Electron. Commun. Probab. **20** (2015), no. 80, 1–3.
DOI: 10.1214/ECP.v20-4531
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

On the dependence of the first exit times on the fluctuations of the domain boundary*

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Abstract

The paper studies first exit times from domains for diffusion processes and their dependence on variations of the boundary. We establish some robustness of the first exit times with respect to the fluctuations of the boundary. More precisely, we present an estimate of the L_1-distance between exit times from two regions via expectations of exit times.

Keywords: diffusion processes; first exit times; variable boundaries.
AMS MSC 2010: Primary 60G40;, Secondary 60G17; 60J50;60J60.
Submitted to ECP on September 2, 2015, final version accepted on October 16, 2015.

1 Introduction

This paper studies path-wise dependence on fluctuations of the boundary for first exit times of diffusion processes. It is known that first exit times from a region for smooth solutions of ordinary equations do not depend continuously on variations of the initial data or on the boundary of the region. On the other hand, first exit times for non-smooth trajectories of diffusion processes have some path-wise regularity with respect to these variations; some results can be found in [1, 2]. In this short note, we present an effective estimate of L_1-distance between exit times from two regions for a diffusion process via expectations of exit times. This means that we established some robustness of the first exit times with respect to the fluctuations of the boundary. This is an extension of the result obtained in [1, 2], where first exits were considered from the fixed domain that was the same for both processes.

2 The result

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a standard probability space, where Ω is a set of elementary events, $\mathcal{F}$ is a complete σ-algebra of events, and $\mathbf{P}$ is a probability measure. Let $w(t)$ be a standard d-dimensional Wiener process. Consider a diffusion process $y(t)$ with the values in $\mathbf{R}^n$ such that

$$\begin{aligned} &dy(t) = f(y(t))dt + \beta(y(t))dw(t), \quad t > 0, \\ &y(0) = a. \end{aligned} \tag{2.1}$$

Here $f : \mathbf{R}^n \to \mathbf{R}^n$ and $\beta : \mathbf{R}^n \to \mathbf{R}^{b\times d}$ are measurable functions, a is a random vector with values in $\mathbf{R}^n$ that is independent on $w(\cdot)$. We assume that all the components of the

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functions f, β are continuously differentiable, and that $\beta(x)\beta(x)^\top \ge cI_n$ for some $c>0$. Here I_n is the unit matrix in $\mathbf{R}^{n\times n}$.

For $x\in\mathbf{R}^n$, we denote by $y^x(t)$ the solution of (1) with the initial condition $y(0)=x$.

For a set $\Gamma\in\mathbf{R}^n$, we denote $\tau(\Gamma)\triangleq\inf\{t:\ y(t)\in\Gamma\}$. This is the first times of achieving Γ for $y(t)$. Similarly, we denote $\tau^x(\Gamma)\triangleq\inf\{t:\ y^x(t)\in\Gamma\}$.

Let D_1, D_2 be two bounded domains in $\mathbf{R}^n$. Let $\Gamma_i=\partial D_i$ be the boundary of D_i.

We assume that the boundaries of D_i are C^1-smooth.

Theorem 2.1. *Let $a\in\bar D_1\cap\bar D_2$ with probability 1. Then*

$$\mathbf{E}|\tau(\Gamma_1)-\tau(\Gamma_2)|\le\max\biggl(\sup_{x\in D_1\cap\Gamma_2}\mathbf{E}\tau^x(\Gamma_1),\ \sup_{x\in D_2\cap\Gamma_1}\mathbf{E}\tau^x(\Gamma_2)\biggr).$$

Note the theorem is oriented on the case where $D_1\backslash D_2\neq\emptyset$ and $D_2\backslash D_1\neq\emptyset$. If $D_1\subset D_2$ or $D_2\subset D_1$ then the estimation (2.2) is obvious.

Theorem 2.1 establishes some robustness in L_1-metric of the first exit times with respect to the fluctuations of the boundary, since the expectations in the right hand part of (2.2) are supposed to be small if $\Gamma_1\approx\Gamma_2$, in a typical case. This can be illustrated as the following.

Example 2.2. Let $n=d=1$, $y(t)=a+w(t)$, $D_0=(0,1)$, $D_\varepsilon=(\varepsilon,1+\varepsilon)$, where $\varepsilon\in[0,1)$. Let $\Gamma_\varepsilon\triangleq\partial D_\varepsilon=\{\varepsilon,1+\varepsilon\}$. Let $\tau^x(\Gamma_\varepsilon)=\inf\{t:\ y^x(t)\in\Gamma_\varepsilon\}$, where $\varepsilon\ge0$ and $x\in D_\varepsilon$. We have that $\mathbf{E}\tau^x(\Gamma_\varepsilon)=(x-\varepsilon)(1+\varepsilon-x)$ for $x\in D_\varepsilon$ and $\varepsilon\ge0$. This can be easily found from the corresponding problems (2.2), (2.5) below that have a trivial quadratic polynomial solution in this case. We have that $\bar D_0\cap\bar D_\varepsilon=[\varepsilon,1]$, $D_0\cap\Gamma_\varepsilon=\{1\}$, and $D_\varepsilon\cap\Gamma_0=\{\varepsilon\}$. It follows from Theorem 2.1 that if $a\in[\varepsilon,1]$ a.s. then

$$\begin{aligned}&\mathbf{E}|\tau(\Gamma_0)-\tau(\Gamma_\varepsilon)|\le\max\left(\mathbf{E}\tau^1(\Gamma_\varepsilon),\,\mathbf{E}\tau^\varepsilon(\Gamma_0)\right)\\&=\max\left((x-\varepsilon)(1+\varepsilon-x)\Big|_{x=1},\ (x-0)(1+0-x)\Big|_{x=\varepsilon}\right).\end{aligned}$$

It gives that $\mathbf{E}|\tau(\Gamma_0)-\tau(\Gamma_\varepsilon)|\le\varepsilon(1-\varepsilon)$.

Proof of Theorem 2.1. Let $\{\mathcal{F}_t\}_{t\ge0}$ be the filtration generated by $w(t)$ and a.

Let e_1 and e_2 be the indicator functions of the random events $\{\tau(\Gamma_1)>\tau(\Gamma_2)\}$ and $\{\tau(\Gamma_2)>\tau(\Gamma_1)\}$ respectively.

Let $\widehat\tau\triangleq\tau(\Gamma_1)\wedge\tau(\Gamma_2)$.

The random variables e_i are measurable with respect to the σ-algebras $\mathcal{F}_{\widehat\tau}$ and $\mathcal{F}_{\tau(\Gamma_i)}$, $i=1,2$, associated with the Markov times $\widehat\tau$ and $\tau(\Gamma_i)$ (Markov times with respect to the filtration $\{\mathcal{F}_t\}$); see, e.g., [3], Section 4.2.

Let $v_i=v_i(x):D_i\to\mathbf{R}$, $i=1,2$, be the solutions in D_i of the Dirichlet problems

$$\mathcal{L}v_i=-1,\quad v_i|_{\Gamma_i}=0. \tag{2.2}$$

Here the differential operator

$$\mathcal{L}=\sum_{k=1}^n f_k\frac{\partial}{\partial y_k}+\frac12\sum_{k,l=1}^n b_{k,l}\frac{\partial^2}{\partial y_k\partial y_l}, \tag{2.3}$$

where f_k, y_l, and $b_{k,l}$ are the components of the vectors f, y and the matrix $b=\beta\beta^\top$.

By Theorem 2.2 from [1] applied to constant in time solutions, it follows that problem (2.2) have solutions $v_i(x)$ such that v_i and $\frac{\partial v_i}{\partial x_k}(x)$ are continuous and bounded, and the norms $\left\|\frac{\partial^2 v_i}{\partial x_k\partial x_m}\right\|_{L_\gamma(D_i)}<+\infty$ for any $\gamma>1$, where $k,m=1,\dots,n$. Therefore, we can apply the generalized Itô's formula given by Theorem II.10.1 from Krylov (1980), p. 122. Using this Itô's formula, we obtain that

$$\begin{aligned}\mathbf{E}\,\{e_1 v_1[y(\tau(\Gamma_2))]\} &= -\mathbf{E}\,\{e_1\{v_1[y(\tau(\Gamma_1))] - v_1[y(\tau(\Gamma_2))]\}\} \\ &= -\mathbf{E}\,\left\{e_1 \int_{\widehat{\tau}}^{\tau(\Gamma_1)} \mathcal{L}_1 v_1[y(t)]dt\right\} \\ &= \mathbf{E}\,\{e_1[\tau(\Gamma_1) - \widehat{\tau}]\} \\ &= \mathbf{E}\,\{e_1[\tau(\Gamma_1) - \tau(\Gamma_2)]\}\,.\end{aligned} \tag{2.4}$$

Similarly, replacing the indices $1, 2$ in (2.4) by $2, 1$, we obtain that

$$\mathbf{E}\,\{e_2 v_2[y(\tau(\Gamma_1))]\} = \mathbf{E}\,\{e_2[\tau(\Gamma_2) - \tau(\Gamma_1)]\}\,.$$

Clearly,

$$\mathbf{E}|\tau(\Gamma_1) - \tau(\Gamma_2)| = \mathbf{E}\,\{e_1[\tau(\Gamma_1) - \tau(\Gamma_2)]\} + \mathbf{E}\,\{e_2[\tau(\Gamma_2) - \tau(\Gamma_1)]\}\,.$$

We have that

$$v_i(x) = \mathbf{E}\,\tau^x(\Gamma_i).$$

Then it follows from (2.4)-(2.5) that

$$\begin{aligned}\mathbf{E}|\tau(\Gamma_1) - \tau(\Gamma_2)| &= \mathbf{E}\,\{e_1[\tau(\Gamma_1) - \tau(\Gamma_2)]\} + \mathbf{E}\,\{e_2[\tau(\Gamma_2) - \tau(\Gamma_1)]\} \\ &= \mathbf{E}\,\{e_1\{v_1[y(\tau(\Gamma_2))]\} + \mathbf{E}\,\{e_2\{v_2[y(\tau(\Gamma_1))]\} \\ &\le \max\left(\sup_{x\in D_1\cap\Gamma_2} v_1(x),\, \sup_{x\in D_2\cap\Gamma_1} v_2(x)\right).\end{aligned}$$

Now the assertion of Theorem 2.1 follows. □

Remark 2.3. We have assumed that the boundaries and coefficients are smooth, the diffusion is non-degenerate, and the domains are bounded. In fact, these conditions can be lifted provided that the right hand part of (2.2) is finite. In particular, a similar result can be obtained for first exit times of a degenerate diffusion process $(y(t), t)$ from cylindrical domains $D_i \times (0, T)$, $i = 1, 2$, $T > 0$; in this case, coefficients f and β can be time dependent.

Acknowledgments. This work was supported by ARC grant of Australia DP120100928 to the author. In addition, the authors would like to thank the anonymous reviewer for the valuable comments and suggestions.

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