Generalized Minimax Inequalities for Set-Valued Mappings[1]

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Abstract: In this paper, we study generalized minimax inequalities in a Hausdorff topological vector space, in which the minimization and the maximization of a two-variable set-valued mapping are alternatively taken in the sense of vector optimization. We establish two types of minimax inequalities by employing a nonlinear scalarization function and its strict monotonicity property. Our results are obtained under weaker convexity assumptions than those existing in the literature. Several illustrative examples are given to illustrate our results.

Keywords: Minimax inequality, set-valued mapping, minimal point, maximal point, nonlinear scalarization function.

1 INTRODUCTION

Throughout the paper, let X, Z and V be real Hausdorff topological vector spaces. Let $S \subset V$ be a closed, convex and pointed cone such that $intS \neq \emptyset$, and let V^* denote the topological dual space of V. Some fundamental terminologies are presented as follows.

Definition 1.1 *Let $A \subset V$ be a nonempty subset.*

(i) *A point $y \in A$ is called a minimal point of A if $A \bigcap (y - S) = \{y\}$; and $MinA$ denotes the set of all minimal points of A.*

(ii) *A point $y \in A$ is called a weakly minimal point of A if $A \bigcap (y - intS) = \emptyset$; and $Min_W A$ denotes the set of all weakly minimal points of A.*

(iii) *A point $y \in A$ is called a maximal point of A if $A \bigcap (y + S) = \{y\}$; and $MaxA$ denotes the set of all maximal points of A.*

(iv) *A point $y \in A$ is called a weakly maximal point of A if $A \bigcap (y + intS) = \emptyset$; and $Max_W A$ denotes the set of all weakly maximal points of A.*

Definition 1.2 *[1] Let $F : X \to 2^V$ be a set-valued mapping.*

(i) *F is said to be upper semicontinuous (u.s.c.) at $x_0 \in X$ if, for any neighborhood $N(F(x_0))$ of $F(x_0)$, there exists a neighborhood $N(x_0)$ of x_0 such that*

$$F(x) \subset N(F(x_0)) \quad \forall x \in N(x_0).$$

(ii) *F is said to be lower semicontinuous (l.s.c.) at $x_0 \in X$ if, for any sequence $\{x_n\} \subset X$ such that $x_n \to x_0$, and any $y_0 \in F(x_0)$, there exists a sequence $y_n \in F(x_n)$ such that $y_n \to y_0$.*

(iii) *F is said to be continuous at $x_0 \in X$ if F is both u.s.c. and l.s.c. at x_0.*

Minimax theorems for real-valued functions were discussed in [5, 6, 12]. Let $X_0 \subset X, Z_0 \subset Z$, and $f : X_0 \times Z_0 \to R$ be a real-valued function. Under suitable conditions, the following equality holds:

$$\inf_{z\in Z_0} \sup_{x\in X_0} f(x,z) = \sup_{x\in X_0} \inf_{z\in Z_0} f(x,z). \tag{1}$$

In recent years, investigations on vector minimax theorems have attracted a lot of attention. Many papers have dealt with this subject under various assumptions (see, e.g., [18, 7, 8, 9, 17, 19, 16]). For vector-valued functions, the two terms in (1) are two sets, not singleton. Thus, the equality in (1) does not, in general, hold. But one may get an inclusion relation between these two sets. In [8, 9], under conditions that the vector-valued function $f(x,.)$ is S-convex for each $x \in X_0$, $-f(.,z)$ is properly S-quasiconvex for each $z \in Z_0$ and $\mathrm{Max} \bigcup_{x\in X_0} \mathrm{Min}_W \bigcup_{z\in Z_0} f(x,z) \subset \mathrm{Min}_W \bigcup_{z\in Z_0} f(x,z) + S \ \forall x \in X_0$, Ferro established that the following vector minimax inequalities:

$$\mathrm{Max} \bigcup_{x\in X_0} \mathrm{Min}_W \bigcup_{z\in Z_0} f(x,z) \subset \mathrm{Min} \bigcup_{z\in Z_0} \mathrm{Max}_W \bigcup_{x\in X_0} f(x,z) + S. \tag{2}$$

In [9], Ferro raised an open problem: (2) may not hold if S-convexity of $f(x,.)$ is relaxed.

In this paper, we study vector minimax inequalities (2) for set-valued mappings. With properties of a strictly monotone function, we derive two types of vector minimax inequalities: one type gives that the max-min set of a two-variable set-valued mapping is contained in the sum of the min-max set of the set-valued mapping and a positive cone, and another type shows that the min-max set of a two-variable set-valued mapping is contained in the sum of the max-min set of the set-valued mapping and the complement of a positive cone. Our results include the corresponding ones for vector-valued functions in [8, 9] as special cases. In particular, we show in Section 3 that Theorem 3.1(i) of [9] is a special case of Corollary 3.1. Therefore, Corollary 3.1 solves a part of the open problem raised by Ferro in [9]. We also show that Theorem 1 of [16] is a special case of our results. Several illustrative examples are given to clarify our results. Some preliminary results are presented as follows.

Lemma 1.1 *[13, Lemma 2.2] Let X_0 and Z_0 be compact subsets of X and Z, respectively. Let $F : X_0 \times Y_0 \to 2^V$ be a continuous set-valued mapping such that for each $(x,z) \in X_0 \times Y_0, F(x,z)$ is a compact set. Then $\Gamma(x) = Min_W \bigcup_{z\in Z_0} F(x,z)$ and $L(z) = Max_W \bigcup_{x\in X_0} F(x,z)$ are u.s.c. on X_0 and Z_0, respectively.*

Lemma 1.2 *[19] Let $A \subset V$ be a nonempty compact subset. Then (i) $MinA \neq \emptyset$; (ii) $A \subset MinA + S$; (iii) $MaxA \neq \emptyset$; and (iv) $A \subset MaxA - S$.*

Remark 1.1 In this paper, S is assumed to be a pointed cone with a nonempty interior. Thus, $\mathrm{Min}A \subset \mathrm{Min}_W A$ and $\mathrm{Max}A \subset \mathrm{Max}_W A$. Consequently, Lemma 1.2 holds for the weakly minimal point set and the weakly maximal point set.

The rest of the paper is organized as follows: In Section 2, we introduce some notation and preliminary results. On this basis, we discuss properties of ξ-function and set-valued mappings. In Section 3, we state two types of minimax theorems for set-valued mappings.

2 SET-VALUED MAPPINGS AND MONOTONE FUNCTIONS

Definition 2.1 *Given $k \in intS$ and $a \in V$, the Gerstewitz's function (see [10, 11]) $\xi_{ka} : V \to R$ is defined by*

$$\xi_{ka}(y) = min\{t \in R | y \in a + tk - S\}.$$

Definition 2.2 *A function $\Psi : V \to R$ is called strictly monotone if*

$$y_1 - y_2 \in intS \Rightarrow \Psi(y_1) > \Psi(y_2).$$

Lemma 2.1 *[3, Theorem 2.1] Let $k \in intS$ and $a \in V$. The following properties hold:*

(i) $\xi_{ka}(y) < r \Longleftrightarrow y \in a + rk - intS$;

(ii) $\xi_{ka}(y) \le r \Longleftrightarrow y \in a + rk - S$;

(iii) $\xi_{ka}(y) = 0 \Longleftrightarrow y \in a - \partial S$, *where* ∂S *is the topological boundary of* S*;*

(iv) $\xi_{ka}(y) > r \Longleftrightarrow y \notin a + rk - S$;

(v) $\xi_{ka}(y) \ge r \Longleftrightarrow y \notin a + rk - intS$;

(vi) $\xi_{ka}(.)$ *is a convex function;*

(vii) $\xi_{ka}(.)$ *is a strictly monotone function;*

(viii) $\xi_{ka}(.)$ *is a continuous function.*

Let $B \subset V$. The cone generated by B is defined by

$$cone(B) := \{tc | t \ge 0, c \in B\}.$$

Lemma 2.2 $C \subset V$ *is a closed and convex cone if and only if there exists a subset* $\Gamma \subset V^* \backslash \{0\}$ *such that*

$$C = \{y \in V | f(y) \le 0 \ \ \forall f \in \Gamma\}. \tag{3}$$

Proof. Assume that C is a closed and convex cone. We take any $\bar{y} \notin C$. Then cone($\bar{y}$) is a pointed, closed and convex cone. Obviously, cone($\bar{y}$) is locally compact (i.e., it has a compact neighborhood base with respect to the relative topology on cone($\bar{y}$)) and

$$cone(\bar{y}) \bigcap C = \{0_V\}.$$

Therefore, by Proposition 3 of [2], there exists $f_{\bar{y}} \in V^*$ such that

$$\begin{aligned} f_{\bar{y}}(z) &> 0 \quad \forall z \in cone(\bar{y}) \backslash \{0_V\}; \\ f_{\bar{y}}(z) &\le 0 \quad \forall z \in C. \end{aligned}$$

Let $\Gamma = \{f_{\bar{y}} \in V^* | \bar{y} \notin C\}$. Define

$$P := \{y \in V | f(y) \le 0 \forall f \in \Gamma\}.$$

Now we prove that $C = P$. Let $y \in C$. By the construction of Γ, we have

$$f(y) \leq 0 \quad \forall f \in \Gamma.$$

Thus, $y \in P$.

Conversely, let $y \in P$ and $y \notin C$. Then there exists a $f_y \in V^*$ such that

$$\begin{aligned} f_y(y) &> 0; \\ f_y(z) &\leq 0 \quad \forall z \in C. \end{aligned}$$

Obviously, $f_y \in \Gamma$, which contradicts $y \in P$. Thus, $P = C$.

If C is defined by (3), it is clear that C is a closed and convex cone. The proof is complete. $\square$

Proposition 2.1 *Let $S \subset V$ be a closed and convex cone with $intS \neq \emptyset$. Let $k \in intS$. Then there exists a $\Gamma \subset V^* \backslash \{0_V\}$ such that*

$$\xi_{ka}(y) = \sup_{f \in \Gamma} \{\frac{f(y) - f(a)}{f(k)}\}.$$

Proof. By Proposition 2.3 of [4] and Lemma 2.2, the conclusion holds. $\square$

Example 2.1 Let $V = R^2, S = \{(x,y) \in V | (-3,1) \begin{pmatrix} x \\ y \end{pmatrix} \geq 0, (1,-3) \begin{pmatrix} x \\ y \end{pmatrix} \geq 0\}$, and $\Gamma = \{f_1, f_2\}$, where

$$\begin{aligned} f_1(z) &= -3x + y, \\ f_2(z) &= x - 3y; \\ z &= (x,y). \end{aligned}$$

Then we have

$$S = \{z \in V | f(z) \geq 0 \forall f \in \Gamma\}.$$

Take $k = (1,1) \in intS$ and $a = 0$. Then,

$$\xi_{k0}(x,y) = \begin{cases} \frac{1}{2}(3x - y), & x \geq y; \\ \frac{1}{2}(3y - x), & y \geq x. \end{cases}$$

Definition 2.3 *Let X_0 be a nonempty convex subset of X and $F : X \to 2^V$ a set-valued mapping.*

(i) *F is said to be properly S-quasiconvex on X_0 if, for any $x_1, x_2 \in X_0$ and $l \in [0, 1]$,*

$$\begin{array}{rrcl} \textit{either} & F(x_1) & \subset & F(lx_1 + (1-l)x_2) - S \\ \textit{or} & F(x_2) & \subset & F(lx_1 + (1-l)x_2) - S. \end{array}$$

(ii) *F is said to be naturally S-quasiconvex on X_0 if for any $x_1, x_2 \in X_0$ and $l \in [0, 1]$,*

$$F(lx_1 + (1-l)x_2) \subset co\{F(x_1) \bigcup F(x_2)\} - S,$$

where coA denotes the convex hull of A.

Remark.2.1 Definition 2.3 is a generalization of the concepts of proper S-quasiconvexity and natural quasiconvexity in [8] and [19]. Note that if $V = R$ and $S = R^+$, then both proper S-quasiconvexity and natural quasiconvexity are reduced to the ordinary quasiconvexity.

Theorem 2.1 *[13, Proposition 2.1] Let $X_0 \subset X$ and $Z_0 \subset Z$ be two nonempty, compact and convex sets. Assume that $F : X_0 \times Z_0 \to 2^R$ is a continuous set-valued mapping and, for each $(x, z) \in X_0 \times Z_0, F(x, z)$ is a compact set and F satisfies the following conditions:*

(i) *for each $x \in X_0, -F(x, .)$ is properly R^+-quasiconvex on Z_0;*

(ii) *for each $z \in Z_0, F(., z)$ is naturally R^+-quasiconvex on X_0;*

(iii) *for each $t \in Z_0$, there exists $x_t \in X_0$ such that*

$$MaxF(x_t, t) \le Max \bigcup_{z \in Z_0} Min_W \bigcup_{x \in X_0} F(x, z).$$

Then

$$Min \bigcup_{x \in X_0} Max_W \bigcup_{z \in Z_0} F(x, z) = Max \bigcup_{z \in Z_0} Min_W \bigcup_{x \in X_0} F(x, z).$$

Remark 2.2 Since X_0 is compact and $F(.,z)$ is compact-valued and u.s.c., $\bigcup_{x\in X_0} F(x,z)$ is compact. Thus, $\mathrm{Min}_W \bigcup_{x\in X_0} F(x,z)$ is well defined. In fact, in the case, it reduces to a single point for each z. By Lemma 1.1, $\mathrm{Min}_W \bigcup_{x\in X_0} F(x,z)$ is u.s.c. with respect to z. Next, $\bigcup_{z\in Z_0} \mathrm{Min}_W \bigcup_{x\in X_0} F(x,z)$ is compact and hence it admits a maximal point. Therefore, all $\mathrm{Max}, \mathrm{Min}, \mathrm{Max}_W$ and Min_W are well defined.

Note that Max and Max_W are the same in 2^R , and so are Min and Min_W . In the rest of paper, max (respectively, min) will be used instead of Max and Max_W (respectively, Min and Min_W) in 2^R

Remark 2.3 In [15], Theorem 2.1 is established, where condition (i) is replaced by the assumption that $F(x,.)$ is R^+-concave.

Lemma 2.3 *Let $F : X_0 \times Z_0 \to 2^V$ be a set-valued mapping, and let, for each $x \in X_0, F(x,.)$ be naturally S-quasiconvex on Z_0. Suppose that, for each $z \in Z_0, -F(.,z)$ is properly S-quasiconvex on X_0. Then, $\xi_{ka}(F(x,.))$ is naturally R^+-quasiconvex for any $x \in X_0$ and $-\xi_{ka}(F(.,z))$ is properly R^+-quasiconvex for any $z \in Z_0$.*

Proof. Take any $z_1, z_2 \in Z_0, \lambda \in [0,1]$ and $y \in F(x, \lambda z_1 + (1-\lambda)z_2)$. By the natural S-quasiconvexity of $F(x,.)$, there exist $y_i \in F(x,z_1)\bigcup F(x,z_2)$ and $\alpha_i \geq 0, i = 1,2,\dots n$, and $s \in S$ such that

$$\sum_{i=1}^{n} \alpha_i = 1,$$

and

$$y = \sum_{i=1}^{n} \alpha_i y_i - s.$$

Therefore,

$$\xi_{ka}(y) = \xi_{ka}(\sum_{i=1}^{n} \alpha_i y_i - s).$$

By Lemmas 2.1(vi)-(viii), we have

$$\xi_{ka}(y) \in \sum_{i=1}^{n} \alpha_i \xi_{ka}(y_i) - R^+ \subset co\{\xi_{ka}(F(x,z_1)) \bigcup \xi_{ka}(F(x,z_2))\} - R^+.$$

Thus, for each $x \in X_0, \xi_{ka}(F(x,.))$ is naturally R^+-quasiconvex. By the Lemma 2.1(vii) and proper S-quasiconvexity of $-F(.,z)$, it is clear that $-\xi_{ka}(F(.,z))$ is properly R^+-quasiconvex for any $z \in Z$. The proof is complete. □

Lemma 2.4 *Let X_0 and Z_0 be compact and convex subsets of X and Z, respectively. Let $F : X_0 \times Z \to 2^V$ be a continuous set-valued mapping with compact values. Suppose that $F(x, z)$ fulfills the following hypothesis:*

(H) *for any $u \in X_0$, there exists $v \in Z_0$ such that*

$$F(u, v) \subset Max \bigcup_{x \in X_0} Min_W \bigcup_{z \in Z_0} F(x, z) - S.$$

Then

$$max\ \xi_{ka}(F(u, v)) \leq max \bigcup_{x \in X_0} min \bigcup_{z \in Z_0} \xi_{ka}(F(x, z)).$$

Proof. By the condition (H), for any $u \in X_0$, there exists $v \in Z_0$ such that

$$F(u, v) \subset \mathrm{Max} \bigcup_{x \in X_0} \mathrm{Min}_W \bigcup_{z \in Z_0} F(x, z) - S.$$

Take any $y \in F(u, v)$. Then there exist $w \in \max \bigcup_{x \in X_0} \min \bigcup_{z \in Z_0} (F(x, z))$ and $s \in S$ such that

$$y = w - s.$$

By Lemma 2.1(vii) and (viii), one has

$$\xi_{ka}(y) \leq \xi_{ka}(w).$$

Thus,

$$\xi_{ka}(y) \leq \max \bigcup_{x \in X_0} \xi_{ka}(\mathrm{Min}_W \bigcup_{z \in Z_0} F(x, z))$$

It follows from Lemma 2.1(vii) that, for any $d \in \mathrm{Min}_W \bigcup_{z \in Z_0} F(x, z)$,

$$\xi_{ka}(d) = \min \bigcup_{z \in Z_0} \xi_{ka} F(x, z).$$

Therefore,

$$\xi_{ka}(y) \leq \max \bigcup_{x \in X_0} \min \bigcup_{z \in Z_0} \xi_{ka}(F(x, z)).$$

This completes the proof. □

Remark 2.4 Clearly, if F is a vector-valued mapping, then hypothesis (H) always holds. Therefore, for a vector-valued function $f(x, z)$, we always have that, for any $u \in X_0$, there exists $v \in Z_0$ such that

$$\max \xi_{ka}(f(u, v)) \leq \max \bigcup_{x \in X_0} \min \bigcup_{z \in Z_0} \xi_{ka}(f(x, z)).$$

3 MINIMAX THEOREMS FOR SET-VALUED MAPPINGS

In this section, we present two types of minimax theorems for set-valued mapping.

Theorem 3.1 *Let X_0 and Z_0 be compact and convex subsets of X and Z, respectively, and let $k \in intS$. Suppose that the following conditions are satisfied:*

(i) $F : X_0 \times Z_0 \to 2^V$ *is a continuous set-valued mapping with compact values;*

(ii) *for each* $x \in X_0, F(x, .)$ *is naturally* S*-quasiconvex on* Z_0*;*

(iii) *for each* $z \in Z_0, -F(., z)$ *is properly* S*-quasiconvex on* X_0*;*

(iv) *there exists an* $x_0 \in X_0$ *such that*

$$Min_W \bigcup_{z\in Z_0} F(x_0, z) \subset Min_W \bigcup_{z\in Z_0} F(x, z) + S \quad \forall x \in X_0;$$

(v) *for any* $u \in X_0$*, there exists* $v \in Z_0$ *such that*

$$F(u, v) \subset Max \bigcup_{x\in X_0} Min_W \bigcup_{z\in Z_0} F(x, z) - S.$$

Then

$$Min_W \bigcup_{z\in Z_0} F(x_0, z) \subset Min \bigcup_{z\in Z_0} Max_W \bigcup_{x\in X_0} F(x, z) + S. \tag{4}$$

Furthermore, if

(vi) $Max \bigcup_{x\in X_0} Min_W \bigcup_{z\in Z_0} F(x, z) \subset Min_W \bigcup_{z\in Z_0} F(x, z) + S, \quad \forall x \in X_0,$

then

$$Max \bigcup_{x\in X_0} Min_W \bigcup_{z\in Z_0} F(x, z) \subset Min \bigcup_{z\in Z_0} Max_W \bigcup_{x\in X_0} F(x, z) + S. \tag{5}$$

Proof. Set $\Gamma(z) = \mathrm{Max}_W \bigcup_{x\in X_0} F(x, z)$. Suppose $\alpha \in V$ and $\alpha \notin \Gamma(Z_0) + S$, i.e., $\Gamma(Z_0) \bigcap (\alpha - S) = \emptyset$. By Lemma 2.1, $\xi_{k\alpha}$ is continuous, convex, strictly monotone and

$$\xi_{k\alpha}(\beta) > 0 \quad \forall \beta \in \Gamma(Z_0). \tag{6}$$

Consider the set-valued mapping:

$$G = \xi_{k\alpha}(F) : X_0 \times Z_0 \to 2^R.$$

It is clear that all conditions in Theorem 2.1 are satisfied for this mapping, and hence we have

$$\min \bigcup_{z\in Z_0} \max \bigcup_{x\in X_0} G(x,z) = \max \bigcup_{x\in X_0} \min \bigcup_{z\in Z_0} G(x,z). \tag{7}$$

By the continuity of $\xi_{k\alpha}$ and $F(.,z)$ and Proposition 6 of Ch.3, Sec.1 in [1], we have that $G(.,z) = \xi_{k\alpha}(F(.,z))$ is upper semicontinuous. Since $\xi_{k\alpha}$ is a scalar-valued continuous function, $G(.,z)$ also is lower semicontinuous. Therefore, $G(.,z) = \xi_{k\alpha}(F(.,z))$ is continuous for each $z \in Z_0$. By the compactness of X_0, there exist $x_z \in X_0$ and $y_z \in F(x_z, z)$ such that

$$\xi_{k\alpha}(y_z) = \max \bigcup_{x\in X_0} \xi_{k\alpha}(F(x,z)).$$

By Lemma 2.1(vii), we have

$$y_z \in \Gamma(z) = \mathrm{Max}_W \bigcup_{x\in X_0} F(x,z).$$

Hence, it follows from (6) that, for each $z \in Z_0$,

$$\max \bigcup_{x\in X_0} G(x,z) = \xi_{k\alpha}(y_z) > 0.$$

Thus,

$$\min \bigcup_{z\in Z_0} \max \bigcup_{x\in X_0} G(x,z) > 0.$$

By (7),

$$\max \bigcup_{x\in X_0} \min \bigcup_{z\in Z_0} G(x,z) > 0.$$

By Lemma 1.1, $\min \bigcup_{z\in Z_0} G(.,z)$ is u.s.c. on X_0. Thus, by the compactness of X_0, there exists an $x' \in X_0$ such that

$$\min \bigcup_{z\in Z_0} G(x',z) > 0.$$

By Lemma 2.1(iv), we have

$$y \notin \alpha - S \quad \forall y \in F(x', z) \text{ and } z \in Z_0.$$

Hence,

$$\alpha \notin \mathrm{Min}_W \bigcup_{z \in Z_0} F(x', z) + S. \tag{8}$$

If

$$\alpha \in \mathrm{Min}_W \bigcup_{z \in Z_0} F(x_0, z),$$

then, by (iv), we have

$$\alpha \in \mathrm{Min}_W \bigcup_{z \in Z_0} F(x, z) + S \forall x \in X_0,$$

which contradicts (8). Thus, $\alpha \in \mathrm{Min}_W \bigcup_{z \in Z_0} F(x_0, z)$ implies

$$\alpha \in \bigcup_{z \in Z_0} \mathrm{Max}_W \bigcup_{x \in X_0} F(x, z) + S.$$

Since $\bigcup_{z \in Z_0} \mathrm{Max}_W \bigcup_{x \in X_0} F(x, z)$ is a compact set, it follows from Lemma 1.2 that

$$\mathrm{Min}_W \bigcup_{z \in Z_0} F(x_0, z) \subset \mathrm{Min} \bigcup_{z \in Z_0} \mathrm{Max}_W \bigcup_{x \in X_0} F(x, z) + S.$$

Thus (4) holds. Clearly, by (vi), (5) holds. □

Remark 3.1 The condition (iv) is similar to the one used in [8]. Clearly, this condition holds if F is a scalar set-valued mapping.

Remark 3.2 Suppose that X and Z are two metric spaces and V is R^p. Assuming that the following conditions are satisfied:

(i) $F : X_0 \times Z_0 \to 2^V$ is a continuous set-valued mapping with compact values;

(ii) for each $x \in X_0, F(x, .)$ is S-convex on Z_0;

(iii) for each $z \in Z_0, F(., z)$ is naturally S-quasiconvex on X_0;

(iv) there exists an $x_0 \in X_0$ such that

$$\mathrm{Min}_W \bigcup_{z \in Z_0} F(x_0, z) \subset \mathrm{Min}_W \bigcup_{z \in Z_0} F(x, z) + S \quad \forall x \in X_0;$$

(v) for any $t \in Z_0$, there exists $x_t \in X_0$ such that

$$F(x_t, t) - \mathrm{Min} \bigcup_{z \in Z_0} \mathrm{Max}_W \bigcup_{x \in X_0} F(x, z) \subset S.$$

Li et al (see Theorem 3.2 in [15]) established that

$$\mathrm{Min}_W \bigcup_{z \in Z_0} F(x_0, z) \subset \mathrm{Min} \left\{ co \left(\bigcup_{z \in Z_0} \mathrm{Max}_W \bigcup_{x \in X_0} F(x, z) \right) \right\} + S.$$

It follows readily that Theorem 3.2 of [15] and Theorem 3.1 obtain a similar result under the different assumption conditions.

Corollary 3.1 *Let X_0 and Z_0 be compact and convex subsets of X and Z, respectively, and let $k \in intS$. Suppose that the following conditions are satisfied:*

(i) *$f : X_0 \times Z_0 \to V$ is a continuous vector-valued mapping;*

(ii) *for each $x \in X_0, f(x, .)$ is naturally S-quasiconvex on Z_0;*

(iii) *for each $z \in Z_0, -f(., z)$ is properly S-quasiconvex on X_0; and*

(iv) *there exists an $x_0 \in X_0$ such that*

$$Min_W \bigcup_{z \in Z_0} f(x_0, z) \subset Min_W \bigcup_{z \in Z_0} f(x, z) + S \quad \forall x \in X_0.$$

Then

$$Min_W \bigcup_{z \in Z_0} f(x_0, z) \subset Min \bigcup_{z \in Z_0} Max_W \bigcup_{x \in X_0} f(x, z) + S. \tag{9}$$

Furthermore, if

(v) $Max \bigcup_{x \in X_0} Min_W \bigcup_{z \in Z_0} f(x, z) \subset Min_W \bigcup_{z \in Z_0} f(x, z) + S \quad \forall x \in X_0,$

then

$$Max \bigcup_{x \in X_0} Min_W \bigcup_{z \in Z_0} f(x, z) \subset Min \bigcup_{z \in Z_0} Max_W \bigcup_{x \in X_0} f(x, z) + S. \tag{10}$$

Proof. Since f is a vector-valued mapping, it is clear that Theorem 3.1 (iv) holds. Thus, by Theorem 3.1, the conclusion follows readily. $\square$

Remark 3.3 If $f(x,.)$ is S-convex for every $x \in X_0$, then it is clear that $f(x,.)$ is naturally S-quasiconvex for every $x \in X_0$. However, the converse is not valid. Thus, Theorem 2.1 (i) of [9] is a special case of Corollary 3.1, which solves a part of the open problem in [9].

Example 3.1 Let $X_0 = [0,1], Z_0 = [0,1]$,

$$f(x,z) = \{(x,y) \in R^2 | y = 1 - (z-1)^2\},$$

and

$$S = \{(x,y) \in R^2 | x \geq 0, y \geq 0\}.$$

Then $f(x,.)$ is naturally S-quasiconvex for every $x \in X$ and $-f(.,z)$ is properly S-quasiconvex for every $z \in Z_0$. Nevertheless, $f(x,.)$ is not S-convex for every $x \in X_0$. Therefore, we cannot claim that (10) holds by Theorem 2.1 (i) of [9]. However, for any $x \in X_0$, we have

$$\mathrm{Min}_W \bigcup_{z \in Z_0} f(x,z) = \{(x,y) \in R^2 | y = 1 - (z-1)^2, z \in [0,1]\}.$$

Take $x_0 = 1 \in X_0$. We have

$$\mathrm{Min}_W \bigcup_{z \in Z_0} f(x_0,z) = \{(1,y) \in R^2 | y = 1 - (z-1)^2, z \in [0,1]\}.$$

Then condition (iv) in Corollary 3.1 holds:

$$\mathrm{Min}_W \bigcup_{z \in Z_0} f(x_0,z) \subset \mathrm{Min}_W \bigcup_{z \in Z_0} f(x,z) + S, \quad \forall x \in X_0.$$

Thus, all conditions of Corollary 3.1 hold. So, the inclusion (9) holds:

$$\mathrm{Min}_W \bigcup_{z \in Z_0} f(x_0,z) \subset \mathrm{Min} \bigcup_{z \in Z_0} \mathrm{Max}_W \bigcup_{x \in X_0} f(x,z) + S.$$

Furthermore, we also have

$$\begin{aligned} \mathrm{Max} \bigcup_{x \in X_0} \mathrm{Min}_W \bigcup_{z \in Z_0} f(x,z) &= \mathrm{Min}_W \bigcup_{z \in Z_0} f(x_0,z) \\ &= \{(1,y) \in R^2 | y = 1 - (z-1)^2, z \in [0,1]\}. \end{aligned}$$

Then condition (v) in Corollary 3.1 holds:

$$\mathrm{Max} \bigcup_{x \in X_0} \mathrm{Min}_W \bigcup_{z \in Z_0} f(x,z) \subset \mathrm{Min}_W \bigcup_{z \in Z_0} f(x,z) + S, \quad \forall x \in X_0.$$

Thus, all conditions of Corollary 3.1 hold. So, the inclusion (10) holds:

$$\mathrm{Max} \bigcup_{x \in X_0} \mathrm{Min}_W \bigcup_{z \in Z_0} f(x,z) \subset \mathrm{Min} \bigcup_{z \in Z_0} \mathrm{Max}_W \bigcup_{x \in X_0} f(x,z) + S.$$

Indeed,

$$\mathrm{Min}_W \bigcup_{z \in Z_0} f(x_0,z) = \{(1,y) \in R^2 | y = 1 - (z-1)^2, z \in [0,1]\},$$

$$\begin{aligned} \mathrm{Max} \bigcup_{x \in X_0} \mathrm{Min}_W \bigcup_{z \in Z_0} f(x,z) &= \mathrm{Min}_W \bigcup_{z \in Z_0} f(x_0,z) \\ &= \{(1,y) \in R^2 | y = 1 - (z-1)^2, z \in [0,1]\}, \end{aligned}$$

$$\mathrm{Min} \bigcup_{z \in Z_0} \mathrm{Max}_W \bigcup_{x \in X_0} f(x,z) = \{(0,0)\},$$

and

$$\mathrm{Min} \bigcup_{z \in Z_0} \mathrm{Max}_W \bigcup_{x \in X_0} f(x,z) + S = S.$$

Hence

$$\mathrm{Min}_W \bigcup_{z \in Z_0} f(x_0,z) \subset \mathrm{Min} \bigcup_{z \in Z_0} \mathrm{Max}_W \bigcup_{x \in X_0} f(x,z) + S$$

and

$$\mathrm{Max} \bigcup_{x \in X_0} \mathrm{Min}_W \bigcup_{z \in Z_0} f(x,z) \subset \mathrm{Min} \bigcup_{z \in Z_0} \mathrm{Max}_W \bigcup_{x \in X_0} f(x,z) + S.$$

Theorem 3.2 *Let X_0 and Z_0 be compact and convex subsets in X and Z, respectively, and let $k \in intS$. Suppose that the following conditions are satisfied:*

(i) *$F : X_0 \times Z_0 \to 2^V$ is a continuous set-valued mapping with compact values;*

(ii) *for each $x \in X_0, -F(x,.)$ is properly S-quasiconvex on Z_0;*

(iii) *for each $z \in Z_0, F(.,z)$ is naturally S-quasiconvex on X_0;*

(iv) *for any $u \in X_0$, there exists $v \in Z_0$ such that*

$$F(u,v) \subset Max \bigcup_{x \in X_0} Min_W \bigcup_{z \in Z_0} F(x,z) - S.$$

Then

$$\mathit{Min} \bigcup_{x \in X_0} \mathit{Max}_W \bigcup_{z \in Z_0} F(x,z) \subset \mathit{Max} \bigcup_{z \in Z_0} \mathit{Min}_W \bigcup_{x \in X_0} F(x,z) + V \backslash (S \backslash \{0\}). \tag{11}$$

Proof. Set

$$L(x) = \mathrm{Max}_W \bigcup_{z \in Z_0} F(x,z),$$

and let

$$y_0 \in \mathrm{Min} \bigcup_{x \in X_0} \mathrm{Max}_W \bigcup_{z \in Z_0} F(x,z) = \mathrm{Min}\ L(X_0).$$

By the definition of a minimal point, we have

$$(L(X_0) - y_0) \bigcap (-S) = \{0\}.$$

That is,

$$(L(X_0) \backslash \{y_0\}) \bigcap (y_0 - S) = \emptyset.$$

By Lemma 2.1 (iii)-(iv), we have

$$\xi_{ky_0}(y) > 0 \quad \forall y \in L(X_0) \backslash \{y_0\}, \tag{12}$$

and

$$\xi_{ky_0}(y_0) = 0. \tag{13}$$

Let $x \in X_0$. By the continuity of ξ_{ky_0} and $F(x,.)$ and the compactness of Z_0, there exist $z_x \in Z_0$ and $y_1 \in F(x, z_x)$ such that

$$\max \bigcup_{z \in Z_0} \xi_{ky_0}(F(x,.)) = \xi_{ky_0}(y_1).$$

By Lemma 2.1 (vii), we have

$$y_1 \in L(x).$$

By (12) and (13),

$$\max \bigcup_{z \in Z_0} \xi_{ky_0}(F(x,.)) \geq 0. \tag{14}$$

Since x is any element of X_0, (14) implies that

$$\min \bigcup_{x \in X_0} \max \bigcup_{z \in Z_0} \xi_{ky_0}(F(x,.)) \geq 0. \tag{15}$$

Consider the set-valued mapping

$$G = \xi_{ky_0}(F) : X_0 \times Z_0 \to 2^R.$$

We see that all conditions of Theorem 2.1 are satisfied for G, and hence we have

$$\min \bigcup_{x \in X_0} \max \bigcup_{z \in Z_0} G(x, z) = \max \bigcup_{z \in Z_0} \min \bigcup_{x \in X_0} G(x, z).$$

So, there exist $x_0 \in X_0, z_0 \in Z_0$ and $y_2 \in F(x_0, z_0)$ such that

$$\begin{aligned} \min \bigcup_{x \in X_0} \max \bigcup_{z \in Z_0} G(x, z) &= \max \bigcup_{z \in Z_0} G(x_0, z) \\ &= \max \bigcup_{z \in Z_0} \min \bigcup_{x \in X_0} G(x, z) \\ &= \min \bigcup_{x \in X_0} G(x, z_0) = \xi_{ky_0}(y_2). \end{aligned} \tag{16}$$

Therefore, by (16) and Lemma 2.1(vii), we have

$$y_2 \in Max_W \bigcup_{z \in Z_0} F(x_0, z) = L(x_0), \tag{17}$$

and

$$y_2 \in \mathrm{Min}_W \bigcup_{x \in X_0} F(x, z_0).$$

By (15) and (16), we get

$$\xi_{ky_0}(y_2) \geq 0.$$

If $y_0 = y_2$, then,

$$y_0 \notin y_2 + S \backslash \{0_V\}. \tag{18}$$

If $y_0 \neq y_2$, then, by (12) and (17), we get

$$\xi_{ky_0}(y_2) > 0.$$

By Lemma 2.1(iv), we have

$$y_2 \notin y_0 - S,$$

i.e.,

$$y_0 \notin y_2 + S \backslash \{0_V\}. \tag{19}$$

¿From (18) and (19) we get

$$\begin{aligned} y_0 &\in y_2 + V\backslash(S\backslash\{0_V\}) \\ &\subset \mathrm{Min}_W \bigcup_{x\in X_0} F(x,z_0) + V\backslash(S\backslash\{0_V\}) \\ &\subset \bigcup_{z\in Z_0} \mathrm{Min}_W \bigcup_{x\in X_0} F(x,z) + V\backslash(S\backslash\{0_V\}). \end{aligned}$$

Since $F(.,.)$ is continuous and X_0 and Z_0 are compact, it follows from Lemma 1.2 that

$$\bigcup_{z\in Z_0} \mathrm{Min}_W \bigcup_{x\in X_0} F(x,z) \subset Max \bigcup_{z\in Z_0} \mathrm{Min}_W \bigcup_{x\in X_0} F(x,z) - S.$$

Thus,

$$\begin{aligned} y_0 &\in Max \bigcup_{z\in Z_0} \mathrm{Min}_W \bigcup_{x\in X_0} F(x,z) - S + V\backslash(S\backslash\{0_V\}) \\ &= Max \bigcup_{z\in Z_0} \mathrm{Min}_W \bigcup_{x\in X_0} F(x,z) + V\backslash(S\backslash\{0_V\}). \end{aligned}$$

Hence, inclusion (11) holds. This completes the proof. □

Corollary 3.2 *Let X_0 and Z_0 be compact and convex subsets in X and Z, respectively, and let $k \in intS$. Suppose that the following conditions are satisfied:*

(i) *$f : X_0 \times Z_0 \to V$ is a continuous vector-valued mapping;*

(ii) *for each $x \in X_0, -f(x,.)$ is properly S-quasiconvex on Z_0; and*

(iii) *for each $z \in Z_0, f(.,z)$ is naturally S-quasiconvex on X_0.*

Then

$$Min \bigcup_{x\in X_0} Max_W \bigcup_{z\in Z_0} f(x,z) \subset Max \bigcup_{z\in Z_0} Min_W \bigcup_{x\in X_0} f(x,z) + V\backslash(S\backslash\{0_V\}). \quad (20)$$

Proof. Since f is a vector-valued mapping, part (iv) of Theorem 3.2 holds. Thus, the conclusion follows readily. □

Remark 3.4 Assuming that the following conditions are satisfied:

(i) S has a compact base;

(ii) for each $z \in Z_0, f(., z)$ is S-convex on X_0; and

(iii) for each $x \in X_0, -f(x, .)$ is properly S-quasiconvex on Z_0,

Li and Wang [16] established that

$$\operatorname{Min}_P \bigcup_{x\in X_0} Max \bigcup_{z\in Z_0} f(x,z) \subset Max \bigcup_{z\in Z_0} \operatorname{Min}_W \bigcup_{x\in X_0} f(x,z) + V\backslash(S\backslash\{0_V\}), \quad (21)$$

where $\operatorname{Min}_P A$ denotes the set of all Benson properly S-minimal points of A. Obviously, if, for any $x \in X_0, Max f(x, Z_0) = Max_W f(x, Z_0)$, then we have

$$\operatorname{Min}_P \bigcup_{x\in X_0} Max \bigcup_{z\in Z_0} f(x,z) \subset \operatorname{Min} \bigcup_{x\in X_0} Max_W \bigcup_{z\in Z_0} f(x,z).$$

Therefore, when $Max f(x, Z_0) = Max_W f(x, Z_0)$ for any $x \in X_0$, Theorem 1 of [16] is a special case of Corollary 3.2.

Example 3.2 Let $X_0 = [0,1], Z_0 = [0,1]$,

$$f(x,z) = \{(yz, yz) \in R^2 | y = 1-(x-1)^2\}, x \in X_0, z \in Z_0,$$

and

$$S = \{(u,v) \in R^2 | u \geq 0, v \geq 0\}.$$

Then $f(., z)$ is naturally S-quasiconvex for every $z \in Z$ and $-f(x, .)$ is properly S-quasiconvex for every $x \in X_0$. Thus, all conditions of Corollary 3.2 hold. So, the inclusion (20) holds:

$$\operatorname{Min} \bigcup_{x\in X_0} \operatorname{Max}_W \bigcup_{z\in Z_0} f(x,z) \subset \operatorname{Max} \bigcup_{z\in Z_0} \operatorname{Min}_W \bigcup_{x\in X_0} f(x,z) + V\backslash(S\backslash\{0_V\}).$$

Indeed,

$$\operatorname{Min} \bigcup_{x\in X_0} \operatorname{Max}_W \bigcup_{z\in Z_0} f(x,z) = \{(0,0)\},$$
$$\operatorname{Max} \bigcup_{z\in Z_0} \operatorname{Min}_W \bigcup_{x\in X_0} f(x,z) = \{(0,0)\},$$

and

$$\operatorname{Max} \bigcup_{z\in Z_0} \operatorname{Min}_W \bigcup_{x\in X_0} f(x,z) + V\backslash(S\backslash\{0_V\}) =$$
$$\{(u,v) \in R^2 | u \geq 0, v < 0, \text{ or } u < 0, v \geq 0, \text{ or } u \leq 0, v \leq 0\}.$$

Thus,

$$\mathrm{Max} \bigcup_{x\in X_0} \mathrm{Min}_W \bigcup_{z\in Z_0} f(x,z) \subset \mathrm{Min} \bigcup_{z\in Z_0} \mathrm{Max}_W \bigcup_{x\in X_0} f(x,z) + V\backslash(S\backslash\{0_V\}).$$

Furthermore, for any $x \in X_0$, we have

$$\mathrm{Max} \bigcup_{z\in Z_0} f(x,z) = \{(x,y)|y = 1-(x-1)^2\},$$

$$\mathrm{Max}_W \bigcup_{z\in Z_0} f(x,z) = \{(x,y)|y = 1-(x-1)^2\},$$

and

$$\mathrm{Max} \bigcup_{z\in Z_0} f(x,z) = \mathrm{Max}_W \bigcup_{z\in Z_0} f(x,z).$$

Thus, it follows from Remark 3.2 that (21) holds. However, $f(.,z)$ is not S-convex for every $x \in X_0\backslash\{0\}$. Therefore, we cannot claim that (21) holds by Theorem 1 of [16].

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