Dimension reduction in maximin setting for bond market *

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Abstract

We study optimal investment problem for a continuous time stochastic market model. The risk-free rate, the appreciation rates, and the volatility of the stocks are all random; they are not necessary adapted to the driving Brownian motion, their distributions are unknown, and they are supposed to be currently observable. To cover fixed income management problems, we assume that the number of risky assets can be larger than the number of driving Brownian motion. The optimal investment problem is stated as a problem with a *maximin* performance criterion to ensure that a strategy is found such that the minimum of expected utility over all possible parameters is maximal. We show that Mutual Fund Theorem holds for this setting. We found also that a saddle point exists and can be found via minimization over a single scalar parameter.

Keywords: continuous time market, optimal portfolio, minimax problems, saddle point, Mutual fund Theorem, fixed income management.

1 Introduction

This paper studies optimal investment problem for a diffusion market consisting of a finite number of risky assets (for example, bonds, stocks or options). Risky assets evolution is described by Ito equation. We assume also that there is a bank account where money grows exponentially according to the short rate (we shall call it risk-free rate).

A typical optimal portfolio selection problem can be formulated as maximization of $\mathbf{E}U(X_T)$ over a class of admissible strategies. Here $\mathbf{E}$ denote the expectation, T is the

terminal time, X_t represents the total wealth at time t, U is a utility function that describes risk preferences.

Optimal strategies for continuous time market models were found for many special models (see, e.g., Merton (1969), Karatzas and Shreve (1998)). Investment problems for fixed income management or for a market with both bonds and stocks available are more difficult to study (see, *e.g.,* Bielecki and Pliska (2001),). The volatility matrix is not invertible in this case, and many standard methods are not applicable; moreover, the model descriptions there are usually cumbersome. These investment problems were solved in dynamic programming approach for known equations for market parameters (see, *e.g.,* Rutkowski (1997), Bielecki and Pliska (2001), Komuro and Konno (2005)).

In some cases, it possible to reduce the dimension of the optimization problem. The most important tool that helps to achieve it for financial models is the so-called *Mutual Fund Theorem.* This theorem says that the distribution of the risky assets in the optimal portfolio does not depend on the investor's risk preferences (or utility function). This means that all rational investors may achieve optimality using the same mutual fund plus a risk free bond portfolio or a saving account, in different proportions. Clearly, calculation of the optimal portfolio is easier in this case because only an one dimensional process has to be selected. Moreover, if Mutual Fund Theorem holds, then, for a typical model, portfolio stays on the *efficient frontier* even if there are errors in the forecast, i.e., it is optimal for some other risk preferences. This reduces the impact of forecast errors.

Mutual Fund Theorem was established first for the discrete time single period mean variance portfolio selection problem, i.e., for the problem with quadratic criterions (Markowitz (1959)). This result was a cornerstone of the modern portfolio theory; in particular, the Capital Assets Pricing Model (CAPM) is based on it. For the multi-period discrete time setting, some versions of Mutual Fund Theorem were obtained so far for problems with quadratic criterions only (Li and Ng (1999), Dokuchaev (2010c)). For the continuous time setting, Mutual Fund Theorem was obtained for portfolio selection problems with quadratic criterions as well as for more general utilities. In particular, Merton's optimal myopic strategies for $U(x) = \delta^{-1}x^\delta$ and $U(x) = \log(x)$ are such that Mutual Fund Theorem holds for the case of random coefficients independent from the driving Brownian motion (Karatzas and Shreve (1998)). It is also known that Mutual Fund Theorem does not hold for power utilities in the presence of correlations (see, e.g., Brennan (1998)). Khanna and Kulldorff (1999) proved that Mutual Fund Theorem theorem holds for a general utility function $U(x)$ for the case of non-random coefficient, and for a setting

with consumption. Lim and Zhou (2002) found some cases when Mutual Fund Theorem theorem holds for problems with quadratic criterions. Dokuchaev and Haussmann (2001) found that Mutual Fund Theorem holds if the scalar value $\int_0^T |\theta(t)|^2 dt$ is non-random, where $\theta(t)$ is the market price of risk process. Schachermayer $et\ al$ (2009) found sufficient conditions for Mutual Fund Theorem expressed via replicability of the European type claims $F(Z(T))$, where $F(\cdot)$ is a deterministic function and $Z(t)$ is the discounted wealth generated by the log-optimal optimal discounted wealth process. The required replicability has to be achieved by trading of the log-optimal mutual fund with discounted wealth $Z(t)$.

It can be summarized that Mutual Fund Theorem was established so far only for several special optimal portfolio selection problems and special market models. The extension on any new case was not trivial; it required significant efforts and variety of mathematical methods. The present paper extend this analysis on the case of maximin problems (or problems with so-called robust performance criterions).

The maximin setting has long history in optimization and optimal control theory. It is presented in robust control, in particular, in H^p-control, and it is related to stochastic game theory. In economics, there is a related setting for investment problems with robust performance criteria. Typically, a solution requires to find a saddle point, or establish a duality theorem (see,e.g., Mou and Yong (2006), Dokuchaev (2006,2008) and references here).

Following Yaari (1987), Dokuchaev and Teo (1999), Cvitanić and Karatzas (1999), Cvitanić (2000), and Dokuchaev (2006,2008), we consider the problem as a $maximin$ problem: Find a strategy which maximizes the infimum of expected over all admissible $(r(\cdot), a(\cdot), \sigma(\cdot))$ from a given class. The present paper presents further development of the results for the maximin setting with currently observable parameters from Dokuchaev (2006,2008): the main result is extended for the case that covers fixed income management problems. The difference is that we assume now that the number of risky assets can be larger than the number of driving Brownian motions. In particular, our setting covers now a case when there are m driving Brownian motions and $N >> m$ bonds which with different maturing times $T_1, \ldots, T_N$. Similarly to the author's paper (2006), we show that the duality theorem holds for the maximin problem and a saddle point exists under some non-restrictive conditions. We show that Mutual Fund Theorem holds for this setting. We found also that the saddle point can be found via minimization over a single scalar parameter and solution of a heat equation.

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2 Definitions and problem statement

We consider a market which consists of a risk free asset or bank account with price $B(t)$, $t \geq 0$ that will be used as the numéraire, and n risky stocks with prices $S_i(t)$, $t \geq 0$, $i = 1, 2, \ldots, n$, where $n < +\infty$ is given.

We assume that

$$B(t) = B(0) \exp \left(\int_0^t r(s) ds \right), \tag{2.1}$$

where $B(0)$ is a given constant, and $r(t)$ is the random process of the risk-free interest rate.

The prices of the stocks evolve according to

$$dS_i(t) = S_i(t) \left[a_i(t) dt + \sum_{j=1}^m \sigma_{ij}(t) dw_j(t) \right], \quad t > 0, \tag{2.2}$$

where the $w_i(t)$ are standard independent Wiener processes, $a_i(t)$ are appreciation rates, and $\sigma_{ij}(t)$ are volatility coefficients. The initial price $S_i(0) > 0$ is a given non-random constant.

We assume that $w(\cdot) = (w_1(\cdot), \ldots, w_m(\cdot))$ is a standard Wiener process on a given standard probability space $(\Omega, \mathbb{F}, \mathbf{P})$, where $\Omega = \{\omega\}$ is a set of elementary events, $\mathbb{F}$ is a complete σ-algebra of events, and $\mathbf{P}$ is a probability measure.

We assume that $r(t)$, $a(t) \triangleq \{a_i(t)\}_{i=1}^n$, and $\sigma(t) \triangleq \{\sigma_{ij}(t)\}_{i,j=1}^{n,m}$ are uniformly bounded, measurable random processes.

We are interested in the case of arbitrage free market with *degenerate* $\sigma(t)\sigma(t)^\top$, because we want to cover fixed income management problems and problems when options on stocks and bonds are considered as tradable assets. In particular, we want to cover the case when the market includes N_1 stocks and $N = n - N_1$ zero-coupon bonds with different maturing times $T_1, \ldots, T_N$, where $N >> m$. Assumption 1 below will be in force throughout this paper and it ensures that the market is arbitrage free, and at the same time it allows us to include bonds and options into consideration (see, e.g., Lamberton and Lapeyre (1996)).

Set $\mu(t) \triangleq (r(t), \widetilde{a}(t), \sigma(t))$, where $\widetilde{a}(t) \triangleq a(t) - r(t)\mathbf{1}$ and $\mathbf{1} \triangleq \{1, 1, \ldots, 1\}^\top \in \mathbf{R}^n$.

Let $\{\mathbb{F}_t^\mu\}_{t \geq 0}$ be the filtration generated by the process $(S(t), \mu(t))$ completed with the null sets of $\mathbb{F}$. Clearly, $\mathbb{F}_t^\mu$ coincides with the filtration generated by the processes $(w(t), \mu(t))$, and with the filtration generated by the processes $(\widetilde{S}(t), \mu(t))$, where

$$\widetilde{S}(t) = [\widetilde{S}_1(t), \ldots \widetilde{S}_n(t)] \triangleq \exp \left(- \int_0^t r(s) ds \right) S(t).$$

We describe now distributions of $\mu(\cdot)$ and what we suppose to know about them.

We assume that there exist a finite-dimensional Euclidean space $\bar{E}$, a compact subset $\mathbb{T} \subset \bar{E}$, and a measurable function $M(\cdot) = (M_r(\cdot), M_a(\cdot), M_\sigma(\cdot)) : [0,T] \times \mathbb{T} \times C([0,T]; \mathbf{R}^n) \to \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^{n \times m}$ that is uniformly bounded and such that $M(t, \alpha, \xi)$ is continuous in $(\alpha, \xi) \in \mathbb{T} \times C([0,t]; \mathbf{R}^n)$ for all t. We assume that there exists a constant $C > 0$ such that

$$\sup_{t \in [0,T]} |\xi_i(t) M_{ai}(t, \alpha, \xi(\cdot))|_{[0,t]}) - \eta_i(t) M_{ai}(t, \alpha, \eta(\cdot)|_{[0,t]})| \leq C \sup_{t \in [0,T]} |\xi(t) - \eta(t)|,$$

$$\sup_{t \in [0,T]} |\xi_i(t) M_{\sigma i}(t, \alpha, \xi(\cdot))|_{[0,t]}) - \eta_i(t) M_{\sigma i}(t, \alpha, \eta(\cdot)|_{[0,t]})| \leq C \sup_{t \in [0,T]} |\xi(t) - \eta(t)|,$$

$$\forall \xi, \eta \in C([0,T]; \mathbf{R}^n), \ \alpha \in \mathbb{T},$$

where ξ_i, η_i, M_{ai} are the ith components of the vectors $\xi = \{\xi_i\}_{i=1}^n$, $\eta = \{\eta_i\}_{i=1}^n$, and $M_a = \{M_{ai}\}$, and where $M_{\sigma i}$ is the ith row of the matrix $M_\sigma = \{M_{\sigma ij}\}_{i,j=1}^{n,m}$.

Under these assumptions, the solution of (2.2) is well defined as the unique strong solution of Ito equation with $\mu(t) = (r(t), \widetilde{a}(t), \sigma(t)) = M(t, \alpha, S(\cdot)|_{[0,t]})$ for any $\alpha \in \mathbb{T}$. Let $S_\alpha(\cdot)$ denote the corresponding solution.

For $\alpha \in \mathbb{T}$, set

$$\bar{M}_r(t, \alpha) \triangleq M_r\left(t, \alpha, S_\alpha(\cdot)|_{[0,t]}\right),$$

$$\bar{M}_a(t, \alpha) \triangleq M_a\left(t, \alpha, S_\alpha(\cdot)|_{[0,t]}\right),$$

$$\bar{M}_\sigma(t, \alpha) \triangleq M_\sigma\left(t, \alpha, S_\alpha(\cdot)|_{[0,t]}\right).$$

Definition 1 *Let $\mathbb{A}(\mathbb{T})$ be a set of all random processes $\mu'(t) = (r'(t), \widetilde{a}'(t), \sigma'(t))$ such that there exists a random vector $\Theta' : \Omega \to \mathbb{T}$ independent of $w(\cdot)$ and such that*

$$\begin{cases} r'(t) \equiv \bar{M}_r\left(t, \Theta'\right) \\ \widetilde{a}'(t) \equiv \bar{M}_a\left(t, \Theta'\right) \\ \sigma'(t) \equiv \bar{M}_\sigma\left(t, \Theta'\right). \end{cases} \tag{2.3}$$

We assume that $\mu(\cdot) \in \mathbb{A}(\mathbb{T})$, and that is the only information available.

Note that the solution of (2.2) is well defined for any $\mu(\cdot) \in \mathbb{A}(\mathbb{T})$, but the market is incomplete because of presence of Θ.

In fact, the solution of investment problem obtained below does not require to know $\bar{E}, \mathbb{T}$, and $M(\cdot)$. It uses only a single scalar characteristic of $(\bar{E}, \mathbb{T}, M(\cdot))$.

Note that $\mathbb{T}$ can be interpreted as a set of unknown parameters.

Example 1 Consider a model when the appreciation rate is random and piecewise constant. Let $n = 1$, let r and σ be given constants, and let $\mathbb{T} \subset \mathbf{R}^N$ be a given set, where $N > 0$ is an integer. Assume that the set of processes $\mu(t) = (r(t), \widetilde{a}(t), \sigma(t))$ consists of all processes such that

$$r(t) \equiv r, \quad \sigma(t) \equiv \sigma,$$
$$\widetilde{a}(t) = \Theta_k, \quad t \in \left[\frac{(k-1)T}{N}, \frac{kT}{N} \right), \quad k = 1, \ldots, N,$$

where $\Theta = (\Theta_1, \ldots, \Theta_N)$ is a N-dimensional random vector independent of $w(\cdot)$ and such that $\Theta \in \mathbb{T}$ a.s. This model can be described as a special case of our model with $\bar{E} = \mathbf{R}^N$, and

$$\left(\bar{M}_r(t, \alpha), \bar{M}_\sigma(t, \alpha) \right) \equiv (r, \sigma), \qquad \alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{T},$$
$$\bar{M}_a(t, \alpha) = \alpha_k, \quad t \in \left[\frac{(k-1)T}{N}, \frac{kT}{N} \right), \qquad k = 1, \ldots, N.$$

$\square$

Example 2 Consider a model when the volatility is changing at random time. Let $n = 1$,

$$(r(t), \widetilde{a}(t)) \equiv (r, \widetilde{a}), \qquad \sigma(t) = \begin{cases} \bar{\sigma}, & t < \tau \\ \xi, & t \geq \tau \end{cases},$$

where $r > 0$, $\widetilde{a}$, and $\bar{\sigma}$ are constants, τ and ξ are random variables such that the pair $\Theta = (\tau, \xi)$ is independent of $w(\cdot)$, and $\xi \in \Delta$, $\tau \in [0, T]$, where $\Delta \subset \mathbf{R}$ is a given subset. This model can be described as a special case of our model with $\bar{E} = \mathbf{R}^2$,

$$\mathbb{T} = \{\alpha\} = \{(\alpha_1, \alpha_2)\} = \Delta \times [0, T],$$

$$(M_r(t, \alpha), M_a(t, \alpha)) \equiv (r, \widetilde{a}), \qquad M_\sigma(t, \alpha) = \begin{cases} \bar{\sigma}, & t < \alpha_2 \\ \alpha_1, & t \geq \alpha_2. \end{cases}$$

$\square$

For $\alpha \in \mathbb{T}$, set

$$\mu_\alpha(t) \triangleq (\bar{M}_r(t, \alpha), \bar{M}_a(t, \alpha), \bar{M}_\sigma(t, \alpha)),$$

where $\bar{M}_r(t, \alpha)$, $\bar{M}_a(t, \alpha)$ and $\bar{M}_\sigma(t, \alpha)$ are as in Definition 1.

Let $X_0 > 0$ be the initial wealth at time $t = 0$, and let $X(t)$ be the wealth at time $t > 0$, $X(0) = X_0$. We assume that

$$X(t) = \pi_0(t) + \sum_{i=1}^{n} \pi_i(t), \tag{2.4}$$

where the pair $(\pi_0(t), \pi(t))$ describes the portfolio at time t. The process $\pi_0(t)$ is the investment in the bond, $\pi_i(t)$ is the investment in the ith stock, $\pi(t) = (\pi_1(t), \ldots, \pi_n(t))^\top$, $t \geq 0$.

Let $\mathbf{S}(t) \triangleq \operatorname{diag}(S_1(t), \ldots, S_n(t))$ and $\widetilde{\mathbf{S}}(t) \triangleq \operatorname{diag}(\widetilde{S}_1(t), \ldots, \widetilde{S}_n(t))$ be diagonal matrices with the corresponding diagonal elements. The portfolio is said to be self-financing, if

$$dX(t) = \pi(t)^\top \mathbf{S}(t)^{-1} dS(t) + \pi_0(t) B(t)^{-1} dB(t).$$

It follows that for such portfolios

$$dX(t) = r(t)X(t)\, dt + \pi(t)^\top \left(\widetilde{a}(t)\, dt + \sigma(t)\, dw(t) \right),$$

$$\pi_0(t) = X(t) - \sum_{i=1}^{n} \pi_i(t),$$

so π alone suffices to specify the portfolio; it is called a self-financing strategy.

The process $\widetilde{X}(t) \triangleq \exp\left(-\int_0^t r(s)ds \right) X(t)$ is called the discounted wealth. It satisfies

$$\widetilde{X}(t) = \frac{B(0)}{B(t)} X(t) = X(0) + \int_0^t \exp\left(-\int_0^t r(s)ds \right) \pi(s)^\top \widetilde{\mathbf{S}}(t)^{-1} d\widetilde{S}(s).$$

Let $\mathbb{G}_t$ be a filtration. Let $\widetilde{\Sigma}(\mathbb{G}.)$ be the class of all $\mathbb{G}_t$-adapted processes $\pi(\cdot) = (\pi_1(t), \ldots, \pi_n(t))$ such that

$$\mathbf{E} \int_0^T |\pi(t)|^2\, dt < \infty.$$

We shall consider classes $\widetilde{\Sigma}(\mathbb{G}.)$ as classes of admissible strategies.

For an Euclidean space E we shall denote by $B([0, T]; E)$ the set of bounded measurable functions $f(t) : [0, T] \to E$. By the definitions of $\widetilde{\Sigma}(\mathbb{F}^\mu)$ and $\mathbb{F}_t^\mu$, any admissible self-financing strategy is of the form

$$\pi(t) = \Gamma(t, [S(\cdot), \mu(\cdot)]|_{[0,t]}), \tag{2.5}$$

where $\Gamma(\cdot)$ is a measurable function, $\Gamma(t, \cdot) : C([0, t]; \mathbf{R}^n) \times B([0, t]; \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^{n \times n}) \to \mathbf{R}^n$, $t \geq 0$.

Clearly, the random processes $\pi(\cdot)$ with the same $\Gamma(\cdot)$ in (2.5) may be different for different $\mu(\cdot) = (r(\cdot), \widetilde{a}(\cdot), \sigma(\cdot))$. Hence we also introduce strategies defined by $\Gamma(\cdot)$: the function $\Gamma(\cdot)$ in (2.5) is said to be a H-strategy (strategy based on historical observations).

Definition 2 *Let $\mathbb{C}$ be the class of all functions $\Gamma(t, \cdot) : C([0,t]; \mathbf{R}^n) \times B([0,t]; \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^{n \times m}) \to \mathbf{R}^n$, $t \geq 0$, such that the corresponding strategy $\pi(\cdot)$ defined by (2.5) belongs to $\widetilde{\Sigma}(\mathbb{F}^{\mu}_{\cdot})$ for any $\mu(\cdot) = (r(\cdot), \widetilde{a}(\cdot), \sigma(\cdot)) \in \mathbb{A}(\mathbb{T})$ and*

$$\sup_{\mu(\cdot) = \mu_\alpha(\cdot) : \ \alpha \in \mathbb{T}} \mathbf{E} \int_0^T |\pi(t)|^2 \, dt < \infty.$$

A function $\Gamma(\cdot) \in \mathbb{C}$ is said to be an admissible H-strategy.

Let the initial wealth $X(0)$ be fixed. For an admissible self-financing strategy $\pi(\cdot)$ such that $\pi(t) = \Gamma(t, [S(\cdot), \mu(\cdot)]|_{[0,t]})$, the process $(\pi(t), X(t))$ is uniquely defined by $\Gamma(\cdot)$ and $\mu(\cdot) = (r(\cdot), \widetilde{a}(\cdot), \sigma(\cdot))$ given $w(\cdot)$. We shall use the notation $X(t, \Gamma(\cdot), \mu(\cdot))$ and $\widetilde{X}(t, \Gamma(\cdot), \mu(\cdot))$ to denote the corresponding total wealth and discounted wealth. Furthermore, we shall use the notation $S(t) = S(t, \mu(\cdot))$ and $\widetilde{S}(t) = \widetilde{S}(t, \mu(\cdot))$ to emphasize that the stock price is different for different $\mu(\cdot)$.

3 Problem statement

Let $T > 0$ and X_0 be given. Let $U(\cdot) : \mathbf{R} \to \mathbf{R} \cup \{-\infty\}$ be a given measurable function such that $U(X_0) < +\infty$. Let $\mathbf{D} \subset \mathbf{R}$ be a given convex set, $X_0 \in \mathbf{D}$.

We may state our general problem as follows: Find an admissible H-strategy $\Gamma(\cdot)$ and the corresponding self-financing strategy $\pi(\cdot) \in \widetilde{\Sigma}(\mathbb{F}^{\mu}_{\cdot})$ that solves the following optimization problem:

$$\text{Maximize} \quad \min_{\mu(\cdot) \in \mathbb{A}(\mathbb{T})} \mathbf{E} U(\widetilde{X}(T, \Gamma(\cdot), \mu(\cdot))) \quad \text{over} \quad \Gamma(\cdot) \tag{3.1}$$

$$\text{subject to} \quad \begin{cases} X(0, \Gamma(\cdot), \mu(\cdot)) = X_0, \\ \widetilde{X}(T, \Gamma(\cdot), \mu(\cdot)) \in \mathbf{D} \quad \text{a.s.} \quad \forall \mu(\cdot) \in \mathbb{A}(\mathbb{T}). \end{cases} \tag{3.2}$$

Clearly, the maximin setting has no sense for the problem with observable $\mu(t)$, if, for example, $\mu(t) \equiv \Theta$, where Θ is a random element of $\mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^{n \times m}$ that is constant in time; one can identify Θ instantly. However, the optimal solution for a general case needs knowledge about distribution of future values of $\mu(\cdot)$. It can be seen from Example 5.1 (Dokuchaev (2006)).

4 Additional assumptions

We assume that $\mu(\cdot) \in \mathbb{A}(\mathbb{T})$.

To proceed further, we assume that Conditions 1-6 remain in force throughout this paper. The first of them ensures that the market is arbitrage free.

Condition 1 For any $\mu(\cdot) \in \mathbb{A}(\mathbb{T})$, there exists a random process $\theta_\mu(t) = (\theta_{\mu 1}(t), \ldots, \theta_{\mu m}(t))^\top$ such that

$$\sigma(t)\theta_\mu(t) \equiv \widetilde{a}(t), \qquad \int_0^T |\theta_\mu(t)|^2 dt < +\infty. \qquad (4.1)$$

Set

$$R_\mu \triangleq \int_0^T |\theta_\mu(t)|^2 dt.$$

Remark 1 Our description of the class of admissible $\mu(\cdot)$ covers a setting when the minimum of R_μ over the class is given, or when the class of admissible $\mu(\cdot)$ is defined by a condition $R_\mu \in [R_{min}, R_{max}]$, where R_{min} and R_{max} are given, $0 \leq R_{min} < R_{max} \leq +\infty$. (It suffices to choose an appropriate pair $(\mathbb{T}, M(\cdot))$.)

Set

$$Z(t, \mu(\cdot)) \triangleq \exp\left(\int_0^t \theta_\mu(s)^\top dw(s) + \frac{1}{2}\int_0^t |\theta_\mu(s)|^2 ds\right). \qquad (4.2)$$

Our standing assumptions imply that $\mathbf{E}Z(T, \mu_\alpha(\cdot))^{-1} = 1$ for all $\alpha \in \mathbb{T}$.

Define the (equivalent martingale) probability measure $\mathbf{P}_*^\alpha$ by

$$\frac{d\mathbf{P}_*^\alpha}{d\mathbf{P}} = Z(T, \mu_\alpha(\cdot))^{-1}.$$

Let $\mathbf{E}_*^\alpha$ be the corresponding expectation.

Condition 2 There exists a measurable set $\Lambda \subseteq \mathbf{R}$, and a measurable function $F(\cdot, \cdot) : (0, \infty) \times \Lambda \to \mathbf{D}$ such that for each $z > 0$, $\widehat{x} = F(z, \lambda)$ is a solution of the optimization problem

$$\text{Maximize} \quad zU(x) - \lambda x \quad \text{over } x \in \mathbf{D}. \qquad (4.3)$$

Moreover, this solution is unique for a.e. $z > 0$.

Condition 2 is easy to verify, since the optimization problem is scalar. Several examples of calculating F can be found in Dokuchaev and Teo (1999), Dokuchaev and Haussmann (2001), and Dokuchaev and Zhou (2001).

Condition 3 For any $\alpha \in \mathbb{T}$, there exist $\widehat{\lambda}_\alpha \in \Lambda$, $C = C_\alpha > 0$, and $c_0 = c_{0,\alpha} \in (0, 1/(2R_{\mu_\alpha}))$ such that $F(\cdot, \widehat{\lambda})$ is piecewise continuous on $(0, \infty)$, $F(Z(T, \mu_\alpha(\cdot)), \widehat{\lambda}_\alpha)$ is $\mathbf{P}_*^\alpha$-integrable, and

$$\mathbf{E}_*^\alpha F(Z(T, \mu_\alpha(\cdot)), \widehat{\lambda}_\alpha) = X_0,$$
$$|F(z, \widehat{\lambda}_\alpha)| \leq C z^{c_0 \log z} \quad \forall z > 0. \qquad (4.4)$$

Examples when imposed above conditions are satisfied can be found in Dokuchaev and Haussmann (2001).

Condition 4 *The function $U(x) : \mathbf{R} \to \mathbf{R}$ is either concave or convex in $x \in \mathbf{D}$, and there exist constants $c > 0$, $p \in (1, 2]$, $q \in (0, 1]$ such that*

$$|U(x)| \leq c \left(|x|^p + 1 \right),$$

$$|U(x) - U(x_1)| \leq c \left(1 + |x| + |x_1| \right)^{2-q} |x - x_1|^q \quad \forall x, x_1 \in \mathbf{D}. \tag{4.5}$$

It can be noted that Condition 4 is not restrictive if $\mathbf{D}$ is a bounded interval. This case is not excluded and it covers the goal achieving problem as well as any problem where an investor wish to avoid big variance for sure. Condition 4 is also satisfied, for example, if $U(x) = \ln x$ or $U(x) = \delta^{-1} x^{\delta}$, $\delta < 1$, $d \neq 0$, and $\mathbf{D} \subset [\varepsilon, +\infty)$, where $\varepsilon > 0$.

In addition, we assume without that there exists a set $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\}$ such that the matrix $\sigma'(t) \triangleq \{\sigma_{i_k j}(t)\}_{k,j=1}^m$ is such that $\sigma'(t)\sigma'(t)^\top \geq c' I_m$, a.s., $\forall t$, where $c' > 0$ is a constants and I_m is the $m \times m$ identity matrix. Without a loss of generality, we replace this condition by the following condition.

Condition 5 $n \geq m$ *and the matrix* $\widehat{\sigma}(t) \triangleq \{\sigma_{i,j}(t)\}_{i,j=1}^m$ *is such that* $\widehat{\sigma}(t)\widehat{\sigma}(t)^\top \geq c I_m$, *a.s., $\forall t$, where $c > 0$ is a constants and I_m is the $m \times m$ identity matrix.*

Condition 6 *There exists a non-random variable R_{min} such that there exists $\widehat{\alpha} \in \mathbb{T}$ such that $R_{\widehat{\mu}} = R_{min}$ and $R_{\widehat{\mu}} \leq R_\mu$ a.s. for all $\mu \in \mathbb{A}(\mathbb{T})$, where $\widehat{\mu} \triangleq \mu_{\widehat{\alpha}}$.*

We have that

$$R_{min} = \inf_{\mu(\cdot) \in \mathbb{A}(\mathbb{T})} R_\mu. \tag{4.6}$$

Remark 2 Condition 6 is satisfied if at least one of the following conditions holds:

(i) $(\widetilde{a}(t), \sigma(t)) \equiv (M_a(t, \Theta), M_\sigma(t, \Theta))$, where $(M_a(\cdot), M_\sigma(\cdot)) : [0, T] \times \mathbb{T} \to \mathbf{R}^n \times \mathbf{R}^{n \times m}$ is a deterministic function ;

(ii) The matrix $\sigma(t)$ is diagonal, and

$$\sigma(t) \equiv M_\sigma(t, \Theta), \quad \widetilde{a}_i(t) = \xi_i(t, \Theta, S(\cdot)|_{[0,t]}) \eta_i(t, \Theta),$$

where $M_\sigma(\cdot) : [0, T] \times \mathbb{T} \to \mathbf{R}^{n \times m}$, $\xi_i(t, \cdot) : \mathbb{T} \times C([0, t]; \mathbf{R}^n) \to \mathbf{R}$ and $\eta_i(\cdot) : [0, T] \times \mathbb{T} \to \mathbf{R}$ are measurable functions such that $|\xi_i(t, \Theta, S(\cdot)|_{[0,t]})| \equiv 1$, $i = 1, \ldots n$.

5   An example: stock and options market

Several examples of models such that Conditions 2-6 are satisfied can be found and be found in Dokuchaev (2006) and Dokuchaev and Haussmann (2001). Let us give an example of a market model where Condition 1 is satisfied.

Consider a risky asset (stock) $S_1(t)$ defined by (2.2) with $i = 1$ and $m = 1$. Let $r(t) \equiv r$ and $\sigma_{11}(t) \equiv \sigma_{11} \neq 0$ be given non-random constants. Further, we assume that there are available European put and call options on that stock with the same expiration time T and different strike prices $K_1, \ldots, K_N$, where N is an integer, possibly a large number. We consider options as additional tradable assets with the Black-Scholes price, i.e., we consider *stock-options market*. We can treat this market as a special case of our market with $m = 1$ and $n = 1 + 2N$.

Let $H_{BS,c}(t, x, K)$ and $H_{BS,p}(t, x, K)$ denote Black-Scholes prices for the put and call options with the claims $(S_1(T) - K)^+$ and $(K - S_1(T))^+$ respectively given condition $\widetilde{S}_1(t) = x$, where $\widetilde{S}_1(t) \triangleq e^{-rt} S_1(t)$. Let

$$\widetilde{H}_{BS,c}(t, x, K) \triangleq e^{-rt} H_{BS,c}(t, x, K), \qquad \widetilde{H}_{BS,p}(t, x, K) \triangleq e^{-rt} H_{BS,p}(t, x, K).$$

We assume that the prices for the risky assets that represent options are $S_i(t) = e^{rt} \widetilde{S}_i(t)$, $i = 2, \ldots, n$, where

$$\widetilde{S}_{1+i}(t) = \widetilde{H}_{BS,c}(t, \widetilde{S}_1(t), K_i), \qquad \widetilde{S}_{1+N+i}(t) \triangleq \widetilde{H}_{BS,p}(t, \widetilde{S}_1(t), K_i), \quad i = 1, \ldots, N.$$

The Black-Scholes formula for the option prices ensures that

$$d\widetilde{S}_{1+i}(t) = \frac{\partial \widetilde{H}_{BS,c}}{\partial x}(t, \widetilde{S}_1(t), K_i) d\widetilde{S}_1(t),$$

$$d\widetilde{S}_{1+N+i}(t) = \frac{\partial \widetilde{H}_{BS,p}}{\partial x}(t, \widetilde{S}_1(t), K_i) d\widetilde{S}_1(t), \quad i = 1, \ldots, N.$$

Then $\mu(t) = (r, \widetilde{a}(t), \sigma(t))$, where

$$\widetilde{a}(t) = (\widetilde{a}_1(t), \ldots, \widetilde{a}_n(t)), \quad \sigma(t) \equiv (\sigma_{11}(t), \ldots, \sigma_{k1}(t))^\top \in \mathbf{R}^{n \times 1},$$

and where $(\widetilde{a}_k(t), \sigma_{k1}(t)) = (\widetilde{a}_1(t), \sigma_{11})$,

$$(\widetilde{a}_k(t), \sigma_{k1}(t)) = \left(\widetilde{a}_1(t) \frac{\partial \widetilde{H}_{BS,c}}{\partial x}(t, \widetilde{S}_1(t), K_i), \sigma_{11} \frac{\partial \widetilde{H}_{BS,c}}{\partial x}(t, \widetilde{S}_1(t), K_i) \right)$$

for $1 < k \leq 1 + N$, and

$$(\widetilde{a}_k(t), \sigma_{k1}(t)) = \left(\widetilde{a}_1(t) \frac{\partial \widetilde{H}_{BS,c}}{\partial x}(t, \widetilde{S}_1(t), K_i), \sigma_{11} \frac{\partial H_{BS,p}}{\partial x}(t, \widetilde{S}_1(t), K_i) \right)$$

for $k > 1 + N$. Condition 1 is satisfied with $\theta_\mu(t) \equiv \sigma_{11}^{-1} \widetilde{a}_1(t)$.

6 The main result: solution of the maximin problem

For given $R \geq 0$, $\lambda \in \Lambda$, $\mu(\cdot)$, let

$$Y(t, \mu(\cdot)) \triangleq \int_0^t \theta_\mu(s)^\top \sigma(s)^{-1} \mathbf{S}(s)^{-1} dS(s), \quad \widehat{F}(y, R, \lambda) \triangleq F(e^{y+R/2}, \lambda).$$

Let the function $u(\cdot) = u(\cdot, R, \lambda) : \mathbf{R}_+ \times [0, T] \to \mathbf{R}$ be the solution of the following Cauchy problem for the heat equation:

$$\frac{\partial u}{\partial t}(x, t, R, \lambda) + \frac{R}{2T} \frac{\partial^2 u}{\partial x^2}(x, t, R, \lambda) = 0 \tag{6.1}$$

$$u(x, T, R, \lambda) = \widehat{F}(x, R, \lambda), \tag{6.2}$$

where $F(\cdot)$ is defined in Condition 2.

We assume that the function u is extended on $t > T$ such that $u(x, R, t) \equiv 0$ if $t > T$. (The same should be assumed in Dokuchaev (2006,2008)).

Note that the equation (6.1) is the heat equation and has known fundamental solution, so the solution(6.1) can be be expressed explicitly.

Lemma 1 *For any* $\mu(\cdot) = (r(\cdot), \widetilde{a}(\cdot), \sigma(\cdot))$, *there exists* $n \times m$-*dimensional* $\mathbb{F}_t$-*adapted matrix process* $D_\mu(t)$ *such that*

$$\theta_\mu(t)^\top D_\mu(t) \sigma(t) \equiv \theta_\mu(t)^\top$$

$$D_\mu(t) \widetilde{a}(t) \equiv \theta_\mu(t). \tag{6.3}$$

Introduce a function $\widetilde{\Gamma}(t, \cdot) : B([0, t]; \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^{n \times n}) \times (0, +\infty) \times \Lambda \to \mathbf{R}^n$ such that

$$\widetilde{\Gamma}(t, [S(\cdot), \mu(\cdot)]|_{[0,t]}, R, \lambda) = B(t) \frac{\partial u}{\partial x} [Y(t, \mu(\cdot)), \tau_\mu(t, R), R, \lambda] \theta_\mu(t)^\top D_\mu(t),$$

where the process $Z(t, \mu(\cdot))$ is defined by (4.2) and where

$$\tau_\mu(t, R) = \tau(t, [S(\cdot), \mu(\cdot)]|_{[0,t]}, R) \triangleq \frac{T}{R} \int_0^t |\theta_\mu(s)|^2 ds.$$

Further, for a given $\alpha \in \mathbb{T}$ and $R \geq 0$, let H-strategy $\widehat{\Gamma}_\alpha(\cdot)$ be defined as

$$\widehat{\Gamma}_\alpha(t, [S(\cdot), \mu(\cdot)]|_{[0,t]}) \triangleq \begin{cases} \widetilde{\Gamma}(t, [S(\cdot), \mu(\cdot)]|_{[0,t]}, R_{\mu_\alpha}, \widehat{\lambda}_\alpha), & R > 0 \\ 0, & R = 0, \end{cases}$$

where $\widehat{\lambda}_a$ is defined from Condition 3.

Definition 3 *Let $\mathbb{C}_0$ be the set of all admissible H-strategies $\Gamma(\cdot) \in \mathbb{C}$ such that*

$$\widetilde{X}(T, \Gamma(\cdot), \mu(\cdot)) \in \mathbf{D} \quad a.s. \quad \forall \mu(\cdot) \in \mathbb{A}(\mathbb{T}).$$

To formulate our main result, we need some generalizations of results from Dokuchaev and Haussmann (2001) for our market when the matrix $\sigma(t)\sigma(t)^\top$ can be degenerate, and these are summarized in the following lemma.

Lemma 2 *(i) For any $R > 0$, $\lambda \in \Lambda$, problem (6.1) has an unique solution $u(\cdot, R, \lambda) \in C^{2,1}(\mathbf{R} \times (0,T))$, with $u(x, t, R, \lambda) \to F(x, \lambda)$ a.e. as $t \to T-$.*

(ii) For any $\alpha \in \mathbb{T}$ such that R_{μ_α} is non-random, the strategy

$$\widehat{\Gamma}_\alpha(t, [S(\cdot), \mu(\cdot)]|_{[0,t]}) = B(t)\frac{\partial u}{\partial x}\left(Y(t, \mu(\cdot)), \tau_{\mu_\alpha}(t, R_{\mu_\alpha}), R_{\mu_\alpha}, \widehat{\lambda}_\alpha\right)\theta_\mu(t)^\top D_\mu(t) \qquad (6.4)$$

belongs to $\mathbb{C}_0$. The corresponding discounted wealth is

$$\widetilde{X}(t, \widehat{\Gamma}_\alpha(\cdot), \mu_\alpha(\cdot))) = u\big(Y(t, \mu_\alpha(\cdot)), \tau_{\mu_\alpha}(t, R_{\mu_\alpha}), R_{\mu_\alpha}, \widehat{\lambda}_\alpha\big),$$

and

$$\mathbf{E}U(\widetilde{X}(T, \widehat{\Gamma}_\alpha(\cdot), \mu_\alpha(\cdot))) \geq \mathbf{E}U(\widetilde{X}(T, \Gamma(\cdot), \mu_\alpha(\cdot))) \quad \forall \Gamma(\cdot) \in \mathbb{C}_0. \qquad (6.5)$$

(iii) The functions $F(\cdot, \widehat{\lambda}_\alpha)$, $u(\cdot, R_{\mu_\alpha}, \widehat{\lambda}_\alpha)$, $\widehat{\Gamma}_\alpha(\cdot, R_{\mu_\alpha})$ as well as the probability distribution of the optimal discounted wealth $\widetilde{X}(T, \widehat{\Gamma}_\alpha(\cdot), \mu_\alpha(\cdot))$ is uniquely defined by R_{μ_α} if R_{μ_α} is non-random.

In this lemma, $C^{2,1}(\mathbf{R} \times (0,T))$ denotes the set of functions defined on $\mathbf{R} \times (0,T)$ that are continuous and have two continuous derivatives in the first variable and one in the second.

Theorem 1 *(i) If $R_{min} = 0$, then the trivial strategy, $\Gamma(\cdot) \equiv 0$, is the unique optimal strategy in $\mathbb{C}$ for problem (3.1)-(3.2).*

(ii) Let $R_{min} > 0$, and let $\widehat{\alpha} \in \mathbb{T}$ be such that $R_{\widehat{\mu}} = R_{min}$, where $\widehat{\mu} \triangleq \mu_{\widehat{\alpha}}$. Then the strategy

$$\widehat{\Gamma}_{\widehat{\alpha}}(t, [S(\cdot), \mu(\cdot)]|_{[0,t]}) \triangleq \widetilde{\Gamma}(t, [S(\cdot), \mu(\cdot)]|_{[0,t]}, R_{min}, \widehat{\lambda}_{\widehat{\alpha}}) \qquad (6.6)$$

belongs to $\mathbb{C}_0$ and is optimal in $\mathbb{C}$ for problem (3.1)-(3.2).

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Corollary 1 *The point $(\widehat{\Gamma}_{\widehat{\alpha}}, \widehat{\mu})$ is a saddle point for problem (3.1)-(3.2). In general, this saddle point is not unique, because $\widehat{\mu}(\cdot)$ can be any such that $R_{\widehat{\mu}} = R_{min}$.*

Corollary 2 *Consider two problems (3.1)-(3.2) such that the corresponding $(\mathbb{T}, M(\cdot)) \triangleq (\mathbb{T}^{(i)}, M^{(i)}(\cdot))$ are different for $i = 1, 2$, but the corresponding n, U, X_0, T, $\mathbf{D}$, and R_{min} are the same. Then these two problems have the same optimal H-strategies.*

Corollary 3 *Under the assumptions of Theorem 1, Mutual Fund Theorem holds, meaning that the vector*

$$\frac{\widetilde{\Gamma}(t, [S(\cdot), \mu(\cdot)]|_{[0,t]}, R_{min}, \widehat{\lambda}_{\widehat{\alpha}})}{|\widetilde{\Gamma}(t, [S(\cdot), \mu(\cdot)]|_{[0,t]}, R_{min}, \widehat{\lambda}_{\widehat{\alpha}})|}$$

does not depend on the choice of U. In other words, the distribution of the risky part of portfolio is independent from risk preferences, and optimal strategies for different U can be constructed using the corresponding mutual fund and the bank account.

7 Proofs

Proof of Lemma 1. Set
$$D_\mu(t) \triangleq \left(\widehat{\sigma}(t)^{-1};\ 0_{n-m,m}\right),$$

where $0_{n-m,m}$ is the nil matrix in $\mathbf{R}^{n-m,m}$, and where $\widehat{\sigma}$ is defined in Condition 5. Clearly, $D_\mu(t)\sigma(t) \equiv I_m$, then the first equation in (6.3) is satisfied. Further, let m-dimensional vector process $\widehat{a}(t)$ be such that $\widehat{a}_i(t) \triangleq \widetilde{a}_i(t)$, $i = 1, \ldots, m$. By the definition of θ, we have that $\widehat{a}(t) = \widehat{\sigma}(t)\theta(t)$. Hence

$$D_\mu(t)\widetilde{a}(t) = \widehat{\sigma}(t)^{-1}\widehat{a}(t) = \widehat{\sigma}(t)^{-1}\widehat{\sigma}(t)\theta_\mu(t) = \theta_\mu(t).$$

This completes the proof. $\square$

Proof of Lemma 2. The statement (i) is obvious. Let $n \leq m$. Then the proof repeats the proof of Lemma 4.1 from Dokuchaev (2006). The proof case when $n > m$ requires small modification, but is similar. It follows from Theorem 5.1 from Dokuchaev and Haussmann (2001) that $F(Z(T, \mu_\alpha(\cdot)), \widehat{\lambda}_\alpha)$ is the optimal claim, and this claim can be replicated by the strategy (6.4). Then the proof follows. $\square$

Proof of Theorem 1 repeats the proof of Theorem 4.1 from Dokuchaev (2006). Proof of Corollary 2 follows from Lemma 2 (iii).

8   Some properties of the optimal strategy

Our problem setting allows to consider the models where the forecast for more distant future is less reliable. It can be taken into account allowing scenarios with smaller $|\theta(t)|$ for future t.

Consider the following example. Assume that $R_{min} > 0$ and that there exists $\varepsilon \in (0, T]$ and $c > 0$ such that

$$|\theta_\mu(t)|^2 = c, \quad \forall t < \varepsilon, \quad \forall \mu(\cdot),$$
$$\min_\mu \int_\varepsilon^T |\theta_\mu(t)|^2 dt = 0.$$

In this case,

$$\tau_\mu(\varepsilon, R_{min}) = \tau_{\widehat{\mu}}(\varepsilon, R_{min}) = T,$$

and

$$\widetilde{X}(t) = \widetilde{X}(t, \widehat{\Gamma}_{\widehat{\alpha}}(\cdot), \mu_\alpha(\cdot))) = \widetilde{X}(\varepsilon, \widehat{\Gamma}_{\widehat{\alpha}}(\cdot), \mu_\alpha(\cdot)), \quad \pi(t) = 0, \quad t > \varepsilon.$$

Assume now that $R_{min} > 0$ and that there exists $\varepsilon \in (0, T]$ and $c > 0$ such that

$$|\theta_{\widehat{\mu}}(t)|^2 = c, \qquad |\theta_\mu(t)|^2 \geq c, \quad \forall t < \varepsilon \quad \forall \mu(\cdot),$$
$$\min_\mu \int_\varepsilon^T |\theta_\mu(t)|^2 dt = 0.$$

In this case, the currently observed (real) $\mu(t)$ is different from the worst case scenario $\widehat{\mu}(t)$. We have that

$$\tau_\mu(\varepsilon, R_{min}) \leq \tau_{\widehat{\mu}}(\varepsilon, R_{min}) = T,$$

and

$$\widetilde{X}(t) = \widetilde{X}(t, \widehat{\Gamma}_{\widehat{\alpha}}(\cdot), \mu_\alpha(\cdot))) = \widetilde{X}(\tau_\mu(\varepsilon, R_{min}), \widehat{\Gamma}_{\widehat{\alpha}}(\cdot), \mu_\alpha(\cdot)),$$
$$\pi(t) = 0, \qquad t > \tau_\mu(\varepsilon, R_{min}).$$

These two examples demonstrate that the setting with increasing uncertainty in the future values $|\theta(t)|$ leads to strategy that correspond to setting with reduced time horizon.

9 On calculation of the optimal strategy

Note that equation (6.1) is the heat equation and has the fundamental solution

$$p(y, \tau, x, t, R) = \frac{1}{\sqrt{2\pi(\tau - t)R/T}} \exp\frac{-(y - x)^2}{2(\tau - t)R/T},$$

and the solution can be expressed explicitly as

$$u(x, t, R, \lambda) = \int_{-\infty}^{+\infty} p(y, T, x, t, R)\widehat{F}(y, R, \lambda)dy.$$

Differentiating, we obtain

$$\frac{\partial u}{\partial x}(x, t, R, \lambda) = \int_{-\infty}^{+\infty} \frac{\partial p}{\partial x}(y, T, x, t, R)\widehat{F}(y, R, \lambda)dy$$

$$= \int_{-\infty}^{+\infty} \frac{y - x}{(T - t)R/T}p(y, T, x, t, R)\widehat{F}(y, R, \lambda)dy$$

$$= \frac{T}{(T - t)R} \int_{-\infty}^{+\infty} y\, p(y, T, x, t, R)\widehat{F}(y, R, \lambda)dy - \frac{xT}{(T - t)R}u(x, t, R, \lambda).$$

If the derivative $\partial\widehat{F}(y, R, \lambda)/\partial y$ exists and satisfies certain regularity conditions, then $\frac{\partial u}{\partial x}$ satisfies the problem (6.1), where $\widehat{F}$ is replaced by $\frac{\partial\widehat{F}}{\partial x}$ in the Cauchy condition. In this case, we have that

$$\frac{\partial u}{\partial x}(x, t, R, \lambda) = \int_{-\infty}^{+\infty} \frac{\partial p}{\partial x}(y, T, x, t, R)\widehat{F}(y, R, \lambda)dy$$

$$= -\int_{-\infty}^{+\infty} \frac{\partial p}{\partial y}(y, T, x, t, R)\widehat{F}(y, R, \lambda)dy$$

$$= \int_{-\infty}^{+\infty} p(y, T, x, t, R)\frac{\partial\widehat{F}}{\partial y}(y, R, \lambda)dy. \tag{9.1}$$

It is required that exists and the integrals in (9.1) converge.

Corollary 4 *If t is such that $\tau = \tau_\mu(t, R_{min}) < T$ then the optimal strategy can be represented as*

$$\widehat{\pi}(t)^\top = \frac{T}{(T - \tau)R_{min}}\Big[\mathbf{E}_*\{Y(T, \mu(\cdot))\widehat{F}(Y(T, \mu(\cdot)), R_{min}, \widehat{\lambda_{\widehat{\mu}}}) \,|\, \mathbb{F}_\tau^\mu\}$$

$$- \widetilde{X}(\tau)\Big]\theta_\mu(t)^\top D_\mu(t),$$

where and $\mathbb{F}_t^\mu$ is the filtration generated by $(S(t), \mu(t))$. If $\partial\widehat{F}/\partial y$ exists and the corresponding integrals in (9.1) converge, then

$$\widehat{\pi}(t)^\top = \mathbf{E}_*\left\{\frac{\partial\widehat{F}}{\partial y}(Y(T, \mu(\cdot)), R_{min}, \widehat{\lambda_{\widehat{\mu}}}) \,\Big|\, \mathbb{F}_\tau^\mu\right\}\theta_\mu(t)^\top D_\mu(t).$$

If t is such that $\tau = \tau_\mu(t, R_{min}) > T$ then $\widehat{\pi}(t) = 0$.

10 Application to bond market

Consider the case when $r(s)$ is a random process. Consider a market with n zero-coupon bonds with bond prices $P_k(t)$, where $t \in [0, T_k]$, and where $\{T_k\}_{k=1}^n$ is a given set of maturing times, $T_k \in (0, T]$, $P(T_k, T_k) = 1$.

We consider the case where there is a driving m-dimensional Wiener process $w(t)$. Let $\mathbb{F}_t^w$ be a filtration generated by this Wiener process $w(t)$. We assume that the process $r(t)$ is adapted to $\mathbb{F}_t^w$. In addition, we assume that we are given a random bounded process $q(t) = q(t, \omega)$ with values in $\mathbf{R}^m$ independent of $w(\cdot)$.

We assume that $q(t) = f(t, \Theta')$, where Θ' is a random (non-observable) vector with values in $\mathbb{T}$, and $F(\cdot) : (0, +\infty) \times \mathbb{T} \to \mathbf{R}^m$ is a given measurable function. It is assumed that the distribution of Θ' is unknown.

Let $\mathbb{F}_t$ denotes the filtration generated by $(w(t), \Theta')$.

Set the bond prices as

$$P_k(t) = \mathbf{E}\left\{ \exp\left(-\int_t^{T_k} r(s)ds + \int_t^{T_k} q(s)^\top dw(s) - \frac{1}{2}\int_t^{T_k} |q(s)|^2 ds\right) \,\Big|\, \mathbb{F}_t \right\}. \qquad (10.1)$$

Clearly, the processes $P_k(t)$ are adapted to $\mathbb{F}_t$, $P_k(T_k) = 1$, $P_k(0) \in [0, 1]$, and $\widetilde{P}_k(t) \triangleq P_k(t)\exp\left(-\int_0^t r(s)ds\right) \in [0, 1]$ a.s.

Note that the (observable) value of $P_k(t)$ depends on the non-observable random variable Θ'. (Clearly, observability $P_k(t)$ does not mean that the value Θ' can be restored from the observations; see, e.g., Example 5.1 from the author's paper (2006)).

Set

$$Z_k(t) \triangleq \exp\left(\int_t^{T_k} q(s)dw(s) - \frac{1}{2}\int_t^{T_k} |q(s)|^2 ds\right).$$

Theorem 2 *Pricing rule (10.1) ensures that, for any k, there exists a $\mathbb{F}_t$-adapted process $\sigma_k(t)$ with values in $\mathbf{R}^n$ such that*

$$d_t P_k(t) = P_k(t)\left(\left[r(t) - \sigma_k(t)^\top q(t)\right]dt + \sigma_k(t)^\top dw(t)\right), \quad t < T_k. \qquad (10.2)$$

Proof. Let k be fixed. We have that

$$P_k(t) = y(t)z(t)\exp\left(\int_0^t r(s)ds\right),$$

where

$$y(t) \triangleq \mathbf{E}\left\{ \exp\left(-\int_0^{T_k} r(s)ds + \int_0^{T_k} q(s)^\top dw(s) - \frac{1}{2}\int_0^{T_k} |q(s)|^2 ds\right) \,\Big|\, \mathbb{F}_t \right\},$$

$$z(t) \triangleq \exp\left(-\int_0^t q(s)^\top dw(s) + \frac{1}{2}\int_0^t |q(s)|^2 ds\right).$$

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It follows from Martingale Representation Theorem (or Clark-Haussmann Theorem) applied on the conditional probability space given Θ' that there exists a square integrable n-dimensional $\mathbb{F}_t$-adapted process $\widehat{y}(t) = \widehat{y}(t, T_k)$ with values in $\mathbf{R}^n$ such that

$$y(T) = \mathbf{E}y(T) + \int_0^T \widehat{y}(t)^\top dw(t)$$

(see, e.g. [6], p. 71). Note that $y(t) > 0$. Set $\delta_k(t) = \widehat{y}(t)/y(t)$. Then

$$y(T) = \mathbf{E}y(T) + \int_0^T y(t)\delta_k(t)^\top dw(t).$$

By Ito formula, it follows that

$$dz(t) = z(t)\Big(|q(t)|^2 dt - q(t)^\top dw(t)\Big).$$

Set $\sigma_k(t) \triangleq \delta_k(t) - q(t)$. Finally, Ito formula applied to (10.3) implies that (10.2) holds. This completes the proof. $\square$

To derive an explicit equation for $P_k(t)$ and $\sigma_k(t)$, we need to specify a model for the evolution of the process $(r(t), q(t))$. The choice of this model defines the model for the bond prices. For instance, let $n = 1$, let the process q be non-random and constant, and let $r(t)$ be an Ornstein-Uhlenbek process. Then this case corresponds to the so-called Vasicek model (see, e.g., Lamerton and Lapeyre (1996), p.127). In that case, $P_k(t)$ can be found explicitly from (10.1).

Let terminal time $T > 0$ be given (we allow $T_k > T$ as well as $T_k \leq T$).

Let us show that this bond market is a special case of the multi-stock market described in Section 2, when the risky assets are $S_k(t) = P_k(t)$, $k = 1, \ldots, N$. By(10.2), $\widetilde{a}(t) \equiv (\widetilde{a}_1(t), \ldots, \widetilde{a}_N(t))^\top \in \mathbf{R}^N$, and $\sigma(t)$ is a matrix process with values in $\mathbf{R}^{N \times n}$ such that its kth row is zero for $t > T_k$ and it is equal to $\sigma_k(t)^\top$ for $t \leq T_k$. The process $\widetilde{a}(t)$ is such that

$$\widetilde{a}_k(t) = \begin{cases} -\sigma_k(t)^\top q(t), & t \leq T_k \\ 0, & t > T_k. \end{cases}$$

Then the corresponding market price of risk process $\theta(t)$ is $\theta(t) \equiv -q(t)$. This process $\theta(t)$ is bounded if $q(t)$ is bounded, since $\theta(t) \equiv -q(t)$. Note that the case $N >> n$ is allowed, and the bond market is still arbitrage free.

Clearly, Theorem 1 is applicable with $\mu(t) = (r(t), \widetilde{a}(t), \sigma(t))$,

$$R_{min} = \inf_\Theta \int_0^T |f(t, \Theta')|^2 dt,$$

where infimum is taken over all possible Θ'.

Conclusion

We consider investment problem for continuous time market model where the market parameters are assumed to be currently observable. It is assumed that the matrix $\sigma\sigma^\top$ can be degenerate, where σ is the volatility matrix. In other words, the number of risky assets can be larger than the number of driving Brownian motions. This feature allows to include market models where different bonds, stocks, and options on the stocks are considered to be tradable assets.

Optimal investment problem for this model is solved in maximin setting (i.e., with robust performance criterion). In this setting, the strategy is selected to maximize the minimal expected utility over all possible market scenarios.

We found that a saddle point exists, and, therefore, it suffices to find the optimal strategy for the worst case scenario. It appears that this strategy can be found via minimization over a single scalar parameter plus solution of a standard one dimensional heat equation. These results were obtained for a general utility function and for market model with random coefficients. Therefore, solution of the problem in maximin setting appears to be easier than for the classical problem where expected utility has to be maximized directly. In addition, we found that Mutual Fund Theorem holds in this maximin setting.

A more difficult problem is to extend these results on models where market parameters are not assumed to be directly observed but have to be estimated from market prices using some filtering procedures (see problem settings in Dokuchaev and Teo (2000) and Dokuchaev (2005)). We leave it for the future research.

Acknowledgments

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