Arbitrary pole placement with the extended Kautsky-Nichols-van Dooren parametric form

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We consider the classic problem of pole placement by state feedback. The well-known eigenstructure assignment algorithm of Kautsky, Nichols and van Dooren (1985) is extended to obtain a parametric formula for the pole-placing feedback matrix that can deliver any desired closed-loop eigenvalues, with any desired multiplicities.

1. Introduction

We consider the classic problem of arbitrary pole placement for linear time-invariant (LTI) systems in state space form

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

where, for all $t \in \mathbb{R}$, $x(t) \in \mathbb{R}^n$ is the state and $u(t) \in \mathbb{R}^m$ is the control input. We assume that the rank of B is m, and that the pair (A,B) is completely reachable. We let $\mathscr{L} = \{\lambda_1, \ldots, \lambda_\nu\}$ be a set of distinct $\nu \leq n$ complex numbers, closed under complex conjugation. For every $i \in \{1, \ldots, \nu\}$, we denote by m_i the multiplicity of λ_i, so that $m_1 + \cdots + m_\nu = n$, and $m_i = m_j$ whenever $\lambda_i = \overline{\lambda}_j$. Let Λ be an $n \times n$ Jordan matrix obtained from the eigenvalues of $\mathscr{L}$, including multiplicities. The problem of *arbitrary exact pole placement (EPP) by state feedback* is to find a real feedback matrix F such that

$$(A + BF)X = X\Lambda, \tag{2}$$

for some real non-singular $n \times n$ matrix X.

The EPP problem has an extensive history. In (Rosenbrock, 1970) it was established that for completely reachable pairs (A,B), the EPP can be solved for any self-conjugate set of eigenvalues with any desired multiplicities, but the possible mini-block orders of the Jordan structure of $A+BF$ are constrained by the *controllability indices* of (A,B). When $m \geq 2$, the EPP admits many solutions for F, and a further problem is to parameterise all the gain matrices F that deliver the desired Jordan structure for $A+BF$. A notable early method for obtaining the required gain matrix F was Ackermann's formula (Ackermann, 1972) – see also (Ogata, 1997) and (Kailath, 1980) – which is applicable to single-input single-output (SISO) systems, and was often found to be numerically inaccurate. In (Varga, 1981) a numerically reliable method was proposed to obtain F for multiple-input multiple-output (MIMO) systems. The classic eigenstructure assignment algorithm of (Moore, 1976) quantified the freedom to simultaneously assign both the closed-loop eigenvalues, and also select the associated eigenvectors.

Early parametric forms for F were given in (Bhattacharyya and de Souza, 1982) and (Fahmy and O'Reilly, 1983); however these methods did not solve the EPP in full generality, as they require the closed-loop eigenvalues to all be distinct from the open-loop ones. The classic pole placement paper of (Kautsky *et al*, 1985) gave a method for obtaining F by employing a QR-factorisation for B and a Sylvester equation for X. In (Byers and Nash, 1989) the method of (Kautsky *et al*, 1985) was adopted to provide a parametric formula for F that was applicable to any desired set of eigenvalues $\mathscr{L}$; this method also did not solve the EPP in full generality, as the algebraic multiplicities of each eigenvalue was limited to at most m, the rank of the B matrix. This corresponds to the case where Λ is a diagonal matrix. The general case where $\mathscr{L}$ contains any desired closed-loop eigenvalues and multiplicities was considered in (Liu and Patton, 1998), (Ait Rami *et al* , 2009) and (Schmid *et al*, 2014a). The main task of this paper is to extend the method of (Kautsky *et al*, 1985) to handle the case of arbitrary multiplicities, thereby placing it on an equal footing, with respect to its generality of application, to the methods of (Liu and Patton, 1998), (Ait Rami *et al* , 2009) and (Schmid *et al*, 2014a).

Such parametric formulae are valuable as they may be used to address optimal control problems, such as the problem of *robust*

exact pole placement, (REPP) which involves obtaining F that solves the EPP and also renders the eigenvalues of $A+BF$ as insensitive to perturbations in A, B and F as possible. The *minimum gain exact pole placement* (MGEPP) problem involves solving the EPP problem and also obtaining the feedback matrix F that has the smallest gain (matrix norm).

The robust pole placement problem was addressed in (Kautsky *et al*, 1985), using the method of pole-placement developed therein. This method is of particular interest as it is the basis of the widely-used MATLAB® `place` command, and is also the basis of MATHEMATICA®'s `KNVD` command. The pole-placing method was also employed by (Tits and Yang, 1996) in their method for robust pole placement, and incorporated it their MATLAB® toolbox known as `robpole`. It has also been employed in the robust pole placement methods of (Byers and Nash, 1989) and (Guo *et al*, 2015). Consequently, all these methods inherit the limitation of the method of (Kautsky *et al*, 1985) that the maximum multiplicity that can be assigned to any eigenvalue is at most m.

Thus, generalising the method of (Kautsky *et al*, 1985) to accommodate any assignable Jordan structure opens the possibility of extending the robust pole placement methods of (Kautsky *et al*, 1985), (Byers and Nash, 1989), (Tits and Yang, 1996) and (Guo *et al*, 2015) to those cases where a defective eigenstructure is desired. In this paper we shall use the pole-placing method of (Kautsky *et al*, 1985) together with the gradient search methods of (Byers and Nash, 1989) to address the problem of robust pole placement for a defective eigenstructure. We shall also consider the MGEPP problem for a defective eigenstructure.

Closing this gap is not only important from a theoretical viewpoint. For discrete-time systems, it is often desirable to assign the closed-loop poles at the origin of the complex plane, so that the closed-loop system will exhibit deadbeat characteristics, in which the zero-input response of the system vanishes within a finite number of time steps. This problem has been traced back to (Kalman, 1964) and has an extensive literature, see for example (Tam and Lam, 1997) and the references therein. Clearly, the deadbeat pole placement problem cannot be solved by the method of (Kautsky *et al*, 1985) and hence this limitation is inherited by the `place`, `robpole` and `KNVD` toolboxes.

We begin with some definitions and notation. We assume the matrix Λ in (2) can be expressed in the Jordan (complex) block diagonal canonical form $\Lambda = \text{blkdiag}\{J(\lambda_1),\cdots,J(\lambda_\nu)\}$, where each $J(\lambda_i)$ is a Jordan matrix for λ_i of order m_i, and may be composed of up to g_i mini-blocks $J(\lambda_i) = \text{blkdiag}\{J_1(\lambda_i),\cdots,J_{g_i}(\lambda_i)\}$, where $1 \le g_i \le m$. We use $\mathscr{P} \stackrel{\text{def}}{=} \{p_{i,k} \,|\, i = 1,\ldots\nu, k = 1,\ldots g_i\}$ to denote the order of each Jordan mini-block $J_k(\lambda_i)$; then $p_{i,k} = p_{j,k}$ whenever $\lambda_i = \overline{\lambda}_j$. If $\mathscr{L}$ and $\mathscr{P}$ satisfy the conditions of the Rosenbrock theorem, we say that the pair $(\mathscr{L},\mathscr{P})$ defines an *assignable Jordan structure* for (A,B).

Given a self-conjugate set of ν complex numbers $\{\lambda_1,\ldots,\lambda_\nu\}$ containing σ complex conjugate pairs, we say that the set is σ-*conformably indexed* if the first 2σ values are complex while the remaining are real, and for all odd $i \le 2\sigma$ we have $\lambda_{i+1} = \overline{\lambda}_i$. We shall assume in the following that $\mathscr{L}$ is σ-conformably indexed. If M is a complex matrix partitioned into ν column matrices $M = [\ M_1 \quad \ldots \quad M_\nu\ ]$, we say that M is σ-*conformably indexed* if the first 2σ column matrices of M are complex while the remaining are real, and for all odd $i \le 2\sigma$ we have $M_{i+1} = \overline{M}_i$. For any matrix X, we denote by $X(\ell)$ the ℓ-th column of X. We denote by I_n the n-dimensional identity matrix. We say that an mn-dimensional parameter matrix $K \stackrel{\text{def}}{=} \text{diag}\{K_1,\ldots,K_\nu\}$ is *compatible* with $(\mathscr{L},\mathscr{P})$ if: (i) for each $1 \le i \le \nu$, K_i is a matrix of dimension $m \times m_i$; (ii) for all odd $1 \le i \le 2\sigma$, the matrix K_i is complex, and such that $K_i = \overline{K}_{i+1}$, while K_i is a real matrix for each $i \ge 2\sigma$; and (iii) each K_i matrix can be partitioned as

$$K_i = [\ K_{i,1} \quad K_{i,2} \quad \ldots \quad K_{i,g_i}\ ], \tag{3}$$

where, for $1 \le k \le g_i$, each $K_{i,k}$ has dimension $m \times p_{i,k}$.

2. Pole placement methods

2.1 *The method of Kautsky, Nichols and van Dooren*

We firstly revisit the algorithm of (Kautsky *et al*, 1985) for the EPP, which requires that Λ in (2) be a diagonal matrix.

Theorem 2.1: [(KAUTSKY *et al*, 1985), THEOREM 3] *Given $\Lambda = \text{diag}\{\lambda_1,\lambda_2,\ldots,\lambda_n\}$ and a non-singular matrix X, then there exists a solution F to (2) if and only if*

$$U_1^\top (AX - X\Lambda) = 0, \tag{4}$$

where $B = [U_0\ U_1]\begin{bmatrix} Z \\ 0 \end{bmatrix}$ with $U = [U_0\ U_1]$ orthogonal and Z non-singular. Then, F is given by

$$F = Z^{-1} U_0^\top (X\Lambda X^{-1} - A). \tag{5}$$

This formulation uses a QR factorisation for B; in (Byers and Nash, 1989) it was pointed out that F may also be obtained

from the singular value decomposition for B. For the case where the desired closed-loop poles in $\mathscr{L}$ are real and distinct, this result was used in (Byers and Nash, 1989) to obtain a parametric form for X and F solving (2), in terms of an mn-dimensional parameter matrix as follows: define, for each $i \in \{1,\ldots,n\}$, the subspace

$$\mathscr{S}_i \stackrel{\text{def}}{=} \ker[U_1^\top (A - \lambda_i I_n)]. \tag{6}$$

Since the pair (A,B) is completely reachable, the dimension of $\mathscr{S}_i$ is equal to m. For $i \in \{1,\ldots,n\}$, let Σ_i be an $n \times m$ basis matrix for $\mathscr{S}_i$. Then, in (Byers and Nash, 1989) matrices X and F are parameterised as follows:

Theorem 2.2: *Let $\Xi = [\xi_{i,j}]_{m\times n}$ be a real $m \times n$ matrix, and for each $i \in \{1,\ldots,n\}$, let $\Xi(i) = \begin{bmatrix} \xi_{i,1} \\ \vdots \\ \xi_{i,m} \end{bmatrix}$ be the i-th column of Ξ. Define*

$$X = [\ \Sigma_1 \quad \ldots \quad \Sigma_n\ ]\,\text{diag}\{\Xi(1),\ldots,\Xi(n)\}, \tag{7}$$

where $\text{diag}\{\Xi(1),\ldots,\Xi(n)\}$ *is an $nm \times n$ block diagonal matrix with $m \times 1$ blocks. Assume X is non-singular and define*

$$F \stackrel{def}{=} Z^{-1} U_0^\top (X \Lambda X^{-1} - A). \tag{8}$$

Then X and F satisfy (2).

Since Ξ is arbitrary, (7)-(8) give a parametrisation of the eigenvector matrices X and feedback matrices F that solve (2). We shall refer to (7)-(8) as the *Kautsky-Nichols-van Dooren parametric form* for X and F. Extending it to the case of any (real or complex) diagonal matrix Λ is straightforward, and thus the parameterisation can accommodate any non-defective eigenstructure.

2.2 *Extending the (Kautsky* et al*, 1985) pole placement method to defective eigenstructures*

The first result of this paper shows how to generalise the Kautsky-Nichols-van Dooren parametric form to accommodate any assignable Jordan structure $(\mathscr{L},\mathscr{P})$ for any reachable pair (A,B).

Theorem 2.3: *Let $(\mathscr{L},\mathscr{P})$ be an assignable Jordan structure for (A,B) and let K be a compatible parameter matrix. For each $i \in \{1,2,...,\nu\}$, let N_i and M_i be full rank matrices satisfying*

$$U_1^\top (A - \lambda_i I_n) N_i = 0, \quad U_1^\top (A - \lambda_i I_n) M_i = I_{n-m}. \tag{9}$$

For each pair $i \in \{1,\ldots,\nu\}$ and $k \in \{1,\ldots,g_i\}$, define the sequence

$$\begin{aligned} x_{i,k}(1) &= N_i K_{i,k}(1), && (10)\\ x_{i,k}(2) &= M_i U_1^\top x_{i,k}(1) + N_i K_{i,k}(2), && (11)\\ &\vdots \\ x_{i,k}(p_{i,k}) &= M_i U_1^\top x_{i,k}(p_{i,k}-1) + N_i K_{i,k}(p_{i,k}), && (12) \end{aligned}$$

as well as the matrices

$$\begin{aligned} X_{i,k} &= [\ x_{i,k}(1) \quad x_{i,k}(2) \quad \ldots \quad x_{i,k}(p_{i,k})\ ], && (13)\\ X_i &= [\ X_{i,1} \quad X_{i,2} \quad \ldots \quad X_{i,g_i}\ ], && (14)\\ X &= [\ X_1 \quad X_2 \quad \ldots \quad X_\nu\ ]. && (15) \end{aligned}$$

Then, for almost all choices of K, the matrix X in (15) is invertible, i.e., X is invertible for every choice of K except those lying in a set of measure zero. The set of all real feedback matrices F such that the closed-loop matrix $A+BF$ has Jordan structure given by $(\mathscr{L},\mathscr{P})$ is parameterised in K by

$$F = Z^{-1} U_0^\top (X \Lambda X^{-1} - A). \tag{16}$$

Proof: The proof will be carried out in three steps. First, we show that if X and F are given by (15) and (16), respectively, then (2) is satisfied, provided X is invertible. Second, we show that the parameterisation given in (16) is comprehensive, i.e., for every feedback matrix F and non-singular square matrix X satisfying (2), there exists a compatible parameter matrix K such that X and F can be recovered from (15) and (16), respectively. Finally, we prove that for almost every compatible parameter matrix K, the matrix X in (15) is non-singular.

Firstly, let K be a compatible input parameter matrix as in (3). By (9), for each $i \in \{1,\dots,\nu\}$, $k \in \{1,\dots,g_i\}$, the vectors $x_{i,k}(1),\dots,x_{i,k}(p_{i,k})$ satisfy

$$\begin{aligned} U_1^\top (A-\lambda_i I_n)x_{i,k}(1) &= 0, && (17)\\ U_1^\top (A-\lambda_i I_n)x_{i,k}(2) &= U_1^\top x_{i,k}(1), && (18)\\ &\vdots \\ U_1^\top (A-\lambda_i I_n)x_{i,k}(p_{i,k}) &= U_1^\top x_{i,k}(p_{i,k}-1), && (19)\end{aligned}$$

and hence form a chain of generalised eigenvectors for the matrix $U_1^\top (A-\lambda_i I_n)$. Thus $U_1^\top (AX_{i,k}-X_{i,k}J_k(\lambda_i))=0$. Hence $U_1^\top (AX_i-X_iJ(\lambda_i))=0$, and finally we have

$$U_1^\top (AX-X\Lambda)=0. \tag{20}$$

Assume X is non-singular and let F be computed from (16). We note that F is a real matrix because for each odd $i \in \{1,\dots,2\sigma\}$, we have $\lambda_{i+1}=\overline{\lambda}_i$ and $X_{i+1}=\overline{X}_i$. Multiplying through by $B=U_0Z$ we obtain $BF=X\Lambda X^{-1}-A$, and hence X and F satisfy (2).

Next, we show that the above parameterisation is exhaustive. We let X and F be any pair of matrices satisfying (2) such that the eigenstructure of $A+BF$ is described by $(\mathscr{L},\mathscr{P})$. Then we can decompose X into block matrices $X = \begin{bmatrix} X_1 & X_2 & \dots & X_\nu \end{bmatrix}$ where for $i \in \{1,\dots,\nu\}$, $X_i = \begin{bmatrix} X_{i,1} & X_{i,2} & \dots & X_{i,k} \end{bmatrix}$ and for $k \in \{1,\dots,g_i\}$ we also define $X_{i,k} = \begin{bmatrix} x_{i,k}(1) & x_{i,k}(2) & \dots & x_{i,k}(p_{i,k}) \end{bmatrix}$; the vectors $x_{i,k}(1)$, $x_{i,k}(2)$, …, $x_{i,k}(p_{i,k}-1)$ form a chain of generalised eigenvectors for $A+BF$ with respect to λ_i. Thus

$$\begin{aligned} (A+BF-\lambda_i I_n)x_{i,k}(1) &= 0, && (21)\\ (A+BF-\lambda_i I_n)x_{i,k}(2) &= x_{i,k}(1), && (22)\\ &\vdots \\ (A+BF-\lambda_i I_n)x_{i,k}(p_{i,k}) &= x_{i,k}(p_{i,k}-1). && (23)\end{aligned}$$

Multiplying (21) by $U_1^\top$, we obtain

$$U_1^\top (A-\lambda_i I_n)x_{i,k}(1) = -U_1^\top BF x_{i,k}(1) = 0, \tag{24}$$

as $U_1^\top B=0$. Hence there exists a compatible parameter matrix $K_{i,k}(1)$ of dimension $m\times 1$ such that (10) holds with respect to N_i and M_i. Multiplying (22) by $U_1^\top$, we have $U_1^\top (A-\lambda_i I_n)x_{i,k}(2) = U_1^\top x_{i,k}(1)$, and hence (11) holds for some parameter matrix $K_{i,k}(2)$. Similarly, we can use (23) to obtain the parameter $K_{i,k}(p_{i,k})$ such that (12) holds. Combining these parameters we obtain an $m\times p_{i,k}$-dimensional parameter matrix $K_{i,k}$; combining these for all $k \in \{1,\dots,g_i\}$ we obtain a parameter matrix K_i of dimension $m\times m_i$, and finally combining these for all $i \in \{1,\dots,\nu\}$ we obtain an mn-dimensional parameter matrix K. It is clear that K constructed in this manner is a compatible parameter matrix for $(\mathscr{L},\mathscr{P})$. Applying the procedure in (10)-(16) with this K, we recover X and F.

Lastly, we show that X is invertible for almost all choices of the parameter matrix K. Let $N_i = \begin{bmatrix} n_{i,1} & \dots & n_{i,m} \end{bmatrix}$ and for each $\mathscr{S}_i$ and $i \in \{1,\dots,\nu\}$ and $k \in \{1,\dots,g_i\}$, we introduce the chain

$$\begin{aligned} v_{i,k}(1) &= n_{i,k}, \\ v_{i,k}(2) &= M_i U_1^\top v_{i,k}(1), \\ &\vdots \\ v_{i,k}(p_{i,k}) &= M_i U_1^\top v_{i,k}(p_{i,k}-1). && (25)\end{aligned}$$

Combining these vectors, we obtain $V_{i,k} = \begin{bmatrix} v_{i,k}(1) & v_{i,k}(2) & \dots & v_{i,k}(p_{i,k}) \end{bmatrix}$. Lastly, we obtain V_i and V as in (14)-(15).

Then, rank$(V) = n$, else no parameter matrix K exists to construct F in (16) that will deliver the desired closed-loop eigenstructure. This contradicts the assumption the pair (A,B) is completely reachable.
Next, let K be any compatible parameter matrix for $(\mathscr{L},\mathscr{P})$, let $X = VK$ and assume X is singular, i.e. rank$(X) \leq n-1$. This means that one column of the matrix

$$\left[\begin{array}{ccc|c|ccc} v_{1,1}(1)K_{1,1}(1) & \dots & v_{1,1}(p_{1,1})K_{1,1}(p_{1,1}) & \dots & v_{\nu,g_\nu}(1)K_{\nu,g_\nu}(1) & \dots & v_{\nu,g_\nu}(p_{\nu,g_\nu})K_{\nu,g_\nu}(p_{\nu,g_\nu}) \end{array}\right]$$

is linearly dependent upon the remaining ones. For simplicity, let us assume this is the last column. This means that there exist coefficients $\{\alpha_{i,k,l} : 1 \leq i \leq \nu, 1 \leq k \leq g_i, 1 \leq l \leq p_{i,k}\}$, not all equal to zero, for which

$$v_{\nu,g_\nu}(p_{\nu,g_\nu})K_{\nu,g_\nu}(p_{\nu,g_\nu}) \;=\; \sum_{i=1}^{\nu-1}\sum_{k=1}^{g_i}\sum_{l=1}^{p_{i,k}} \alpha_{i,k,l}v_{i,k}(l) + \sum_{k=1}^{g_\nu-1}\sum_{l=1}^{p_{\nu,k}} \alpha_{\nu,k,l}v_{\nu,k}(l) + \sum_{\ell=1}^{p_{\nu,g_\nu}-1} \alpha_{\nu,g_\nu,\ell}v_{\nu,g_\nu}(\ell).$$

This implies that rank$(VK) = n$ may fail only when $K_{\nu,g_\nu}(p_{\nu,g_\nu})$ lies on an $(m-1)$-dimensional hyperplane in the m-dimensional parameter space. Thus the set of compatible parameter matrices K that can lead to a loss of rank in X is given by the union of a finite number of hyperplanes of dimension at most $nm-1$ within the nm-dimensional parameter space. Since hyperplanes have Lebesgue measure zero on the nm-dimensional parameter space (Rudin, 1987), we conclude the set of parameter matrices K leading to singular X has zero Lebesgue measure. ■
For the case of real and distinct eigenvalues in $\mathscr{L}$, we have $\nu = n$ and $m_i = 1$ for all i, and hence (15)-(16) reduce to (7)-(8). Hence we shall refer to the parametric formulae (15)-(16) as the *extended Kautsky-Nichols-van Dooren parametric form* for X and F. Theorem 2.3 should be compared with Theorem 2.1 of (Schmid *et al*, 2014a), which also provides a parametric form for all F solving the arbitrary EPP. Both methods can accommodate any assignable eigenstructure and utilise an nm-dimensional parameter matrix.

2.3 *Implementation of the pole placement method*

To implement the above pole placement method on any reachable pair (A,B), for any desired assignable Jordan structure $(\mathscr{L},\mathscr{P})$, we proceed as follows:

Algorithm 2.1:

(i) Sort $\mathscr{L}$ so that it is σ-conformably indexed. Choose any parameter matrix K that is compatible with $(\mathscr{L},\mathscr{P})$.
(ii) For each $i \in \{1,\dots,\nu\}$, compute full rank matrices M_i and N_i satisfying (9). For computational reliability, an orthonormal choice is to be preferred.
(iii) Use K, M_i and N_i to compute (10)-(14) and hence obtain X in (15).
(iv) If X is singular then it cannot be used, and an alternative parameter matrix must be obtained in Step (i).
(v) For a non-singular X, obtain the feedback matrix F using (16).

Theorem 2.3 then assures that F and X satisfy (2) with respect to the matrix Λ with Jordan structure defined by $(\mathscr{L},\mathscr{P})$. The utility of parametric forms such as (15)-(16) is that they naturally lend themselves to the consideration of optimal pole placement problems, which we next consider.

3. Optimal Pole placement Problems

For systems with multiple inputs ($m \geq 2$), solutions to the EPP problem (2) are non-unique and this invites the consideration of optimal pole placement problems in which one seeks F and X to solve the EPP while also possessing some other desirable properties. Among many possible optimal control problems, we shall consider the robust exact pole placement problem (REPP) and also minimum gain exact pole placement problem (MGEPP). Both problems have an extensive literature; see (Schmid *et al*, 2014a) for a recent survey. Numerical experiments offering performance comparisons of several methods for these two problems appeared in (Pandey *et al*, 2014) and (Pandey *et al*, 2015).

3.1 *Robustness measures*

For a square matrix M with simple eigenvalues, the first order sensitivity of each individual λ_i to uncertainty in M is given by the *eigenvalue condition number* (Williamson, 1965)

$$c_i(X) \stackrel{\text{def}}{=} \frac{\|y_i\|_2\|x_i\|_2}{|y_i^\top x_i|}, \tag{26}$$

where y_i and x_i are the left and right eigenvectors of M associated with λ_i and X is the matrix of right eigenvectors; $c_i(X)$ is the Frobenius norm of the gradient of $\lambda_i(M)$ with respect to M under the (natural) trace inner product. We use

$$c_\infty(X) \stackrel{\text{def}}{=} \max_i \; c_i(X) \tag{27}$$

to denote the worst-case eigenvalue condition number. In (Bauer and Fike, 1960) it is established that $c_\infty(X)$ is upper-bounded by the *spectral condition number* $\kappa_2(X) \stackrel{\text{def}}{=} \|X\|_2\|X^{-1}\|_2$. The *Frobenius condition number* $\kappa_{\text{FRO}}(X) \stackrel{\text{def}}{=} \|X\|_{\text{FRO}}\|X^{-1}\|_{\text{FRO}}$ is also used as a robustness measure. Minimising the measures $c_\infty(X)$, $\kappa_2(X)$ and $\kappa_{\text{FRO}}(X)$ corresponds to superior robustness, with perfect robustness being achieved only when the eigenvector matrix is unitary, i.e., when M is normal. Finally, in (Tits and Yang, 1996) the orthogonality measure

$$|\det(X)| = \sqrt{\det(XX^*)}, \tag{28}$$

was considered, which represents the volume of the box spanned by the (unit length) column vectors of X, and was used as the robustness measure.

3.2 *Robust Pole placement methods for non-defective eigenstructures*

For the case of a non-defective eigenstructure, two heuristic methods were proposed in (Kautsky *et al*, 1985) to iteratively select sets of closed-loop eigenvectors from the corresponding matrices N_i in (9) so as to increase their mutual orthogonality. The first of these methods (known as Method 0) has been implemented within MATLAB® as the `place` command. In (Tits and Yang, 1996) it was pointed out that these methods are equivalent up to a sequential maximisation of $|\det(X)|$, and offered some improvements, which were implemented in their `robpole` toolbox.
As noted earlier, in (Byers and Nash, 1989) the pole placement method of (Kautsky *et al*, 1985) was developed into a parametric formula, given here as Theorem 2.2; this method was used to address the robust pole placement problem. Noting that $\kappa_{\text{FRO}}(X) \geq \kappa_2(X)$, the authors of (Byers and Nash, 1989) considered the unconstrained optimisation problem

$$(\mathscr{P}_1): \quad \min_K \kappa_{\text{FRO}}(X_K), \tag{29}$$

where K is any compatible parameter matrix, and X_K is the matrix obtained from using K in the pole placement procedure. The Frobenius matrix norm enjoys the virtue of being differentiable with respect to the parameter matrix K, and (Byers and Nash, 1989) solved $(\mathscr{P}_1)$ by gradient search methods. Recently (Schmid *et al*, 2014d) developed a MATLAB® toolbox known as `byersnash`, to implement the method of (Byers and Nash, 1989). Performance comparisons conducted on large collections of sample systems with non-defective eigenstructures showed that the `byersnash` toolbox gave consistently superior robustness performance than the MATLAB® `place` command, when $\kappa_{\text{FRO}}(X)$ was used as the robustness measure. However, further performance comparisons in (Pandey *et al*, 2014) showed that both `place` and `robpole` outperformed `byersnash` when $|\det(X)|$ was used as the robustness measure.

3.3 *Extending the (Byers and Nash, 1989) robust pole placement method to defective eigenstructures*

Having extended the pole placement of (Kautsky *et al*, 1985) for the EPP to the case of a defective eigenstructure, it is natural to consider whether the method can also be employed to achieve a robust eigenstructure in the defective case. It is well-known that closed-loop eigenvalues corresponding to large Jordan blocks may be highly sensitive to parameter uncertainty. In (Chatelin, 1993), the following result was presented on eigenvalue sensitivity for matrices with a defective eigenstructure: let A and X be such that $A = XJX^{-1}$, where J is the Jordan form of A, and let $A' = A + H$, where H represents a perturbation in the entries of

A. Then, for each eigenvalue λ' of A', there exists an eigenvalue λ of A such that

$$\frac{|\lambda-\lambda'|}{(1+|\lambda-\lambda'|)^{\ell-1}} \leq \kappa_2(X)\|H\|_2, \tag{30}$$

where ℓ is the size of the largest Jordan mini-block associated with λ.
The bound (30) suggests that the robustness of the closed-loop eigenvalues depends upon the spectral condition number of X, and thus to obtain a more robust eigenstructure, we should seek F that minimises this condition number. It should be noted, however, that the measure (30) is only useful as a local sensitivity measure, as it relies on the first order expression of the perturbation of the eigenvalues with respect to H and the second order terms are neglected. Consequently, it is only a good criterion when the perturbation is not too large.
In order to simultaneously consider the REPP and MGEPP problems, we introduce the unconstrained optimisation problem

$$(\mathscr{P}_2): \quad \min_K \ \alpha\|X_K\|\|X_K^{-1}\| + (1-\alpha)\|F_K\|^2 \tag{31}$$

where K is any compatible parameter matrix, and X_K and F_K are the matrices obtained from using K in the pole placement procedure of Algorithm 2.1. Also $\|\cdot\|$ is any suitable matrix norm, and α is a weighting factor, with $0 \leq \alpha \leq 1$.
The particular case $\alpha = 0$ corresponds to the minimum gain pole placement problem, while $\alpha = 1$ corresponds to the robust pole placement problem. In seeking to address the robust pole placement problem with a defective eigenstructure, we decided to employ the gradient search method of (Byers and Nash, 1989), rather than seek to extend the heuristic methods of (Kautsky *et al*, 1985). While the spectral norm $\|\cdot\|_2$ might be a natural first choice for the matrix norm in (31), it has the drawback of not being differentiable with respect to the matrix K. Since we shall seek to solve $(\mathscr{P}_2)$ via gradient search methods, we instead employ the Frobenius matrix norm. Moreover, the Frobenius condition number $\kappa_{\text{FRO}}(X) = \|X\|_{\text{FRO}}\|X^{-1}\|_{\text{FRO}}$ satisfies $\kappa_2(X) \leq \kappa_{\text{FRO}}(X)$, and this offers an upper bound for the sensitivity measure in (30). Gradient methods require computation of the first and second order derivatives of $\kappa_{\text{FRO}}(X)$ with respect to K, and from these the gradient and Hessian matrices are easily obtained. Unconstrained nonlinear optimisation methods can then be used to seek local minima. Computation of the matrix derivatives required for $\kappa_{\text{FRO}}(X)$ appeared in (Schmid *et al*, 2014b), and the matrix derivatives required for $\|F_K\|^2$ appeared in (Schmid *et al*, 2014c).

4. Example

The following pair (A,B) appeared as Example 5 in (Byers and Nash, 1989):

$$A = \begin{bmatrix} -0.1290 & 0 & 0.3960 & 0.2500 & 0.0019 \\ 0.0329 & 0 & -0.0078 & 0.0122 & -0.6210 \\ 0.0072 & 0 & -0.1000 & 0.0009 & -0.0385 \\ 0.0041 & 0 & 0-0.0822 & 0 & \\ 0.0035 & 0 & 0.0035 & 0.0043 & -0.0743 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0.1390 \\ 0 & 0.0359 \\ 0 & -0.0989 \\ 0.0249 & 0 \\ 0 & -0.0053 \end{bmatrix}.$$

We seek to solve a deadbeat pole placement problem, in which all the closed-loop poles are to be located at zero by a suitable feedback matrix. The controllability indices of the pair (A,B) are $3,2$. We seek to obtain a feedback matrix F that assigns a Jordan 3-block and a Jordan 2-block for Λ in (2). Thus, for a deadbeat pole placement, we choose $\mathscr{L}_1 = \{0\}$ and $\mathscr{P}_1 = \{3,2\}$. We consider problem $(\mathscr{P}_1)$ above with the values (i) $\alpha = 0$, (ii) $\alpha = 0.1$ and (iii) $\alpha = 1$. We obtain the feedback matrices

$$\begin{aligned} F_1 &= \begin{bmatrix} -2.540687212462339 & -0.000000000000003 & 5.487493841163132 & 8.586630902936868 & -2.853030947488984 \\ 0.325963828864331 & 0 & -1.162961250550408 & -0.254554054443773 & -2.128661165785518 \end{bmatrix}, \\ F_2 &= \begin{bmatrix} -2.553915367684512 & -0.000000000000001 & 5.498853573068885 & 8.602766525170175 & -2.778402259766297 \\ 0.324382385896125 & -0.000000000000000 & -1.161603179992470 & -0.252625020055386 & -2.119739210195851 \end{bmatrix}, \\ F_3 &= 100\begin{bmatrix} 0.271552650427021 & -0.000000000000000 & -0.200140255044157 & -0.276363086831048 & -1.703873477329505 \\ 0.038761531099346 & -0.000000000000000 & -0.042117006832431 & -0.045850530646197 & -0.221576041864537 \end{bmatrix}. \end{aligned}$$

To establish a comparison with the performance of MATLAB®'s `place` command, which cannot handle this example as the desired multiplicity exceeds the rank of B, we consider the set $\mathscr{L}_2 = \{0, \varepsilon, -\varepsilon\}$, where $\varepsilon > 0$, and seek to assign these poles with multiplicities 2, 2 and 1, respectively, within a non-defective eigenstructure. Progressively reducing the value of ε, we observe that the smallest value for which `place` can assign these poles is $\varepsilon = 1.9 \times 10^{-5}$, as any further reduction in ε leads to error messages. The matrix given by `place` is

$$F_p = \begin{bmatrix} -2.348710003776051 & 0.000000000000001 & -1.289569095518355 & -2.623566456109491 & 30.444406414898477 \\ -0.910497545855925 & 0.000000000000000 & 1.664828571992231 & 0.967446078653003 & 5.427245473038826 \end{bmatrix}.$$

To compare the performance of these for feedback matrices, we computed several performance indices. Firstly, the closed-loop spectral radius is given by

$$\rho(A+BF) \stackrel{\text{def}}{=} \max\{|\text{eig}_i(A+BF)| : i \in \{1,\ldots,n\}\} \tag{32}$$

where, for any square matrix Z, $\text{eig}_i(Z)$ denotes the i-th eigenvalue. Thus, the smaller the closed-loop spectral radius, the closer the feedback matrix comes to achieving the deadbeat control objective. We also computed $\|F\|_2$ and $\kappa_2(X)$ from each closed loop eigenstructure. The results are shown in Table 1.

Table 1. Performance comparisons

	$\rho(A+BF)$	$\kappa_2(X)$	$\|F\|_2$
F_1 $(\alpha = 0)$	7.8×10^{-7}	852.0533	10.89
F_2 $(\alpha = 0.1)$	8.5×10^{-7}	48.7176	10.9
F_3 $(\alpha = 1)$	5.2×10^{-7}	23.0818	177.4
F_p (`place`)	1.9×10^{-5}	4.2×10^{8}	31.1

We observe that the use of the extended Kautsky-Nichols-van Dooren algorithm allows us to obtain a closed-loop spectral radius that is 24 times smaller than that obtainable with the original Kautsky-Nichols-van Dooren algorithm. Moreover, by employing the gradient search methods introduced in (Byers and Nash, 1989), the deadbeat pole placement was achieved with substantially smaller gain and greatly improved robustness.

5. Conclusion

We have extended the classic pole placement method of Kautsky, Nichols and van Dooren to address the problem of exact pole placement for any desired eigenstructure with arbitrary multiplicities. The parametric form was shown to include all the matrices X and F satisfying (2), for any given assignable Jordan eigenstructure $(\mathscr{L}, \mathscr{P})$ for (A,B). The set of parameter matrices leading to singular X, for which the algorithm does not yield a pole-placing F, has been shown to have measure zero within the parameter space. These aspects of the parameterisation were not considered by (Byers and Nash, 1989). The algorithm has been shown to be readily amenable to problems of optimal pole placement. The method provides an interesting parallel to the parametric formula given in the recent paper (Schmid *et al*, 2014a) that also achieved arbitrary pole placement, but was derived from the Klein-Moore parametric form.

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