the ambiguities is mainly driven by the precision of the DD ranges. The average precision of the ambiguities will therefore be rather poor, unless a sufficient number of samples are taken or a sufficiently long observation time-span is used.

Apart from the fact that this last theorem shows how the average precision is effected by the observation weights, by the number of satellites tracked, by the number of observation epochs used, and by a change in the receiver-satellite geometry, it also provides for a fast and easy way to compute the *average gain* in baseline precision. Based on the average variance of either the L_1 or the L_2 ambiguities, the average gain in baseline precision follows from Theorem 4 as

$$\bar{\delta} = 1 + \frac{m-1}{6}\left[\frac{\lambda_j^2\bar{\sigma}^2_{\hat{a}_j} - \sigma^2_{\bar{\phi}_j}}{\sigma^2_{\bar{\phi}}}\right], \qquad j = 1,2 \;, \tag{14}$$

where $\sigma^2_{\bar{\phi}} = 1/(\alpha_1+\alpha_2)k$ is the variance of the weighted average of the L_1 and L_2 time-averaged phases. Since in practice the second term of Eq.(14) will be much larger than 1 and the average variance of the ambiguities, when expressed in units of range rather than cycles, much larger than the variance of the time-averaged phases, the average gain can be approximated as $\bar{\delta} \simeq \lambda_j^2\frac{m-1}{6}\bar{\sigma}^2_{\hat{a}_j}/\sigma^2_{\bar{\phi}}$.

Equation (14) holds for both the phase-only case and the phase-and-code case. In the phase-only case, $\epsilon = 0$ and Eq.(14) directly gives the average gain number $\bar{\gamma}$ and thus the average change in the receiver-satellite geometry. In order to obtain the average gain number from Eq.(14) when both phase and code data are used, one can use the following approximation

$$\bar{\delta} - 1 \simeq \frac{\bar{\gamma}-1}{1+\epsilon\bar{\gamma}} \;. \tag{15}$$

This equation is exact when $\epsilon = 0$ and a good approximation when $\epsilon \neq 0$. To see this, we develop the right-hand side of Eq.(11) into its Taylor series around $\bar{\gamma}$ up to and including the second-order term. From this expansion we obtain the approximation

$$\bar{\delta} - 1 \simeq \frac{\bar{\gamma}-1}{1+\epsilon\bar{\gamma}}\left[1 - \frac{1}{3}\left(\frac{\bar{\gamma}-1}{1+\epsilon\bar{\gamma}}\right)\left(\frac{\epsilon(1+\epsilon)}{1+\epsilon\gamma}\right)\sum_{i=1}^{3}\left(\frac{\bar{\gamma}-\gamma_i}{\bar{\gamma}-1}\right)^2\right] .$$

We know that $0 \leq \epsilon \leq 1$. For $\epsilon = 0$, we of course obtain Eq.(15) again. But also for $\epsilon = 1$, the second term within the square brackets can be neglected when the average gain number is large.

Thus in order to compute the average gain number $\bar{\gamma}$ one proceeds as follows. From the ambiguity variance matrix, one first computes the average variance using the last equation of Eq.(9). The average gain in baseline precision is then obtained from Eq.(14) and from it, by inversion of Eq.(15), the average gain number is obtained.

4 The widelane precision

In this section we will study the precision of the widelane (L_w) ambiguities in relation to the precision of the DD ambiguities. The least-squares estimate of the widelane ambiguity vector is defined as $\hat{a}_w = \hat{a}_1 - \hat{a}_2$. In order to compare the precision of the widelane ambiguities with the precision of the L_1 ambiguities, we compare the variances of the two linear functions $f^T\hat{a}_w$ and $f^T\hat{a}_1$. The variance matrices of the L_1 ambiguities and the widelane ambiguities of the geometry-based model follow from Theorem 1 as

$$Q_{\hat{a}_1} = D^T\left[\frac{1}{\alpha_1 k\lambda_1^2}I_m + \frac{1}{\lambda_1^2}\bar{A}Q_{\hat{b}}(\phi,p)\bar{A}^T\right]D \ ,$$

$$Q_{\hat{a}_w} = D^T\left[\left(\frac{1}{\alpha_1 k\lambda_1^2} + \frac{1}{\alpha_2 k\lambda_2^2}\right)I_m + \frac{1}{\lambda_w^2}\bar{A}Q_{\hat{b}}(\phi,p)\bar{A}^T\right]D \ ,$$

where λ_w is the wavelength of the widelane. Thus $\frac{1}{\lambda_w} = \frac{1}{\lambda_1} - \frac{1}{\lambda_2}$. The variance of the widelane function $f^T\hat{a}_w$ is smaller than or at most equal to the variance of $f^T\hat{a}_1$, when

$$\frac{f^TQ_{\hat{a}_w}f}{f^TQ_{\hat{a}_1}f} \le 1 \ . \tag{16}$$

If this holds for all $f \in R^{m-1}$, then all functions of the widelane ambiguities have a precision which is better than the precision of the same functions of the L_1 ambiguities. In that case, the individual widelane ambiguities themselves also have a precision which is better than their L_1 counterparts. This is not guaranteed however, if the inequality does not hold.

The following theorem allows one to deduce the conditions for which the precision of the widelane ambiguities is better than that of the L_1 ambiguities.

Theorem 5 (L_w/L_1 *variance ratio*)

The stationary values of the variance ratio Eq.(16) are the roots of the characteristic equation $| Q_{\hat{a}_w} - vQ_{\hat{a}_1} |= 0$ and they are given in descending order as

$$\begin{cases} v_{1,i} = 1 + \dfrac{\alpha_1}{\alpha_2}\dfrac{\lambda_1^2}{\lambda_2^2} \ , \\ v_{2,j} = (1 - \dfrac{\lambda_1}{\lambda_2})^2 + \dfrac{\lambda_1}{\lambda_2}\dfrac{2 + (\frac{\alpha_1}{\alpha_2} - 1)\frac{\lambda_1}{\lambda_2}}{\frac{\alpha_1}{\alpha_1+\alpha_2}\delta_j + \frac{\alpha_2}{\alpha_1+\alpha_2}} \ , \end{cases} \tag{17}$$

with $\delta_j = (1+\epsilon)\gamma_j/(1+\epsilon\gamma_j)$ and for $i = 1, \ldots, (m-4)$, $j = 1, 2, 3$.

Proof: see Appendix. □

The theorem shows that the variance ratios or eigenvalues can be divided in two groups; a first group of $(m-4)$ eigenvalues which are all equal, and a second group of three smaller eigenvalues, which are generally not equal. The eigenvalues of the first group are independent of the receiver-satellite geometry (γ_i) and independent of the weight ratio ϵ. This is not the case for the three smallest eigenvalues.

Let us first have a closer look at the eigenvalues of the first group. Since these eigenvalues are clearly larger than 1, there exist $(m-4)$ linear functions of $\hat{a}_w$ that have a precision which is poorer than the precision of the same functions of $\hat{a}_1$. This already shows that it is *not* guaranteed that the widelane ambiguities are of a better precision than the L_1 ambiguities. The functions that produce these large variance ratios are those which are invariant to changes in the baseline; that is, those functions for which $f \in N(\bar{A}^TD)$. Fortunately, since these functions are of a very high precision when applied to $\hat{a}_1$ and since the $(m-4)$ largest variance ratios are only 1.61 when $\alpha_1 = \alpha_2$, the corresponding widelane functions will still be very precise.

Let us now consider the three smallest eigenvalues. They become identical to the $(m-4)$ largest eigenvalues when $\gamma_i = 1$ or when $\epsilon = \infty$. In that case *every* function of the widelane ambiguities has a precision which is poorer than the same functions of the L_1 ambiguities. The individual widelane ambiguities themselves will then also be poorer than their DD counterparts. In practice however, these conditions will not be satisfied, since $\epsilon = \infty$ corresponds with having exact code data and $\gamma_i = 1$ corresponds with a situation that requires an unrealistically long observation time-span. But the three smallest eigenvalues can still be larger than 1 for $\gamma_i \neq 1$. It is therefore also of importance to know how large the change in the receiver-satellite geometry must be, in order for the three smallest eigenvalues to be equal to 1. It follows from Eq.(17) that

$$v_{2,i} = 1 \Leftrightarrow \frac{\gamma_i - 1}{\epsilon\gamma_i + 1} = \frac{\alpha_1 + \alpha_2}{\alpha_2}\left[2\frac{\lambda_2}{\lambda_1} - 1\right]^{-1}, \ \ i = 1, 2, 3 \ .$$

Since the right-hand side of the second equality is about 1.28, it follows for small ϵ that the gain numbers must be $\gamma_i = 2.28$. This value is so small and the corresponding change in receiver-satellite geometry so large, that we may conclude in practice that the three smallest eigenvalues will indeed be smaller than 1.

Since the three smallest eigenvalues are monotone decreasing functions of the gain numbers γ_i, the corresponding precision of the widelane ambiguities will get better in relation to that of the L_1 ambiguities, when there is less change in the receiver-satellite geometry. The limiting case $\gamma_i = \infty$ holds for the single-epoch geometry-based model and for the time-averaged model. In this case, the second term on the right-hand side of the expression for $v_{2,i}$, $i = 1, 2, 3$, can be neglected and the three smallest variance ratios can be approximated as

$$v_{2,i} \simeq \left(1 - \frac{\lambda_1}{\lambda_2}\right)^2 \simeq 0.05, \qquad i = 1, 2, 3 \tag{18}$$

which is considerably smaller than 1. The functions that produce these small eigenvalues are those lying in the range space of $D^+\bar{A}$, where D^+ is the pseudo-inverse of D. These are also the functions, when applied to the L_1 ambiguities, that produce the poorest precision. For the time-averaged model, the single-epoch geometry-based model and the geometry-based model with a short observation time-span, the conclusion therefore reads that by using the widelane ambiguities, functions of the L_1 ambiguities that have a very poor precision have their precision improved by a factor of about 20, while functions of the L_1 ambiguities that already have a very high precision, have their precision degraded by a factor of only 1.61.

So far we have considered the geometry-based model and the time-averaged model. In order to obtain the results for the geometry-free model, we simply have to set $\gamma_i = \infty$ and $m = 4$ in Eq.(17). Hence, in this case all eigenvalues are smaller than 1 and approximately equal to 0.05. Thus for the geometry-free model, it is guaranteed that every function of the widelane ambiguities has a precision better than the precision of the same functions of the L_1 ambiguities.

When comparing the results for the three models, we thus observe that the 'stronger' the model becomes in terms of change in receiver-satellite geometry (smaller γ_i), amount of satellite redundancy [larger $(m-4)$], and in terms of the relative precision of code with respect to phase (larger ϵ), the less relevant the widelane ambiguities become where it concerns their precision in relation to that of the DD ambiguities.

5 Ambiguity correlation

In the previous two sections we studied the precision of the ambiguities, both of the DD ambiguities as well as of the widelane ambiguities. In this section, we will study the correlation between the ambiguities. First we will analyze the correlation between the L_1 and the L_2 ambiguities and then consider the correlation between the widelane ambiguities and the L_2 ambiguities. The following theorem provides the canonical correlation structure between the L_1 and the L_2 ambiguities for the geometry-based model. The canonical correlations that hold for the time-averaged model and the geometry-free model can also be obtained from the theorem.

Theorem 6 (*L_1/L_2 correlation*)

Let $\rho(p, q)$ be the correlation coefficient of $p^T\hat{a}_1$ and $q^T\hat{a}_2$. Then for the geometry-based model, the solution to

$$\rho_i = \rho(p_i, q_i) = \max_p \max_q \rho(p, q) \ ,$$

subject to

$$p^T Q_{\hat{a}_1} p_j = 0 \text{ and } q^T Q_{\hat{a}_2} q_j = 0, \quad j = 1, \ldots, (i-1) \ ,$$

is given as

$$\rho_i = \frac{\gamma_{4-i} - 1}{\sqrt{\left(\gamma_{4-i}\frac{\alpha_1+\beta_1+\beta_2}{\alpha_1} + \frac{\alpha_2}{\alpha_1}\right)\left(\gamma_{4-i}\frac{\alpha_2+\beta_1+\beta_2}{\alpha_2} + \frac{\alpha_1}{\alpha_2}\right)}} \tag{19}$$

for $i = 1, 2, 3$ and $\rho_i = 0$ for $i = 4, \ldots, (m-1)$.

Proof: see Appendix. □

In order to obtain the results that hold true for the geometry-free model, we simply have to set $m = 4$ and $\gamma_{4-i} = \infty, \ i = 1, 2, 3$. This shows that all ambiguity correlation coefficients of the geometry-free model are non-zero and equal to

$$\rho = \sqrt{\frac{\alpha_1\alpha_2}{(\alpha_1+\beta_1+\beta_2)(\alpha_2+\beta_1+\beta_2)}} \ . \tag{20}$$

These correlation coefficients are very close to 1, due to the very high precision of the phase data and the relatively poor precision of the code data. Note that the correlation coefficient is somewhat pulled away from 1, due to the presence of the code data. This effect however, will be very small in practice. Hence we must conclude that in case of the geometry-free model, the L_1 and the L_2 ambiguities are highly correlated indeed.

The results that hold true for the time-averaged model follow by setting $\gamma_{4-i} = \infty$. Thus in this case, again the three largest correlation coefficients are very close to one. The remaining correlation coefficients however are identically zero, provided that satellite redundancy is present. This can be explained by the dimension of the null space of $D^T\bar{A}$, which equals $(m-4)$. That is, the zero correlations correspond with ambiguity functions that are invariant to changes in the baseline. These are also the functions which have a very high precision.

For the geometry-based model the results are identical to that of the time-averaged model if only one single observation epoch is used. They will differ however, when more than one epoch of data is used. This difference will be small though, when the gain numbers are large, that is, when only a short observation time-span is used. In that case, the three largest correlation coefficients can be approximated by Eq.(20).

Since the correlation is close to 1 when the gain numbers are large, it is of interest to consider the sensitivity of the correlation to changes in the receiver-satellite geometry. If we assume equally weighted phase data and equally weighted code data, the derivative of the correlation coefficient with respect to its gain number follows from the theorem as

$$\frac{\mathrm{d}\rho}{\mathrm{d}\gamma} = \frac{2(1+\epsilon)}{(\gamma(1+2\epsilon)+1)^2} \ .$$

This allows us to give a sketch of the curve $\rho(\gamma)$. It starts at zero for $\gamma_i = 1$, having a slope of somewhat less than 30°, since ϵ is small. It then increases as the gain gets larger. Finally, it slowly approaches 1 when the gain tends to infinity, at which point it has a horizontal tangent. This shows that the canonical correlation coefficient is *not* very sensitive to changes in the gain, when the gain is large. Only when $\gamma_i \simeq 1$ will the change in ρ_i be equal to about one half of the gain number. The conclusion reads therefore, that in practice one can forget about being able to *decorrelate* the ambiguities, by taking advantage of small changes in the receiver-satellite geometry. Very small gain numbers are needed to push the correlation coefficient significantly towards zero. For instance, for $\rho_i \leq 1/2$, one must have $\gamma_i \leq 3$, ($\alpha_1 = \alpha_2, \epsilon = 0$).

From the preceding discussion we can thus draw the conclusion that for all practical purposes, all three models provide least-squares ambiguities that are extremely correlated. We will therefore now consider the widelane (L_w) ambiguities and study how they effect the correlation structure. The following theorem relates the L_w/L_2-correlation to the L_1/L_2-correlation. We will only consider the magnitude of the correlation and not its sign.

Theorem 7 (*L_w/L_2 correlation*)

Assume that the L_1 and L_2 phase data are equally precise ($\alpha_1 = \alpha_2$), and let $\rho_{12,i}$ and $\rho_{w2,i}$, $i = 1, \ldots, (m-1)$, be the L_1/L_2 and L_w/L_2 canonical correlation coefficients. Then

$$\rho^2_{w2,i} = \frac{\left(\rho_{12,i} - \frac{\lambda_1}{\lambda_2}\right)^2}{\left(\rho_{12,i} - \frac{\lambda_1}{\lambda_2}\right)^2 + \left(1 - \rho^2_{12,i}\right)} \tag{21}$$

for $i = 1, \ldots, (m-1)$.

Proof: The proof goes along similar lines to that of Theorem 6. □

Note that the L_w/L_2-correlation coefficient is *not* a monotone function of the L_1/L_2-correlation on the complete interval $[0, 1]$. We therefore have to discriminate between two intervals. For the interval $\rho_{12,i} \in [0, \frac{\lambda_1}{\lambda_2}]$, the L_w/L_2-correlation is a monotone decreasing function, and for the interval $\rho_{12,i} \in [\frac{\lambda_1}{\lambda_2}, 1]$ it is a monotone increasing function. In the first interval, the function starts at

$$\rho_{w2,i} = 1 \Big/ \sqrt{1 + \frac{\lambda_2^2}{\lambda_1^2}} \simeq 0.61 \ , \tag{22}$$

then decreases, passing the line $\rho_{w2,i} = \rho_{12,i}$ at $\rho_{12,i} = \frac{1}{2}\frac{\lambda_1}{\lambda_2}$ and then ends at zero. In the second interval, it starts at zero and ends at 1. Thus in this second interval the L_w/L_2-correlation is always less than the L_1/L_2-correlation, unless $\rho_{12,i} = 1$. In the first interval however, this is only the case for the second half of the interval. Thus the interval in which *decorrelation* occurs is given by

$$\rho_{12,i} \in \left(\frac{1}{2}\frac{\lambda_1}{\lambda_2}, 1\right) \simeq (0.39, 1) \ . \tag{23}$$

The decorrelation that takes place inside this interval, in particular in the interval $[\frac{\lambda_1}{\lambda_2}, 1]$, can be very significant. It follows from the theorem that $\rho_{w2,i} \leq 0.5$ for $\rho_{12,i} \leq 0.9534$. However, the decorrelation is much smaller for values of $\rho_{12,i}$ that are closer to one. For instance, for $\rho_{12,i} = 0.999$, we get $\rho_{w2,i} = 0.980$ and for $\rho_{12,i} = 0.99$, we get $\rho_{w2,i} = 0.83$. Thus for DD ambiguities that are extremely correlated, the reduction in correlation can be expected to be marginal. We will return to this matter in the following section.

From Theorem 6 we know that for the L_1 and the L_2 ambiguities, the large correlation coefficients are close to 1 and the small correlation coefficients identical to zero. Hence for all practical purposes, the transformation to the widelane ambiguities can be seen as a decorrelating transformation, in the sense that it pushes the large correlation coefficients down to smaller values. The price to be paid, in case satellite redundancy is present in the time-averaged model and in the geometry-based model, is that the zero correlation coefficients are pulled upwards to the level of Eq.(22). We thus see here a similar mechanism at work as that which we saw when studying the effect of the widelane transformation on the precision of the ambiguities. In the precision case, the ambiguity functions having a poor precision were improved at the expense of the very precise ambiguity functions. In the correlation case, it is the ambiguity functions that are correlated which get improved, at the expense of the functions that are not correlated.

6 Paired L_1 and L_2 ambiguities

In the theory of integer least-squares ambiguity estimation, a central role is played by the ambiguity search space. This is the case for both the *estimation* as well as for the *validation* of the integer ambiguities. The ambiguity search space is a scaled version of the confidence ellipsoid of the (real-valued) least-squares ambiguities. Apart from its location and size, the ambiguity search space is uniquely determined by the variance-covariance matrix of the least-squares ambiguities and thus by the precision and correlation of the ambiguities.

In the previous sections we analyzed the precision and correlation of the DD ambiguities and of the widelane ambiguities. We have seen that in general, the DD ambiguities are of a poor precision, while highly correlated. We have also seen that for those ambiguity functions that are highly correlated and of a poor precision, the introduction of the widelane ambiguities generally results in a decorrelation and in an improvement of precision.

In this section we will study the relation between precision and correlation on the one hand, and the ambiguity search space on the other. As a result we will be able to place the transformation to the widelane ambiguities, within the class of the decorrelating ambiguity transformations. In order to keep the analysis tractable, it suffices for the present purposes to restrict our attention to the two-dimensional case. We will therefore consider the ambiguity search spaces of paired L_1 and L_2 ambiguities. In fact this is not unlike how in practice the ambiguity estimation problem is tackled in case of the geometry-free model. In that case the dual-frequency data corresponding to single satellite pairs are used to solve for the L_1 and L_2 ambiguities as single pairs. See for instance Hatch (1982), Euler and Goad (1990), Euler and Hatch (1994), and Teunissen (1996a).

In the present case, we will also consider the L_1 and L_2 ambiguities as single pairs, but now we will have the receiver-satellite geometry included as well. We will assume that the dual frequency-phase data are equally precise and also that the dual-frequency code data are equally precise. Thus $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = \beta$.

Let us consider the ith pair of L_1 and L_2 ambiguities. From Theorem 1, the 2×2 variance matrix of the ith pair then follows as

$$\begin{bmatrix} \sigma_{1i}^2 & \sigma_{1i,2i} \\ \sigma_{2i,1i} & \sigma_{2i}^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\lambda_1^2}\left(\frac{2}{\alpha k} + \frac{\kappa_i}{\beta k}\right) & \frac{\kappa_i}{\lambda_1 \lambda_2 \beta k} \\ \frac{\kappa_i}{\lambda_1 \lambda_2 \beta k} & \frac{1}{\lambda_2^2}\left(\frac{2}{\alpha k} + \frac{\kappa_i}{\beta k}\right) \end{bmatrix} , \tag{24}$$

where κ_i is equal to $k\beta$ times the variance of the ith DD range. Thus for the geometry-based model, we have $\kappa_i = k\beta\sigma_{\hat{r}}^2$. Note that

$$\kappa_i \leq 1 \ . \tag{25}$$

The equality sign holds for the geometry-free model. For the geometry-based model and the time-averaged model, the equality sign holds only when there is no satellite redundancy and the gain numbers become infinite.

Since the precision of the ambiguities is rather poor, their individual confidence intervals will be rather large. Also, the box that encloses the ambiguity search space and which has its sides parallel to the grid axes will have a rather large area. Its area can be compared to the area of the box that best fits the search space. The box that best fits the search space has its sides parallel to the two principal axes of the search space. This best-fitting box has an area which is proportional to the square root of the product of the two eigenvalues of the variance matrix of Eq.(24). Since this product is also equal to the product of the two ambiguity variances multiplied by 1 minus the square of the correlation coefficient, it follows that the ratio of the area of the best fitting box and the area of the box having its sides parallel to the grid axes is equal to the square root of 1 minus the square of the correlation coefficient. The correlation coefficient follows from Eq.(24) as

$$\rho_i = [1 + 2\epsilon/\kappa_i]^{-1} \ . \tag{26}$$

For $\kappa_i = 1$ this is of course identical to Eq.(20), when the phase data are assumed equally precise and likewise the code data. Since the correlation coefficient will generally be very close to 1, it follows that the ratio of the areas of the two boxes will be very close to zero. Hence, there exists then a large disparity between the two areas, the area of the box with its sides parallel to the grid axes being very large due to the poor precision, and the area of the best fitting box being small due to the large correlation.

Since the area of the best-fitting box is so much smaller than the area of the box having its sides parallel to the grid axes, one can expect the search space to be elongated and orientated away from the grid axes. The following theorem makes this clear.

Theorem 8 (*Orientation and elongation*)

Let ω_i be the orientation of the major axis of the search space as measured counter clockwise from the first grid axis and let $e_i \geq 1$ be the elongation of the search space as measured by the ratio of the lengths of the major and minor axes. Then

$$\begin{aligned} &\text{(i)} \quad \omega_i = \tfrac{1}{2}\arctan\left[2\rho_i\left(\tfrac{\lambda_2}{\lambda_1} - \tfrac{\lambda_1}{\lambda_2}\right)^{-1}\right] , \\ &\text{(ii)} \quad e_i \ = \left(\left[1 + \sqrt{(1-\chi)}\,\right] \Big/ \left[1 - \sqrt{(1-\chi)}\,\right]\right)^{1/2} , \end{aligned} \tag{27}$$

with

$$\chi = 4\left(1 - \rho_i^2\right)\left(\frac{\lambda_2}{\lambda_1} + \frac{\lambda_1}{\lambda_2}\right)^{-2} \ .$$

Proof: see Appendix. □

This result shows that both orientation and elongation can be expressed in terms of the correlation coefficient. The search space will be orientated under an angle of $\omega_i \simeq 38°$ if ϵ/κ_i is so small that the correlation coefficient can be approximated by 1. Better code, poorer phase and/or a stronger receiver-satellite geometry, will rotate the search space clockwise. A full alignment with the grid axes however, is only reached in the limiting case $\epsilon/\kappa_i \to \infty$.

Since the elongation e_i is a monotone decreasing function of χ, the theorem also shows that the elongation gets larger when the correlation gets larger, and thus ϵ/κ_i gets smaller. For small ϵ/κ_i, we have the approximation

$$e_i \simeq \frac{1}{2}\sqrt{\frac{\kappa_i}{\epsilon}}\left(\frac{\lambda_2}{\lambda_1} + \frac{\lambda_1}{\lambda_2}\right) ,$$

which shows that the elongation is as large as 103 when $\epsilon/\kappa_i = 10^{-4}$.

The orientation and shape of the ambiguity search space as just determined has an important impact on one's ability to solve efficiently for the integer least-squares ambiguities. It will be clear that the search for the integer least-squares ambiguities becomes trivial in case the ambiguities are uncorrelated. In that case the search space has its principal axes parallel to the grid axes and the sought-for integer ambiguities simply follow from rounding the *real-valued* least-squares ambiguities to their nearest integer. The search becomes nontrivial, however, when $\rho_i \neq 0$. In fact, the more the ambiguities are correlated, the more outstretched the search space becomes, failing to have its principal axes aligned with the grid axes. Hence the more troublesome the search becomes.

From the fact that the high correlation complicates the computational process of estimating the integer least-squares ambiguities, one should of course not conclude that it is more advantageous to abandon the concept of the search space and to concentrate on the two ambiguities separately. From a computational point of view this may seem attractive, since scalar integer least-squares estimation is trivial. But by considering the ambiguities on an individual basis, one is in fact disregarding essential information. As a consequence, one will then not be computing the sought-for integer least-squares solution, but at best only an approximation to it. Also, the validation of the integer ambiguities will be seriously affected by it. By considering the ambiguities separately, one is namely also disregarding information which could have been put to a good use in the validation step.

Thus it is important not to disregard the correlation, but instead to make a full use of it. In the computational process of estimating the integer least-squares ambiguities, a good use of the correlation can be made in defining new ambiguities which have the property that they are less correlated and more precise than the original DD ambiguities. This is the concept of the least-squares ambiguity decorrelation adjustment (LAMBDA), with which, by means of a *decorrelating* ambiguity transformation, the original search space is transformed into a new, more circular search space Teunissen (1993). The decorrelating ambiguity transformation is constructed from a sequence of Gaussian transformations of the following two types

$$Z_1^T = \begin{bmatrix} 1 & z_{12} \\ 0 & 1 \end{bmatrix} \text{ and } Z_2^T = \begin{bmatrix} 1 & 0 \\ z_{21} & 1 \end{bmatrix} , \tag{28}$$

in which the integers z_{12} and z_{21} are chosen such that the transformed ambiguities become less correlated and more precise. They are taken as -1 multiplied by the nearest integer of the ratio of the covariance and variance. Hence, the transformations are integer approximations to the fully decorrelating *conditional least-squares* transformations.

In order for the transformations of Eq.(28) to be admissible, they need to be area preserving, having entries which are all integer. The area-preserving property is automatically satisfied by means of their structure. Hence, the only remaining condition is that both z_{12} and z_{21} need to be integer. An example of a transformation from the class of Eq.(28), is given by

$$Z_w^T = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} . \tag{29}$$

This is the transformation that replaces the L_1 ambiguity by the widelane ambiguity. Due to the area-preserving property of the transformation, a direct relation can be established between, on the one hand, the precision of the original and transformed ambiguities, and on the other, their correlation coefficients. This is shown in the following corollary.

Corollary (*Precision and correlation*)

Let $\sigma^2_{w_i}/\sigma^2_{1,i}$ denote the variance ratio of the widelane ambiguity and the L_1 ambiguity of the ith pair, and let $\rho_{w,i}$ and ρ_i denote the two correlation coefficients of, respectively, the widelane ambiguity and the L_1 ambiguity with the L_2 ambiguity. Then

$$\text{(i)} \quad \rho^2_{w,i} = \left(\rho_i - \tfrac{\lambda_1}{\lambda_2}\right)^2 \Big/ \left(\left(\rho_i - \tfrac{\lambda_1}{\lambda_2}\right)^2 + \left(1 - \rho_i^2\right)\right) ,$$

$$\text{(ii)} \quad \sigma^2_{w,i}/\sigma^2_{1,i} = \left(1 - \rho_i^2\right) \Big/ \left(1 - \rho^2_{w,i}\right) .$$

Proof: Case (i) is a direct consequence of Theorem 7. Case (ii) follows from the area-preserving property of the ambiguity transformation. Since the transformation to the widelane ambiguity is area preserving, it leaves the determinant of the variance matrix unchanged. The stated result follows then from recognizing that the determinant equals 1 minus the square of the correlation coefficient, times the product of the variances. □

This result shows the coupling between correlation and precision. It shows that precision improves when the correlation gets smaller. It also shows that the amount in which the precision improves can be inferred from the correlation between the original DD ambiguities.

From the results of the corollary one can also infer that the transformed search space must be more circular than the original DD search space. This can be seen as follows. The area of the box that encloses the DD search space and which has its sides parallel to the grid axes is proportional to $\sigma_{1,i}\sigma_{2,i}$ and the area of the corresponding box for the transformed search space is proportional to $\sigma_{w,i}\sigma_{2,i}$. Since the proportionality constant is the same for both areas, the ratio of the two areas equals $\sigma_{w,i}/\sigma_{1,i}$. This shows that if the ratio is smaller than 1, the box that encloses the transformed search space will have a smaller area than the box that encloses the original search space. But this implies, since both search spaces have the same area, that the transformed search space must be more circular than its original counterpart.

Since the widelane transformation Eq.(29), is a member of the class of ambiguity transformations, Eq. (28), and since the introduction of the widelane ambiguity reduces the correlation, it becomes of interest to understand how the widelane transformation fits into the general framework of the decorrelating ambiguity transformations that are constructed by means of the LAMBDA method. The following theorem makes this relation precise.

Theorem 9 (*Ambiguity decorrelation*)

(i) Ambiguity decorrelation is possible if and only if

$$\kappa_i > \frac{2\lambda_1}{2\lambda_2 - \lambda_1}\epsilon \;;$$

(ii) if ambiguity decorrelation is possible, then the *first* ambiguity transformation is always

$$Z_w^T = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \;;$$

(iii) further decorrelation is possible if and only if

$$\kappa_i > \frac{2\left(3\lambda_1^2 + \lambda_2^2\right)}{\lambda_1^2 - \left(2\lambda_1 - \lambda_2\right)^2}\epsilon \ .$$

Proof: see Appendix. □

This result is quite remarkable. It shows that if decorrelation is possible, then the first step towards decorrelation *always* goes via the widelane transformation. Hence, the LAMBDA method will always automatically produce the widelane ambiguity as its *initialization*. With $\frac{\lambda_2}{\lambda_1} = \frac{77}{60}$, it follows from the theorem that decorrelation is impossible when $\kappa_i \leq 1.28\epsilon$. But as we observed already in the previous section, this is not likely to happen, since it would require a very strong receiver-satellite geometry.

Also, the second condition of the theorem, $\kappa_i > 19.2\epsilon$ for $\frac{\lambda_2}{\lambda_1} = \frac{77}{60}$, will be satisfied in most practical cases. This can also be understood in the light of the results of the previous section, where it was pointed out that if the original correlation coefficient is very close to 1, the introduction of the widelane ambiguity will only give a marginal decorrelation of the ambiguities. Hence, a further decorrelation must be possible. This shows that the decorrelating ambiguity transformations as constructed by means of the LAMBDA method will go beyond the widelane transformation in their effort to obtain more decorrelated ambiguities.

7 Summary

In this contribution we studied the precision and correlation of the ambiguities for the geometry-based model, the time-averaged model, and the geometry-free model. We showed how the average ambiguity precision is effected by the observation weights, by the number of satellites tracked, by the number of observation epochs used, and by the change over time in the relative receiver-satellite geometry. From it, the conclusion was reached that in practice the ambiguities are generally of a very poor precision indeed. This holds for all three models, although the precision of the ambiguities of the geometry-based model is somewhat better than that of the time-averaged model, which in turn produces ambiguities that are of a better precision than those of the geometry-free model.

It was also shown that it is generally not guaranteed that the widelane ambiguities are of a better precision than their DD counterparts. For the geometry-free model it is guaranteed, but for the other two models it depends on the change in receiver-satellite geometry, on the number of satellites tracked, and on the phase-code variance ratio. Fortunately, in practice the situation is less dramatic as it may seem at first sight. We could namely show for the two models, that by using the widelane ambiguities, functions of the L_1 ambiguities that have a very poor precision have their precision improved by a factor of about 20, while functions of the L_1 ambiguities that already have a very high precision have their precision degraded by a factor of only 1.61.

In addition to the poor precision of the DD ambiguities, it was shown that they are highly correlated as well. And when the gain numbers are large, one can not expect to decorrelate the ambiguities significantly simply by relying on further changes in the receiver-satellite geometry. The gain numbers have to be unrealistically small, in order to be able to push the correlation coefficients significantly away from 1. Since one cannot rely on the strength of the model to decorrelate the ambiguities, one has to use other means to come up with less correlated ambiguities. This is possible if one reparametrizes the ambiguities. The simplest reparametrization is provided by the widelane ambiguity. It was shown when the transformation to the widelane ambiguity decorrelates and when not. Here we saw a similar mechanism at work as that which we saw when studying the ambiguity precision. That is, the ambiguity functions that are correlated get less correlated by the widelane transformation, but this is achieved at the expense of the functions that are not correlated.

For the two-dimensional case, which is the case that parallels the approach used in practice for the geometry-free model, it was shown that decorrelation, if possible, always starts off with the widelane ambiguities. This implies that the decorrelating ambiguity transformation as it is constructed by the least-squares ambiguity decorrelation adjustment is always initialized by the widelane transformation.

As was shown already for the two-dimensional case, the poor precision and high correlation of the DD ambiguities have important consequences for the size, shape, and orientation of the ambiguity search space. In Part III we will study the geometry of the ambiguity search space, but now for the multivariate case, in detail. This will be done for the single-frequency case as well as for the dual-frequency case. The intrinsic properties of the various search spaces will be revealed and their implications for the computation of the integer ambiguities will be discussed.

Appendix

Proof of Theorem 1 (*Least-squares DD ambiguities*)

In any linear model of observation equations which is partitioned in two sets of parameter vectors, the least-squares estimator of one of the two parameter vectors can be obtained by performing a least-squares adjustment on the residual vector formed from the observables and the least-squares estimator of the other parameter vector. Thus in order to derive the least-squares estimator of the ambiguity vector, the adjustment should in our case be performed on the residual vectors $D^T(\phi_j(i) - A_i\hat{b})$, $D^T(p_j(i) - A_i\hat{b})$, $j = 1, 2$, $i = 1, \ldots, k$. And since the corresponding design matrix consists of scaled unit matrices, the resulting ambiguity estimator simply follows as a scaled version of the time average of these residual vectors.

End of proof. □

Proof of Theorem 2 (*Average DD precision*)

Let D_s^T denote the DD operator having satellite $s \in \{1, \ldots, m\}$ as reference. The variance of the ith component of the DD estimator then reads

$$\sigma_{is}^2 = c_i^T D_s^T Q D_s c_i = c_i^T D_s^T Q P D_s c_i \ ,$$

where the $(m-1)$-vector c_i is the canonical unit vector having 1 as its ith entry. The second equality, with the projector $P = I_m - \frac{1}{m} e_m e_m^T$ inserted, holds, since $D_s^T e_m = 0$ and thus $D_s^T P = D_s^T$. Since the $m \times (m-1)$ matrix D_s is given as

$$D_s = \begin{bmatrix} I_{s-1} & 0 \\ -e_{s-1}^T & -e_{m-s}^T \\ 0 & I_{m-s} \end{bmatrix} \ ,$$

it follows that the ith diagonal element of the matrix $D_s^T Q P D_s$ equals $(c_i - c_s)^T Q P (c_i - c_s)$, where the m-vectors c_i and c_s are again canonical unit vectors. Taking the sum of the $(m-1)$ diagonal entries of $D_s^T Q P D_s$ gives therefore

$$\begin{aligned} \sum_{i=1}^{m} \sigma_{is}^2 &= \sum_{i=1}^{m} \Big[(c_i - c_s)^T Q P (c_i - c_s) \Big] \\ &= \sum_{i=1}^{m} \Big[c_i^T Q P c_i - 2 c_s^T Q P c_i + c_s^T Q P c_s \Big] \\ &= \text{trace } QP + m \, c_s^T Q P c_s \ . \end{aligned}$$

The last equality follows, since $P \sum_{i=1}^{m} c_i = P e_m = 0$. If we now divide by $(m-1)$ and take the average over all m satellites, we get

$$\bar{\sigma}^2 = \frac{2}{m-1} \text{trace } QP$$

This is the first equation of the theorem. Inserting $P = I_m - \frac{1}{m} e_m e_m^T$ proves the second equation of the theorem. To prove the third equation, we insert $P = D(D^T D)^{-1} D^T$. This allows us to write

$$\bar{\sigma}^2 = \frac{2}{m-1} \text{trace } D^T Q D (D^T D)^{-1} \ .$$

Inserting $(D^T D)^{-1} = I_{m-1} - \frac{1}{m} e_{m-1} e_{m-1}^T$ proves the last equation of the theorem.
End of proof. □

Proof of Theorem 3 (*Average precision of DD range*)

Case (i): since the least-squares estimator of the time-averaged DD range vector of the geometry-based model is given as $\hat{r} = D^T \bar{A} \hat{b}(\phi, p)$, its variance matrix follows as $Q_{\hat{r}} = D^T \bar{A} Q_{\hat{b}}(\phi, p) \bar{A}^T D$. Using the first equation of Theorem 2 shows that

$$\begin{aligned} \bar{\sigma}_{\hat{r}}^2 &= \frac{2}{m-1} [\text{trace } \bar{A} Q_{\hat{b}}(\phi, p) \bar{A}^T P] \\ &= \frac{2}{m-1} [\text{trace } Q_{\hat{b}}(\phi, p) \bar{A}^T P \bar{A}] \ . \end{aligned} \tag{30}$$

According to Theorem 7 of *Part I*, we have

$$Q_{\hat{b}}(\phi, p) \bar{A}^T P \bar{A} = \frac{1}{(\beta_1 + \beta_2) k} F^{-T} \epsilon (\Gamma - I_3)(\epsilon \Gamma + I_3)^{-1} F^T \ .$$

Substitution into Eq.(30) proves case (i).
Case (ii): in order to obtain the average variance for the time-averaged model, we only need to take the limits $\gamma_i \to \infty$ of $\bar{\sigma}_{\hat{r}}^2$. As a result we obtain $\bar{\sigma}_{\hat{r}}^2$.
Case (iii): in order to obtain the average variance for the geometry-free model, we only need to insert $m = 4$ into the expression of $\bar{\sigma}_{\hat{r}}^2$. As a result we obtain $\bar{\sigma}_{\hat{r}}^2$.
End of proof. □

Proof of Theorem 4 (*Average precision of ambiguities*)

We will only prove the widelane case. The cases for L_1 and L_2 follow from taking the limits $\lambda_2 \to \infty$ and $\lambda_1 \to \infty$. The widelane ambiguity follows from taking the difference of the L_1 and L_2 ambiguities

$$\hat{a}_w = \hat{a}_1 - \hat{a}_2 = D^T \left[\frac{1}{\lambda_1} \bar{\phi}_1 - \frac{1}{\lambda_2} \bar{\phi}_2 - \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2} \right) \bar{A} \hat{b}(\phi, p) \right] \ ,$$

Noting that the time-averaged phase data do not correlate with the baseline estimator, an application of the error propagation law gives for the ith component, having satellite s as reference

$$\sigma_{\hat{a}_{w_{is}}}^2 = \frac{1}{k} \left(\frac{1}{\alpha_1 \lambda_1^2} + \frac{1}{\alpha_2 \lambda_2^2} \right) + \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2} \right)^2 \sigma_{\hat{r}_{is}}^2 \ .$$

Taking the average and using the results of the previous theorem gives the stated result.
End of proof. □

Proof of Theorem 5 (L_w/L_1 *Variance ratio*)

The variance matrices of the L_1 and the L_w ambiguities are given as

$$\begin{aligned} Q_{\hat{a}_1} &= D^T \left[\frac{1}{\alpha_1 k \lambda_1^2} I_m + \frac{1}{\lambda_1^2} \bar{A} Q_{\hat{b}}(\phi, p) \bar{A}^T \right] D \ , \\ Q_{\hat{a}_w} &= D^T \left[\left(\frac{1}{\alpha_1 k \lambda_1^2} + \frac{1}{\alpha_2 k \lambda_2^2} \right) I_m + \frac{1}{\lambda_w^2} \bar{A} Q_{\hat{b}}(\phi, p) \bar{A}^T \right] D \ . \end{aligned} \tag{31}$$

where λ_w is the wavelength of the widelane. The null space of the matrix $\bar{A}^T D$ and the range space of the matrix $D^+ \bar{A}$, where D^+ is the pseudo-inverse of D, are complementary and together they span the space R^{m-1}. We thus have the direct sum

$$R^{m-1} = N(\bar{A}^T D) \oplus R(D^+ \bar{A}) \ ,$$

with the dimensions $\dim N(\bar{A}^T D) = m - 4$ and $\dim R(D^+ \bar{A}) = 3$. Let the $(m-1) \times (m-4)$ matrix N be a basis matrix of $N(\bar{A}^T D)$ and define the $(m-1) \times (m-1)$ matrix $M = (N, D^+ \bar{A})$. It follows then with Eq. (31) that

$M^T Q_{\hat{a}_1} M =$

$$\begin{bmatrix} pN^T D^T DN & 0 \\ 0 & \bar{A}^T P\bar{A}\left[pI_3 + qQ_{\hat{b}}(\phi,p)\bar{A}^T P\bar{A}\right] \end{bmatrix}$$

and

$M^T Q_{\hat{a}_w} M =$

$$\begin{bmatrix} rN^T D^T DN & 0 \\ 0 & \bar{A}^T P\bar{A}\left[rI_3 + sQ_{\hat{b}}(\phi,p)\bar{A}^T P\bar{A}\right] \end{bmatrix}$$

with

$$p = \frac{1}{\alpha_1 k\lambda_1^2}\ , \qquad q = \frac{1}{\lambda_1^2}\ ,$$
$$r = \frac{1}{\alpha_1 k\lambda_1^2} + \frac{1}{\alpha_2 k\lambda_2^2}\ , \qquad s = \frac{1}{\lambda_w^2}\ .$$

Since the matrix M is square and of full rank, it follows from the block diagonal matrices that the characteristic equation $\mid Q_{\hat{a}_w} - vQ_{\hat{a}_1} \mid = 0$ can be decomposed into the following two equations

(a) $\mid (r - vp)N^T D^T DN \mid\ = 0\ ,$

(b) $\mid (r - vp)I_3 + (s - vq)Q_{\hat{b}}(\phi,p)\bar{A}^T P\bar{A} \mid = 0\ .$

From the first equation, the $(m-4)$ largest eigenvalues of the theorem follow. To solve the second equation, we recall from Theorem 7 of *Part I*, that

$$Q_{\hat{b}}(\phi,p)\bar{A}^T P\bar{A} = \frac{1}{(\alpha_1+\alpha_2)k} F^{-T}(\epsilon\Gamma + I_3)^{-1}(\Gamma - I_3)F^T\ .$$

Substitution of this canonical decomposition into the second characteristic equation just given, gives the remaining three eigenvalues.
End of proof. □

Proof of Theorem 6 (L_1/L_2 *correlation*)

The proof of this theorem consists of two steps. We will first bring $\rho(p,q)$ into a simpler form. For that purpose we introduce the $(m-1)\times 3$ matrix

$$M = (D^T D)^{-1/2} D^T \bar{A} Q_{\hat{b}}(\phi,p)^{1/2}\ .$$

Let its singular value decomposition be given as

$$M = U\begin{bmatrix}\Lambda_3 \\ 0\end{bmatrix}V^T\ .$$

Introducing the reparametrizations

$$p = (D^T D)^{-1/2} U \begin{bmatrix} (\Lambda_3 + \frac{1}{\alpha_1 k}I_3)^{-1/2} & 0 \\ 0 & \sqrt{\alpha_1 k} I_{m-4}\end{bmatrix} u$$

and

$$q = (D^T D)^{-1/2} U \begin{bmatrix} (\Lambda_3 + \frac{1}{\alpha_2 k}I_3)^{-1/2} & 0 \\ 0 & \sqrt{\alpha_2 k} I_{m-4}\end{bmatrix} v$$

it follows, using the results of Theorem 1, that

$$p^T Q_{\hat{a}_1\hat{a}_2} q = u^T\Sigma v\ ,\ p^T Q_{\hat{a}_1} p = u^T u\ ,\ q^T Q_{\hat{a}_2} q = v^T v$$

with

$$\Sigma = \begin{bmatrix} (\Lambda_3 + \frac{1}{\alpha_1 k}I_3)^{-1/2}\Lambda_3(\Lambda_3 + \frac{1}{\alpha_2 k}I_3)^{-1/2} & 0 \\ 0 & 0\end{bmatrix}\ . \qquad (32)$$

This shows that the correlation coefficients ρ_i follow from solving

$$\rho_i = \max_u \max_v \frac{u^T\Sigma v}{\sqrt{u^T u v^T v}} = \frac{u_i^T \Sigma v_i}{\sqrt{u_i^T u_i v_i^T v_i}}\ , \qquad (33)$$

subject to

$u^T u_j = 0 \quad i = 1,\ldots,(i-1)\ ,$

$v^T v_j = 0 \quad i = 1,\ldots,(i-1)\ .$

In order to solve this problem, we still need the entries of the diagonal matrix Σ and thus the entries of the diagonal matrix Λ_3. Since

$$M^T M = Q_{\hat{b}}(\phi,p)^{1/2}\bar{A}^T P\bar{A} Q_{\hat{b}}(\phi,p)^{1/2} = V\Lambda_3^2 V^T\ ,$$

and since the eigenvalues of this matrix are those of $Q_{\hat{b}}(\phi,p)\bar{A}^T P\bar{A}$, it follows that the entries $\lambda_i^2, i = 1,2,3$, of Λ_3^2 are given by the solution of the characteristic equation

$$\mid Q_{\hat{b}}(\phi,p)\bar{A}^T P\bar{A} - \lambda^2 I_3 \mid = 0\ .$$

With $Q_{\hat{b}}(\phi,p)\bar{A}^T P\bar{A} = Q_{\hat{b}}(\phi,p)Q_{\hat{b}}(\bar{p})^{-1}/k(\beta_1+\beta_2)$, it follows from the canonical decompositions of Theorem 7 of *Part I*, that the roots of the above characteristic equation are given as

$$\lambda_i^2 = \frac{1}{(\alpha_1+\alpha_2)k}\frac{\gamma_i - 1}{\epsilon\gamma_i + 1}\ , \quad i = 1,2,3\ ,$$

with $\epsilon = (\beta_1+\beta_2)/(\alpha_1+\alpha_2)$. This result, together with Eq.(32), shows that the solution of Eq.(33) is given by that stated in the theorem.
End of proof. □

Proof of Theorem 8 (*Orientation and elongation*)

The eigenvalue decomposition of the variance matrix

$$\begin{bmatrix} \frac{1}{\lambda_1^2}(\frac{2}{\alpha k} + \frac{\kappa_i}{\beta k}) & \frac{\kappa_i}{\lambda_1\lambda_2\beta k} \\ \frac{\kappa_i}{\lambda_1\lambda_2\beta k} & \frac{1}{\lambda_2^2}(\frac{2}{\alpha k} + \frac{\kappa_i}{\beta k}) \end{bmatrix}$$

reads

$$\begin{bmatrix}\cos\omega_i & -\sin\omega_i \\ \sin\omega_i & \cos\omega_i\end{bmatrix}\begin{bmatrix}\mu_1 & 0 \\ 0 & \mu_2\end{bmatrix}\begin{bmatrix}\cos\omega_i & \sin\omega_i \\ -\sin\omega_i & \cos\omega_i\end{bmatrix}$$

with

$$\tan 2\omega_i = 2\rho_i\left(\frac{\lambda_2}{\lambda_1} - \frac{\lambda_1}{\lambda_2}\right)^{-1}$$

and the eigenvalues

$$\mu_{1,2} = \kappa_i \frac{\left(\frac{\lambda_2}{\lambda_1} + \frac{\lambda_1}{\lambda_2}\right) \pm \sqrt{\left(\frac{\lambda_2}{\lambda_1} - \frac{\lambda_1}{\lambda_2}\right)^2 + 4\rho_i^2}}{2\lambda_1\lambda_2\rho_i\beta k} \quad ,$$

and where the correlation coefficient is given as

$$\rho_i = [1 + 2\epsilon/\kappa_i]^{-1} \quad .$$

The elongation follows then from $e_i = \sqrt{\mu_1/\mu_2}$.
End of proof. □

Proof of Theorem 9 (*Ambiguity decorrelation*)

The first decorrelation step is performed with the transformation

$$Z_1^T = \begin{bmatrix} 1 & -[\sigma_{1i,2i}\sigma_{2i}^{-2}] \\ 0 & 1 \end{bmatrix} \quad ,$$

where '[.]' denotes rounding to the nearest integer and where

$$\sigma_{1i,2i}\sigma_{2i}^{-2} = \rho_i\lambda_2/\lambda_1 = (1 + 2\epsilon/\kappa_i)^{-1}\lambda_2/\lambda_1 \quad . \tag{34}$$

Case (i): in order for the transformation Z_1^T not to reduce to the trivial identity transformation, the following condition needs to be fulfilled

$$\mid \sigma_{1i,2i}\sigma_{2i}^{-2} \mid > \frac{1}{2} \quad . \tag{35}$$

This together with Eq.(34) gives the stated result.
Case (ii): assuming that Eq.(35) is fulfilled and noting that Eq.(34) is positive, it follows that $[\sigma_{1i,2i}\sigma_{2i}^{-2}] = 1$ if and only if $\sigma_{1i,2i}\sigma_{2i}^{-2} < 3/2$. But since $\rho_i \leq 1$ and $\lambda_2/\lambda_1 = 77/60$ in case of GPS, this condition is identically fulfilled.

Case (iii): after the widelane transformation Z_w^T has been applied, the variance matrix of the transformed ambiguities reads

$$\begin{bmatrix} \sigma_w^2 & \sigma_{2w} \\ \sigma_{w2} & \sigma_2^2 \end{bmatrix} \quad ,$$

with

$$\sigma_{2w}\sigma_w^{-2} = -\frac{1 + \kappa_i(1 - \lambda_2/\lambda_1)/2\epsilon}{(1 + \lambda_2^2/\lambda_1^2) + \kappa_i(1 - \lambda_2/\lambda_1)^2/2\epsilon} \quad .$$

Since

$$\sigma_{2w}\sigma_w^{-2} > -1/2 \Leftrightarrow \epsilon/\kappa_i > -1/2 \quad ,$$

$$\sigma_{2w}\sigma_w^{-2} < +1/2 \Leftrightarrow \epsilon/\kappa_i > -1/2 + 2\frac{\lambda_1\lambda_2}{3\lambda_1^2 + \lambda_2^2} \quad ,$$

the stated result follows.
End of proof. □

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