

Optimal replication of random claims by ordinary integrals with applications in finance*

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Abstract

By the classical Martingale Representation Theorem, replication of random vectors can be achieved via the stochastic integrals or solutions of the stochastic differential equations. We introduce a new approach to replication of random vectors via adapted differentiable processes generated by a controlled ordinary differential equation. We found that a control that ensures replication exists and is not unique. This leads to a new optimal control problem: find a replicating control that is minimal in an integral norm. We found an explicit solution of this problem. Possible applications to the portfolio selection problems and to the bond pricing are suggested.

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1 Introduction

By the classical Martingale Representation Theorem, random variables generated by a Wiener process can be represented via stochastic integrals. This means that it is possible to find a Ito process such that the terminal value matches a given random vector at a fixed terminal time. This key result leads to the theory of backward stochastic differential equations and has many applications in Mathematical Finance.

We introduce a new approach where the replication of random vectors is achieved via adapted differentiable processes generated by a controlled ordinary differential equation. We found that the solution of this version of the replication problem exists and is not unique. Therefore, an optimal control problem arises: find a replicating control process that is minimal in a certain norm. We found an explicit solution of this problem in the linear quadratic setting. Possible applications to the portfolio selection problems and to the bond pricing are suggested.

2 The control problem

Consider a standard probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Let $w(t)$ be a standard d -dimensional Wiener

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process which generates the filtration $\mathcal{F}_t = \sigma\{w(r) : 0 \leq r \leq t\}$ augmented by all the \mathbf{P} -null sets in \mathcal{F} ; we assume that $w(0) = 0$.

Let $f \in L_2(\Omega, \mathcal{F}_T, \mathbf{P}; \mathbf{R}^n)$ be a random vector.

Let $a \in \mathbf{R}^n$, $A \in \mathbf{R}^{n \times n}$ be given, and let $b \in \mathbf{R}^{n \times n}$ be a non-degenerate matrix. Let $\Gamma : [0, +\infty) \rightarrow \mathbf{R}^{n \times n}$ be a measurable matrix valued function.

We consider the following linear quadratic optimal stochastic control problem

$$(2.1) \text{ Minimize } \mathbf{E} \int_0^T u(t)^\top \Gamma(t) u(t) dt \text{ over } u(\cdot)$$

(2.2) subject to

$$\frac{dx}{dt}(t) = Ax(t) + bu(t), \quad t \in (0, T)$$

$$x(0) = a, \quad x(T) = f \quad \text{a.s.}$$

This is a stochastic control problem with the equality type constraints on the terminal value of the plant process that must hold almost surely. The only known approach to this problem is based on the so-called backward stochastic differential equations (BSDEs), where the terminal value for the plant process is fixed and the diffusion coefficient has to be selected. In the corresponding control setting for BSDEs, a non-zero diffusion coefficient is presented in the plant equation as an auxiliary control process. The first problem of this kind was introduced in [7]. Problem (2.1)-(2.3) is different: a non-zero diffusion coefficient is not allowed. This setting was introduced in [8],[9].

2.1 Admissible weights Γ and controls. For $p \geq 1$ and $q \geq 1$, we denote by $L_{p,q}^{n \times m}$ the class of random processes $v(t)$ adapted to \mathcal{F}_t with the values in $\mathbf{R}^{n \times m}$ such that $\mathbf{E} \left(\int_0^T |v(t)|^q dt \right)^{p/q} < +\infty$. We denote by $|\cdot|$ the Euclidean norm for vectors and the Frobenius norm for matrices.

Let $g : [0, T) \rightarrow \mathbf{R}$ be a given measurable function such that there exist $c > 0$ and $\alpha \in (0.5, 1)$ such that

$$(2.3) \quad 0 < g(t) \leq c(T-t)^\alpha,$$

$$g(t)^{-1} \leq c(1 + (T-t)^{-\alpha}), \quad t \in [0, T).$$

An example of such a function is $g(t) = 1$ for $t < T - T_1$, $g(t) = (T-t)^\alpha$ for $t \geq T - T_1$, where $T_1 \in (0, T]$ can be any number.

Let U be the set of all processes from $L_{2,1}^{n \times 1}$ such that

$$(2.4) \quad \mathbf{E} \int_0^T g(t) |u(t)|^2 dt < +\infty.$$

By the definition of $L_{2,1}^{n \times 1}$, it follows that, for $u \in U$,

$$(2.5) \quad \mathbf{E} \left(\int_0^T |u(t)| dt \right)^2 < +\infty.$$

We consider U as the set of admissible controls.

We assume that $\Gamma(t)$ is a measurable matrix valued function in $\mathbf{R}^{n \times n}$, such that $\Gamma(t) = g(t)G(t)$, where $G(t) > 0$ is a symmetric positively defined matrix such that the matrices $G(t)$ and $G(t)^{-1}$ are both bounded. By the definitions, $\mathbf{E} \int_0^T u(t)^\top \Gamma(t) u(t) dt < +\infty$ for $u \in U$.

Restrictions on the choice of $\Gamma(t) = g(t)G(t)$ mean that the penalty for the large size of $u(t)$ vanishes as $t \rightarrow T$. Thus, we do not exclude fast growing $u(t)$ as $t \rightarrow T$ such that $u(t)$ is not square integrable.

2.2 The optimal control. By the Martingale Representation Theorem, there exists a unique $k_f \in L_{2,2}^{n \times d}$ such that

$$f = \mathbf{E}f + \int_0^T k_f(t) dw(t).$$

(See, e.g., Theorem 4.2.4 in [14], p.67).

We assume that there exists $\tau \in (0, T)$ such that

$$\operatorname{ess\,sup}_{t \in [\tau, T]} \mathbf{E} |k_f(t)|^2 < +\infty.$$

Let

$$\begin{aligned} \widehat{k}_\mu(t) &= R(t)^{-1} k_f(t), & R(s) &\triangleq \int_s^T Q(t) dt, \\ Q(t) &= e^{A(T-t)} b \Gamma(t)^{-1} b^\top e^{A^\top(T-t)}. \end{aligned}$$

By Lemma 1 from [9], it follows that $\widehat{k}_\mu(\cdot) \in L_{2,2}^{n \times d}$.

THEOREM 2.1. *Let*

$$\widehat{\mu}(t) = R(0)^{-1} (\mathbf{E}f - e^{AT}a) + \int_0^t \widehat{k}_\mu(s) dw(s)$$

and

$$\widehat{u}(t) = \Gamma(t)^{-1} b^\top e^{A^\top(T-t)} \widehat{\mu}(t).$$

Then this u belongs to U and it is a unique optimal solution of problem (2.1)-(2.3) in the class $u \in U$.

In [8], a related problem was considered for a simpler case when it was required to ensure that $x(T) = \mathbf{E}\{f | \mathcal{F}_\theta\}$ for some $\theta < T$.

3 Applications to finance

Replication on the basis of Martingale Representation Theorem is the main tool in the modern Mathematical Finance. The replication on the basis of Theorem 2.1 can also be applied to problems arising in finance. Some possible applications are suggested below, including optimal cash accumulation policy and modeling of the bond prices.

3.1 Optimal cash accumulation policy.

Consider a market model where there is a risky asset with the price $S(t)$ which is a random continuous time process with positive values such that

$$dS(t) = S(t)[a(t)dt + \sigma(t)dw(t)],$$

where $a(t)$ and $\sigma(t)$ are some \mathcal{F}_t -adapted bounded processes such that $\sigma(t) \geq C$ a.e., where $C > 0$ is a constant.

Assume that an investor wishes to accumulate gradually an amount of cash that allows to purchase a share of this risky asset at the given terminal time T . Let $u(t)$ be the process describing the density of the cash deposits/withdrawals at time $t \in (0, T)$, such that $u(t)\Delta t$ is the amount of cash deposited/withdrawn during the time interval $(t, t + \Delta t)$, for a small $\Delta t > 0$. Assume that it is preferable that the cash flow will be as smooth as possible.

Let us assume first that the bank interest rate is zero, for both loans and savings. In this case, the total amount of cash at the terminal time will be $\int_0^T u(t)dt$. Theorem 2.1 can be applied now for $n = 1$, $A = 0$, $b = 1$, $f = S(T)$. In this case,

$$R(s) \triangleq \int_s^T \Gamma(t)^{-1} dt.$$

If $a(t) \equiv 0$ then Theorem 2.1 ensures that the process

$$\begin{aligned} (3.1) \quad u(t) &= \\ &= \Gamma(t)^{-1} \left[R(0)^{-1} S(0) + \int_0^t R(s)^{-1} dS(s) \right] \end{aligned}$$

is such that

$$(3.2) \quad \int_0^T u(t)dt = f \quad \text{a.e.}$$

Moreover, this $u(t)$ is optimal in U in the sense of the optimality criterion from Theorem 2.1, i.e., $\mathbf{E} \int_0^T \Gamma(t) u(t)^2 dt$ is minimal.

Let us consider a more general model where the bank interest rate is $r \geq 0$, for both loans and savings. In this case, the total amount of cash at the terminal time will be $\int_0^T e^{r(T-t)} u(t) dt$.

Theorem 2.1 can be applied now for $n = 1$, $A = r$, $b = 1$, $f = S(T)$. In this case,

$$R(s) \triangleq \int_s^T Q(t)dt, \quad Q(t) = e^{2r(T-t)}\Gamma(t)^{-1}.$$

If $a(t) \equiv 0$ then Theorem 2.1 ensures that the corresponding process (3.1) is such that

$$(3.3) \quad \int_0^T e^{r(T-t)}u(t)dt = f \quad \text{a.e.}$$

Again, this $u(t)$ is optimal in U in the sense of the optimality criterion from Theorem 2.1, i.e., $\mathbf{E} \int_0^T \Gamma(t)u(t)^2 dt$ is minimal.

If $a(\cdot) \neq 0$, then conditions (3.2) are still satisfied for $u(t)$ defined by (3.1) but the value $\mathbf{E} \int_0^T \Gamma(t)u(t)^2 dt$ is not minimal over u anymore. Instead, $\mathbf{E}_Q \int_0^T \Gamma(t)u(t)^2 dt$ is minimal, where \mathbf{E}_Q is the expectation defined by an equivalent probability measure Q such that $S(t)$ is a martingale; we will call it a martingale measure. This still means that deviations of u are minimal but in a different metric. It can be also noted that the definition of the class U for the original measure has to be adjusted for the new measure Q , with the expectations \mathbf{E} replaced by \mathbf{E}_Q .

Let us consider a modification of the cash accumulating problem where the accumulated cash amount has to be a given proportion of the excess achieved by the equity at the terminal time. This problem arises for a writer of a naked or a partially naked call option. To cover this case, it suffices to apply Theorem 2.1 with $f = c \max(S(T) - K, 0)$, where $c > 0$ is the prescribed proportion. We assume that $\sigma(t)$ is non-random and that the bank interest rate is $r > 0$. In this case, it is well known that

$$f = H(S(0), 0) + \int_0^T \frac{\partial H}{\partial x}(S(t), t)dS(t).$$

Therefore, $k_f(t) = \frac{\partial H}{\partial x}(S(t), t)\sigma(t)S(t)$ and $\mathbf{E}_Q f = H(S(0), 0)$. By Theorem 2.1, (3.3) is ensured for this f with

$$u(t) = c\Gamma(t)^{-1} \left[R(0)^{-1}H(S(0), 0) + \int_0^t R(s)^{-1} \frac{\partial H}{\partial x}(S(s), s)dS(s) \right],$$

where $H(x, t) = \mathbf{E}_Q \{ \max(S(T) - K, 0) | S(t) = x \}$. This expectation is under the martingale measure Q again; if $\sigma(t)$ is constant, it can be calculated by the standard Black-Scholes formula. In addition, $\mathbf{E}_Q \int_0^T \Gamma(t)u(t)^2 dt$ is minimal among all u such that (3.2) is satisfied.

Another possible choices of f in this setting may include $f = c \min_{t \in [0, T]} S(t)$ (a given proportion of the minimum achieved by the equity during the time period $[0, T]$), or $f = \frac{c}{T} \int_0^T S(t)dt$ (a given proportion of the average equity value over the time period $[0, T]$). Here $K > 0$ and $c > 0$ are some constants. In these cases, $u(t)$ also can be represented explicitly for non-random $\sigma(t)$.

The model described above can also be applied to the problems of optimal dividend flow selection. In particular, it can be applied to the setting where the manager of a firm with the capitalization $S(t)$ wishes to pay dividends during the time period $[0, T]$ such that the total payoff $\int_0^T u(t)dt$ over this interval will be, say 5% of the equity $S(T)$ at time T . The typical approach is a barrier criterion of dividend payments or analysis of ruin times; the methods are usually based on dynamic programming (see, e.g., [5], [11]) and the bibliography here). Theorem 2.1 leads to a new approach to this problem.

3.2 Modelling of the bond prices. Consider continuous time bond pricing model for zero coupon

bonds. Let $B(t, T)$ be the bond price at time t for the zero coupon bond with payoff \$1 at time T , where $T > t$. Let $r(t)$ be the short rate. We assume that the process $r(t)$ is \mathcal{F}_t -adapted. Here \mathcal{F}_t is the same as above; it is the filtration generated by a Wiener process.

We assume that the probability measure \mathbf{P} is a measure used for the pricing such that, for a given process $r(t)$,

$$B(t, T) = \mathbf{E}\left\{\exp\left(-\int_t^T r(s)ds\right) \middle| \mathcal{F}_t\right\}.$$

Let $\tilde{B}(t, T)$ be the discounted bond price defined as

$$\tilde{B}(t, T) = \exp\left(-\int_0^t r(s)ds\right)B(t, T).$$

In particular,

$$\tilde{B}(T, T) = \exp\left(-\int_0^T r(s)ds\right).$$

Usually, a model evolution of $r(t)$ is being suggested first and then the distributions of $\tilde{B}(t, T)$ and $B(t, T)$ are derived. Theorem 2.1 gives an alternative: to model first the distribution of the random variable $\xi = \tilde{B}(T, T)$, and derive the evolution of $r(t)$ from the distribution of ξ . It appears that it can be done for a quite wide class of random variables ξ with the values in $(0, 1)$ such that $f = -\log \xi$ satisfies conditions of Theorem 2.1. Since $\xi \in (0, 1)$, we have that $f > 0$ a.s. By Theorem 2.1, there exists an adapted process $r(t)$ such that

$$f = \int_0^T r(s)ds, \quad \xi = \exp\left(-\int_0^T r(s)ds\right).$$

Moreover, the process $r(t)$ can be selected to be optimal meaning that it has minimal deviations (in the sense of the optimality criterion from Theorem 2.1).

This approach can be extended on the case of a bond market where there are bonds with different

non-random maturity times $T_k, k = 1, \dots, N, T_k > T_{k+1}$ for all k . Assume that we are given \mathcal{F}_{T_k} -measurable random variables ξ_k with values in $(0, 1)$. Let $f_1 = -\log \xi_1, f_2 = \log \xi_1 - \log \xi_2, \dots, f_k = \log \xi_k - \log \xi_{k-1}$. Applying Theorem 2.1 modified for the positive initial times, we obtain that there exists an adapted process $r(t)$ such that

$$f_1 = \int_0^{T_1} r(s)ds, \dots, f_k = \int_{T_{k-1}}^{T_k} r(s)ds,$$

$k = 2, \dots, N$, and

$$\xi_k = \exp\left(-\int_0^{T_k} r(s)ds\right), \quad k = 1, \dots, N.$$

This leads to a bond market model such that

$$\tilde{B}(T_k, T_k) = \xi_k, \quad k = 1, \dots, N,$$

for an arbitrarily chosen set $\{\xi_k\}$ such that the corresponding random variables f_k has final second moments and that condition (2.6) is satisfied for $T = T_k$ and $f = f_k$. As we had mentioned before, the conventional approach is to select a model for the process $r(t)$ first and then to derive $\tilde{B}(T_k, T_k)$. The possibility to start with a model for $\tilde{B}(T_k, T_k)$ is established here. This could give new opportunities for bond pricing models.

4 Proof of Theorem 2.1

For the sake of completeness, we give below the proof of Theorem 2.1; this proof follows the proof of Theorem 1 from [9].

Clearly, equation (2.3) gives that

$$(4.1) \quad x(t) = \int_0^t e^{A(t-s)}bu(s)ds + e^{At}a.$$

By (2.5), $x(T) \in L_2(\Omega, \mathcal{F}_T, \mathbf{P}; \mathbf{R}^n)$ for any $u \in U$.

Let the function $L(u, \mu) : U \times L_2(\Omega, \mathcal{F}_T, \mathbf{P}; \mathbf{R}^n) \rightarrow \mathbf{R}$ be defined as

$$L(u, \mu)$$

$$\triangleq \frac{1}{2} \mathbf{E} \int_0^T u(t)^\top \Gamma(t) u(t) dt + \mathbf{E} \mu^\top (f - x(T)). \quad \leq C_2 \sup_{t \in [0, T]} \mathbf{E} |\mu(t)|^2 \int_0^T g(t)^{-1} dt < +\infty.$$

For a given μ , consider the following problem:

$$(4.2) \quad \text{Minimize } L(u, \mu) \text{ over } u \in U.$$

This problem does not have constraints on terminal value $x(T)$. Therefore, it can be solved by usual stochastic control methods for the forward plant equations. We solve problem (4.2) using the so-called stochastic maximum principle that gives a necessary condition of optimality; see, e.g., [1]-[4], [6]-[7], [12]-[13], [15]-[16]). For our problem (4.2), all versions of the stochastic maximum principle from the cited papers are equivalent and can be formulated as the following: if $u = u_\mu \in U$ is optimal then

$$(4.3) \quad \begin{aligned} \psi(t)^\top b u_\mu(t) - \frac{1}{2} u_\mu(t)^\top \Gamma(t) u_\mu(t) \\ \geq \psi(t)^\top b v - \frac{1}{2} v^\top \Gamma(t) v \end{aligned}$$

for a.e. t for all $v \in \mathbf{R}^n$ a.s., where $\psi(t)$ is a process from $L_{2,2}^{n \times 1}$ such that

$$\begin{aligned} d\psi(t) &= -A^\top \psi(t) dt + \chi(t) dw(t), \\ \psi(T) &= \mu, \end{aligned}$$

for some process $\chi \in L_{2,2}^{n \times n}$. (See, e.g., Theorem 1.5 from [4], p.609). The only solution of the backward equation for ψ is

$$\psi(t) = e^{A^\top(T-t)} \mu(t), \quad \mu(t) = \mathbf{E}\{\mu | \mathcal{F}_t\}.$$

Necessary conditions of optimality (4.3) are satisfied for a unique up to equivalency process $u = u_\mu$ defined as

$$(4.4) \quad u_\mu(t) = \Gamma(t)^{-1} b^\top \psi(t).$$

Let us show that $u_\mu \in U$ for any μ . We have that

$$\mathbf{E} \left(\int_0^T |u_\mu(t)| dt \right)^2 \leq C_1 \mathbf{E} \left(\int_0^T |\Gamma(t)^{-1}| |\mu(t)| dt \right)^2$$

In addition,

$$\begin{aligned} \mathbf{E} \int_0^T g(t) |u_\mu(t)|^2 dt &\leq C_3 \mathbf{E} \int_0^T g(t) |\Gamma(t)^{-1} \mu(t)|^2 dt \\ &\leq C_4 \mathbf{E} \int_0^T g(t)^{-1} |\mu(t)|^2 dt \\ &\leq C_4 \sup_{t \in [0, T]} \mathbf{E} |\mu(t)|^2 \int_0^T g(t)^{-1} dt < +\infty. \end{aligned}$$

Here $C_i > 0$ are constants defined by A, b, n , and T . Hence $u_\mu \in U$.

Clearly, the function $L(u, \mu)$ is strictly concave in u , and this minimization problem has a unique solution. Therefore, this $u = u_\mu$ is the unique solution of (4.2).

Further, we consider the following problem:

$$(4.5) \quad \begin{aligned} &\text{Maximize } L(u_\mu, \mu) \\ &\text{over } \mu \in L_2(\Omega, \mathcal{F}_T, \mathbf{P}; \mathbf{R}^n). \end{aligned}$$

For $u = u_\mu$, equation (4.1) gives

$$x(T) = \int_0^T e^{A(T-t)} b u_\mu(t) dt + e^{AT} a.$$

Hence

$$\begin{aligned} L(u_\mu, \mu) &= \frac{1}{2} \mathbf{E} \int_0^T u_\mu(t)^\top \Gamma(t) u_\mu(t) dt \\ &\quad - \mathbf{E} \mu^\top \int_0^T e^{A(T-t)} b u_\mu(t) dt - \mathbf{E} \mu^\top e^{AT} a + \mathbf{E} \mu^\top f. \end{aligned}$$

We have that

$$\begin{aligned} &\mathbf{E} \mu^\top \int_0^T e^{A(T-t)} b u_\mu(t) dt \\ &= \mathbf{E} \mu^\top \int_0^T e^{A(T-t)} b \Gamma(t)^{-1} b^\top \psi(t) dt \\ &= \mathbf{E} \mu^\top \int_0^T e^{A(T-t)} b \Gamma(t)^{-1} b^\top e^{A^\top(T-t)} \mu(t) dt \\ &= \mathbf{E} \mu^\top \int_0^T Q(t) \mu(t) dt \\ &= \mathbf{E} \int_0^T \mathbf{E} \{ \mu^\top Q(t) \mu(t) | \mathcal{F}_t \} dt \\ &= \mathbf{E} \int_0^T \mu(t)^\top Q(t) \mu(t) dt. \end{aligned}$$

The fifth equality here holds by Fubini's Theorem. Further, we have that

$$\begin{aligned}
& \mathbf{E} \int_0^T u_\mu(t)^\top \Gamma(t) u_\mu(t) dt \\
&= \mathbf{E} \int_0^T (\Gamma(t)^{-1} b^\top \psi(t))^\top \Gamma(t) \Gamma(t)^{-1} b^\top \psi(t) dt \\
&= \mathbf{E} \int_0^T \psi(t)^\top b \Gamma(t)^{-1} b^\top \psi(t) dt \\
&= \mathbf{E} \int_0^T (e^{A^\top(T-t)} \mu(t))^\top b \Gamma(t)^{-1} b^\top e^{A^\top(T-t)} \mu(t) dt \\
&= \mathbf{E} \int_0^T \mu(t)^\top e^{A(T-t)} b \Gamma(t)^{-1} b^\top e^{A^\top(T-t)} \mu(t) dt \\
&= \mathbf{E} \int_0^T \mu(t)^\top Q(t) \mu(t) dt.
\end{aligned}$$

It follows that

$$\begin{aligned}
L(u_\mu, \mu) &= \mathbf{E} \mu^\top (f - e^{AT} a) - \frac{1}{2} \mathbf{E} \int_0^T \mu(t)^\top Q(t) \mu(t) dt.
\end{aligned}$$

By the Martingale Representation Theorem, there exists $k_\mu \in L_{2,2}^{n \times d}$ such that

$$\mu = \bar{\mu} + \xi(T),$$

where $\bar{\mu} \triangleq \mathbf{E} \mu$ and $\xi(t) \triangleq \int_0^t k_\mu(s) dw(s)$. It follows that

$$\begin{aligned}
& \mathbf{E} \int_0^T \mu(t)^\top Q(t) \mu(t) dt \\
&= \mathbf{E} \int_0^T (\bar{\mu} + \xi(t))^\top Q(t) (\bar{\mu} + \xi(t)) dt \\
&= \int_0^T \bar{\mu}^\top Q(t) \bar{\mu} dt + \mathbf{E} \int_0^T \xi(t)^\top Q(t) \xi(t) dt \\
&= \bar{\mu}^\top \left(\int_0^T Q(t) dt \right) \bar{\mu} + \int_0^T \mathbf{E} \xi(t)^\top Q(t) \xi(t) dt \\
&= \bar{\mu}^\top R(0) \bar{\mu} + \int_0^T dt \mathbf{E} \int_0^t k_\mu(s)^\top Q(t) k_\mu(s) ds \\
&= \bar{\mu}^\top R(0) \bar{\mu} + \mathbf{E} \int_0^T ds \int_s^T k_\mu(s)^\top Q(t) k_\mu(s) dt \\
&= \bar{\mu}^\top R(0) \bar{\mu} + \mathbf{E} \int_0^T k_\mu(s)^\top R(s) k_\mu(s) ds.
\end{aligned}$$

We have used Fubini's Theorem again to change the order of integration. Similarly,

$$\mathbf{E} \mu^\top f = \bar{\mu}^\top \bar{f} + \mathbf{E} \int_0^T k_\mu(t)^\top k_f(t) dt,$$

and $\mathbf{E} \mu^\top e^{AT} a = \bar{\mu}^\top e^{AT} a$. It follows that

$$\begin{aligned}
L(u_\mu, \mu) &= \bar{\mu}^\top (\bar{f} - e^{AT} a) - \frac{1}{2} \bar{\mu}^\top R(0) \bar{\mu} \\
&\quad - \frac{1}{2} \mathbf{E} \int_0^T k_\mu(\tau)^\top R(\tau) k_\mu(\tau) d\tau \\
&\quad + \mathbf{E} \int_0^T k_\mu(t)^\top k_f(t) dt.
\end{aligned}$$

Clearly, the maximum of this quadratic form is achieved for

$$\bar{\mu} = R(0)^{-1} (\bar{f} - e^{AT} a), \quad \hat{k}_\mu(t) = R(t)^{-1} k_f(t).$$

This means that the optimal solution $\hat{\mu}$ of problem (4.6) is

$$\hat{\mu} = R(0)^{-1} (\bar{f} - e^{AT} a) + \int_0^T \hat{k}_\mu(t) dw(t).$$

Let $\hat{u}(t)$ and $\hat{\mu}(t)$ be defined by (4.4)-(4.4) for $\mu = \hat{\mu}$, i.e., $\hat{u} = u_{\hat{\mu}}$. By Lemma 1 from [9], it follows that $\hat{k}_\mu(\cdot) \in L_{2,2}^{n \times d}$. It follows that

$$\mathbf{E} \int_0^T |\hat{k}_\mu(t)|^2 dt < +\infty.$$

It follows that $\sup_{t \in [0, T]} \mathbf{E} |\hat{\mu}(t)|^2 < +\infty$.

We found that $\sup_\mu \inf_u L(u, \mu)$ is achieved for $(\hat{u}, \hat{\mu})$. We have that $L(u, \mu)$ is strictly convex in $u \in U$ and affine in $\mu \in L_2(\Omega, \mathcal{F}, \mathbf{P}, \mathbf{R}^n)$. In addition, $L(u, \mu)$ is continuous in $u \in L_{2,2}^{n \times 1}$ given $\mu \in L_2(\Omega, \mathcal{F}, \mathbf{P}, \mathbf{R}^n)$, and $L(u, \mu)$ is continuous in $\mu \in L_2(\Omega, \mathcal{F}, \mathbf{P}, \mathbf{R}^n)$ given $u \in U$. By Proposition 2.3 from [10], Chapter VI, p. 175, it follows that

$$(4.6) \quad \inf_{u \in U} \sup_\mu L(u, \mu) = \sup_\mu \inf_{u \in U} L(u, \mu).$$

Therefore, $(\hat{u}, \hat{\mu})$ is the unique saddle point for (4.6).

Let U_f be the set of all $u(\cdot) \in U$ such that (2.3) holds. It is easy to see that

$$\begin{aligned} \inf_{u \in U_f} \frac{1}{2} \mathbf{E} \int_0^T u(t)^\top \Gamma(t) u(t) dt \\ = \inf_{u \in U} \sup_{\mu} L(u, \mu), \end{aligned}$$

and any solution (u, μ) of (4.6) is such that $u \in U_f$. It follows that $\hat{u} \in U_f$ and it is the optimal solution for problem (2.1)-(2.3). Then the proof of Theorem 2.1 follows. \square

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