

A Simple Method to Improve the Geoid from a Global Geopotential Model (or Coarse Geoid Estimation Using Only the Innermost Zone Contribution of Stokes's Formula)

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Abstract

This paper presents the results of curiosity-driven experiments to determine how well, if at all, only the innermost zone contribution of Stokes's formula can model the geoid for GPS height transformation. Terrestrial gravity and terrain data are used over Australia, and the results compared with GPS-derived ellipsoidal and Australian Height Datum heights. The results are largely as expected in that the innermost zone contribution to the total geoid height is small. While this approach does not improve upon a regional gravimetric geoid model, it does make a small improvement upon a global geopotential model. Therefore, in areas where regional gravity and terrain data are limited, some small improvements may be made to existing global geopotential models by calculating the innermost zone contribution from localised gravity and terrain data. This is an extraordinarily simple calculation. It may prove to be of value to those who do not have access to regional gravity data (i.e., due to restricted field access or confidentiality) or resources (i.e., personnel, software and high-powered computers) to compile a locally improved geoid model. Accordingly, the method can provide an interim geoid model until the issues of data availability and resources have been resolved to allow the compilation of a regional gravimetric geoid model.

1. Introduction

As is well known, a gravimetric geoid model should be computed using a global integration of terrestrial gravity anomalies via Stokes's formula. This requirement has been alleviated to a large extent by using a high-degree global geopotential model, which provides much of the contribution of the remote zones to the geoid. The contribution of the inner zones to the geoid is provided by a regional integration of terrestrial gravity data in an adapted form of Stokes's integral. Over the last forty years, considerable effort has been directed towards optimally combining a global geopotential model with terrestrial gravity and terrain data over a region.

This paper presents the results of curiosity-driven experiments where the geoid is *very coarsely* approximated by only the innermost zone contribution of Stokes's formula and the quadratic term of the primary indirect effect. Three specific approaches are considered as follows:

- (1) the innermost zone contribution from terrestrial gravity anomalies only;
- (2) the innermost zone contribution from residual gravity anomalies based on a satellite-only global geopotential model; and
- (3) the innermost zone contribution from residual gravity anomalies based on a combined global geopotential model.

These three coarse approximations will be compared with an Australia-wide set of 1013 co-located GPS-derived ellipsoidal heights and Australian Height Datum (AHD) heights

(Featherstone and Guo 2001), the EGM96S satellite-only and EGM96 combined global geopotential models (Lemoine *et al.* 1998), and the AUSGeoid98 regional gravimetric geoid model of Australia (Featherstone *et al.* 2001). This will give an empirical indication of the relative performance of these approximations of the geoid for the transformation of GPS-derived heights to orthometric heights.

Another motivation for these experiments comes from the many examples where GPS users do not have access to a regional gravimetric geoid model, but do have access to terrestrial gravity data over very limited and/or localised areas (due to restricted field access or confidentiality). In addition, these data may not have been used to construct combined global geopotential models. Moreover, the GPS users may not have the resources (i.e., personnel, software and computers) or access to homogeneous regional gravity data coverage with which to compute a regional gravimetric geoid model. An example of the latter occurs in South East Asia where bordering countries rarely exchange gravity data to allow a regional geoid model to be computed properly. This raises the question as to how well a geoid model can be constructed from a very restricted amount of gravity and terrain data for the transformation of GPS-derived ellipsoidal heights to orthometric heights. This very approximate geoid model from only the innermost zone can be adopted as an interim geoid model until the data and resources become available to properly compile a regional gravimetric geoid model.

2. The Techniques and Their Limitations

It is essential to point out that a geoid model coarsely approximated from only terrestrial gravity data in the innermost zone is theoretically incorrect since it violates the primary requirement for a proper solution of the geodetic boundary-value problem. Essentially, the innermost zone contribution can be conceptualised as a highly truncated form of Stokes's formula, with a correspondingly large truncation error. However, a proportion of this truncation bias can be provided by a global geopotential model, as will be discussed below.

Due to the singularity in Stokes's kernel (when expressed in terms of spherical polar coordinates centred at the computation point), alternative methods are required to practically evaluate Stokes's formula in the innermost zone (eg. Heiskanen and Moritz, 1967, p.120-123). Importantly, however, if the mean gravity anomaly in the innermost zone is small, its contribution to the geoid will also be small. Conversely, in areas outside the innermost zone where the mean gravity anomalies may be large, these will make no contribution to the geoid approximated by only the innermost zone. Accordingly, the geoid approximation from the innermost zone is generally expected to be small.

2.1 The innermost zone and terrestrial gravity anomalies

The first mathematical model experimented with uses an innermost zone contribution evaluated using only the mean terrestrial gravity anomaly (i.e., one referred to the GRS80 ellipsoid) in a square element. Here, the very coarsely approximated geoid height is

$$N_1 = \frac{2a \ln(\sqrt{2} + 1)}{\pi \gamma} \Delta g - \frac{\pi G \rho H^2}{\gamma} \quad (1)$$

where the first term on the right-hand-side is the contribution of the innermost zone to the geoid as given by Haagmans *et al.* (1993, appendix), which is more rigorous for a square

element than the spherical-cap approximation given by Heiskanen and Moritz (1967, p.120-123). In Eq. (1), a is the side-length (in metres) of the square element comprising the innermost zone, Δg is the mean Faye (i.e., terrain-corrected, free-air as an approximation of the Helmert) terrestrial gravity anomaly, and γ is normal gravity on the surface of the normal ellipsoid. The second term on the right-hand-side of Eq. (1) is the quadratic approximation of the primary indirect effect on the geoid (Wichiencharoen, 1982), in which G is the Universal gravitational constant, ρ is the topographic mass-density (assumed constant), and H is the orthometric height of the computation point. This term will also be used for the subsequent geoid approximations.

2.2 A satellite-only global geopotential model and residual gravity anomalies

In this mathematical model, a satellite-only global geopotential model is used to contribute the very long-wavelength components of the geoid. It is also used to compute the gravity anomaly in the innermost zone, which is subtracted from the terrestrial gravity anomaly to yield the residual gravity anomaly. This is necessary to avoid the innermost zone contribution from adding [some of] the long-wavelength components to the geoid solution twice. The residual gravity anomaly is used to compute the innermost zone contribution to the geoid, which is then added to the geoid height implied by the global geopotential model. This is analogous with the remove-compute-restore approach to regional geoid computation. Accordingly, the very coarsely approximated geoid height is given by

$$N_2 = N_{SGGM} + \frac{2a \ln(\sqrt{2} + 1)}{\pi \gamma} (\Delta g - \Delta g_{SGGM}) - \frac{\pi G \rho H^2}{\gamma} \quad (2)$$

where N_{SGGM} is the low-frequency contribution to the geoid from the satellite-only global geopotential model, and Δg_{SGGM} are the gravity anomalies implied by the same degree of expansion of the same global geopotential model. The benefit of using this remove-compute-restore-type approach is that the global geopotential model provides a large proportion of the large truncation bias that results when using only Eq. (1).

2.3 A combined global geopotential model and residual gravity anomalies

In the third mathematical model experimented with, the very coarsely approximated geoid height is given by

$$N_3 = N_{CGGM} + \frac{2a \ln(\sqrt{2} + 1)}{\pi \gamma} (\Delta g - \Delta g_{CGGM}) - \frac{\pi G \rho H^2}{\gamma} \quad (3)$$

where N_{CGGM} is the geoid contribution of a combined global geopotential model and Δg_{CGGM} is the gravity anomaly implied by the same degree of expansion of the same model. The argument presented above regarding the truncation bias is now more valid because the increased degree of expansion of the combined global geopotential model (compared to the satellite-only model) provides a larger proportion of the truncation bias. Accordingly, this mathematical model is expected to deliver the best results. However, this is contingent on terrestrial gravity data in and around the area of interest having been used to construct the combined global geopotential model.

3. Data and Results

The experiments were conducted over Australia, where a reasonably dense coverage of regional terrestrial gravity, terrain and GPS-levelling data are available.

- The Australian Surveying and Land Information Group (now the National Mapping Division of Geoscience Australia) supplied the GPS and spirit levelling data at 1013 points covering most of Australia (Figure 1; Featherstone and Guo, 2001). The spirit levelling data refer to the Australian Height Datum (AHD).

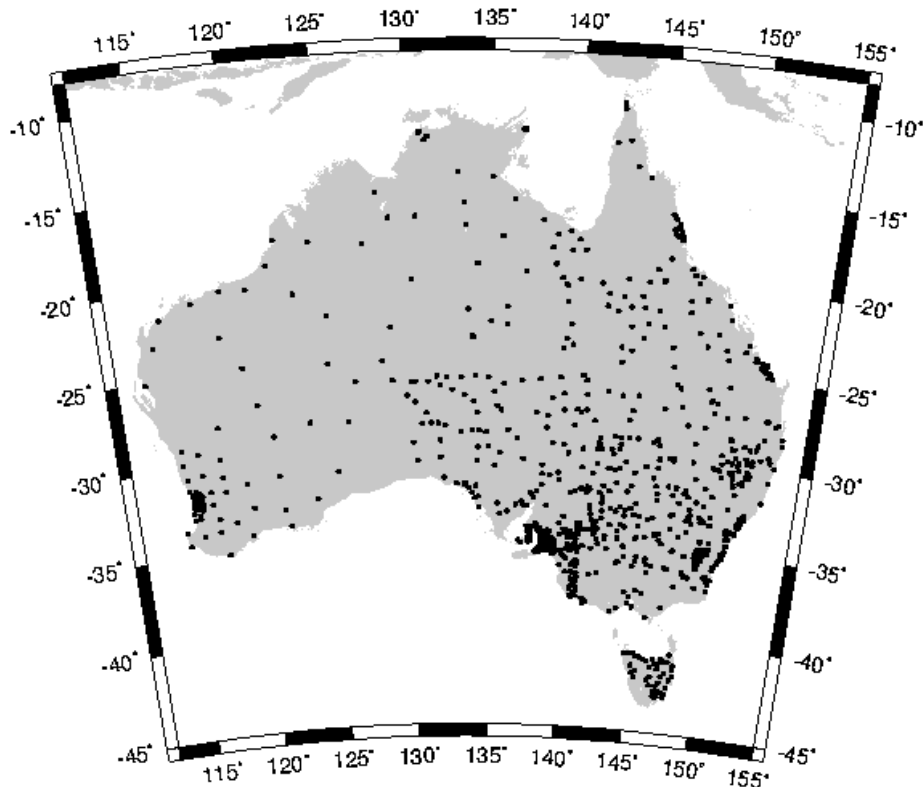


Figure 1. Spatial coverage of the 1013 GPS-AHD stations [Lambert conical projection]

- The terrestrial gravity data used for these experiments are exactly the same as those used to compute AUSGeoid98 (Featherstone *et al.* 2001). These were provided by the Australian Geological Survey Organisation (now Geoscience Australia) and have been supplemented with Sandwell and Smith's (1997) satellite altimeter-derived gravity anomalies in marine areas using least squares collocation (Kirby and Forsberg 1998). Gravimetric terrain corrections have been applied to these data using version 1 of the Australian digital elevation model (Kirby and Featherstone 1999). The resulting set of mean Faye gravity anomalies is given on a 2' by 2' grid over the area bound by $46^{\circ}\text{S} \leq \phi \leq 8^{\circ}\text{S}$ and $108^{\circ}\text{E} \leq \lambda \leq 160^{\circ}\text{E}$ (Figure 2).

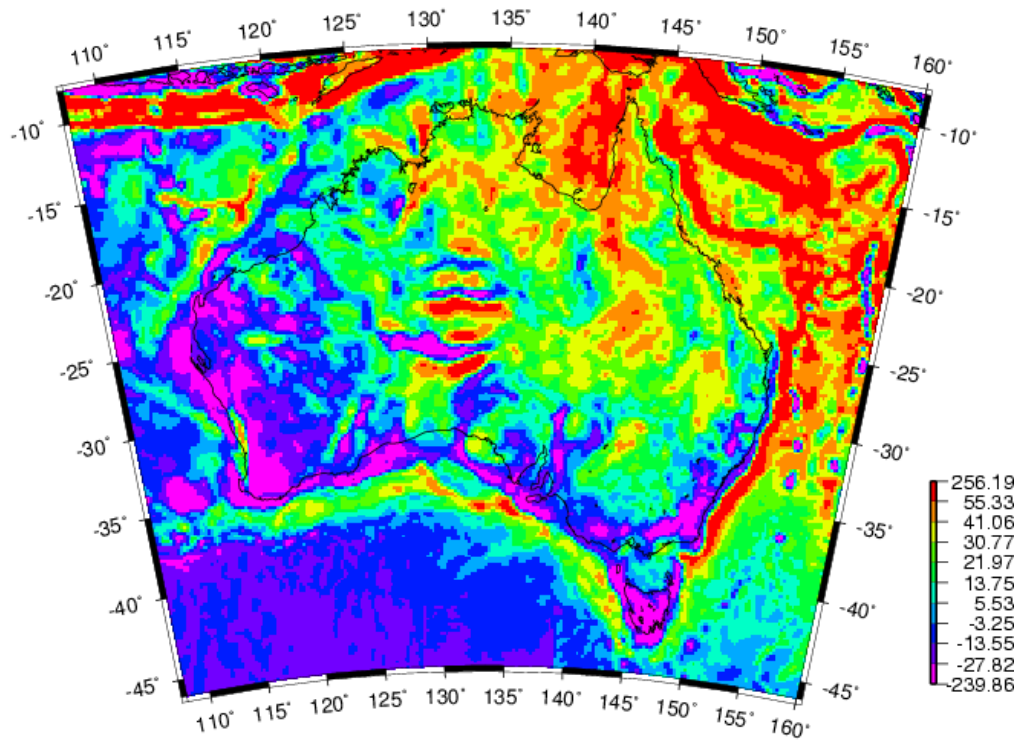


Figure 2. Terrestrial Faye gravity anomalies over Australia [units in mGal]

- Version 1 of the Australian digital elevation model (Carroll and Morse, 1996) was used to compute the approximate primary indirect effect (second term on the right-hand-side of Eqs. (1) to (3)), which was generalised onto the same 2' by 2' grid as the gravity data over land areas. Due to the lack of a digital density model over Australia, a constant topographic mass-density of 2670 kgm^{-3} had to be used in this computation.
- The EGM96S (satellite-only) and EGM96 (combined) global geopotential models (Lemoine *et al.* 1998) were used in Eqs. (2) and (3), respectively. The expansion of EGM96S was truncated at degree 20 in Eq. (2), which is the point beyond which satellite-only global geopotential models become heavily contaminated by noise. This can be demonstrated simply by plotting the error degree variances of the EGM96S coefficients as a proportion of their degree variances. The complete degree-360 expansion of EGM96 was used in Eq. (3).

Table 1 shows the descriptive statistics of only the innermost zone contributions (first term on the right-hand-side of Eqs. (1) to (3)), as well as the approximate primary indirect effect, which is common to all equations. From Table 1, it can be seen that the innermost zone contributions to the geoid from each of the three approximate mathematical models are small and reasonably similar. The approximate primary indirect effect is small, which is to be expected because the tallest mountain in Australia is only 2,228 m high. Given these small values, it is to be expected that Eq. (1), which does not include a global geopotential model and thus has the largest truncation error, will deliver poor results.

<i>Geoid model</i>	<i>Max</i>	<i>Min</i>	<i>Mean</i>	<i>STD</i>
1st term in Eq. 1 with terrestrial gravity anomalies	0.624	-0.488	-0.006	±0.062
2nd term in Eq. 2 with residual gravity anomalies	0.624	-0.505	-0.002	±0.059
2nd term in Eq. 3 with residual gravity anomalies	0.416	-0.597	-0.002	±0.032
Approximate primary indirect effect	0.000	-0.221	-0.002	±0.006

Table 1. Descriptive statistics of the innermost zone contributions from Eqs. (1) to (3) and the approximate primary indirect effect [units in metres]

Table 2 shows the descriptive statistics of the three very approximate geoid models (all terms in Eqs. (1) to (3)). By way of comparison, the AUSGeoid98 regional geoid model [based on the degree-360 expansion of EGM96] is also shown in Table 2. From Table 2, the approximate geoid heights computed from Eqs. (2) and (3) appear reasonably similar to AUSGeoid98, though the maximum and minimum differences are greater than the contribution of the innermost zone (cf. Table 1). This demonstrates the relatively large contribution that the regional integration makes to AUSGeoid98. Also from Table 2, the approximate geoid height computed from Eq. (1) is very different. As stated, this is due to the very large truncation error (up to ~80 m) that results from using only an innermost zone computation of the original Stokes formula (i.e., no inclusion of a global geopotential model to provide an proportion of the truncation bias).

<i>Geoid model</i>	<i>Max</i>	<i>Min</i>	<i>Mean</i>	<i>STD</i>
N_1 (Eq. 1 with terrestrial gravity anomalies)	0.624	-0.488	-0.008	±0.066
N_2 (Eq. 2 with residual gravity anomalies)	75.806	-39.244	10.902	±33.263
N_3 (Eq. 3 with residual gravity anomalies)	85.906	-40.476	10.852	±33.406
AUSGeoid98 (Featherstone <i>et al.</i> , 2001)	80.644	-40.583	10.802	±33.334

Table 2. Descriptive statistics of the geoid models experimented with [units in metres]

Figures 4 and 5 show images of the differences between the AUSGeoid98 regional geoid model (Figure 3), and the very approximate geoid solutions based on EGM96S and EGM96 (Eqs. 2 and 3, respectively). The difference between the very approximate geoid solution based on Eq. (1) and AUSGeoid98 is so small (cf. Table 1) as to be imperceptible in a figure, and would simply resemble Figure 3. As such, it is not shown here.

Figure 4 shows that Eq. (2), based on the satellite-only EGM96S global geopotential model, omits most of the medium wavelength geoid undulations that are provided by regional integration [a one-degree integration radius was used for AUSGeoid98]. Figure 5 shows that Eq. (3), based on the combined EGM96 global geopotential model, omits only some of the medium wavelength geoid undulations that are provided by AUSGeoid98. This is as expected because the higher degree expansion of EGM96 contributes more to the evaluation of the truncation bias at medium wavelengths (cf. Figures 4 and 5).

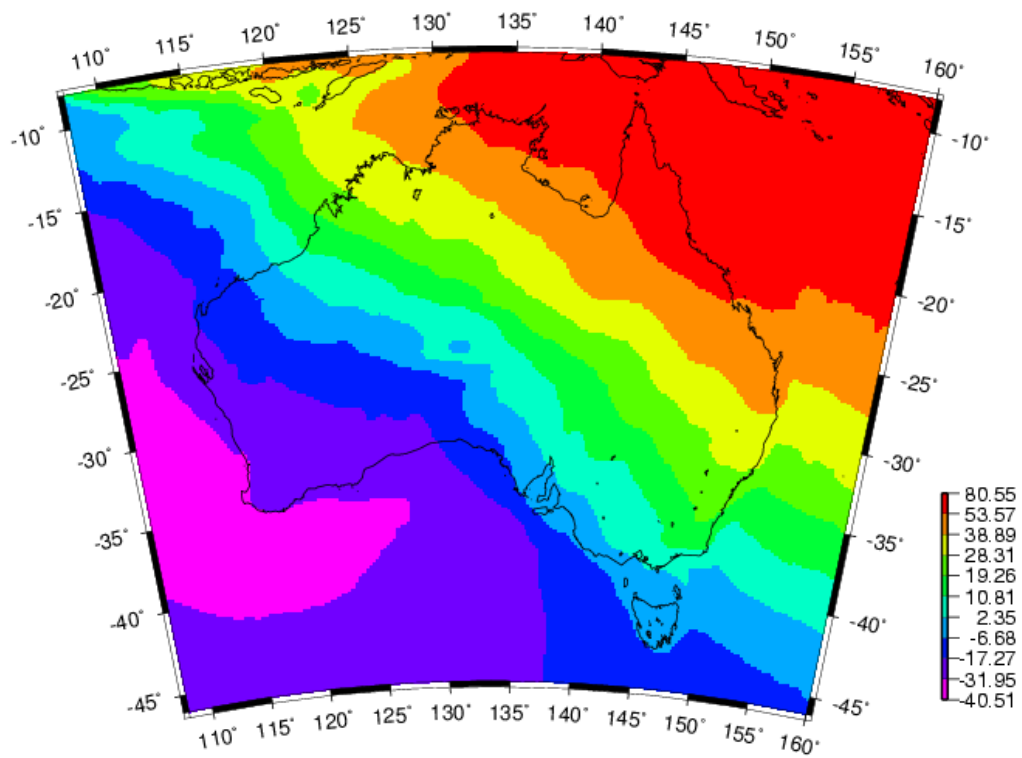


Figure 3. AUSGeoid98 [units in metres]

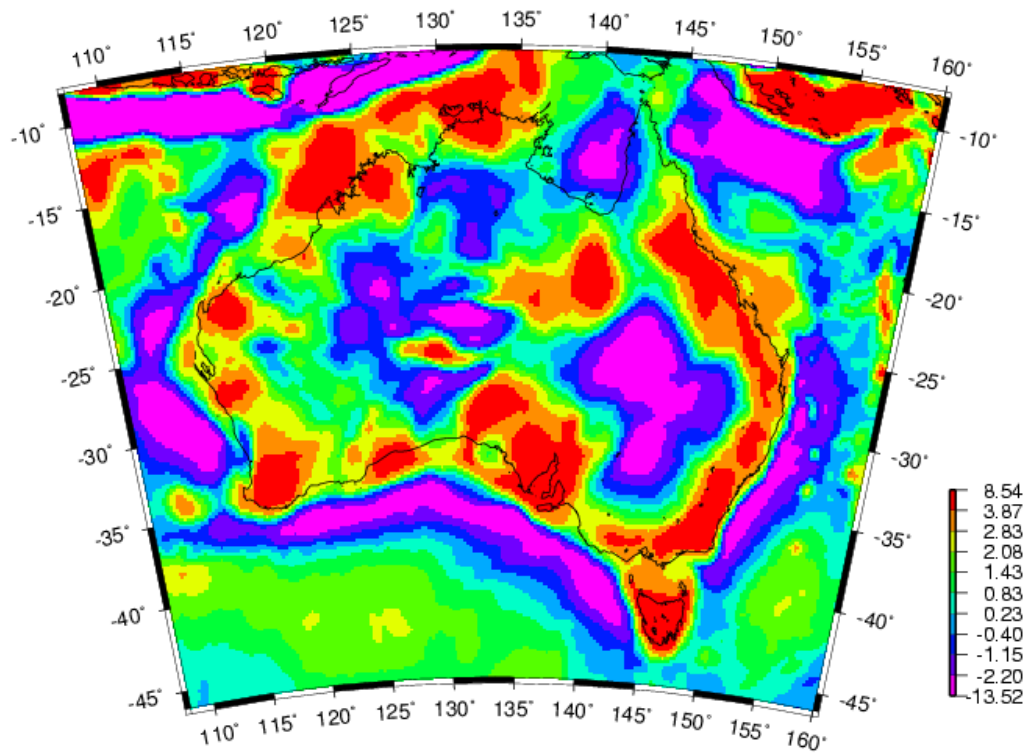


Figure 4. Differences between AUSGeoid98 and the geoid from Eq. 2 [units in metres]

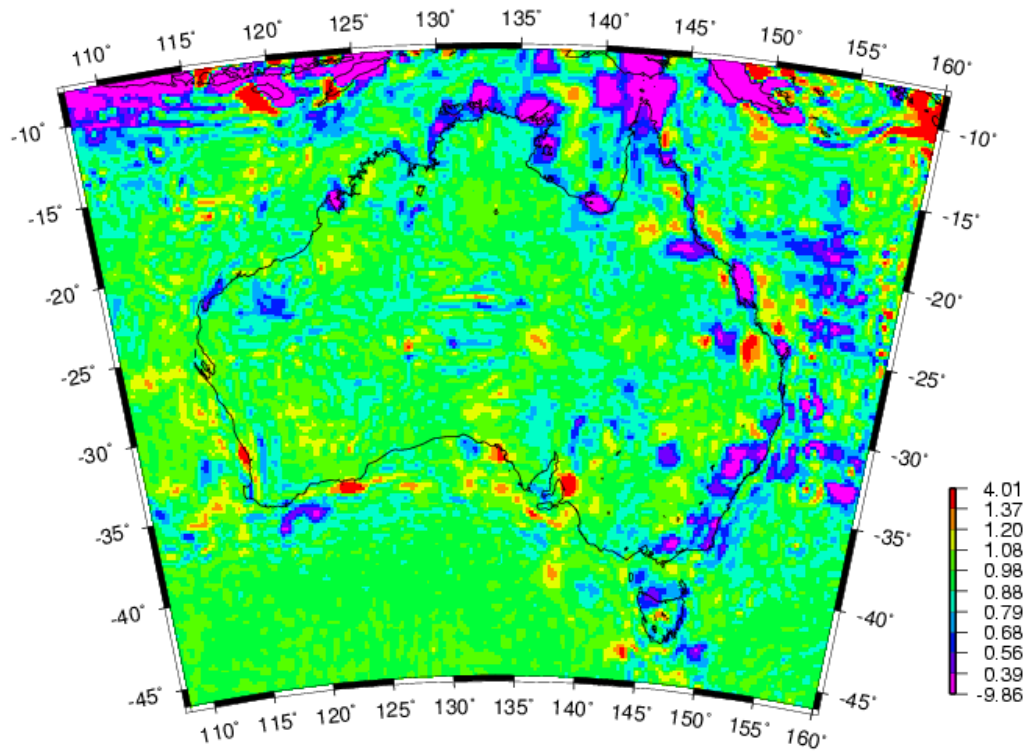


Figure 5. Differences between AUSGeoid98 and the geoid from Eq. 3 [units in metres]

As is customary with most validations of gravimetric geoid models on land, the three very approximate geoid models (Eqs. 1 to 3) were differenced with the 1013 Australian GPS-levelling data (cf. Featherstone and Guo, 2001). By way of comparison, the EGM96S, EGM96 and AUSGeoid98 geoid models were also differenced with the same GPS-levelling data. The results of the fit of each geoid model to the GPS-levelling data are summarised by the descriptive statistics in Table 3. Of course, it is acknowledged that such comparisons are equivocal because of the error budget of the GPS-levelling data.

<i>Geoid model</i>	<i>Max</i>	<i>Min</i>	<i>Mean</i>	<i>STD</i>
N_1 (Eq. 1 with terrestrial gravity anomalies)	71.268	-32.747	11.284	± 23.074
EGM96S truncated to degree 20	6.967	-6.515	1.528	± 2.505
N_2 (Eq. 2 with residual gravity anomalies)	6.767	-6.449	1.521	± 2.470
EGM96 complete to degree 360	3.537	-2.441	-0.015	± 0.441
N_3 (Eq. 3 with residual gravity anomalies)	3.510	-2.464	-0.008	± 0.422
AUSGeoid98 regional geoid model	3.558	-2.572	-0.002	± 0.314

Table 3. Descriptive statistics of the difference between 1013 GPS-AHD data and the various geoid models experimented with [units in metres]

From Table 3, the approximate geoid solution based only on terrestrial gravity anomalies (Eq. 1) is in extremely poor agreement with the GPS-levelling data, which is as

expected. As such, it should never be used. Moreover, given the ready and free availability of global geopotential models that give much better fits (Table 3), it would be pointless to trial this approach any further. Next, it is interesting to observe from Table 3 that the approximate geoid solutions based on the global geopotential models (Eqs. 2 and 3) give marginally better fits to the Australian GPS-levelling data than the global geopotential models alone. Entirely as expected, they do not make as much of an improvement as the AUSGeoid98 regional gravimetric geoid model (Table 3).

4. Concluding Remarks

The results of these curiosity-driven experiments are largely as expected. That is, the contribution of mean terrestrial gravity anomalies from the innermost zone to the geoid is small. Nevertheless, the very coarse approximations of the geoid height given by Eqs. (2) and (3) can add some localised geoid information to a global geopotential model, but as expected, this is not as good as the improvement made by a regional gravimetric geoid model. This very approximate approach may prove to be of some value in areas or countries where access to regional gravity and terrain data is restricted, or where the resources of personnel, software and high-powered computers are not available. This simple method can be adopted to yield an interim geoid solution for use until the issues of data and resources have been resolved.

From the results presented for Australia, Eqs (2) and (3) can be used to make some small improvements to the global geopotential model that, in turn, improve upon the transformation of GPS-derived ellipsoidal heights to orthometric heights. Of these two approaches, Eq. (3) delivers the best results because the high-degree combined global geopotential model provides a larger proportion of the truncation bias. However, this may be contingent on terrestrial gravity data being used in EGM96 (as is the case in Australia). Finally, the calculation of the very approximate geoid height is extraordinarily simple and can be achieved using spreadsheets instead of the need to use high-powered computers, as is the case when computing regional gravimetric geoid models. This simplicity may be of value to GPS users in developing countries.

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