Input-to-State Stability for A Class of Hybrid Dynamical Systems via Hybrid Time Approach

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Abstract—This paper studies the ISS (input-to-state stability) for a class of HDS (hybrid dynamical systems). By using the concept of hybrid time for HDS, two kinds of ISS notions are proposed. They are called the first ISS property and the second ISS property of HDS. By employing the ISS properties on continuous and/or discrete dynamics in the HDS, the first and second ISS properties for the whole HDS are investigated. We show that there exists the first ISS property for the whole HDS if any one of dynamics in the HDS has the ISS property and the dwell time on this dynamics is sufficiently long even if there is no ISS property for the other dynamics. Moreover, the second ISS property of HDS is derived for the cases in which one kind of dynamics without external inputs has a stability property while the other one has no stability and ISS properties. Finally, one example is given for illustration.

I. INTRODUCTION

ISS (input-to-state stability) analysis of nonlinear systems aims to investigate how external disturbances affect the system stability. Since the notion of ISS was proposed in [1] in the late 1980s, ISS property analysis of nonlinear systems with external disturbances has quickly become one of the more active research topics in nonlinear analysis and design. ISS has been successfully employed in the stability analysis and control synthesis of nonlinear systems. These works mainly focused on the following topics related to the study of the ISS property: feedback ISS stabilization for nonlinear systems [2, 4, 9-10, 13], ISS nonlinear small gain results for nonlinear systems [6, 17, 18], ISS for sampled-data systems [8], ISS problem using averaging technique [7], ISS properties of networked control systems [5], ISS for discrete-time systems [3], ISS for impulsive systems in [11-12,37], and ISS for time-delay systems in [14-16,39].

Hybrid dynamical systems (HDS) provide a natural framework for mathematical modeling of many physical phenomena and practical engineering problems. Their study has received considerable attention for the last two decades, see [26-36, 38] and the references cited therein. Due to many factors such as the hybrid structure, multiple dynamics, time-delays, and external disturbances, etc, it is now recognized that the dynamical behaviors (for example, stability and ISS) for HDS may be complex and hence it remains a difficult task to investigate their dynamics despite progress so far.

Recently, stability and \( \mathcal{X} \mathcal{L} \mathcal{Z} \)-stability properties based on set-value stability theory, invariance principles, and smooth Lyapunov functions are investigated for more general HDS in the literature, see [20-25]. In these references, concepts such as hybrid time domain, hybrid trajectory for HDS were proposed. It has been shown that these concepts of HDS contribute to the analysis of stability for HDS. The mentioned research on the stability of HDS assumes the absence of external disturbance inputs. More recently, ISS analysis for HDS has been reported in [19] by using \( \mathcal{X} \mathcal{L} \mathcal{Z} \)-functions and smooth ISS-Lyapunov function method. However, the ISS property studied in [19] requires that every subsystem in a HDS has the corresponding ISS property and thus it can not reflect completely the diversity of dynamics for subsystems in a HDS. Hence, the ISS of HDS has not been fully developed.

In this paper, we focus on the study of ISS for a class of HDS. In the HDS, there exist two kinds of dynamics (continuous-time and discrete-time dynamics). And these two kinds of dynamics appear alternately. The continuous-time dynamics can be looked upon as the natural process of HDS while the discrete-time dynamics is due to data-sampling or discrete control inputs. In [37], we have noted that the hybrid time variables, specially the discrete-time variable in the hybrid time domain, play an important role in the analysis of ISS for HDS. On account of this fact, in this paper, we propose two kinds of ISS notions for HDS, i.e., the first and the second ISS property of HDS. These two ISS notions are more general than that in [19]. By means of ISS properties allocating to the different types of dynamics in the HDS, sufficient conditions for the first and second ISS property of such HDS are established.

II. PRELIMINARIES

In the sequel, let \( R = (-\infty, \infty), R_+ = [0, +\infty), \mathbb{R}^n \) be the \( n \)-dimensional Euclidean space, and \( \mathbb{N} = \{0, 1, 2, \cdots\} \). For \( t_1, t_2 \in R_+ \), satisfying \( t_1 \leq t_2 \) and \( t_2 - t_1 \in \mathbb{N} \), denote \( \mathcal{N}[t_1, t_2] = \{t_1 + k : k \in \mathbb{N}, t_1 \leq t_1 + k \leq t_2\} \).

A function \( \gamma : R_+ \to R_+ \) is of class-\( \mathcal{X} \) (\( \gamma \in \mathcal{X} \)) if it is continuous, zero at zero and strictly increasing. It is of class-\( \mathcal{X}_m \) if it is of class-\( \mathcal{X} \) and is unbounded. A continuous function \( \beta : R_+ \times R_+ \to R_+ \) is of class-\( \mathcal{X} \mathcal{L} \) if \( \beta(\cdot, t) \) is of class-\( \mathcal{X} \) for \( t \geq 0 \) and \( \beta(s, \cdot) \) is monotonically decreasing to zero for \( s > 0 \). A continuous function \( \beta : R_+ \times R_+ \to R_+ \) is of class-\( \mathcal{X} \mathcal{Z} \) if \( \beta(\cdot, t) \) is of class-\( \mathcal{X} \mathcal{L} \) for \( t \geq 0 \) and \( \beta(s, \cdot) \) is of class-\( \mathcal{X} \mathcal{Z} \) for \( s \geq 0 \). A function \( \beta : R_+ \times \mathbb{N} \to R_+ \) is...
of class $K$ if $\beta(s,i) \in K$ for $i \in \mathbb{N}$, and for any given $s > 0$, the series \{\beta(s,i),i \in \mathbb{N}\} is monotonically decreasing to zero.

Denote: $\mathcal{K}^+ \triangleq \{\gamma \in \mathcal{K} : \gamma(a+b) \leq \gamma(a) + \gamma(b), a,b \in \mathbb{R}_+\}$, $\mathcal{K}^+ \cap \mathcal{L}^+ \triangleq \{\beta \in \mathcal{K} \cap \mathcal{L} : \beta(a+b) \leq \beta(a) + \beta(b), a,b \in \mathbb{R}_+\}$, $\mathcal{K}^+ \cap \mathcal{L}^+ \triangleq \{\beta \in \mathcal{K} \cap \mathcal{L} : \beta(a,s(i)) \leq \beta(a+s(i),s), a,s(i) \in \mathbb{R}_+\}$, and $\mathcal{K}^+ \cap \mathcal{L}^+ \triangleq \{\alpha \in \mathcal{K}^+ \cap \mathcal{L}^+ : \alpha(a+b) \leq \alpha(a) + \alpha(b), a,b \in \mathbb{R}_+\}$.

Given a $n \times n$ matrix $A$, $||A||$ denotes the norm of $A$ induced by the Euclidean vector norm, i.e., $||A|| = \lambda_{\text{max}}(A^T A)^{1/2}$. For any function $u : R_+ \rightarrow R_+$, we denote $||u||_s = \sup\{||u(s)|| : 0 \leq s \leq t, t \in R_+\}$, and $||u||_\infty = \sup\{||u(s)|| : s \in R_+\}$.

Consider the HDS under external inputs:

$$
\begin{align*}
\{x(t) \in S_{ci} & \{x(t) : \dot{x} = f_{ci}(t,x,w_{ci}), \text{ } t \in I_{ci}, \} \\
\{x(t) \in S_{di} & \{x(t) : \dot{x} = f_{di}(t-1,x(t-1),w_{di}), \text{ } t \in I_{di}, \}
\end{align*}
$$

where $x(t) \in R^n$ is the state of HDS; $w_{ci} : R_+ \rightarrow R^{n_0}$, $w_{di} : R_+ \rightarrow R^{n_0}$, with $w_{ci}(0) = 0$, $w_{di}(0) = 0$, represent the external disturbance inputs satisfying $||w_{ci}||_\infty < \infty$, $||w_{di}||_\infty < \infty$, where $||w_{ci}||_\infty \triangleq \sup_{s \in \mathbb{R}_+} \{||w_{ci}(s)|| : 0 \leq s \leq t, t \in R_+\}$, $||w_{di}||_\infty \triangleq \sup_{s \in \mathbb{R}_+} \{||w_{di}(s)|| : 0 \leq s \leq t, t \in R_+\}$.

Consider the HDS under external inputs:

$$
\begin{align*}
\{x(t) & \in S_{ci} \text{ } x(t) : \dot{x} = f_{ci}(t,x,w_{ci}), \text{ } t \in I_{ci}, \} \\
\{x(t) & \in S_{di} \text{ } x(t) : \dot{x} = f_{di}(t-1,x(t-1),w_{di}), \text{ } t \in I_{di}, \}
\end{align*}
$$

where $x(t) \in R^n$ is the state of HDS; $w_{ci} : R_+ \rightarrow R^{n_0}$, $w_{di} : R_+ \rightarrow R^{n_0}$, with $w_{ci}(0) = 0$, $w_{di}(0) = 0$, represent the external disturbance inputs satisfying $||w_{ci}||_\infty < \infty$, $||w_{di}||_\infty < \infty$, where $||w_{ci}||_\infty \triangleq \sup_{s \in \mathbb{R}_+} \{||w_{ci}(s)|| : 0 \leq s \leq t, t \in R_+\}$, $||w_{di}||_\infty \triangleq \sup_{s \in \mathbb{R}_+} \{||w_{di}(s)|| : 0 \leq s \leq t, t \in R_+\}$.

**Definition 2.1.** (i) The HDS (1) is said to have the first ISS property if there exist a function $\beta \in \mathcal{K}^+ \cap \mathcal{L}^+$, and functions $\gamma, \gamma_i, \gamma_d \in \mathcal{K}$ such that, for any $(t,i) \in I_{ci}, i \in \mathbb{N}$, $t \geq t_0$,

$$
\phi(||x(t,i)||) \leq \beta(||x_0||, t-t_0) + \gamma(||w_{ci}(i)||) + \gamma_i(||w_{di}(i)||).
$$

(ii) The HDS (1) is said to have the second ISS property if there exist a function $\beta \in \mathcal{K}^+ \cap \mathcal{L}^+$, and functions $\gamma, \gamma_i, \gamma_d \in \mathcal{K}$ such that, for any $(t,i) \in I_{ci}, i \in \mathbb{N}$,

$$
\phi(||x(t,i)||) \leq \beta(||x_0||, i) + \gamma(||w_{ci}(i)||) + \gamma_i(||w_{di}(i)||).
$$

(iii) The HDS (1) is said to have the first exponential ISS property if there exist constants $\alpha > 0, K > 0$, and $\mathcal{K}$-functions $\gamma$ and $\gamma_i$, such that for any $(t,i) \in I_{ci}, i \in \mathbb{N}$, $t \geq t_0$,

$$
||x(t,i)|| \leq K e^{-\alpha(t-t_0)} ||x_0|| + \gamma(||w_{ci}(i)||) + \gamma_i(||w_{di}(i)||).
$$

(iv) The HDS (1) is said to have the second exponential ISS property if there exist constants $\alpha > 0, K > 0$, and $\mathcal{K}$-functions $\gamma$ and $\gamma_i$, such that for any $(t,i) \in I_{ci}, i \in \mathbb{N}$,

$$
||x(t,i)|| \leq K e^{-\alpha i} ||x_0|| + \gamma(||w_{ci}(i)||) + \gamma_i(||w_{di}(i)||).
$$
Proof. First, by the following Claims 1◦-2◦, we claim that β ∈ K + L +.

Claim 1◦: if β c, β d ∈ K + L , then, β ∈ K + L ... for any a ∈ R+, there exists a K+-class function ˆγ satisfying (9).

Then, HDS (1) has the first ISS property.

Thus, from conditions (i)-(ii) and β c, β d ∈ K + L +, it yields that
\[ \|x(t_{2i+1}, i)\| ≤ β(\|x(t_0), t - t_0\|) + \sum_{j=1}^{i} β(\|w_d(t_{2j}), t - t_{2j}\|) \]

Noting \( a_0 = \|x(t_i, 0)\| \) ≤ β(\( \|x(t_0), t_1 - t_0\| \) + γ(\( \|w_c\| t_1 \)), and thus by (12), it yields that
\[ a_i ≤ β(\|x_0\|, t_{2i+1} - t_0) + \sum_{j=1}^{i} β(\|w_d(t_{2j}), t_{2i+1} - t_{2j}\|) \]

Hence, for any \( (t, i) \in \mathcal{A} \), if \( t \in I_{di} = [t_{2i}, t_{2i+1}] \), then, by condition (i) and (13), we get
\[ \|x(t, i)\| ≤ β(\|x_0\|, t - t_0) + \sum_{j=1}^{i} β(\|w_d(t_{2j}), t - t_{2j}\|) \]

and condition (i), we get
\[ \|x(t, i)\| ≤ β(\|x_0\|, t - t_0) + \sum_{j=1}^{i} β(\|w_d(t_{2j}), t - t_{2j}\|) \]

Since β ∈ K + L +, thus, for any a ∈ R+ and i, j ∈ N with j ≤ i, we have \( β(a, t_{2i+1} - t_{2j}) ≤ β(a, t_{2i} - t_{2j}) \), \( β(a, t_{2i} - t_{2j-1}) ≤ β(a, t_{2i-1} - t_{2j-1}) \), and \( β(a, t_{2i} - t_{2j}) ≤ β(a, t_{2i-1} - t_{2j}) \). Hence, it follows from (14)-(15) and (9) that
\[ \|x(t, i)\| ≤ β(\|x_0\|, t - t_0) + \sum_{j=1}^{i} β(\|w_d(t_{2j}), t_{2i+1} - t_{2j}\|) \]

Case III: \( S_{di} \) has the ISS property while \( S_{di} \) may have no ISS property.

Theorem 3.2. Suppose \( \Delta_{sup}^d < \infty \) and assume that there exist functions \( V^c, V^d_i \in C[\mathcal{R}_+ \times \mathcal{R}_+, \mathcal{R}_+] \), \( i \in N \), such that the following conditions are satisfied:

(i) there exist \( \mathcal{H} \)-class functions \( \Phi_1, \Phi_2 \), such that
\[ \Phi_1(\|x\|) ≤ V^d(t, x) ≤ \Phi_2(\|x\|) \]
ii) there exist \( \mathcal{L} \)-class function \( β_c \), and \( \mathcal{H} \)-class function \( γ_c \) such that for any \( t \in \mathcal{I}_c, i \in N \),
\[ V^c(t, x(t, i)) ≤ β_c(V^c(t_{2j}, x(t_{2j}, i), t - t_0) + \gamma_c(\|w_c\| t)) \]

(iii) there exist \( \mathcal{H} \)-class functions \( \psi, \gamma \) with \( \psi(s) ≥ s \) and \( \psi \in \mathcal{H}^+ \), such that for any \( t \in \mathcal{I}_c, i \in N \),
\[ V^d_i(t, x(t, i)) ≤ \psi(V^d(t - 1, x(t, i))) + \gamma(\|w_d\| t) \]

(iv) for any \( i ≥ 1, i \in N \), the solution \( x(t) \) satisfies
\[ V^c(t_{2j}, x(t_{2j}, i), t - t_0) ≤ V^d_i(t_{2j}, x(t_{2j}, i), t - t_0) \]
\[ V^d_i(t_{2j}, x(t_{2j}, i), t - t_0) ≤ V^c(t_{2j}, x(t_{2j}, i), t - t_0) \]

(v) there exists a function \( β ∈ \mathcal{H}^+ \), \( \Delta_{sup}^d \) satisfying, for any \( a, s ∈ R_+ \) and any \( m ∈ N [0, \Delta_{sup}^d] \),
\[ \psi^m(β(a, s)) ≤ β(a, m + s) \]

Then, HDS (1) has the first ISS property.
Proof. Let $a_i = V_{i+1}(t_2, x(t_2), i - 1)$ for any $i \geq 1$, and for $i \in N$, let $v_i = \psi^{\Delta t}(\gamma(\|w_c\|_{t_2(i)})) + \gamma(\|w_d\|_{t_2(i)}) + \psi(\gamma(\|w_d\|_{t_2(t_2+i)}) + \psi(\gamma(\|w_c\|_{t_2(t_2+i)}) + \ldots + \psi(\gamma(\|w_d\|_{t_2(t_2+i)}))).$ By conditions (ii)-(iii), it yields that

$$a_{i+1} \leq \psi^{\Delta t}(V_{i+1}(t_2(i), x(t_2(i)))).$$

By condition (i), it derives from (23) and conditions (ii)-(iii) that

$$a_{i+1} \leq \bar{\beta}(V(t_0, x_0), t_2(i)) - t_0)
+ \sum_{j=0}^{i-1} \beta(v_j, t_2(i) - t_2(j+1)) + v_i, i \in N.$$  

Thus, it yields that

$$a_{i+1} \leq \beta(V(t_0, x_0), t_2(i) - t_0)
+ \sum_{j=0}^{i-1} \beta(v_j, t_2(i) - t_2(j+1)) + v_i, i \in N.$$  

By (22), we have $\beta_0(a, s) \leq \beta(a, s)$ for any $a, s \in R_+$. Thus, for any $(t, i) \in \mathcal{F}$, if $t \in L_i$, then, it follows from (20) and (25) that

$$V^c(t, x(t), i)) \leq \beta(a_i, t - t_2) + \gamma(\|w_c\|_{t_0})
+ \sum_{j=0}^{i-1} \beta(v_j, t_2(i) - t_2(j+1)) + v_i, i \in N.$$  

Thus, by using (26)-(27) and condition (i), we obtain that for any $(t, i) \in \mathcal{F}, i \in N$,

$$\varphi_i(\|x(t, i)\|) \leq \tilde{\beta}(\|x_0\|, t - t_0) + \gamma(\|w_c\|_{t_0}) + \gamma(\|w_d\|_{t_0}),$$

where $\tilde{\beta}(a, s) = \bar{\beta}(\psi^{\Delta t}(\gamma(\|w_c\|_{t_0}))) + \gamma(\|w_d\|_{t_0}) + \gamma(\gamma(\|w_d\|_{t_0}) + \psi(\gamma(\|w_d\|_{t_0}) + \ldots + \psi(\gamma(\|w_d\|_{t_0})$).

Hence, HDS (1) has the first ISS property. □

Case III: $S_{di}$ has the ISS property while $S_{ai}$ may have no ISS property.

Theorem 3.3. Suppose $\Delta_{up}^{\tau} < \infty, \Delta_{inf}^{\tau} > 0$ and assume that there exist functions $V^c, V^i \in C[R_+ \times R^n, R_+], i \in N$, such that the conditions (i) and (iv) of Theorem 3.2 hold and the following conditions are satisfied:

(i) there exist $\mathcal{H}^+\mathcal{H}^-$-class function $\alpha_0^c$ and $\mathcal{H}^-$-class function $\gamma_c$ such that for any $t \in I_d$, $V^c(t, x(t), i) \leq \alpha_0^c(t, x(t), i)) + \gamma_c(\|w_c\|_{t_0}); (29)$

(ii) there exist $\mathcal{H}^-$-class functions $\gamma_0^c$ with $\psi \in \mathcal{H}^+$ and $\gamma(s) < s$ for any $s > 0$, such that for any $t \in I_d, i \in N$, $V^i(t, x(t), i)) \leq \psi(\gamma(t, x(t), i)) + \gamma(\|w_c\|_{t_0}); (30)$

(iii) there exists a function $\beta \in \mathcal{H}^+\mathcal{H}^-$ satisfying (9) and, for any $a, s \in R_+ and any $m \in \mathcal{N}(\mathcal{H}^{inf}, +\infty)$,

$$\alpha_q(a, s) \leq \beta(a, m + s). (31)$$

Then, HDS (1) has the first ISS property.

Proof. By using the similar proof of Theorem 3.2, we can derive the result. The details are omitted here. □

Remark 3.1. In Theorem 3.2, if $\beta_0(a, s) = ae^{-\alpha s}, \psi(s) = qs$ for some positive constants $a > 0, q > 1$, and any $a, s \in R_+$, then, we define $\beta(a, s) = ae^{-\alpha s}$, where $\alpha$ satisfies $0 < \alpha < \alpha^*$ and

$$\frac{\Delta_{ci}}{\Delta_{di}} \geq \frac{\ln q + \alpha^*}{\alpha^* - \alpha^*}, i \in N.$$  

Similarly, in Theorem 3.3, if $\alpha_0(a, s) = ae^{\alpha s}, \psi(s) = qs$ for some positive constants $a > 0, q < 1$, and any $a, s \in R_+$, then, we define $\beta(a, s) = ae^{-\alpha s}$, where $\alpha^*$ satisfies $0 < \alpha^* < \ln q$ and

$$\frac{\Delta_{di}}{\Delta_{ci}} \geq \frac{\alpha^*}{\alpha^* - |\ln q + \alpha^*|}, i \in N.$$  

It follows from (32)-(33) and Theorem 3.2-3.3 that there exists the first ISS property for the whole HDS if any one of these two kinds of dynamics has the first ISS property and the dwell time on the dynamics with the ISS property is enough long.

Theorem 3.4. Suppose $0 < \Delta_{inf} \leq \Delta_{up} < +\infty$ and assume that there exist functions $V^c, V^i \in C[R_+ \times R^n, R_+], i \in N$, such that the condition (iv) of Theorem 3.2 holds and the following conditions are satisfied:

(i) there exist positive constants $c_1, c_2, r > 0$, such that

$$c_1\|x\|^r \leq V^c(t, x) \leq c_2\|x\|^r, V^c = V^i or V^i = V^d, t \in R_+;$$

(ii) there exist $\mathcal{H}^-$-class function $\gamma_c$, and constants $p_i$ with $0 < p \leq |p_i| \leq \bar{p}, i \in N$, for some positive constants $p_i$, such that for any $t \in L_i, i \in N$,

$$D^+V^c(t, x(t), i)) \leq pV^c(t, x(t), i)) + \gamma_c(\|w_c(t, i))|^r; (35)$$

(iii) there exist $\mathcal{H}^-$-class function $\gamma_0^c$ and constants $q_i$ with $q_i > -1 and 0 < q \leq |q_i| \leq \bar{q}, i \in N$, for some positive constants $q_i$, such that for any $t \in L_i, i \in N$,

$$V^i(t, f_{di}(t, x(t), i), w_{di}(t - 1)) \leq (1 + q_i)V^i(t, x(t), i)) + \gamma(\|w_{di}(t - 1)); (36)$$

(iv) there exists a constant $\alpha > 0$ such that

$$\sigma_t \triangleq \Delta_{ci} + \Delta_{di} (1 + q_i) \leq -\alpha \Delta^+, i \in N.$$  

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Then, HDS (1) has the first exponential ISS property. 

Proof. Due to space limitations, the details are omitted here. \hfill \Box

B. The Second ISS Property of HDS

In this part, we investigate the second ISS property of HDS (1) under two special cases: \(w_c \equiv 0\), and \(w_d \equiv 0\). Due to space limitations, we include only an outline for the proof.

**Theorem 3.5.** Let \(w_d \equiv 0\). Suppose \(A^{up}_d < \infty\), \(A^{inf}_d > 1\), and assume that there exist functions \(V^c, V^d \in C[R_+ \times R^n, R_+]\), \(i \in N\), such that the conditions (i) and (iv) of Theorem 3.2 holds and the following conditions are satisfied:

(i) there exist \(\mathcal{K}\)-class functions \(c_1, c_2, \gamma \in \mathcal{K}\) and function \(p \in C[R_+, R_+]\), such that for any \(t \in I_{ci}, i \in N\),
\[
D^+ V^c(t, x(t, i)) \leq p(t)c_1(V^c(t, x(t, i))) + \gamma(||x||)c_2(V(t, x(t, i))); 
\]
(ii) there exists a \(\mathcal{K}\)-class function \(\psi\) satisfying \(\psi \in \mathcal{K}^+\) and \(\psi(s) < s\) for all \(s > 0\), such that for any \(t \in I_{di}, i \in N\),
\[
D^+ V^d(t, x(t, i)) \leq \gamma(||x||)V(t, x(t, i)), 
\]
where for any \(a \in R_+\), there exists a \(\hat{\gamma} \in \mathcal{K}\) such that
\[
\sum_{i=0}^{\infty} \psi^{i}(a) \leq \hat{\gamma}(a); 
\]
(iii) there is a \(\mathcal{K}\)-class function \(\gamma \in \mathcal{K}\) such that \(\sum_{i=0}^{\infty} \psi^{i}(s) \leq \psi^{-1}(s)\).

Then, HDS (1) has the second ISS property.

Proof. Let \(m_i^c(t) = V^c(t, x(t, i)), m_i^d(t) = V^d(t, x(t, i)), \) for any \((t, i) \in J, i \in N,\) and \(a_i = V^c(t_{i+1}, x(t_{i+1}, i)), V_i(t) = \gamma(||x||), \) for \((t, i) \in J, v_i = V_i(t_{i+1}), i \in N.\)

Claim 1°: for any \((t, i) \in J, t \in I_{di}, i \in N,\) we claim that
\[
m_i^c(t) \leq \psi^i(a_0) + \sum_{j=0}^{i-1} \psi^j(\gamma^j), t \in I_{di}, i \in N, 
\]
where for any \(0 \leq k \leq i, \gamma_k = \{
\)
\begin{align*}
&v_i(t), k = i; \\
&v_k, k < i.
\end{align*}

Claim 2°: if \(t \in I_{di} = N[t_{2i+1} + 1, t_{2i+1}],\) we claim that
\[
m_i^d(t) \leq \psi^i(a_0) + \sum_{j=0}^{i-1} \psi^j(\gamma^j), t \in I_{di}, i \in N. 
\]

Claim 3°: by Lemma 2.1, there exists a function \(\beta \in \mathcal{K}\) such that
\[
a_0 = m_0^c(t_1) \leq \beta(m_0^c(t_{i+1}), t_{i+1} - t_0) 
\]
Hence, by Claims 1°-3°, we get \(\beta(a, i) \leq \psi(\beta(\varphi(a), A^{up}_d)),\) for any \(a \in R_+, i \in N, \) and \(\tilde{\gamma}(s) = \tilde{\gamma}(\gamma(s))\) for any \(s \in R_+,\) then, \(\beta \in \mathcal{K}, \tilde{\gamma} \in \mathcal{K},\) and for any \((t, i) \in J, i \in N,\)
\[
\varphi^i(||x||) \leq \beta(||x||) + \tilde{\gamma}( ||x||). 
\]

Hence, HDS (1) has the second ISS property. \hfill \Box

**Theorem 3.6.** Let \(w_c \equiv 0\). Suppose \(A^{up}_d < \infty\) and assume that there exist functions \(V^c, V^d \in C[R_+ \times R^n, R_+]\), \(i \in N\), such that the conditions (i) and (iv) of Theorem 3.2 holds and the following conditions are satisfied:

(i) there exist functions \(c \in \mathcal{K}, p \in C[R_+, R_+]\), such that for any \(t \in I_{ci}, i \in N,\)
\[
D^+ V^c(t, x(t, i)) \leq -p(t)c(V^c(t, x(t, i))); 
\]
(ii) there exist \(\mathcal{K}\)-class functions \(\psi, \gamma\) with \(\psi \in \mathcal{K}^+\) and \(\psi(s) > s\) for any \(s \in R_+,\) such that for any \(t \in I_{di}, i \in N,\)
\[
V_i^d(t, x(t, i)) \leq \gamma(||x||) + \gamma(||w_d(t, i)||); 
\]
(iii) there exists a increasing function \(\psi\) with \(\psi^{-1} \in \mathcal{K}^+\) and \(\psi(s) > s\) for any \(s > 0,\) such that for any positive constant \(z > 0,\)
\[
\int_{s}^{\infty} p(s)ds + \int_{s}^{\infty} \frac{\psi^{-1}(\gamma(s))}{c(s)}ds \leq 0, i \in N, 
\]
where for any \(a \in R_+,\) there exists a \(\mathcal{K}\)-class function \(\gamma\) such that \(\sum_{i=0}^{\infty} \psi^{-i}(a) \leq \gamma(a).\)

Then, HDS (1) has the second ISS property.

Proof. It can be derived by using the similar proof as in Theorem 3.5. The details are thus omitted. \hfill \Box

**Remark 3.2.** Similar results on dwell time as in Remark 3.1 can be derived for the second ISS property of HDS. Moreover, one can see from the results and conditions of Theorems 3.5-3.6 that the second ISS property for the whole HDS exists when one kind of dynamics without external inputs in the HDS has a stability property while the other one with external inputs has no stability and ISS properties.

IV. Example

In this section, we give one example for illustration.

**Example 4.1.** Consider the HDS:
\[
\dot{x} = f_c(t, x) + w_c(t) = A_c(t)x + w_c(t),\quad t \in I_{ci};
\]
\[
x(t) = A_{di}(t-1)x(t-1) + \varphi_{di}(t-1, x(t-1)) + w_{di}(t-1),\quad t \in I_{di}, i \in N, 
\]
where \(A_c(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.1 & 0.1(1 - e^{-t}) \end{bmatrix}, \) and \(A_{di}(t) = \begin{bmatrix} 0.4 + 0.1 \sin(t) & 0 \\ 0 & 0.1 \sin(t) \end{bmatrix}, \)
\[
\varphi_{di}(t, x(t)) = \begin{bmatrix} 0.1(1 - e^{-t}) & 0 \\ 0 & 0.5 \cos(t) \end{bmatrix}, \quad x(t) \in \mathbb{R}^2, 
\]
and assume that \(\|w_c\|_\infty < \infty, \|w_d\|_\infty < \infty\) and the time intervals: \(I_{ci} = [t_{2i+1}, t_{2i+1}],\) and \(I_{di} = [t_{2i+1} + 1, t_{2i+1}],\) are chosen as: \(t_0 = 0; t_{2i+1} = t_{2i+1} + 1; t_{2i+1} = t_{2i+1} + 6, i \in N.\)

Let \(V(t, x) = ||x||,\) we get
\[
D^+ V \leq pV + d_1\|w_c\|_\infty, \quad t \in I_{ci};
\]
\[
V + (1 + q_i)V(t, x(t)) + d_2\|w_d\|_\infty, t \in I_{di},
\]
where $p = 0.5, q_i = -0.25, d_1 = d_2 = 1$. Thus, we get
\[ p\Delta c_i + \Delta d_i \ln(1 + q_i) \leq -0.0301(\Delta c_i + \Delta d_i). \]
By Theorem 3.4, we get that the HDS has the first exponential ISS property. Moreover, we can get that
\[ \|x(t, i)\| \leq 2.5487\|x_0\|e^{-0.0301t} + 6.9634\|w_c\|_i + 8.4680\|w_d\|_i, \quad (t, i) \in \mathcal{F}_i, i \in N. \]
In the simulation, we take the initial condition $t_0 = 0, x_0 = (1, 3 - 5)^T$. The external disturbance inputs are in the form of: $w_c(t) = \text{rand}(1)(-0.1, -0.1, 0.1)^T, w_d(t) = 0.5(\sin(t), \sin(t), \cos(t))^T, \quad t \in \mathbb{R}_+$. The result of numerical simulations is given in Fig.1.

**Fig.1.** The first ISS property of HDS (49).

### REFERENCES


