The Power Spectral Density of a Correlated and Jittered Pulse Train

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Abstract: - A general power spectral density expression for a pulse train subject to correlated amplitude variations and correlated timing jitter is established and verified by simulation for the multivariate Gaussian case. Six distinct components in the power spectral density are identified: three associated with the independent case and three associated with the correlated amplitude variations and the correlated timing jitter. It is shown that correlated jitter increases the spectral level around harmonic frequencies whilst correlated amplitude variations, in part, leads to an increase in the spectrum at low frequencies. The combined effect of correlated amplitude and correlated timing jitter is shown to be small.

Key-Words: - pulse train, jitter, correlation, power spectral density.

1 Introduction

The importance of correlation in random phenomena has been well elucidated by, for example, Stanley [1]. In engineering and science, both correlated and uncorrelated random phenomena occur widely and an important case is that of a periodic pulse train being subject to external influences such that the pulses exhibit amplitude variations and jittered commencement times, e.g. [2] - [5]. One important way of characterizing the randomness inherent in practical pulse trains is via an autocorrelation function and a power spectral density. As the uncorrelated case is easier to analyse, it is often considered, e.g. [3], [4], but there is increasingly interest in the correlated case, e.g. [2].

Early results include those of Banta [6] and Mazzetti [7] leading to work by Leneman and Beutler [8]-[10] where the emphasis was on pulse trains based on stationary point processes and general results. Related results include those of van der Elsen [11] and the broad discussion in [12]. Recent results include [4] where emphasis was on the independent jitter case and [3], ch. 5 [13], where the uncorrelated jitter case was considered. In [2] the correlated case was considered for the situation of low levels of amplitude and jitter variations. In general, the independent jitter case has usually been considered due to the relative complication that correlation introduces. For specific correlated cases a custom approach is likely to lead to more insightful results than a general approach.

This paper details the power spectral density for the specific case of correlated amplitude variations and correlated jitter correlation; the amplitude and jitter variations are assumed to be independent of one another. A direct Fourier approach is used (rather than an autocorrelation function approach, e.g. [4]) and six distinct components of the power spectral density are identified. Simulation results for a high level of amplitude and jitter variation, and consistent with that found in high speed optical pulse trains, e.g. [14], and based on correlated Gaussian random variables, are used to confirm theoretical results.

2 Background

Consider an experiment whose outcomes define a sample space

$$S = \{((a_1, j_1), \ldots, (a_N, j_N)):\ a_i \in S_{A_i}, j_i \in S_{J_i}\} \quad (1)$$

and a corresponding matrix of random variables

$$\Omega = ((A_1, J_1), \ldots, (A_N, J_N)). \quad (2)$$

The sample spaces of A_i and J_i are, respectively, S_{A_i} and S_{J_i}. The outcomes of these random variables are, denoted, respectively, a_i and j_i and have density functions as given by f_{A_i} and f_{J_i}. The random variables $A_1, \ldots, A_N$ are assumed to be correlated but independent of the correlated random variables $J_1, \ldots, J_N$. The probability of a single outcome from the experiment is governed by the joint probability

$$\begin{aligned} &P[A_1 \in (a_1 + da_1), \ldots, A_N \in (a_N + da_N), \\ &\qquad J_1 \in (j_1 + dj_1), \ldots, J_N \in (j_N + dj_N)] \end{aligned} \quad (3)$$

which can be approximated by the joint probability density function

$$f_{A_1 \ldots A_N}(a_1, \ldots, a_N) f_{J_1 \ldots J_N}(j_1, \ldots, j_N) \cdot da_1 \ldots da_N dj_1 \ldots dj_N \quad (4)$$

assuming $da_1, \ldots, dj_N$ are sufficiently small. A random process $X: S \to S_X$, consistent with a jittered periodic pulse train on the interval $[0, ND]$, can be defined on this experiment according to

$$X(((a_1, j_1), \ldots, (a_N, j_N)), t) = \sum_{k=1}^{N} a_k p(t - kD - j_k) \quad (5)$$
$$((a_1, j_1), \ldots, (a_N, j_N)) \in S, t \in [0, ND]$$

where $p: \boldsymbol{R} \to \boldsymbol{R}$ is the pulse function, and the set of signals defined by the random process is

$$S_X = \left\{ \begin{array}{l} X(((a_1, j_1), \ldots, (a_N, j_N)), t): \\ \qquad ((a_1, j_1), \ldots, (a_N, j_N)) \in S \end{array} \right\}. \quad (6)$$

Convenient notation for the random process is

$$X(\Omega, t) = \sum_{k=1}^{N} A_k p(t - kD - J_k). \quad (7)$$

One signal defined by the random process is illustrated in Figure 1.

2.1 Power Spectral Density

For the finite interval $[0, T]$, the power spectral density is defined as (e.g. [13], [15])

$$G(T, f) = \boldsymbol{E}\left[\frac{|X(\Omega, T, f)|^2}{T}\right] \quad (8)$$

where $\boldsymbol{E}$ is the expectation operator and $X(\Omega, T, f)$ is the Fourier transform of $X(\Omega, t)$ evaluated on the interval $[0, T]$, i.e.

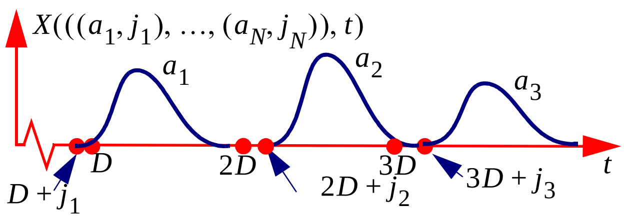

Fig. 1 Illustration of one outcome of the random process defining a jittered pulse train.

$$X(\Omega, T, f) = \int_0^T X(\Omega, t) e^{-j2\pi ft} dt. \quad (9)$$

For the infinite interval $[0, \infty]$, the power spectral density is

$$G(\infty, f) = \lim_{T \to \infty} G(T, f). \quad (10)$$

3 Power Spectral Density

The following results can be proved for the correlated pulse train:

Theorem 1 PSD: Correlated Pulse Train

The power spectral density, on $[0, T]$, $T = ND$, of a jittered pulse train, as specified by Eqn 7, is

$$G_X(T, f) = \frac{|P(f)|^2}{T}\left[\sum_{i=1}^{N} \boldsymbol{E}[|A_i|^2] + \sum_{i=1}^{N} \sum_{k=1, k \neq i}^{N} e^{-j2\pi f(i-k)D} \boldsymbol{E}[A_i A_k^*] \boldsymbol{E}[e^{-j2\pi f(J_i - J_k)}] \right] \quad (11)$$

where $*$ is the conjugate operator and $P(f)$ is the Fourier transform of $p(t)$ evaluated on $[0, \infty]$. The assumption here is the usual one, e.g. [10], that the interval $[0, T]$ does not significantly affect the Fourier transform of the pulses. For the case where the correlation depends only on the spacing between the pulses, and not on their absolute location, consistent with the assumptions:

$$\begin{array}{l} f_{A_i A_k}(a_i, a_k) = f_{A_{i+1} A_{k+1}}(a_i, a_k) \\ f_{J_i J_k}(j_i, j_k) = f_{J_{i+1} J_{k+1}}(j_i, j_k) \\ \qquad i, k, i+1, k+1 \in \boldsymbol{Z}^+ \end{array} \quad (12)$$

and where the individual random variables are identically distributed, i.e. $f_{A_i}(a_i) = f_{A_{i+r}}(a_i)$ and $f_{J_i}(j_i) = f_{J_{i+r}}(j_i)$, the power spectral density is:

$$G_X(T, f) = r|P(f)|^2 \overline{|A|^2} + 2r|P(f)|^2 \sum_{i=1}^{N-1} \left[1 - \frac{i}{N}\right] \cdot Re\left[e^{j2\pi iDf} \boldsymbol{E}[A_1 A_{1+i}^*] \cdot \boldsymbol{E}[e^{-j2\pi f(J_1 - J_{1+i})}]\right]. \quad (13)$$

Here $\overline{|A|^2} = \boldsymbol{E}[|A_i|^2]$ is the mean squared amplitude and $r = 1/D$ is the pulse rate. For the infinite interval $[0, \infty]$, this power spectral density can be written as:

ISBN: 978-1-61804-141-8

$$G_X(\infty, f) = r|P(f)|^2\Big[\ \overline{|A|^2} - |\mu_A|^2|F_J(f)|^2 + r|\mu_A|^2|F_J(f)|^2 \cdot \sum_{n=-\infty}^{\infty}\delta(f-nr) + 2\sum_{i=1}^{M-1} Re\Big[\ e^{j2\pi iDf}\cdot[\boldsymbol{E}[A_1A_{1+i}{}^*] - |\mu_A|^2]\cdot \Big[\boldsymbol{E}[e^{-j2\pi f(J_1-J_{1+i})}] - |F_J(f)|^2\Big]\ \Big] + 2|\mu_A|^2 \cdot \sum_{i=1}^{M-1} Re\Big[e^{j2\pi iDf}\Big[\boldsymbol{E}[e^{-j2\pi f(J_1-J_{1+i})}] - |F_J(f)|^2\Big]\Big] + 2|F_J(f)|^2\sum_{i=1}^{M-1} Re[e^{j2\pi iDf}(\boldsymbol{E}[A_1A_{1+i}{}^*] - |\mu_A|^2)]\ \Big]. \tag{14}$$

Here δ is the Dirac delta, $\mu_A = \boldsymbol{E}[A_i]$, $F_{J_i}(f) = \boldsymbol{E}[e^{-j2\pi fJ_i}] = \phi_{J_i}[-f]$ is the Fourier transform of f_{J_i} with ϕ_{J_i} being the characteristic function of J_i, and the correlation is assumed to extend over M pulses consistent with A_1 and A_{M+1}, as well as J_1 and J_{M+1}, being independent.

Proof
A detailed proof is given in Appendix 1.

3.1 Notes

The detailed nature of the proof given in the appendix facilitates the generalization of the above result, e.g. for the case where all random variables are correlated and/or when the pulse functions are randomly chosen.

For the case of independent amplitude variations, and zero jitter variations consistent with $F_J(f) = 1$, the power spectral density, assuming the infinite interval and with $\sigma_A^2 = \mathrm{var}[A_i]$, is

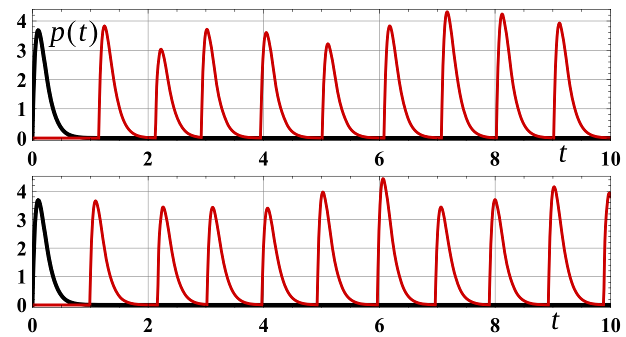

Fig. 2 One waveform from the jittered pulse train random process. Top: Independent case. Bottom: Correlated case.

$$G_X(\infty, f) = r|P(f)|^2[\sigma_A^2 + r|\mu_A|^2\sum_{n=-\infty}^{\infty}\delta(f-nr)] \tag{15}$$

For the deterministic case $\sigma_A^2 = 0$. For the non-zero jitter case, and when all the amplitude and jitter random variables are independent, the power spectral density on the infinite interval is

$$G_X(\infty, f) = r|P(f)|^2\Big[\ \overline{|A|^2} - |\mu_A|^2|F_J(f)|^2 + r|\mu_A|^2|F_J(f)|^2\sum_{n=-\infty}^{\infty}\delta(f-nr)\ \Big]. \tag{16}$$

Apart from the first three terms in Eqn 14, which are associated with the independent random variable case, the other terms are, respectively, the contribution due to both the amplitude and jitter random variables being correlated, the jitter random variables being correlated and the amplitude random variables being correlated.

4 Example

To examine the power spectral density expression of a jittered pulse train, the normalized case of $D = 1/r = 1$, $\mu_A = A_o = 1$ is considered along with a pulse function defined by

$$p(t) = \frac{t^{i-1}e^{-t/\tau}}{\tau^i(i-1)!} \Leftrightarrow P(f) = \frac{1}{(1+j2\pi f\tau)^i}. \tag{17}$$

Assumed values are $\tau = 0.1D$ and $i = 2$. The amplitude and jitter variances are chosen as $\sigma_A^2 = (0.1A_o)^2$ and $\sigma_J^2 = (0.1D)^2$ and are consistent with the high levels of jitter found in ultra high speed optical pulse trains, e.g. [14]. Gaussian probability density functions for the timing jitter and amplitude variations are assumed, e.g.

$$f_J(j) = \frac{1}{\sqrt{2\pi}\sigma_J}\cdot\frac{\exp[-(j-\mu_J)^2/2\sigma_J^2]}{\sqrt{2\pi}\sigma_J} \Leftrightarrow F_J(f) = \exp[-j2\pi f\mu_J - 2\pi^2 f^2\sigma_J^2] \tag{18}$$

Correlated and uncorrelated signals from a random pulse train are shown in Figure 2.

4.1 Independent Case

For the independent case the power spectral density, as given by Eqn 16, is:

ISBN: 978-1-61804-141-8

$$G_X(\infty, f) = \frac{r}{(1+4\pi^2 f^2\tau^2)^i}\left[\sigma_A^2 + \mu_A^2 - \mu_A^2 e^{-4\pi^2 f^2 \sigma_J^2} + r\mu_A^2 e^{-4\pi^2 f^2\sigma_J^2}\sum_{n=-\infty}^{\infty}\delta(f-nr)\right] \quad (19)$$

and this power spectral density, along with the power spectral density for the zero jitter case, is shown in Figure 3.

4.2 Correlated Case

For the correlated case, the amplitude and jitter random variables are assumed to be multivariate normal, i.e. with $X = (X_1, \ldots, X_N)$, $x = (x_1, \ldots, x_N)$

$$f_X(x) = \frac{1}{D}\cdot\exp\left[\frac{-(x-\mu)^T\Sigma^{-1}(x-\mu)}{2}\right] \qquad D = (2\pi)^{M/2}|\Sigma|^{1/2} \quad (20)$$

where $\mu = \{\mu_1, \ldots, \mu_N\}$, $\mu_i = \boldsymbol{E}[X_i]$ and

$$\Sigma = [C_{ij}] = [\rho_{ij}\sigma_i\sigma_j] \qquad i, j \in \{1, \ldots, N\}. \quad (21)$$

Here $C_{ij} = \boldsymbol{E}[(X_i - \mu_i)(X_i - \mu_j)]$, $\sigma_i^2 = \mathrm{var}(X_i)$ and ρ_{ij} is the correlation coefficient between X_i and X_j. The marginal density functions are normal consistent with Eqn 18 and the joint probability density function is the bivariate normal:

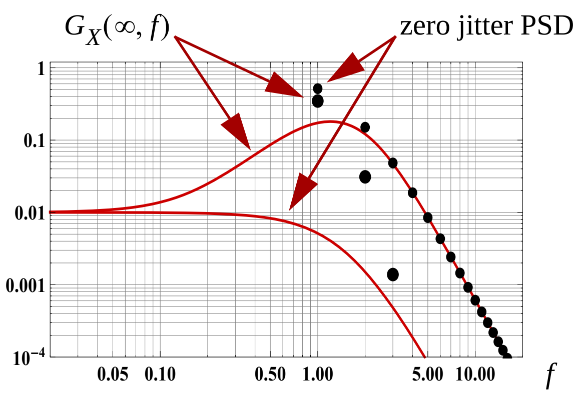

Fig. 3 Power spectral density of a jittered pulse train defined on $[0, \infty]$. The power in the impulses at $f = 0, r, 2r, 3r, 4r$, as represented by the larger dots, are, respectively, 1, 0.346, 0.031, $1.38x10^{-3}$, $3.37x10^{-5}$. The power in the impulses for the zero jitter case, as represented by the smaller dots, are, respectively, 1, 0.514, 0.150, $4.82x10^{-2}$, $1.86x10^{-2}$.

$$f_{X_iX_j}(x_i, x_j) = \frac{1}{2\pi\sigma_i\sigma_j\sqrt{1-\rho_{ij}^2}}\cdot\exp\left[\frac{-1}{1-\rho_{ij}^2}\cdot\left[\frac{(x_i-\mu_i)^2}{2\sigma_i^2} - \frac{\rho_{ij}(x_i-\mu_i)(x_j-\mu_j)}{\sigma_i\sigma_j} + \frac{(x_j-\mu_j)^2}{2\sigma_j^2}\right]\right] \quad (22)$$

It then follows that $\boldsymbol{E}[X_iX_j^*] = \mu_i\mu_j + \rho_{ij}\sigma_i\sigma_j$ and

$$\boldsymbol{E}[e^{-j2\pi f(X_i - X_k)}] = \exp[-j2\pi f(\mu_i - \mu_k)]\cdot \exp[-2\pi^2 f^2(\sigma_i^2 + \sigma_k^2 - 2\rho_{ik}\sigma_i\sigma_k)]. \quad (23)$$

For the amplitude random variables $\mu_i = \mu_j = \mu_A$, $\sigma_i = \sigma_j = \sigma_A$ and exponential correlation according to $\rho_{ij} = \exp(-|i-j|)$ is assumed consistent with $\Sigma_A = [\rho_{ij}\sigma_A^2]$ where

$$\Sigma_A = \begin{bmatrix} \sigma_A^2 & e^{-1}\sigma_A^2 & \ldots & e^{-N+1}\sigma_A^2 \\ \ldots & & & \ldots \\ e^{-N+1}\sigma_A^2 & \ldots & e^{-1}\sigma_A^2 & \sigma_A^2 \end{bmatrix} \quad (24)$$

Analogous definitions, and results, are assumed for the jittered random variables $J_1, \ldots, J_N$.

It then follows from Eqn 13 that the power spectral density is:

$$G_X(T, f) = r|P(f)|^2\overline{|A|^2} + 2r|P(f)|^2\sum_{i=1}^{N-1}\left[1-\frac{i}{N}\right]\cdot (\mu_A^2 + \rho_{1,1+i}^A\sigma_A^2)\exp[-4\pi^2 f^2\sigma_J^2(1-\rho_{1,1+i}^J)]\cdot \cos[2\pi i D f] \quad (25)$$

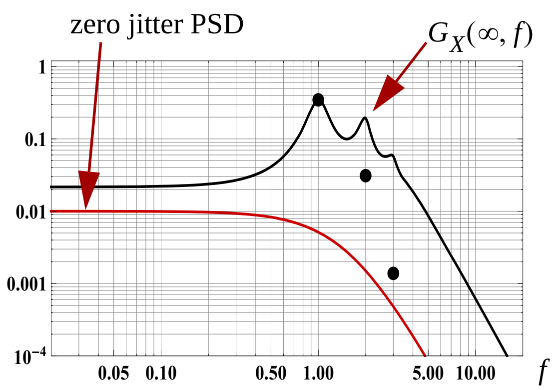

Fig. 4 Power spectral density of a correlated and jittered pulse train defined on $[0, \infty]$. The power in the impulses are as specified in Figure 3. The power in the impulses for the zero jitter case are as shown in Figure 3.

ISBN: 978-1-61804-141-8

where superscripts have been used to distinguish between the amplitude and jitter correlation coefficients. For the infinite interval the power spectral density, as given by Eqn 14, is

$$G_X(\infty, f) = r|P(f)|^2\Big[\overline{|A|^2} - \mu_A^2 e^{-4\pi^2 f^2 \sigma_J^2} + r\mu_A^2 e^{-4\pi^2 f^2 \sigma_J^2} \sum_{n=-\infty}^{\infty} \delta(f - nr) + 2\sigma_A^2 \sum_{i=1}^{M-1} \cos(2\pi i D f) \rho_{1,1+i}^A e^{-4\pi^2 f^2 \sigma_J^2} \cdot [\exp(4\pi^2 f^2 \sigma_J^2 \rho_{1,1+i}^J) - 1] + 2\mu_A^2 \cdot \sum_{i=1}^{M-1}\Big[\cos(2\pi i D f) e^{-4\pi^2 f^2 \sigma_J^2}[\exp(4\pi^2 f^2 \sigma_J^2 \rho_{1,1+i}^J) - 1]\Big] + 2e^{-4\pi^2 f^2 \sigma_J^2} \sigma_A^2 \sum_{i=1}^{M-1} \cos(2\pi i D f) \rho_{1,1+i}^A\Big] \quad (26)$$

The same parameters as for the independent case, as well as $\rho_{1,1+i}^A = \rho_{1,1+i}^J = e^{-i}$ are assumed and one signal is shown in Figure 2. The power spectral density is graphed in Figure 4.

Evidence for the validity of the power spectral density, as given by Eqn 26, is given in Figure 5 where simulation results using the FFT are displayed after averaging 100 jittered signals on the interval $[0, 200D]$.

4.3 Discussion

As is clearly evident in the power spectral densities shown in Figure 3, the transition from the non-jittered case to the jittered case leads to a transfer of energy from the periodic components to the non-periodic components. As is evident in Figure 4, and consistent with the nature of correlation, correlation leads to the strengthening of the power spectral density levels around the periodic components.

The five non-impulsive components of the power spectral density, as specified by Eqn 26, are shown in Figure 6. The first two components are associated with the independent random variable case. For the parameters chosen, the dominant term (4th term) is due to the correlated jitter which leads to spectral broadening around the harmonic components. The correlated amplitudes (term 5) lead to an increase in the spectrum level at low frequencies. The third term, due to both the amplitude and jitter random variables being correlated, is small.

5 Conclusion

A power spectral density expression for a correlated jittered pulse train has been established and verified by simulation for the multivariate Gaussian case. It was shown that correlated amplitude and jitter leads to three distinct additional components in the power spectral density. The effects of jitter, when compared to the zero jitter case, have been clearly identified for the independent and correlated jitter cases. It was shown that correlated jitter leads to spectral spread near the harmonic components whilst correlated amplitude jitter mainly leads to an increase in the power spectral density at low frequencies.

Appendix 1: Proof of Theorem

Lemma 1 Summation of Hermitian Matrix

For the case where $F_{ik} = F_{ki}^*$ and $F_{ik} = F_{i+1,k+1}$, $i, k, i+1, k+1 \in \{1, \ldots, N\}$, it follows that

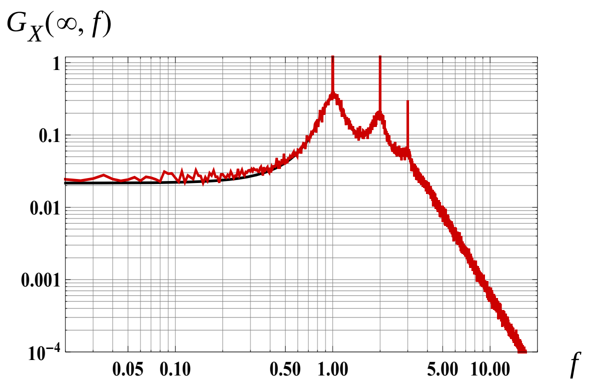

Fig. 5 Simulation and theoretical results for the power spectral density of a correlated and jittered pulse train.

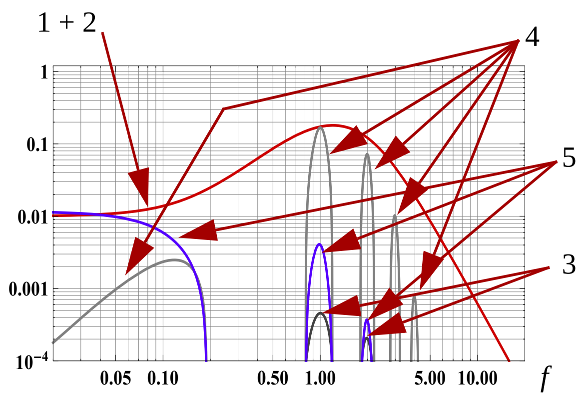

Fig. 6 Non-impulsive components of power spectral density.

$$S = \sum_{i=1}^{N} \sum_{k=1, k \neq i}^{N} F_{ik} = 2N\sum_{i=1}^{N-1}\left[1-\frac{i}{N}\right]Re[F_{1i}] \qquad (27)$$

Proof

This result follows by noting that there are $N-1$ identical terms where $k = i+1$ and $k = i-1$, $N-2$ identical terms where $k = i+2$ and $k = i-2$ ... and 1 term where $k = i+(N-1)$, $k = i-(N-1)$.

Proof of Theorem

The Fourier transform of one outcome of the random process, as given by Eqn 5, is

$$X(((a_1, j_1), \ldots, (a_N, j_N)), T, f) = P(f)\sum_{k=1}^{N} a_k e^{-j2\pi f[(k-1)D + j_k]}. \qquad (28)$$

It then follows, after splitting the double summation into its diagonal and off-diagonal terms, that

$$\left|X(((a_1, j_1), \ldots, (a_N, j_N)), T, f)\right|^2 = |P(f)|^2 \sum_{i=1}^{N} |a_i|^2 + |P(f)|^2 \sum_{i=1}^{N} \sum_{k=1, k\neq i}^{N} a_i a_k^* e^{-j2\pi f[(i-k)D + j_i - j_k]}. \qquad (29)$$

Consistent with Eqn 3 and Eqn 8, the power spectral density of X is

$$G_X(T, f) = \frac{|P(f)|^2}{T}\Bigg[\sum_{i=1}^{N}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} |a_i|^2 f_{A_1\ldots A_N}(a_1, \ldots, a_N)da_1\ldots da_N + \sum_{i=1}^{N}\sum_{k=1, k\neq i}^{N} e^{-j2\pi f(i-k)D}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} a_i a_k^* e^{-j2\pi f(j_i - j_k)} \cdot f_{A_1\ldots A_N}(a_1, \ldots, a_N) \cdot f_{J_1\ldots J_N}(j_1, \ldots, j_N) \cdot da_1\ldots da_N dj_1\ldots dj_N \Bigg] \qquad (30)$$

This result follows after the order of summation and integration has been interchanged. Use of the results from conditional probability theory:

$$f_{A_1\ldots A_N}(a_1, \ldots, a_N) = f_{A_i}(a_i) \cdot f_{A_1\ldots A_{i-1}A_{i+1}\ldots A_N|_{A_i = a_i}}(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_N)$$

$$f_{A_1\ldots A_N}(a_1, \ldots, a_N) = f_{A_iA_k}(a_i, a_k) \cdot f_{A_{1N/ik}}(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{k-1}, a_{k+1}, \ldots, a_N) \qquad (31)$$

where the following notation has been used:

$$A_{1N/ik} = A_1\ldots A_{i-1}A_{i+1}\ldots A_{k-1}A_{k+1}\ldots A_N / A_i = a_i, A_k = a_k \qquad (32)$$

yields

$$G_X(T, f) = \frac{|P(f)|^2}{T}\Bigg[\sum_{i=1}^{N}\int_{-\infty}^{\infty} |a_i|^2 f_{A_i}(a_i)da_i + \sum_{i=1}^{N}\sum_{k=1, k\neq i}^{N} e^{-j2\pi f(i-k)D} \cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a_i a_k^* f_{A_iA_k}(a_i, a_k)da_i da_k \cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j2\pi f(j_i - j_k)} f_{J_iJ_k}(j_i, j_k)dj_i dj_k \Bigg]. \qquad (33)$$

Eqn 11 and, after using Lemma 1, Eqn 13 then follow. To identify the terms associated with the independent case consider the double summation in the power spectral density, as given by Eqn 33, which can be written as:

$$S_2 = \sum_{i=1}^{N}\sum_{k=1, k\neq i}^{N} e^{-j2\pi f(i-k)D} \cdot \boldsymbol{E}[A_iA_k^*] \cdot \boldsymbol{E}[e^{-j2\pi f(J_i - J_k)}] = 2N\sum_{i-1}^{N-1}\left[1-\frac{i}{N}\right] \cdot Re\left[e^{j2\pi iDf} \cdot \boldsymbol{E}[A_1A_{1+i}^*] \cdot \boldsymbol{E}[e^{-j2\pi f(J_1 - J_{1+i})}]\right] \qquad (34)$$

by using the result stated in Lemma 1, the assumptions leading to Eqn 12, and the result $f_{\Omega_i\Omega_k}(\omega_i, \omega_k) = f_{\Omega_k\Omega_i}(\omega_k, \omega_i)$. By adding and subtracting terms associated with the independent case, S_2 can be written as

$$S_2 = I_1 + I_2 + I_3 + 2N \cdot \sum_{i-1}^{N-1}\left[1-\frac{i}{N}\right] \cdot Re\Big[e^{j2\pi iDf} \cdot (\boldsymbol{E}[A_1A_{1+i}^*] - |\mu_A|^2) \cdot \left(\boldsymbol{E}[e^{-j2\pi f(J_1 - J_{1+i})}] - |F_J(f)|^2\right) \Big] \qquad (35)$$

where

$$I_1 = 2N|F_J(f)|^2\sum_{i=1}^{N-1}\left[1-\frac{i}{N}\right] \cdot Re[e^{j2\pi iDf} \cdot [\boldsymbol{E}[A_1A_{1+k}^*] - |\mu_A|^2]] \qquad (36)$$

ISBN: 978-1-61804-141-8

$$I_2 = 2N|\mu_A|^2 \sum_{i=1}^{N-1}\left[1-\frac{i}{N}\right] \cdot$$
$$Re\left[e^{j2\pi iDf} \cdot \left[\boldsymbol{E}[e^{-j2\pi f(J_1 - J_{1+i})}] - |F_J(f)|^2\right]\right] \tag{37}$$

$$\begin{aligned} I_3 &= 2N|\mu_A|^2|F_J(f)|^2 \sum_{i=1}^{N-1}\left[1-\frac{i}{N}\right] \cdot Re[e^{j2\pi iDf}] \\ &= |\mu_A|^2|F_J(f)|^2 \sum_{i=1}^{N}\sum_{k=1,\,k\neq i}^{N} e^{-j2\pi f(i-k)D} \\ &= N|\mu_A|^2|F_J(f)|^2 \cdot \left[\frac{1}{N}\cdot\frac{\sin(\pi Nf/r)^2}{\sin(\pi f/r)^2} - 1\right]. \end{aligned} \tag{38}$$

In Eqn 38 the second equality arises from Lemma 1 and the final form arises from p. 45 [13]. For the case of $N \to \infty$, the first term in the last equation in Eqn 38 becomes a sequence of impulses, p. 45 [13], and the power spectral density for the infinite interval, as given by Eqn 14, results after substituting the above results into Eqn 33.

Acknowledgement:

Prof. A. Zoubir hosted, and supported, a visit to the Technical University of Darmstadt, Darmstadt, Germany, where the research underpinning the paper was completed.

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