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Frequency-dependent Effective Hydraulic Conductivity in Randomly Layered Porous Medium - Strong Contrast Approximation

E. Caspari* (CO2CRC / Curtin University), T.M. Müller (CSIRO) & B. Gurevich (CO2CRC / Curtin University / CSIRO)

SUMMARY

Upscaling of transport properties of heterogeneous porous rocks, such as hydraulic conductivity, is controlled not only by the spatial heterogeneity, but also by the temporal variations of the flow field. Thus effective hydraulic conductivity and permeability of a heterogeneous porous medium is frequency-dependent. Existing upscaling methods for this property are often limited to small spatial variations of permeability, which is a significant limitation. In this study we apply a strong contrast expansion method, originally suggested in the context of the dynamic dielectric constant, to the problem of upscaling hydraulic conductivity in a 1D random medium. A closed-form expression for the effective hydraulic conductivity, which depends on the second order statistics, is derived. Unlike the small contrast approximations, in the low- and high-frequency limits this new expression yields harmonic and arithmetic averages of permeability, respectively, which are known to be exact low- and high-frequency limits in 1D.

Introduction

Upscaling physical properties of heterogeneous rocks is important for reservoir characterisation. Backus averaging for effective seismic velocities in a layered elastic medium is an example of a widely used up-scaling technique (Backus, 1962). However, even in a simple approximation, reservoir rocks should be modelled as two-phase composites consisting of an elastic frame and a fluid phase. If this fluid phase is continuous, then the fluid has the ability to flow and thus introduces a time-dependent process. Consequently, proper upscaling of reservoir rocks should account for this time dependency and gives rise to dynamic-equivalent, i.e. frequency-dependent transport properties. For porous rocks, existing methods for the evaluation of dynamic-equivalent transport properties are typically limited to low-contrast heterogeneities (Sanchez-Vila, 2006). In this paper we explore an approach that can potentially be used for high-contrast. For simplicity, we focus on upscaling the hydraulic conductivity of a 1D random medium.

Computing an effective hydraulic conductivity is an integral part for the characterisation of fluid flows in heterogeneous, porous media. Natural porous media are heterogeneous on many length scales such as the pore, formation and regional scales. Here we focus on the formation scale which is sometimes referred to as mesoscopic heterogeneities. This means that its characteristic length scale is much larger than its pore scale but much smaller than its regional scale, which is a proxy for the volume of investigation such as the sample size or the seismic wavelength. If, in addition to this spatial heterogeneity, the temporal variation of the flow field is of importance, then the effective conductivity becomes a time-dependent, or equivalently a frequency-dependent quantity (Dagan, 1982; Indelman, 1996). Oscillating, or generally transient flows arise in hydrogeology in a number of contexts. On a much shorter time scale, oscillating flows can be seismic wave induced and are known to affect seismic signals (Müller et al., 2007).

Previous studies of frequency-dependent, hydraulic conductivity include averaging Darcy's law (Indelman, 1996) and upscaling Biot's equation of poroelasticity (Müller and Gurevich, 2006). These results are based on weak-contrast perturbation theories and therefore limit their application to materials with weak conductivity fluctuations. However, contrasts in permeability are usually very large, and thus high contrasts are particularly important for hydraulic conductivity problems. Improved perturbation theories have been reported by Hristopulos and Christakos (1997). Our analysis is based on a pore-pressure diffusion equation, derived from Biot's equation of poroelasticity, in a randomly layered porous medium. We apply a strong-contrast expansion method for a two-component, composite random medium, suggested earlier in the context of the dynamic-equivalent dielectric constant (Rechtsman and Torquato, 2008).

Effective hydraulic conductivity - strong contrast approximation

Time-dependent flow in fluid-saturated porous media can be analysed by pore pressure diffusion equations. Various methods exist to estimate an effective hydraulic conductivity for quasistatic flow based on an averaged Darcy's law. Chandler and Johnson (1981) have shown that an equivalent pressure diffusion equation is obtained from Biot's equations of poroelasticity in the quasistatic limit. This pore pressure p diffusion equation is given in the frequency-domain by

$$\frac{d^2 p(x)}{dx^2} + k_0^2 p(x) = 0, \tag{1}$$

where k_0 is the slow wavenumber of the homogeneous medium, defined as $k_0 = \sqrt{\frac{i\omega}{\kappa_0 N}}$ with a hydraulic conductivity κ_0 and a poroelastic parameter N. N preliminary depends on the fluid compressibility and porosity of the medium.

Random medium

In a heterogeneous medium, the medium parameters are functions of position and therefore they can be seen as random parameters. Here we consider only inhomogeneities in hydraulic conductivity. For the following analysis, a simple one-dimensional, two-component medium is chosen. The composite

medium consists of alternating layers of random thickness with hydraulic conductivities κ_1 and κ_2. The local conductivity of the medium is given by $\kappa(x) = \kappa_1 I^{(1)}(x) + \kappa_2 I^{(2)}(x)$, where $I^{(1)}(x)$ and $I^{(2)}(x)$ are indicator functions for component one and two, respectively. The indicator function $I^{(j)}(x)$ for a component j can be written as

$$I^{(j)}(x) = \begin{cases} 1 & \text{x in } \Omega_j \\ 0 & \text{otherwise,} \end{cases}$$

where the domain Ω_j is occupied by the component j. Averaging the indicator functions lead to n-point correlation functions $S_n^{(j)} = \langle I^{(j)}(x_1) I^{(j)}(x_2)....I^{(j)}(x_n) \rangle$ which contain micro-structural information of the random heterogeneous medium (Torquato, 2002). In the following, we take only the one-point $S_1^{(j)}$ and two-point correlation functions $S_2^{(j)}$ into account. $S_1^{(j)}$ gives the probability that a randomly chosen point belongs to component j, which is equivalent to the volume concentration of this component, whereas $S_2^{(j)}$ reflects the extent to which two points are correlated in the system.
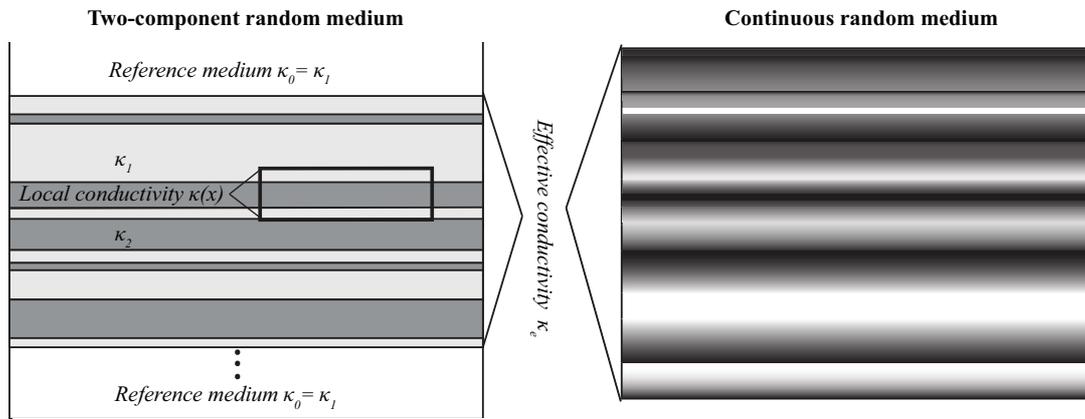

Two-component random medium

Reference medium $\kappa_0 = \kappa_1$

κ_1

Local conductivity $\kappa(x)$

κ_2

Reference medium $\kappa_0 = \kappa_1$

Effective conductivity κ_e

Continuous random medium

Figure 1 *Comparison between a discrete two-component and a continuous random medium*

Strong contrast expansion

To derive an approximation for high contrasts, we utilise the strong contrast expansion of Rechtsman and Torquato (2008). The method is based on the assumption that the composite medium is embedded in an infinite reference medium. Hence the composite medium causes perturbations with respect to the reference medium, which can be expressed by a perturbing operator

$$\tilde{L} = \tilde{k}^2 = k^2(x) - k_0^2 = \frac{i\omega}{N}\left(\frac{1}{\kappa_j} - \frac{1}{\kappa_0}\right) I^{(j)}(x) = \frac{i\omega}{\kappa_0 N}\left(\frac{\kappa_0 - \kappa_j}{\kappa_j}\right) I^{(j)}(x),$$

where the conductivity κ_j is either associated with component one or two and κ_0 is the reference medium. In general, the reference medium is a free parameter in the theory. For simplicity we choose the reference medium to be component one. Introducing this perturbing operator as a source term in equation (1) leads to the differential equation for the randomly layered medium,

$$\frac{d^2 p(x)}{dx^2} + k_0^2 p(x) = \tilde{L}p(x). \tag{2}$$

The formal solution of equation (2) can be written as an integral equation

$$p(x) = p_0(x) + \int G(x, x') P(x') dx', \tag{3}$$

where p_0 is the solution of the reference medium, $G(x, x')$ is the Green's function of the infinite medium, defined as $G = (e^{ik_0|x|})/(2k_0 i)$ and $P(x')$ is the so-called polarisation field, given by

$$P(x') = \tilde{L}p(x'). \tag{4}$$

In order to find an expression for an effective perturbing operator L_e, or equivalently, an effective hydraulic conductivity κ_e, we seek relationships between the averaged polarisation $\langle P \rangle$ and pressure $\langle p \rangle$ fields. A relationship of this form can be found by ensemble averaging of equation (4) which implicitly defines an effective perturbing operator

$$\langle P(x') \rangle = L_e \langle p(x') \rangle, \quad L_e = \frac{i\omega}{\kappa_0 N} \left(\frac{\kappa_0 - \kappa_e}{\kappa_e} \right). \tag{5}$$

A second relationship is obtained by solving equation (3) for the polarisation field in iterations

$$P = \tilde{L} p_0 + \tilde{L} G P = \tilde{L} p_0 + \tilde{L} G \tilde{L} p_0 ... = S p_0, \tag{6}$$

where the integral equation is written in operator form and S is a scattering series. Introducing the scattering series back into equation (3) and ensemble averaging, then yields

$$\langle p \rangle = [\langle S \rangle^{-1} + G] \langle P \rangle = \frac{\langle P(x) \rangle}{\langle \tilde{L}(x) \rangle} - \int \frac{\langle \tilde{L}(x)\tilde{L}(x') \rangle - \langle \tilde{L}(x) \rangle \langle \tilde{L}(x') \rangle}{\langle \tilde{L}(x) \rangle \langle \tilde{L}(x') \rangle} \langle P(x,x') \rangle G(x,x') ... \tag{7}$$

Note that the solution p_0 of the reference medium is eliminated from the final equation. A comparison between these two relationships equations (5) & (7), or more precisely of $[\langle S \rangle^{-1} + G]$ and L_e^{-1} results in the searched-for expression of the effective hydraulic conductivity

$$\kappa_e = \frac{\phi_2 \kappa_0 - L' A_2 \kappa_0}{L' \phi_2^2 + \phi_2 - L' A_2}, \quad L' = \frac{\kappa_0 - \kappa_2}{\kappa_2}, \quad A_2 = k_0^2 \int_{-\infty}^{\infty} (S_2 - \phi_2^2) G(x,x') dx'. \tag{8}$$

Only the one- and two-point correlation functions are considered and S_2 denotes the two-point correlation function associated with component 2.

Comparison between weak and high contrast perturbation methods

The underlying medium is a principal difference between the strong contrast expansion and the perturbation method applied by Müller and Gurevich (2006). While the first approach is based on a discrete, two-component random medium, the second is based on a continuous random medium (Figure 1). However, the two methods are comparable if equivalent values for the medium parameters are chosen.
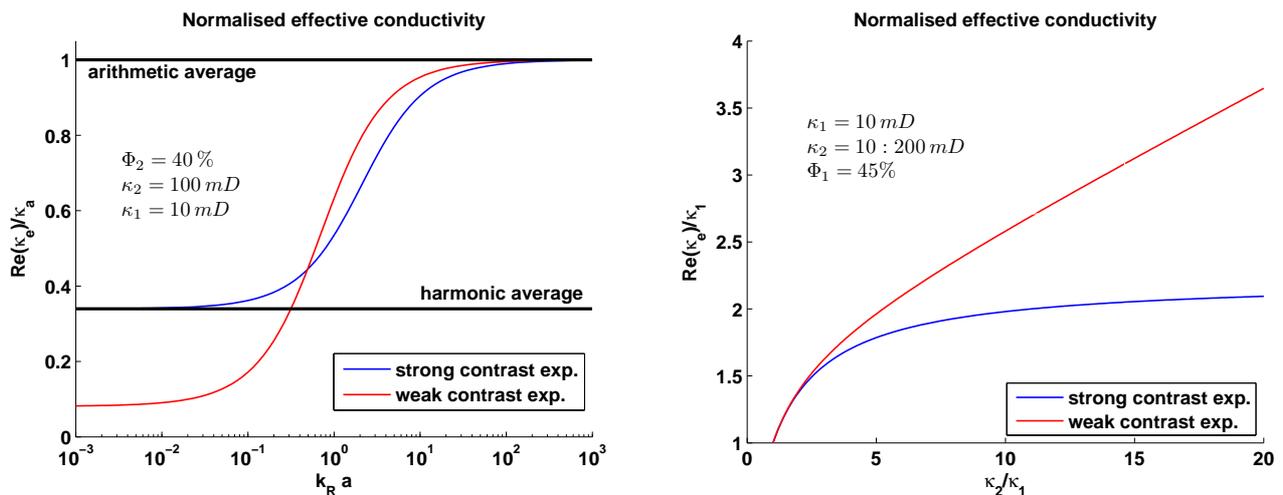

Figure 2 *a) Normalised effective conductivity vs dimensionless frequency $k_R a$, where k_R is the real part of k_0 and $a = 0.01\,m$; b) Normalised effective conductivity vs conductivity contrast ($f = 100\,Hz$)*

In order to compare the two approximations we choose an exponential correlation function $e^{(-|x/a|)}$, where a is the correlation length, describing a characteristic thickness of the layers. For this correlation

function the expressions of the effective hydraulic conductivity can be written as

$$\kappa_e = \kappa_a \left(1 - \sigma^2 + \frac{\sigma^2 k_0 a}{k_0 a + 1i}\right) \qquad \text{Müller and Gurevich (2006),} \qquad (9)$$

$$\kappa_e = \frac{\kappa_2 \kappa_h + \kappa_a \kappa_h k_0 a}{\kappa_2 + \kappa_h k_0 a} \qquad \text{Strong contrast expansion,} \qquad (10)$$

where κ_a, κ_h and σ^2 are the arithmetic average, harmonic average and variance of κ, respectively.

Figure 2a illustrates the frequency dependence of the effective hydraulic conductivity. In 1D the effective hydraulic conductivity is bounded by the harmonic average in the low-frequency limit and by the arithmetic average in the high-frequency limit (Indelman, 1996). It can be observed that the weak contrast expansion violates the low-frequency limit, while the strong contrast expansion coincides with the exact bounds. In fact, the strong contrast expansion converges to the bounds for arbitrary contrasts. This can be shown by the frequency limits of equation (10)

$$\kappa_e(\omega \to 0) = \kappa_h, \quad \kappa_e(\omega \to \infty) = \kappa_a,$$

which agree with the bounds. Conversely the low-frequency limit of equation (9) yields the lower bound for weak contrasts only (Müller et al., 2007).

In Figure 2b the effective hydraulic conductivity is plotted against the conductivity contrast ratio for a frequency of 100 Hz and a correlation length of 0.01 m. For low contrast the two approximations are in good agreement as expected, but afterwards they increasingly diverge.

Conclusions

The application of the strong-contrast expansion method of Rechtsman and Torquato (2008) yields a closed-form expression for the effective hydraulic conductivity in a randomly layered medium, which depends on the second order statistics. We find that this method coincides with the exact bounds in a 1D medium for arbitrarily high contrasts. Furthermore the method fulfills these bounds for any choice of the reference medium in 1D.

Acknowledgements

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