Journal of the Australian Ceramic Society Volume 44[2], 2008, 38-41

A Reassessment of the Strength Distributions of Advanced Ceramics

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Available Online at: www.austceram.com/ACS-Journal-2008vol2.asp

Abstract
Three versatile functions (the Weibull, normal and lognormal distributions) are used to the fit of fracture strengths of advanced ceramics. The best distribution for each ceramics is determined in terms of its main distribution features and a minimum information criterion. The results show that there seems to be not a universal distribution. Special focus is on the skewed distribution on the side of low strength values in order to provide deep insights into the relationship between fracture and defect statistics.

Keywords: Strength distribution, Weibull, Normal, Lognormal, Minimum information criterion, Zinc oxide

INTRODUCTION

The major attraction of advanced ceramics, in contrast to traditional ones such as brick and porcelain, is their high strength, high operating temperatures, and excellent resistance to corrosion and wear. But, the inherent brittleness of ceramics makes them sensitive to process-related defects [1,2]. Without a complete characterization and the use of correct statistics, the measured strengths may have little applicability or even result in misleading conclusions. Therefore, successful use of advanced ceramics for demanding applications requires accurate and reliable information to improve processing technology, eliminate critical defects, and increase toughness.

Fracture strengths of brittle materials, i.e., the maximum stresses they can withstand, are distributed over a wide range of values even if a set of nominally identical specimens are tested under the same conditions. The strength of a brittle material is therefore not a well-defined quantity and has to be described with fracture statistics. Since the pioneering work of Weibull [3], it has been shown that the Weibull distribution with two parameters is very successful in fitting a large body of strength data, especially for brittle materials. Although the Weibull distribution is usually suggested to be considered first, more and more evidence indicates that there seems to be not a universal distribution [4,5]. Especially, in the case of small specimens, the Weibull distribution should be considered as an empirical one on an equal footing with other functions such as normal, lognormal, power law, type I extreme value distributions etc [6]. On the other hand, the correct choice for an optimal strength distribution of ceramics is pivotal for us to understand their underlying failure mechanisms. In this paper, a simple quantitative method based on the minimum information criterion is proposed, which can be used to determine the best strength distribution. Together with analyses of the main features of these distributions, the relationship between fracture and defect statistics is discussed.

STRENGTH DISTRIBUTIONS

Among various possible candidates, the following three functions are most widely used in the fit of strength data of brittle materials: the Weibull, normal and lognormal distributions.

Weibull Distribution

Let us image that a small volume in a brittle material is like a chain of links, and if any link breaks, the material will fail. In terms of the so-called weakest-link principle and a simply empirical function, the cumulative failure probability of a brittle material subjected to a load σ can be represented as $F(\sigma) = 1 - \exp(-[(\sigma - \sigma_{th})/\sigma_0]^m)$, where σ_0 is a normalized strength, σ_{th} is a threshold stress (below which no failure occurs), and m is the Weibull modulus or shape factor, a measure of the degree of strength dispersion [3]. For simplicity, σ_{th} is usually taken as zero in most real applications. Thus, the probability density function of a two-parameter Weibull distribution, $f(\sigma) = dF(\sigma)/d\sigma$, is given by

$$f(\sigma) = \frac{m}{\sigma_0}\left(\frac{\sigma}{\sigma_0}\right)^{m-1} \exp\left[-\left(\frac{\sigma}{\sigma_0}\right)^m\right] \quad (1)$$

Normal Distribution

Without special care in manufacture and handling, strengths of brittle materials usually exhibit more or less symmetrical distributions, so it is not surprised that normal distribution could be a natural choice to these data. For normal distribution, its probability density function is

$$f(\sigma) = \frac{1}{\sqrt{2\pi}\alpha}\exp\left[-\frac{(\sigma - \overline{\sigma})^2}{2\alpha^2}\right] \quad (2)$$

where $\overline{\sigma}$ and α are the mean and standard deviation, respectively.

Lognormal Distribution

According to the definition, lognormal distribution is the distribution of a random variable whose logarithm is normally distributed. Thus, similar to normal distribution, its probability density function follows

$$f(\sigma) = \frac{1}{\sqrt{2\pi}\sigma\alpha}\exp\left[-\frac{(\ln\sigma - \overline{\sigma})^2}{2\alpha^2}\right] \quad (3)$$

Its mean and variance are $\exp(\alpha + \overline{\sigma}^2)$ and $\exp(2\alpha + \overline{\sigma}^2)(\exp(\overline{\sigma}^2) - 1)$, respectively. Relative to the additive transformation of normal distribution, $\overline{\sigma} \pm \alpha$, the multiplicative transformation of lognormal distribution can be expressed as, $\overline{\sigma}^* \cdot / s^*$, where $\overline{\sigma}^*$ is an estimator of the median, s^* estimates the multiplicative standard deviation, and the sign "·/" indicates "times or divided by" [7].

MINIMUM INFORMATION CRITERION

To find the unknown parameters in distributions in Eqns 1 to 3, a usual way is the linear regression (least-squares) procedure. However, the best estimate of these parameters is by the maximum likelihood method, which shows the smallest coefficient of variation (the ratio of the standard deviation to mean of a random variable). The likelihood of a probability density function is defined as $L = \Pi_{i=1}^{N} f(\sigma_i)$, where σ_i is strength of the i-th specimen and N is the total number of specimens (or experiments). Thus, for the given function, its log-likelihood function is $\ln L = \sum_{i=1}^{N} \ln f(\sigma_i)$.

Generally, we identify an appropriate distribution for strength data using goodness-of-fit tests. However, for small sample sizes, it is usually difficult to distinguish between two functions such as the Weibull and normal distributions. The likelihood ratio appears to be the most promising for use in obtaining confidence bounds. Following the similar consideration, the likelihood approach can be extended to make comparisons between models by a minimum or Akaike information criterion (AIC) [8], which links the likelihood to a distance between true and estimated distributions, and is defined as

$$\text{AIC} = -2\ln\hat{L} + 2k \quad (4)$$

where $\ln\hat{L}$ is the maximum log-likelihood for a given model, and k is the number of parameters to be fitted in the model. This represents a rough way of compensating for additional parameters and is a useful measure of the relative effectiveness of different distributions. The best distribution is that for which AIC has the smallest value [9,10]. In typical cases, model differences which would be significant at around the 5% confidence level correspond to differences in AIC values of 1.5–2 [8].

RESULTS AND DISCUSSION

Table 1 lists the parameters obtained by fitting each distribution to strength data of three advanced ceramics such as silicon nitride (Si_3N_4), silicon carbide (SiC), and zinc oxide (ZnO) [5]. For comparison, strength data of soda-lime were also analysed [4]. As indicated in Table 1, ZnO exhibits the smallest strength scatter, and soda-lime has the largest one. This is easy to see from the Weibull moduli and additive (or multiplicative) standard deviations in normal (or lognormal) distribution. As for three advanced ceramics, the degree of scatter (from large to small) follows the order SiC, Si_3N_4, and ZnO in terms of the Weibull moduli and multiplicative standard deviations. But, it is worth noting that the mean strength of Si_3N_4 is much larger than that of SiC or ZnO.

Table 2 summarizes the AIC values calculated by three distributions. For Si_3N_4 and SiC, the Weibull distribution fits the data better than normal or lognormal, but for ZnO, it is just opposite. However, the difference of AIC values between normal and lognormal distributions for ZnO is not large enough to distinguish the better one since ΔAIC < 1.5. In other words, we cannot say normal distribution is superior to lognormal distribution although people are usually in favour of the former. Also it is surprising to see that, for soda-lime, lognormal distribution is better than the Weibull distribution. Thus, additional mechanical analysis is clearly needed in order to determine an optimal distribution of strength data.

Table 1: The fitted parameters using the Weibull, normal, and lognormal distributions

Materials	Weibull		Normal		Lognormal	
	M	σ_0	α	$\overline{\sigma}$	s^*	$\overline{\sigma}^*$
Soda-lime	5.74	128.70	20.59	119.74	1.18	118.09
Si_3N_4	14.89	933.56	80.49	899.42	1.10	895.56
SiC	9.62	376.20	42.26	357.87	1.13	355.18
ZnO	20.92	104.81	5.17	102.37	1.05	102.24

It is well-known that size effect is a direct consequence of the Weibull distribution. That is, the larger the specimen, the smaller the mean strength of the corresponding sample. In a general term, size effect can be described as, $\overline{\sigma} \sim V^{-1/m}$, where V is the effective volume of a specimen [2]. Thus, we have another method to evaluate the results in Table 2. As shown in Fig. 1, for Si_3N_4 and SiC, size effect is clearly observed as expected, but in the case of ZnO, there is no size effect [11].

Table 2: The difference of *AIC values calculated by three distributions, where ΔAIC = max{AIC} – min{AIC}*

Materials	N	min{AIC}	Max{AIC}	ΔAIC
Soda-lime	24	Lognormal	Weibull	6.54
Si_3N_4	55	Weibull	Lognormal	12.59
SiC	75	Weibull	Lognormal	7.29
ZnO	109	Normal	Weibull	9.76

Further, microscopic observations showed that there are very different microstructures or defects in these materials. In Si_3N_4 and SiC, crack-like flaws are sparsely distributed, and thus it is not surprising that their strengths yield the Weibull distribution. But flaws in ZnO are approximately spherical pores with various sizes and sharp grooves (see insets in Fig. 1). As a typical kind of electroceramics, ZnO ceramics are applied for varistors and designed with respect to electrical rather than mechanical properties. As a consequence of high porosity (about 5 vol.%) and pronounced R-curve behaviour before fracture in ZnO, a group of pores as well as the interaction between them would affect the final fracture rather than only the largest one as postulated in the weakest-link model [12].

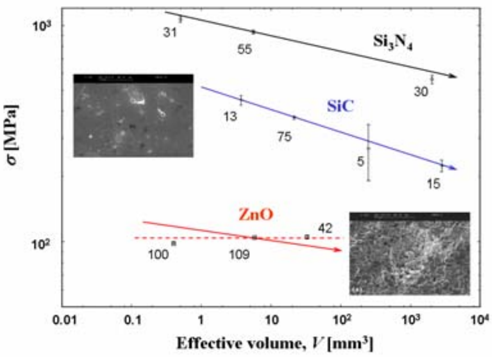

Fig. 1: Dependence of mean strength on effective volume, where numerals are the total number of tests and error bars refer to the 95% confidence band. The slope of solid line is –1/m with the Weibull moduli determined by the largest numbers (55 for Si_3N_4, 75 for SiC, and 109 for ZnO). The dashed line indicates the mean strength of 109 specimens of ZnO. Insets are typical defects or fracture origin in SiC and ZnO, respectively.

Next, let us have a little more discussion on the strength distribution of ZnO. For simplicity, the interaction between pores has been ignored. The failure probability of n defects can be written as $P_N(n) = [N!/(n!(N-n)!)]\ p^n(1-p)^{N-n}$, where p is the failure probability of individual defect and N is the total number of defects. In the limit of large N, there are two extreme cases for the binomial distribution [13]. If p is not too small (corresponding to ZnO), the binomial distribution approaches normal distribution. On the contrary, if $p \ll 1$, we have the Poisson distribution, $P_N(n) = a^n\exp(-a)/n!$, where $a = Np$. Further, let $n = 0$, one can easily obtain $P_N(0) = \exp(-Np)$, and then the weakest-link model can be described in the form, $F = 1 - P_N(0) = 1 - \exp(-Np)$. Based on few assumptions, a similar formula $F_S(\sigma) = 1 - \exp[-\langle N_{c,S}(\sigma)\rangle]$ was proposed, where $\langle N_{c,S}(\sigma)\rangle$ indicates the mean number of critical defects in a specimen of size S. So, the Weibull distribution is only a special case of this general distribution function [14].

Intuitively, real strength data show more or less skewed distribution rather than symmetrical one as predicted by normal distribution. Also, strength values cannot be negative. In fact, the probability of a flaw being critical in ZnO is dependent on a lot of factors such as size, shape, pore/gain interaction etc. This assumption has been supported by fracture statistics and finite element modelling [12,15]. Thus, the failure probability in ZnO can be estimated by $p = \Pi\ p_i$, where p_i is the failure probability of the i-th influential driver. Then, we have $\ln p = \sum \ln p_i$. In other words, the failure probability follows lognormal distribution since its logarithm has normal distribution provided individual factors are independent of each other. Here, it is of interest to note that grain size distributions measured in brittle materials such as ceramics are approximately lognormally distributed [16]. Thus, in addition to the Weibull distribution, it seems that lognormal distribution is preference over normal distribution in describing strength data.

CONCLUSIONS

Strength data of soda-lime and three advanced ceramics have been reassessed using three versatile statistical distributions. Based on the main features of strength distributions, a minimum information criterion, and microscopic observations, the best distribution for each material has been determined. The results show that there seems to be not a universal distribution that can be used in all cases. In addition to the Weibull distribution, the skewed strength distribution on the side of smaller strengths can be well described by lognormal distribution. These findings could shed light into a better understanding of the correlation between fracture and defect statistics.

ACKNOWLEDGMENTS

This work was supported by the Australian Research Council and the Lise Meitner Program of the Austrian Science Fund.

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