

## Research Article

# Estimation of Unknown Functions of Iterative Difference Inequalities and Their Applications

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We establish some new nonlinear retarded finite difference inequalities. The results that we propose here can be used as tools in the theory of certain new classes of finite difference equations in various difference situations. We also give applications of the inequalities to show the usefulness of our results.

## 1. Introduction

An integral inequality that provides an explicit bound to the unknown function furnishes a handy tool to investigate qualitative properties of solutions of differential and integral equations. One of the best known and widely used inequalities in the study of nonlinear differential equations is Gronwall-Bellman inequality [1, 2], which can be stated as follows. If  $u$  and  $f$  are nonnegative continuous functions on an interval  $[a, b]$  satisfying

$$u(t) \leq c + \int_a^t f(s)u(s)ds, \quad t \in [a, b], \quad (1)$$

for some constant  $c \geq 0$ , then

$$u(t) \leq c \exp\left(\int_a^t f(s)ds\right), \quad t \in [a, b]. \quad (2)$$

Being an important tool in the study of qualitative properties of solutions of differential equations and integral equations, various generalizations of Gronwall inequalities [1, 2] and their applications have attracted great interests of many mathematicians [3–5]. Some recent works can be found in [6–12] and the references therein. Along with the development of the theory of integral inequalities and the theory of difference equations, more and more attentions are paid to discrete

versions of Gronwall-type inequalities; see [13–36] and the references cited therein.

Sugiyama [13] established the most precise and complete discrete analogue of the Gronwall inequality [1] in the following form. Let  $u(n)$  and  $f(n)$  be nonnegative functions defined for  $n \in \mathbf{N}$ , and suppose that  $f(n) \geq 0$  for every  $n \in \mathbf{N}$ . If

$$u(n) < u_0 + \sum_{s=n_0}^{n-1} f(s)u(s), \quad n \in \mathbf{N}, \quad (3)$$

where  $\mathbf{N}$  is the set of points  $n_0 + k$  ( $k = 0, 1, 2, \dots$ ),  $n_0 \geq 0$  is a given integer, and  $u_0$  is a nonnegative constant, then

$$u(n) < u_0 \prod_{s=n_0}^{n-1} [1 + f(s)], \quad n \in \mathbf{N}. \quad (4)$$

Pachpatte [15] established some generalized discrete analogue of the Gronwall inequality in the following form. Let  $m(s)$  be a positive and monotone nondecreasing function on  $\mathbf{N}$ , and let  $a(s), b(s)$  be nonnegative functions on  $\mathbf{N}$ . If  $u(n)$  satisfies

$$u(n) \leq m(s) + \sum_{s=n_0}^{n-1} a(s) \left( u(s) + \sum_{\tau=n_0}^{s-1} b(\tau)u(\tau) \right), \quad \forall n \in \mathbf{N}, \quad (5)$$

then

$$u(n) \leq P(n) m(s), \quad \forall n \in \mathbf{N}, \quad (6)$$

where

$$P(n) = 1 + \sum_{s=n_0}^{n-1} a(s) \prod_{\tau=n_0}^{s-1} [1 + a(\tau) + b(\tau)], \quad \forall n \in \mathbf{N}. \quad (7)$$

**Lemma 1** (see [16]). *Suppose that  $u_0$  is a nonnegative constant and  $u(n), a(n), b(n), c(n),$  and  $d(n)$  are nonnegative functions defined on  $\mathbf{N}$ ,  $1 + a(n) - b(n) \geq 0$  for all  $n \in \mathbf{N}$ . If  $u(n)$  satisfies the inequality*

$$u(n) \leq u_0 + \sum_{s=n_0}^{n-1} a(s) u(s) + \sum_{s=n_0}^{n-1} b(s) \left( \sum_{\tau=n_0}^{s-1} c(\tau) u(\tau) \left( \sum_{t=n_0}^{\tau-1} d(t) u(t) \right) \right), \quad (8)$$

$$\forall n \in \mathbf{N},$$

then

$$u(n) \leq u_0 \prod_{s=n_0}^{n-1} [1 + a(s) - b(s)] + \sum_{s=n_0}^{n-1} b(s) R(s) \prod_{t=s+1}^{n-1} [1 + a(t) - b(t)], \quad (9)$$

$$\forall n \in \mathbf{N},$$

where

$$R(n) \leq \frac{u_0 \prod_{s=n_0}^{n-1} [1 + a(s) + b(s) + c(s) Q(s)]}{1 + u_0 \sum_{s=n_0}^{n-1} c(s) \prod_{t=n_0}^s [1 + a(t) + b(t) + c(t) Q(t)]}, \quad (10)$$

$$\forall n \in \mathbf{N},$$

in which

$$Q(n) \leq \frac{u_0 \prod_{s=n_0}^{n-1} [1 + a(s) + b(s) + d(s)]}{1 - u_0 \sum_{s=n_0}^{n-1} c(s) \prod_{t=n_0}^s [1 + a(t) + b(t) + d(t)]}, \quad (11)$$

$$\forall n \in \mathbf{N},$$

and  $\sum_{s=n_0}^{n-1} c(s) \prod_{t=n_0}^s [1 + a(t) + b(t) + d(t)] < u_0^{-1}$  for all  $n \in \mathbf{N}$ .

**Lemma 2** (see [14, 18]). *Let  $w(n, r)$  be a real-valued function defined for  $n \in \mathbf{N}$ ,  $0 \leq r < \infty$  and monotone nondecreasing with respect to  $r$  for any fixed  $n \in \mathbf{N}$ . Let  $u(n)$  be a real-valued function defined for  $n \in \mathbf{N}$  such that*

$$\Delta u(n) \leq w(n, u(n)), \quad \forall n \in \mathbf{N}. \quad (12)$$

Let  $r(n)$  be a solution of

$$\Delta r(n) = w(n, r(n)), \quad r(0) = r_0, \quad \forall n \in \mathbf{N}, \quad (13)$$

such that  $u(0) \leq r(0)$ . Then

$$u(n) \leq r(n), \quad \forall n \in \mathbf{N}. \quad (14)$$

Pachpatte [18, 19] also established some difference inequalities of product form as follows. Let  $u, a, b$  be nonnegative functions defined on  $\mathbf{N}$  and let  $c$  be a nonnegative constant. Let  $w(n, r)$  be a nonnegative function defined for  $n \in \mathbf{N}$ ,  $0 \leq r < \infty$  and monotone nondecreasing with respect to  $r$  for any fixed  $n \in \mathbf{N}$ . If  $u(n)$  satisfies

$$u^2(n) \leq c^2 + 2 \sum_{s=n_0}^{n-1} u(s) \left[ a(s) \left( u(s) + \sum_{t=n_0}^{s-1} b(t) u(t) \right) + w(s, u(s)) \right], \quad \forall n \in \mathbf{N}, \quad (15)$$

then

$$u(n) \leq P(n) r(n), \quad \forall n \in \mathbf{N}, \quad (16)$$

where  $P(n)$  is defined by (7), and  $r(n)$  is a solution of

$$\Delta r(n) = w(n, P(n) r(n)), \quad r(0) = c, \quad \forall n \in \mathbf{N}. \quad (17)$$

Let  $u, a, b$  be nonnegative functions defined for  $n \in \mathbf{N}$  and let  $c$  be a nonnegative constant. Let  $w(n, r)$  be a nonnegative function defined for  $n \in \mathbf{N}$ ,  $0 \leq r < \infty$  and monotone nondecreasing with respect to  $r$  for any fixed  $n \in \mathbf{N}$ . If  $u(n)$  satisfies

$$u^2(n) \leq c^2 + \sum_{s=n_0}^{n-1} a(s) (u(s+1) + u(s)) \times \left[ \left( u(s) + \sum_{\tau=n_0}^{s-1} b(\tau) u(\tau) \right) + w(s, u(s)) \right], \quad \forall n \in \mathbf{N}, \quad (18)$$

then

$$u(n) \leq P(n) r(n), \quad \forall n \in \mathbf{N}, \quad (19)$$

where  $P(n)$  is defined by (7), and  $r(n)$  is a solution of the difference equation

$$\Delta r(n) = a(n) w(n, P(n) r(n)), \quad r(0) = c, \quad \forall n \in \mathbf{N}. \quad (20)$$

Motivated by the results given in [16, 18, 19], in this paper, we discuss new nonlinear finite difference inequalities:

$$\begin{aligned}
 u^2(n) &\leq c^2 + \sum_{s=n_0}^{n-1} f(s) (u(s+1) + u(s)) \\
 &\quad \times \left[ \left( u(s) + \sum_{t=n_0}^{s-1} g(t) u(t) \right. \right. \\
 &\quad \quad \left. \left. \times \sum_{\tau=n_0}^{t-1} h(\tau) u(\tau) \right) \right. \\
 &\quad \left. + w(s, u(s)) \right], \quad \forall n \in \mathbf{N}.
 \end{aligned} \tag{21}$$

Our inequalities can be used as tools in the study of certain classes of finite difference equations. We also present some immediate applications to show the importance of our results to study the various problems in the theory of finite difference equations.

### 2. Main Results

Throughout this paper, let  $\mathbf{R} = (-\infty, +\infty)$ ,  $\mathbf{R}_+ = [0, +\infty)$ . Let  $\mathbf{N} := \{n_0, n_0 + 1, n_0 + 2, \dots\}$  and  $\mathbf{N}_T := \{n_0, n_0 + 1, n_0 + 2, \dots, T\}$ ,  $T \in \mathbf{N}$ . For function  $u(n)$ ,  $n \in \mathbf{N}$ , we define the operator  $\Delta$  by  $\Delta u(n) = u(n+1) - u(n)$ . Obviously, the linear difference equation  $\Delta u(n) = f(n)$  with the initial condition  $u(n_0) = 0$  has the solution  $u(n) = \sum_{s=n_0}^{n-1} f(s)$ . For convenience, in the sequel we complementarily define that  $\sum_{s=n_0}^{n_0-1} f(s) = 0$  and  $\prod_{s=n_0}^{n_0-1} f(s) = 1$ .

**Theorem 3.** *Let  $\beta > 0$  be a constant,  $x, y$  positive functions defined on  $\mathbf{N}$ ,  $x$  a monotone increasing function, and  $y$  a monotone decreasing function. Let  $\phi$  be a nonnegative function defined on  $\mathbf{N}$  such that*

$$\begin{aligned}
 x(n+1)y(n+1) - x(n)y(n) &\leq \phi(n) [x(n)y(n+1)]^\beta, \\
 &\quad \forall n \in \mathbf{N}.
 \end{aligned} \tag{22}$$

(i) *Suppose  $\beta > 1$ . If  $1 + (1 - \beta)[x(n_0)y(n_0)]^{\beta-1} \sum_{s=n_0}^{n-1} \phi(s) > 0$ , then*

$$\begin{aligned}
 x(n) &\leq \frac{x(n_0)y(n_0)y(n)}{\left\{ 1 + (1 - \beta) [x(n_0)y(n_0)]^{\beta-1} \sum_{s=n_0}^{n-1} \phi(s) \right\}^{1/(\beta-1)}}, \\
 &\quad \forall n \in \mathbf{N}.
 \end{aligned} \tag{23}$$

(ii) *Suppose  $0 < \beta < 1$ . Then*

$$\begin{aligned}
 x(n) &\leq y^{-1}(n) \left\{ [x(n_0)y(n_0)]^{1-\beta} \right. \\
 &\quad \left. + \sum_{s=n_0}^{n-1} (1 - \beta) \phi(s) \right\}^{1/(1-\beta)}, \quad \forall n \in \mathbf{N}.
 \end{aligned} \tag{24}$$

*Proof.* (i) We apply mean value theorem for differentiation to the function

$$F(z) = \frac{z^{1-\beta}}{(1-\beta)}, \quad z > 0, \tag{25}$$

and then there exists  $\xi$  between  $x(n)y(n)$  and  $x(n+1)y(n+1)$  such that

$$\begin{aligned}
 [x(n+1)y(n+1)]^{1-\beta} - [x(n)y(n)]^{1-\beta} \\
 = (1 - \beta) \xi^{-\beta} [x(n+1)y(n+1) - x(n)y(n)].
 \end{aligned} \tag{26}$$

Because  $x(n)$  is monotone increasing and  $y(n)$  is monotone decreasing and  $-\beta < 0$ , we see that  $[x(n)y(n+1)]^{-\beta} \geq [x(n+1)y(n+1)]^{-\beta}$  and  $[x(n)y(n+1)]^{-\beta} \geq [x(n)y(n)]^{-\beta}$ . So for all values of  $\xi$  between  $x(n)y(n)$  and  $x(n+1)y(n+1)$  we have

$$[x(n)y(n+1)]^{-\beta} \geq \xi^{-\beta}. \tag{27}$$

From (22) and (27), we have

$$\begin{aligned}
 \xi^{-\beta} [x(n+1)y(n+1) - x(n)y(n)] \\
 \leq [x(n)y(n+1)]^{-\beta} \phi(n) [x(n)y(n+1)]^\beta \\
 = \phi(n).
 \end{aligned} \tag{28}$$

Since  $1 - \beta < 0$ , from (26) and (28) we have

$$[x(n+1)y(n+1)]^{1-\beta} - [x(n)y(n)]^{1-\beta} \geq (1 - \beta) \phi(n). \tag{29}$$

Taking  $n = s$  in (29) and summing up over  $s$  from  $n_0$  to  $n - 1$ , we obtain

$$[x(n)y(n)]^{1-\beta} \geq [x(n_0)y(n_0)]^{1-\beta} + \sum_{s=n_0}^{n-1} (1 - \beta) \phi(s). \tag{30}$$

From (30), we obtain our required estimation (23).

(ii) Now by following the same steps as in the proof of (i) before (29) we have

$$[x(n+1)y(n+1)]^{1-\beta} - [x(n)y(n)]^{1-\beta} \leq (1 - \beta) \phi(n), \tag{31}$$

because  $1 - \beta > 0$ . Taking  $n = s$  in (31) and summing up over  $s$  from  $n_0$  to  $n - 1$ , we obtain

$$[x(n)y(n)]^{1-\beta} \leq [x(n_0)y(n_0)]^{1-\beta} + \sum_{s=n_0}^{n-1} (1 - \beta) \phi(s). \tag{32}$$

From (32), we obtain our required estimation (24).  $\square$

**Theorem 4.** Let  $m(s)$  be a positive and monotone nondecreasing function defined on  $\mathbf{N}$  and  $f(s), g(s), h(s)$  nonnegative functions defined on  $\mathbf{N}$ . If  $u(n)$  satisfies

$$u(n) \leq m(n) + \sum_{s=n_0}^{n-1} f(s) u(s) + \sum_{s=n_0}^{n-1} f(s) \left( \sum_{t=n_0}^{s-1} g(t) u(t) \times \left( \sum_{\tau=n_0}^{t-1} h(\tau) u(\tau) \right) \right), \quad \forall n \in \mathbf{N}, \tag{33}$$

then

$$u(n) \leq m(n) + \sum_{s=n_0}^{n-1} f(s) U(s), \quad \forall n \in \mathbf{N}, \tag{34}$$

where

$$U(n) \leq \frac{m(n) \prod_{s=n_0}^{n-1} [1 + f(s) + g(s) V(s)]}{1 + m(n) \sum_{s=n_0}^{n-1} g(s) \prod_{t=n_0}^s [1 + f(t) + g(t) V(t)]}, \quad \forall n \in \mathbf{N}, \tag{35}$$

in which

$$V(n) = \frac{m(n) \prod_{s=n_0}^{n-1} [1 + f(s) + h(s)]}{1 - m(n) \sum_{s=n_0}^{n-1} g(s) \prod_{t=n_0}^s [1 + f(t) + h(t)]}, \quad \forall n \in \mathbf{N}, \tag{36}$$

and  $\sum_{s=n_0}^{n-1} g(s) \prod_{t=n_0}^s [1 + f(t) + h(t)] < m^{-1}(n)$  for all  $n \in \mathbf{N}$ .

*Proof.* Fix  $T \in \mathbf{N}$ , where  $T$  is chosen arbitrarily, since  $m(t)$  is a nonnegative and monotone nondecreasing function, from (33), we have

$$u(n) \leq m(T) + \sum_{s=n_0}^{n-1} f(s) u(s) + \sum_{s=n_0}^{n-1} f(s) \left( \sum_{t=n_0}^{s-1} g(t) u(t) \times \left( \sum_{\tau=n_0}^{t-1} h(\tau) u(\tau) \right) \right), \quad \forall n \in N_T. \tag{37}$$

Define a function  $z(n)$  by the right-hand side of (37). Then  $z(n)$  is a positive and monotone nondecreasing function defined on  $\mathbf{N}$ . We have

$$z(n_0) = m(T), \quad u(n) \leq z(n), \quad \forall n \in \mathbf{N}. \tag{38}$$

Using the definitions of the operator  $\Delta$  and  $z$ , we obtain

$$\Delta z(n) = f(n) \left\{ u(n) + \left[ \sum_{t=n_0}^{n-1} g(t) u(t) \left( \sum_{\tau=n_0}^{t-1} h(\tau) u(\tau) \right) \right] \right\} \leq f(n) \left\{ z(n) + \left[ \sum_{t=n_0}^{n-1} g(t) z(t) \left( \sum_{\tau=n_0}^{t-1} h(\tau) z(\tau) \right) \right] \right\}, \quad \forall n \in N_T. \tag{39}$$

Let

$$z_1(n) = z(n) + \left[ \sum_{t=n_0}^{n-1} g(t) z(t) \left( \sum_{\tau=n_0}^{t-1} h(\tau) z(\tau) \right) \right], \quad \forall n \in N_T. \tag{40}$$

Then

$$z_1(n_0) = z(n_0) = m(T), \quad z(n) \leq z_1(n), \quad \forall n \in N_T. \tag{41}$$

It follows that

$$\Delta z_1(n) = \Delta z(n) + g(n) z(n) \left( \sum_{\tau=n_0}^{n-1} h(\tau) z(\tau) \right) \leq f(n) z_1(n) + g(n) z_1(n) \left( \sum_{\tau=n_0}^{n-1} h(\tau) z_1(\tau) \right), \quad \forall n \in N_T. \tag{42}$$

Adding  $g(n)z_1^2(n)$  to both sides of the above inequality we have

$$\Delta z_1(n) + g(n) z_1^2(n) \leq f(n) z_1(n) + g(n) z_1(n) \times \left[ z_1(n) + \left( \sum_{\tau=n_0}^{n-1} h(\tau) z_1(\tau) \right) \right], \quad \forall n \in N_T. \tag{43}$$

Put

$$z_2(n) = z_1(n) + \sum_{\tau=n_0}^{n-1} h(\tau) z_1(\tau), \quad \forall n \in N_T, \tag{44}$$

and then  $z_1(n) \leq z_2(n)$ ,  $z_2(n_0) = z_1(n_0) = m(T)$  and

$$\Delta z_2(n) = \Delta z_1(n) + h(n) z_1(n) \leq f(n) z_2(n) + g(n) z_2^2(n) + h(n) z_2(n), \quad \forall n \in N_T. \tag{45}$$

We see that the inequality

$$z_2(n+1) - (1 + f(n) + h(n)) z_2(n) \leq g(n) z_2^2(n), \quad \forall n \in N_T. \tag{46}$$

Define a function

$$P_1(n) = \prod_{s=n_0}^{n-1} (1 + f(s) + h(s))^{-1}, \quad \forall n \in N_T. \quad (47)$$

Multiplying by  $P_1(n+1)$  to both sides of (46) we obtain

$$\begin{aligned} z_2(n+1) P_1(n+1) - z_2(n) P_1(n) \\ \leq P_1^{-1}(n+1) g(n) [z_2(n) P_1(n+1)]^2, \end{aligned} \quad (48)$$

$$\forall n \in N_T.$$

Let  $x(n) = z_2(n)$ ,  $y(n) = P_1(n)$ ,  $\phi(n) = P_1^{-1}(n+1)g(n)$ , and  $\beta = 2$ . Because  $z_2(n)$  is monotone increasing,  $P_1(n)$  is monotone decreasing and  $2 > 0$ ; applying Theorem 3 to (48) we obtain

$$\begin{aligned} z_2(n) &\leq \frac{z_2(n_0) P_1^{-1}(n)}{1 - z_2(n_0) \sum_{s=n_0}^{n-1} g(s) P_1^{-1}(s+1)} \\ &= \frac{m(T) \prod_{s=n_0}^{n-1} [1 + f(s) + h(s)]}{1 - m(T) \sum_{s=n_0}^{n-1} g(s) \prod_{t=n_0}^s [1 + f(t) + h(t)]}, \end{aligned} \quad (49)$$

$$\forall n \in N_T,$$

where  $P_1(n_0) = 1$ ,  $z_2(n_0) = m(T)$  are used. Define a function  $\tilde{V}$  of the right-hand side of (49). Substituting (49) in (43) we obtain

$$\begin{aligned} z_1(n+1) - (1 + f(n) + g(n) \tilde{V}(n)) z_1(n) &\leq -g(n) z_1^2(n), \\ \forall n \in N_T. \end{aligned} \quad (50)$$

Performing the same derivation as in (46)–(49), we obtain from (50) that

$$\begin{aligned} z_1(n) &\leq \frac{m(T) \prod_{s=n_0}^{n-1} [1 + f(s) + g(s) \tilde{V}(s)]}{1 + m(T) \sum_{s=n_0}^{n-1} g(s) \prod_{t=n_0}^s [1 + f(t) + g(t) \tilde{V}(t)]}, \\ \forall n \in N_T. \end{aligned} \quad (51)$$

Define a function  $\tilde{U}$  of the right-hand side of (51). Substituting (51) in (39) we obtain

$$\Delta z(n) = f(n) \tilde{U}(n), \quad \forall n \in N_T. \quad (52)$$

Using (38), from (52) it follows that

$$u(n) \leq m(T) + \sum_{s=n_0}^{n-1} f(s) \tilde{U}(s), \quad \forall n \in N_T. \quad (53)$$

Since  $T \in \mathbf{N}$  is arbitrary, from (53), we get the required estimate (35).  $\square$

**Theorem 5.** Let  $u, f, g, h$  be nonnegative functions defined for  $n \in \mathbf{N}$  and  $c$  a nonnegative constant. Let  $w(n, r)$  be a real-valued function defined for  $n \in \mathbf{N}$ ,  $0 \leq r < \infty$ , and monotone

nondecreasing with respect to  $r$  for any fixed  $n \in \mathbf{N}$ . If  $u(n)$  satisfies (21), then

$$u(n) \leq v(n) + \sum_{s=n_0}^{n-1} f(s) W_1(s), \quad \forall n \in \mathbf{N}, \quad (54)$$

where

$$\begin{aligned} W_1(n) \\ \leq \frac{v(n) \prod_{s=n_0}^{n-1} [1 + f(s) + g(s) W_2(s)]}{1 + v(n) \sum_{s=n_0}^{n-1} g(s) \prod_{t=n_0}^s [1 + f(t) + g(t) W_2(t)]}, \end{aligned}$$

$$\forall n \in \mathbf{N}, \quad (55)$$

in which

$$\begin{aligned} W_2(n) = \frac{v(n) \prod_{s=n_0}^{n-1} [1 + f(s) + h(s)]}{1 - v(n) \sum_{s=n_0}^{n-1} g(s) \prod_{t=n_0}^s [1 + f(t) + h(t)]}, \end{aligned} \quad (56)$$

$$\forall n \in \mathbf{N},$$

and  $v(n)$  is a solution of the difference equation

$$\begin{aligned} \Delta r(n) &= f(n) w \left( n, r(n) + \sum_{s=n_0}^{n-1} f(s) W_3(s) \right), \\ r(0) &= c, \quad \forall n \in \mathbf{N}, \end{aligned} \quad (57)$$

where

$$\begin{aligned} W_3(n) &\leq \frac{r(n) \prod_{s=n_0}^{n-1} [1 + f(s) + g(s) W_4(s)]}{1 + r(n) \sum_{s=n_0}^{n-1} g(s) \prod_{t=n_0}^s [1 + f(t) + g(t) W_4(t)]}, \\ \forall n \in \mathbf{N}, \end{aligned} \quad (58)$$

in which

$$\begin{aligned} W_4(n) = \frac{r(n) \prod_{s=n_0}^{n-1} [1 + f(s) + h(s)]}{1 - r(n) \sum_{s=n_0}^{n-1} g(s) \prod_{t=n_0}^s [1 + f(t) + h(t)]}, \end{aligned} \quad (59)$$

$$\forall n \in \mathbf{N},$$

and  $\sum_{s=n_0}^{n-1} g(s) \prod_{t=n_0}^s [1 + f(t) + h(t)] < v^{-1}(n)$  for all  $n \in \mathbf{N}$ .

*Proof.* We first assume that  $c > 0$  and define a function  $z(n)$  by the right-hand side of (21). Then  $z(n)$  is a positive and monotone nondecreasing function defined on  $\mathbf{N}$ . We have

$$z(0) = c^2, \quad u(n) \leq \sqrt{z(n)}, \quad \forall n \in \mathbf{N}. \quad (60)$$

Using the definitions of the operator  $\Delta$  and  $z$ , we obtain

$$\begin{aligned} \Delta z(n) &= f(n)(u(n+1) + u(n)) \\ &\times \left[ \left( u(n) + \sum_{t=n_0}^{n-1} g(t)u(t) \sum_{\tau=n_0}^{t-1} h(\tau)u(\tau) \right) \right. \\ &\quad \left. + w(n, u(n)) \right] \\ &\leq f(n) \left( \sqrt{z(n+1)} + \sqrt{z(n)} \right) \\ &\times \left[ \left( \sqrt{z(n)} + \sum_{t=n_0}^{n-1} g(t)\sqrt{z(t)} \right. \right. \\ &\quad \left. \left. \times \sum_{\tau=n_0}^{t-1} h(\tau)\sqrt{z(\tau)} \right) + w(n, \sqrt{z(n)}) \right], \end{aligned} \tag{61}$$

$\forall n \in \mathbf{N}$ .

From (61) it follows that the inequality

$$\begin{aligned} \Delta(\sqrt{z(n)}) &= \frac{\Delta z(n)}{\sqrt{z(n+1)} + \sqrt{z(n)}} \\ &\leq f(n) \left[ \left( \sqrt{z(n)} + \sum_{t=n_0}^{n-1} g(t)\sqrt{z(t)} \right. \right. \\ &\quad \left. \left. \times \sum_{\tau=n_0}^{t-1} h(\tau)\sqrt{z(\tau)} \right) \right. \\ &\quad \left. + w(n, \sqrt{z(n)}) \right] \end{aligned} \tag{62}$$

holds for all  $n \in \mathbf{N}$ . Setting  $n = s$  in (62) and substituting  $s = n_0, 1, 2, \dots, n-1$ , successively, we get

$$\begin{aligned} \sqrt{z(n)} &\leq c + \sum_{s=n_0}^{n-1} f(s) \\ &\times \left[ \left( \left( \sqrt{z(s)} + \sum_{t=n_0}^{s-1} g(t)\sqrt{z(t)} \right) \right. \right. \\ &\quad \left. \left. \times \sum_{\tau=n_0}^{t-1} h(\tau)\sqrt{z(\tau)} \right) \right. \\ &\quad \left. + w(s, \sqrt{z(s)}) \right], \end{aligned} \tag{63}$$

$\forall n \in \mathbf{N}$ .

Define a function  $z_1(n)$  by

$$z_1(n) = c + \sum_{s=n_0}^{n-1} f(s)w(s, \sqrt{z(s)}), \quad \forall n \in \mathbf{N}. \tag{64}$$

Then  $z_1(n) = c$  and

$$\Delta z_1(n) = f(n)w(n, \sqrt{z(n)}), \quad \forall n \in \mathbf{N}. \tag{65}$$

Using (64), the inequality (63) can be written as

$$\begin{aligned} \sqrt{z(n)} &\leq z_1(n) \\ &+ \sum_{s=n_0}^{n-1} f(s) \left( \sqrt{z(s)} + \sum_{t=n_0}^{s-1} g(t)\sqrt{z(t)} \right. \\ &\quad \left. \times \sum_{\tau=n_0}^{t-1} h(\tau)\sqrt{z(\tau)} \right), \end{aligned} \tag{66}$$

$\forall n \in \mathbf{N}$ .

Since  $z_1(n)$  is positive and monotone nondecreasing for  $n \in \mathbf{N}$ ,  $f(s), g(s), h(s)$  satisfy the conditions in Theorem 4. Now an application of Theorem 4 to (66) yields

$$\sqrt{z(n)} \leq z_1(n) + \sum_{s=n_0}^{n-1} f(s)\widetilde{W}_1(s), \quad \forall n \in \mathbf{N}, \tag{67}$$

where

$$\begin{aligned} \widetilde{W}_1(n) &\leq \frac{z_1(n) \prod_{s=n_0}^{n-1} [1 + f(s) + g(s)\widetilde{W}_2(s)]}{1 + z_1(n) \sum_{s=n_0}^{n-1} g(s) \prod_{t=n_0}^s [1 + f(t) + g(t)\widetilde{W}_2(t)]}, \end{aligned} \tag{68}$$

$\forall n \in \mathbf{N}$ ,

in which

$$\widetilde{W}_2(n) = \frac{z_1(n) \prod_{s=n_0}^{n-1} [1 + f(s) + h(s)]}{1 - z_1(n) \sum_{s=n_0}^{n-1} g(s) \prod_{t=n_0}^s [1 + f(t) + h(t)]}, \tag{69}$$

$\forall n \in \mathbf{N}$ .

Since  $w(n, r)$  is monotone nondecreasing with respect to  $r$  for any fixed  $n \in \mathbf{N}$ , from (65) and (67), we have

$$\Delta z_1(n) \leq f(n)w\left(n, z_1(n) + \sum_{s=n_0}^{n-1} f(s)\widetilde{W}_1(s)\right), \quad \forall n \in \mathbf{N}. \tag{70}$$

Now as a suitable application of Lemma 2, we obtain

$$z_1(n) \leq v(n), \quad \forall n \in \mathbf{N}, \tag{71}$$

where  $v(n)$  is a solution of (57). Using (60), (67), and (71), we obtain our required estimation (54).

If  $c$  is nonnegative, we can carry out the above procedure with  $c + \epsilon$  instead of  $c$  where  $\epsilon$  is an arbitrary small number. Letting  $\epsilon \rightarrow 0$ , we obtain (54).  $\square$

### 3. Application to Finite Difference Equations

In this section, we consider the following difference equation:

$$\begin{aligned} \Delta x(n) &= f(n) \\ &\times \left[ F \left( n, x(n), \sum_{t=n_0}^{n-1} g(t) x(t) \right. \right. \\ &\quad \left. \left. \times \sum_{\tau=n_0}^{t-1} H(t, \tau, x(\tau)) \right) + K(n, x(n)) \right], \\ &\quad \forall n \in \mathbf{N}, \end{aligned} \tag{72}$$

$$x(n_0) = x_0, \tag{73}$$

where  $F, H, K$  are real-valued functions defined, respectively, on  $\mathbf{N} \times \mathbf{R}^2, \mathbf{N}^2 \times \mathbf{R}, \mathbf{N} \times \mathbf{R}$ ,  $f$  is as defined in Theorem 5, and  $x_0$  is a constant. We assume that

$$\begin{aligned} |K(n, x(n))| &\leq w(n, |x(n)|), \\ |H(n, t, x(t))| &\leq \sum_{t=n_0}^{n-1} h(t) |x(t)|, \\ |F(n, x(n), y(n))| &\leq |x(n)| + |y(n)|, \end{aligned} \tag{74}$$

where  $g, h, w$  are as defined in Theorem 5. Using the definitions of the operator  $\Delta$ , from (72), we see that the inequality

$$\begin{aligned} x(n+1) - x(n) &= f(n) \left[ F \left( n, x(n), \sum_{t=n_0}^{n-1} g(t) x(t) \right. \right. \\ &\quad \left. \left. \times \sum_{\tau=n_0}^{t-1} H(t, \tau, x(\tau)) \right) + K(n, x(n)) \right] \end{aligned} \tag{75}$$

holds for all  $n \in \mathbf{N}$ . It follows that

$$\begin{aligned} x^2(n+1) - x^2(n) &= f(n) [x(n+1) - x(n)] \\ &\times \left[ F \left( n, x(n), \sum_{t=n_0}^{n-1} g(t) x(t) \sum_{\tau=n_0}^{t-1} H(t, \tau, x(\tau)) \right) \right. \\ &\quad \left. + K(n, x(n)) \right], \quad \forall n \in \mathbf{N}. \end{aligned} \tag{76}$$

From (76), we have

$$\begin{aligned} x^2(n) &= x^2(n_0) + \sum_{s=n_0}^{n-1} f(s) [x(s+1) - x(s)] \\ &\quad \times \left[ F \left( s, x(s), \sum_{t=n_0}^{s-1} g(t) x(t) \right. \right. \\ &\quad \left. \left. \times \sum_{\tau=n_0}^{t-1} H(t, \tau, x(\tau)) \right) \right. \\ &\quad \left. + K(s, x(s)) \right], \quad \forall n \in \mathbf{N}. \end{aligned} \tag{77}$$

Using the conditions (74), we obtain

$$\begin{aligned} |x(n)|^2 &= x^2(n_0) + \sum_{s=n_0}^{n-1} f(s) [|x(s+1)| - |x(s)|] \\ &\quad \times \left[ |x(s)| + \sum_{t=n_0}^{s-1} g(t) |x(t)| \right. \\ &\quad \left. \times \sum_{\tau=n_0}^{t-1} h(\tau) |x(\tau)| \right. \\ &\quad \left. + w(s, |x(s)|) \right], \quad \forall n \in \mathbf{N}. \end{aligned} \tag{78}$$

Now an application of Theorem 5 to (78) yields the estimation of the difference equation (72), that is,

$$|x(n)| \leq v(n) + \sum_{s=n_0}^{n-1} f(s) W_5(s), \quad \forall n \in \mathbf{N}, \tag{79}$$

where

$$W_5(n) \leq \frac{v(n) \prod_{s=n_0}^{n-1} [1 + f(s) + g(s) W_6(s)]}{1 + v(n) \sum_{s=n_0}^{n-1} g(s) \prod_{t=n_0}^s [1 + f(t) + g(t) W_6(t)]}, \tag{80}$$

$\forall n \in \mathbf{N}$ ,

in which

$$W_6(n) = \frac{v(n) \prod_{s=n_0}^{n-1} [1 + f(s) + h(s)]}{1 - v(n) \sum_{s=n_0}^{n-1} g(s) \prod_{t=n_0}^s [1 + f(t) + h(t)]}, \tag{81}$$

$\forall n \in \mathbf{N}$ ,

and  $v(n)$  is a solution of the difference equation

$$\begin{aligned} \Delta r(n) &= f(n) w \left( n, r(n) + \sum_{s=n_0}^{n-1} f(s) W_7(s) \right), \\ r(0) &= |x_0|, \quad \forall n \in \mathbf{N}, \end{aligned} \tag{82}$$



where

$$W_7(n) \leq \frac{r(n) \prod_{s=n_0}^{n-1} [1 + f(s) + g(s) W_8(s)]}{1 + r(n) \sum_{s=n_0}^{n-1} g(s) \prod_{t=n_0}^s [1 + f(t) + g(t) W_8(t)]}, \quad \forall n \in \mathbf{N}, \quad (83)$$

in which

$$W_8(n) = \frac{r(n) \prod_{s=n_0}^{n-1} [1 + f(s) + h(s)]}{1 - r(n) \sum_{s=n_0}^{n-1} g(s) \prod_{t=n_0}^s [1 + f(t) + h(t)]}, \quad \forall n \in \mathbf{N}, \quad (84)$$

and  $\sum_{s=n_0}^{n-1} g(s) \prod_{t=n_0}^s [1 + f(t) + h(t)] < v^{-1}(n)$  for all  $n \in \mathbf{N}$ .

### Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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