

On forward and backward SPDEs with non-local boundary conditions

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Abstract

We study linear stochastic partial differential equations of parabolic type with non-local in time or mixed in time boundary conditions. The standard Cauchy condition at the terminal time is replaced by a condition that mixes the random values of the solution at different times, including the terminal time, initial time and continuously distributed times. For the case of backward equations, this setting covers almost surely periodicity. Uniqueness, solvability and regularity results for the solutions are obtained. Some possible applications to portfolio selection are discussed.

Keywords: SPDEs, backward SPDEs, periodic conditions, non-local conditions, portfolio selection.

1 Introduction

Stochastic partial differential equations (SPDEs) are well studied in the existing literature for the case of Cauchy boundary conditions at the initial time or at the terminal time. Forward parabolic SPDEs are usually considered with a Cauchy condition at initial time, and backward parabolic SPDEs are usually considered with a Cauchy condition at terminal time. A backward SPDE cannot be transformed into a forward equation by a simple time change. Usually, a backward SPDE is solvable in the sense that there exists a diffusion term being considered as a

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part of the solution that helps to ensure that the solution is adapted to the driving Brownian motions. In addition, there are results for the pairs of forward and backward equations with separate Cauchy conditions at initial time and the terminal time respectively. The results for solvability and regularity of forward and backward SPDEs can be found in [1, 2, 7, 8, 10, 12, 14, 16, 17, 20, 22, 25, 26, 30, 34, 36, 37, 38].

There are also results for SPDEs with boundary conditions that mix the solution at different times that may include initial time and terminal time. This category includes stationary type solutions for forward SPDEs; see, e.g., [3, 4, 6, 18, 27, 28, 35], and the references therein. Related results were obtained for periodic solutions of SPDEs [5, 19, 21]. As was mentioned in [19], it is difficult to expect that, in general, a SPDE has a periodic in time solution $u(\cdot, t)|_{t \in [0, T]}$ in a usual sense of exact equality $u(\cdot, 0) = u(\cdot, T)$ that holds almost surely.

To overcome this, the periodicity of the solutions of stochastic equations was usually considered in some relaxed sense. The most common approach was to consider the periodicity in the sense of the distributions [5, 21, 29]. In [19], the periodicity was established in a stronger sense as a "random periodic solution" (see Definition 1.1 in [19]); still, this definition does not assume that the equality $u(\cdot, 0) = u(\cdot, T)$ holds almost surely. For ordinary stochastic equations, some periodic solutions were obtained in [32] in a setting with relaxed adaptivity requirements, and in [33] in some asymptotic sense for a setting with time decaying random noise.

The present paper addresses these and related problems again. We consider SPDEs with the Dirichlet condition at the boundary of the state domain; the equations are of a parabolic type and are not necessary self-adjoint. The standard boundary value Cauchy condition at the one fixed time is replaced by a non-local in time condition that mixes in one equation the values of the solution at different times over given time interval, including the terminal time and continuously distributed times. This is a novel setting comparing with the periodic conditions for the distributions, or with conditions from [21, 19]. We obtained sufficient conditions for existence and regularity of the solutions in L_2 -setting for forward and backward SPDEs, with more emphasise on backward equation (Theorems 3.1-3.3 and 4.1-4.2 below). For backward equations, our non-local conditions include, for instance, almost surely periodicity conditions $u(\cdot, T) = u(\cdot, 0)$, as well as more general conditions $\kappa u(\cdot, T) = u(\cdot, 0) + \xi$ with $\kappa \in [-1, 1]$ and some given ξ (Theorem 4.2 below). These almost surely periodicity conditions were not considered in the existing literature. In particular, these conditions were not allowed in [13], where forward SPDEs were considered with conditions that were similar to the conditions $\mathbf{E}u(\cdot, T) = u(\cdot, 0)$.

Some possible applications to portfolio selection problems are discussed (Section 5 and Theorem 5.1).

2 The problem setting and definitions

We are given a standard complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and a right-continuous filtration \mathcal{F}_t of complete σ -algebras of events, $t \geq 0$. We are given also a N -dimensional Wiener process $w(t)$ with independent components; it is a Wiener process with respect to \mathcal{F}_t .

Assume that we are given an open bounded domain $D \subset \mathbf{R}^n$ with a C^2 -smooth boundary ∂D . Let $T > 0$ be given, and let $Q \triangleq D \times [0, T]$.

We consider the following boundary value problems for forward equations in Q

$$\begin{aligned} d_t u &= (\mathcal{A}u + \varphi) dt + \sum_{i=1}^N [B_i u + h_i] dw_i(t), \quad t \geq 0, \\ u(x, t, \omega) |_{x \in \partial D} &= 0 \\ u(\cdot, 0) - \Gamma u &= \xi. \end{aligned}$$

and the following boundary value problems for backward equations in Q

$$\begin{aligned} d_t u + (\mathcal{A}u + \varphi) dt + \sum_{i=1}^N B_i \chi_i dt &= \sum_{i=1}^N \chi_i dw_i(t), \quad t \geq 0, \\ u(x, t, \omega) |_{x \in \partial D} &= 0 \\ u(\cdot, T) - \Gamma u &= \xi. \end{aligned}$$

Here $u = u(x, t, \omega)$, $\chi_i = \chi_i(x, t, \omega)$, $h_i = h_i(x, t, \omega)$, $\varphi = \varphi(x, t, \omega)$, $\xi = \xi(x, \omega)$, $(x, t) \in Q$, $\omega \in \Omega$.

In these boundary problems, Γ is a linear operator that maps functions defined on $Q \times \Omega$ to functions defined on $D \times \Omega$. The operator \mathcal{A} is defined as

$$\mathcal{A}v \triangleq \sum_{i=1}^n \frac{\partial}{\partial x_i} \sum_{j=1}^n \left(b_{ij}(x, t, \omega) \frac{\partial v}{\partial x_j}(x) \right) + \sum_{i=1}^n f_i(x, t, \omega) \frac{\partial v}{\partial x_i}(x) + \lambda(x, t, \omega)v(x), \quad (2.1)$$

where b_{ij}, f_i, x_i are the components of b, f , and x respectively, and

$$B_i v \triangleq \frac{dv}{dx}(x) \beta_i(x, t, \omega) + \bar{\beta}_i(x, t, \omega) v(x), \quad i = 1, \dots, N. \quad (2.2)$$

We assume that the functions $b(x, t, \omega) : \mathbf{R}^n \times [0, T] \times \Omega \rightarrow \mathbf{R}^{n \times n}$, $\beta_j(x, t, \omega) : \mathbf{R}^n \times [0, T] \times \Omega \rightarrow \mathbf{R}^n$, $\bar{\beta}_i(x, t, \omega) : \mathbf{R}^n \times [0, T] \times \Omega \rightarrow \mathbf{R}$, $f(x, t, \omega) : \mathbf{R}^n \times [0, T] \times \Omega \rightarrow \mathbf{R}^n$, $\lambda(x, t, \omega) : \mathbf{R}^n \times [0, T] \times \Omega \rightarrow \mathbf{R}$, $\chi_i(x, t, \omega) : \mathbf{R}^n \times [0, T] \times \Omega \rightarrow \mathbf{R}$, and $\varphi(x, t, \omega) : \mathbf{R}^n \times [0, T] \times \Omega \rightarrow \mathbf{R}$ are progressively measurable with respect to \mathcal{F}_t for all $x \in \mathbf{R}^n$, and the function $\xi(x, \omega) : \mathbf{R}^n \times \Omega \rightarrow \mathbf{R}$ is \mathcal{F}_0 -measurable for all $x \in \mathbf{R}^n$. In fact, we will also consider φ and ξ from wider classes. In particular, we will consider generalized functions φ .

We do not exclude an important special case when the functions b , f , λ , φ , and ξ , are deterministic, and $h_i \equiv 0$, $B_i \equiv 0$ ($\forall i$). In this case, the equations are deterministic, and $\chi_i \equiv 0$ ($\forall i$) for backward equations.

Spaces and classes of functions

We denote by $\|\cdot\|_X$ the norm in a linear normed space X , and $(\cdot, \cdot)_X$ denote the scalar product in a Hilbert space X .

We introduce some spaces of real valued functions.

Let $G \subset \mathbf{R}^d$ be an open domain. For $q \geq 1$, we denote by $L_q(G)$ the usual Banach spaces of classes of equivalency of measurable by Lebesgue functions $v : G \rightarrow \mathbf{R}$, with the norms $\|v\|_{L_q(G)} = \left(\int_G |v(x)|^q dx\right)^{1/q}$. For integers $m \geq 0$, we denote by $W_q^m(G)$ the Sobolev spaces of functions that belong to $L_q(G)$ together with the distributional derivatives up to the m th order, $q \geq 1$, with the norms $\|v\|_{W_q^m(G)} = \left(\sum_{k: |k| \leq m} \|\Delta^k v\|_{L_q(G)}^q\right)^{1/q}$. Here $\Delta^k = \Delta_{k_1} \dots \Delta_{k_d}$ is the partial derivative of the order $|k| = \sum_{i=1}^d k_i$, $0 \leq k_i \leq |k|$.

We denote by $|\cdot|$ the Euclidean norm in \mathbf{R}^k , and \bar{G} denote the closure of a region $G \subset \mathbf{R}^k$.

Let $H^0 \triangleq L_2(D)$, and let $H^1 \triangleq W_2^1(D)$ be the closure in the $W_2^1(D)$ -norm of the set of all smooth functions $u : D \rightarrow \mathbf{R}$ such that $u|_{\partial D} \equiv 0$. Let $H^2 = W_2^2(D) \cap H^1$ be the space equipped with the norm of $W_2^2(D)$. The spaces H^k and $W_2^k(D)$ are called Sobolev spaces; they are Hilbert spaces, and H^k is a closed subspace of $W_2^k(D)$, $k = 1, 2$.

Let H^{-1} be the dual space to H^1 , with the norm $\|\cdot\|_{H^{-1}}$ such that if $u \in H^0$ then $\|u\|_{H^{-1}}$ is the supremum of $(u, v)_{H^0}$ over all $v \in H^1$ such that $\|v\|_{H^1} \leq 1$. H^{-1} is a Hilbert space.

Let $C_0(\bar{D})$ be the Banach space of all functions $u \in C(\bar{D})$ such that $u|_{\partial D} \equiv 0$ equipped with the norm from $C(\bar{D})$.

We shall write $(u, v)_{H^0}$ for $u \in H^{-1}$ and $v \in H^1$, meaning the obvious extension of the bilinear form from $u \in H^0$ and $v \in H^1$.

We denote by $\bar{\ell}_k$ the Lebesgue measure in \mathbf{R}^k , and we denote by $\bar{\mathcal{B}}_k$ the σ -algebra of Lebesgue sets in \mathbf{R}^k .

We denote by $\bar{\mathcal{P}}$ the completion (with respect to the measure $\bar{\ell}_1 \times \mathbf{P}$) of the σ -algebra of subsets of $[0, T] \times \Omega$, generated by functions that are progressively measurable with respect to \mathcal{F}_t .

We introduce the spaces

$$\begin{aligned} X^k(s, t) &\triangleq L^2([s, t] \times \Omega, \bar{\mathcal{P}}, \bar{\ell}_1 \times \mathbf{P}; H^k), \\ Z_t^k &\triangleq L^2(\Omega, \mathcal{F}_t, \mathbf{P}; H^k), \end{aligned}$$

$$\begin{aligned}
\mathcal{C}^k(s, t) &\triangleq C\left([s, T]; Z_T^k\right), \quad k = -1, 0, 1, 2, \\
\mathcal{X}_c^k &= L^2([0, T] \times \Omega, \bar{\mathcal{P}}, \bar{\ell}_1 \times \mathbf{P}; C^k(\bar{D})), \quad k \geq 0, \\
\mathcal{Z}_c^k &\triangleq L_2(\Omega, \mathcal{F}_T, \mathbf{P}; C^k(D)), \quad k \geq 0.
\end{aligned}$$

The spaces $X^k(s, t)$ and Z_t^k are Hilbert spaces.

In addition, we introduce the spaces

$$Y^k(s, t) \triangleq X^k(s, t) \cap \mathcal{C}^{k-1}(s, t), \quad k = 1, 2,$$

with the norm $\|u\|_{Y^k(s, T)} \triangleq \|u\|_{X^k(s, t)} + \|u\|_{\mathcal{C}^{k-1}(s, t)}$.

For brevity, we shall use the notations $X^k \triangleq X^k(0, T)$, $\mathcal{C}^k \triangleq \mathcal{C}^k(0, T)$, and $Y^k \triangleq Y^k(0, T)$.

Proposition 2.1 *Let $\zeta \in X^0$, and let a sequence $\{\zeta_k\}_{k=1}^{+\infty} \subset L^\infty([0, T] \times \Omega, \ell_1 \times \mathbf{P}; C(D))$ be such that all $\zeta_k(\cdot, t, \omega)$ are progressively measurable with respect to \mathcal{F}_t , and $\|\zeta - \zeta_k\|_{X^0} \rightarrow 0$ as $k \rightarrow +\infty$. Let $t \in [0, T]$ and $j \in \{1, \dots, N\}$ be given. Then the sequence of the integrals $\int_0^t \zeta_k(x, s, \omega) dw_j(s)$ converges in Z_t^0 as $k \rightarrow \infty$, and its limit depends on ζ , but does not depend on $\{\zeta_k\}$.*

Proof follows from completeness of X^0 and from the equality

$$\begin{aligned}
&\mathbf{E} \int_0^t \|\zeta_k(\cdot, s, \omega) - \zeta_m(\cdot, s, \omega)\|_{H^0}^2 ds \\
&= \int_D dx \mathbf{E} \left[\int_0^t (\zeta_k(x, s, \omega) - \zeta_m(x, s, \omega)) dw_j(s) \right]^2.
\end{aligned}$$

Definition 2.1 For $\zeta \in X^0$, $t \in [0, T]$, $j \in \{1, \dots, N\}$, we define $\int_0^t \zeta(x, s, \omega) dw_j(s)$ as the limit in Z_t^0 as $k \rightarrow \infty$ of a sequence $\int_0^t \zeta_k(x, s, \omega) dw_j(s)$, where the sequence $\{\zeta_k\}$ is such as in Proposition 2.1.

Conditions for the coefficients

To proceed further, we assume that Conditions 2.1-2.3 remain in force throughout this paper.

Condition 2.1 *The matrix b is symmetric. In addition, there exists a constant $\delta > 0$ such that*

$$y^\top b(x, t, \omega) y - \frac{1}{2} \sum_{i=1}^N |y^\top \beta_i(x, t, \omega)|^2 \geq \delta |y|^2 \quad \forall y \in \mathbf{R}^n, (x, t) \in D \times [0, T], \omega \in \Omega. \quad (2.3)$$

Condition 2.2 *The functions $b(x, t, \omega)$, $f(x, t, \omega)$, $\lambda(x, t, \omega)$, $\beta_i(x, t, \omega)$, and $\bar{\beta}_i(x, t, \omega)$, are bounded and differentiable in x for a.e. t, ω , and the corresponding derivatives are bounded.*

It follows from this condition that there exist modifications of β_i such that the functions $\beta_i(x, t, \omega)$ are continuous in x for a.e. t, ω . We assume that β_i are such functions.

Condition 2.3 *There exists an integer $m \geq 0$, a set $\{t_i\}_{i=1}^m \subset [0, T]$, and linear continuous operators $\bar{\Gamma} : L_2(Q) \rightarrow H^0$, $\bar{\Gamma}_i : H^0 \rightarrow H^0$, $i = 1, \dots, N$, such that the operators $\bar{\Gamma} : L_2([0, T]; \mathcal{B}_1, \ell_1, H^1) \rightarrow W_2^1(D)$ and $\bar{\Gamma}_i : H^1 \rightarrow W_2^1(D)$ are continuous and*

$$\Gamma u = \mathbf{E}\{\bar{\Gamma}u + \sum_{i=1}^m \bar{\Gamma}_i u(\cdot, t_i)\}.$$

By Condition 2.3, the mapping $\Gamma : Y^1 \rightarrow Z_T^0$ is linear and continuous. This condition covers $\Gamma = \mathbf{E}\{u(\cdot, 0)|\mathcal{F}_0\}$, $\Gamma = \mathbf{E}\{u(\cdot, T)|\mathcal{F}_0\}$, as well as the cases where

$$\bar{\Gamma}u = \int_0^T k_0(t)u(\cdot, t)dt, \quad \bar{\Gamma}_i u(\cdot, t_i) = k_i u(\cdot, t_i),$$

where $k_0(\cdot) \in L_2(0, T)$ and $k_i \in \mathbf{R}$. It covers also Γ such that

$$\bar{\Gamma}u = \int_0^T dt \int_D k_0(x, y, t)u(y, t)dx, \quad \bar{\Gamma}_i u(\cdot, t_i)(x) = \int_D k_i(x, y)u(y, t_i)dy,$$

where $k_i(\cdot)$ are some regular enough kernels.

For the boundary value problems discussed below, we will express the dependence of the solutions on $(n, D, T, \Gamma, b, f, \lambda)$ via dependence of the solutions via the following set of parameters

$$\mathcal{P} \triangleq \left(n, D, T, \Gamma, \delta, \text{ess sup}_{x,t,\omega,i} \left[|b(x, t, \omega)| + |f(x, t, \omega)| + |\lambda(x, t, \omega)| + |\beta_i(x, t, \omega)| + |\bar{\beta}_i(x, t, \omega)| \right] \right).$$

Sometimes we shall omit ω . It appears that the backward SPDEs with non-local in time boundary conditions have some interesting features that are absent for the related results for forward SPDEs with non-local conditions. The present paper focuses on on these features for backward SPDEs that are absent for forward SPDEs. In the next section, we review all known results for forward SPDEs with non-local boundary conditions described above, with the purpose to compare these results with the corresponding results for BSPDEs with non-local conditions.

3 Forward SPDEs

We consider the following boundary value problem in Q

$$d_t u = (\mathcal{A}u + \varphi) dt + \sum_{i=1}^N [B_i u + h_i] dw_i(t), \quad t \geq 0, \quad (3.1)$$

$$u(x, t, \omega)|_{x \in \partial D} = 0 \quad (3.2)$$

$$u(x, 0, \omega) - \Gamma u(\cdot) = \xi(x, \omega). \quad (3.3)$$

Definition 3.1 Let $u \in Y^1$, $\varphi \in X^{-1}$, and $h_i \in X^0$. We say that equations (3.1)-(3.2) are satisfied if

$$\begin{aligned} u(\cdot, t, \omega) = u(\cdot, 0, \omega) &+ \int_0^t (\mathcal{A}u(\cdot, s, \omega) + \varphi(\cdot, s, \omega)) ds \\ &+ \sum_{i=1}^N \int_0^t [B_i u(\cdot, s, \omega) + h_i(\cdot, s, \omega)] dw_i(s) \end{aligned} \quad (3.4)$$

for all $t \in [0, T]$, and this equality is satisfied as an equality in Z_T^{-1} .

Note that the condition on ∂D is satisfied in the sense that $u(\cdot, t, \omega) \in H^1$ for a.e. t, ω . Further, $u \in Y^1$, and the value of $u(\cdot, t, \omega)$ is uniquely defined in Z_T^0 given t , by the definitions of the corresponding spaces. The integrals with dw_i in (3.4) are defined as elements of Z_T^0 . The integral with ds in (3.4) is defined as an element of Z_T^{-1} . In fact, Definition 3.1 requires for (3.1) that this integral must be equal to an element of Z_T^0 in the sense of equality in Z_T^{-1} .

Theorem 3.1 *There exists a number $\kappa = \kappa(\mathcal{P}) > 0$ such that problem (3.1)-(3.3) has a unique solution in the class Y^1 , for any $\varphi \in X^{-1}$, $h_i \in X^0$, $\xi \in Z_0^0$, and any Γ such that $\|\Gamma\| \leq \kappa$, where $\|\Gamma\|$ is the norm of the operator $\Gamma : Y^1 \rightarrow Z_0^T$.*

Theorem 3.2 *Assume that Condition 2.3 is satisfied with $\{t_k\}_{k=1}^m \subset (0, T]$. Let \mathcal{F}_0 be the \mathbf{P} -augmentation of the set $\{\emptyset, \Omega\}$. Assume that at least one of the following conditions is satisfied:*

- (i) *the function b is non-random, or*
- (ii) *$\beta_i(x, t, \omega) = 0$ for $x \in \partial D$, $i = 1, \dots, N$.*

Further, assume that problem (3.1)-(3.3) with $\varphi \equiv 0$, $h_i \equiv 0$, $\xi \equiv 0$, does not admit non-zero solutions in the class Y^1 . Then problem (3.1)-(3.3) has a unique solution u in the class Y^1 for any $\varphi \in X^{-1}$, $h_i \in X^0$, and $\xi \in H^0$.

By the assumptions on the coefficients, the operator \mathcal{A} can be represented as

$$\mathcal{A}v = \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} (b_{ij}(x, t)v(x)) + \sum_{i=1}^n \frac{\partial}{\partial x_i} (\widehat{f}_i(x, t)v(x)) + \widehat{\lambda}(x, t)v(x), \quad (3.5)$$

for measurable bounded functions b, \widehat{f} and $\widehat{\lambda}$.

Theorem 3.3 *Let the functions b, \widehat{f} and $\widehat{\lambda}$ in (3.5) be non-random, and let $\widehat{\lambda}(x, t) \leq 0$. Further, let*

$$\Gamma u = \mathbf{E} \left\{ \int_0^T k_0(t)u(\cdot, t)dt + \sum_{i=1}^m k_i u(\cdot, t_i) \middle| \mathcal{F}_0 \right\},$$

where $t_i > 0$, and where $k_i \in \mathbf{R}$, $k_0(\cdot) \in L_2(0, T)$ are such that

$$\int_0^T |k_0(t)| dt + \sum_{i=1}^m |k_i| \leq 1.$$

Then problem (3.1)-(3.3) has a unique solution u in the class Y^1 for any $\varphi \in X^{-1}$, $h_i \in X^0$, and $\xi \in H^0$.

So far, the existence results for forward SPDEs does not cover the case of almost surely periodicity in time. For these equations, non-local boundary conditions have to include expectations of the solutions only, such as $u(x, 0, \omega) = \mathbf{E}u(x, T, \omega)$ a.e.. "True" periodic conditions such as $u(x, 0, \omega) = u(x, T, \omega)$ were not allowed. It appears that existence results for almost surely periodicity and for more general non-local conditions can be obtained for backward SPDEs considered in the next section.

4 Backward SPDEs

For backward SPDEs, we will study the following boundary value problem in Q

$$d_t u + (\mathcal{A}u + \varphi) dt + \sum_{i=1}^N B_i \chi_i dt = \sum_{i=1}^N \chi_i dw_i(t), \quad t \geq 0, \quad (4.1)$$

$$u(x, t, \omega) |_{x \in \partial D} = 0 \quad (4.2)$$

$$u(\cdot, T) - \Gamma u = \xi. \quad (4.3)$$

Here $u = u(x, t, \omega)$, $\varphi = \varphi(x, t, \omega)$, $\chi_i = \chi_i(x, t, \omega)$, $(x, t) \in Q$, $\omega \in \Omega$.

In (4.3), Γ is a linear operator that maps functions defined on $Q \times \Omega$ to functions defined on $D \times \Omega$. For instance, the case where $\Gamma u = u(\cdot, 0)$ is not excluded; this case corresponds to the periodic type boundary condition $u(\cdot, T) - u(\cdot, 0) = \xi$.

Definition 4.1 Let $u \in Y^1$, $\chi_i \in X^0$, $i = 1, \dots, N$, and $\varphi \in X^{-1}$. We say that equations (4.1)-(4.2) are satisfied if

$$\begin{aligned} u(\cdot, t, \omega) = u(\cdot, T, \omega) &+ \int_t^T (\mathcal{A}u(\cdot, s, \omega) + \varphi(\cdot, s, \omega)) ds \\ &+ \sum_{i=1}^N \int_t^T B_i \chi_i(\cdot, s, \omega) ds - \sum_{i=1}^N \int_t^T \chi_i(\cdot, s) dw_i(s) \end{aligned}$$

for all r, t such that $0 \leq r < t \leq T$, and this equality is satisfied as an equality in Z_T^{-1} .

Similarly to Definition 3.1, the integral with ds in (4.4) is defined as an element of Z_T^{-1} . Definition 4.1 requires for (4.1) that this integral must be equal to an element of Z_T^0 in the sense of equality in Z_T^{-1} .

Starting from now and up to the end of this section, we assume that Condition 4.1 holds.

Condition 4.1 (i) $\beta_i(x, t, \omega) = 0$ for $x \in \partial D$, $i = 1, \dots, N$.

(ii) \mathcal{F}_0 is the \mathbf{P} -augmentation of the set $\{\emptyset, \Omega\}$.

(iii) Condition 2.3 is satisfied with $\{t_i\}_{i=1}^m \subset [0, T]$.

Note that the assumptions on Γ imposed in Condition 4.1 allows to consider $\Gamma u = u(\cdot, 0)$, i.e., the periodic type boundary conditions $u(\cdot, T) = u(\cdot, 0)$ a.s.; it suffices to assume that $t_1 = 0$, \bar{G}_1 is identical operator, $\bar{\Gamma} = 0$, $\bar{G}_i = 0$, $i > 1$.

Theorem 4.1 Assume that problem (4.1)-(4.3) with $\varphi \equiv 0$, $\xi \equiv 0$, does not admit non-zero solutions $(u, \chi_1, \dots, \chi_N)$ in the class $Y^1 \times (X^0)^N$. Then problem (4.1)-(4.3) has a unique solution $(u, \chi_1, \dots, \chi_N)$ in the class $Y^1 \times (X^0)^N$, for any $\varphi \in X^{-1}$, and $\xi \in H^0$. In addition,

$$\|u\|_{Y^1} + \sum_{i=1}^N \|\chi_i\|_{X^0} \leq C (\|\varphi\|_{X^{-1}} + \|\xi\|_{H^0}), \quad (4.4)$$

where $C > 0$ does not depend on φ and ξ .

By the assumptions, we have that the functions b, f and λ are such that the operator \mathcal{A} can be represented as

$$\mathcal{A}v = \sum_{i,j=1}^n b_{ij}(x, t, \omega) \frac{\partial^2 v}{\partial x_i \partial x_j}(x) + \sum_{i=1}^n \hat{f}_i(x, t, \omega) \frac{\partial v}{\partial x_i}(x) + \hat{\lambda}(x, t, \omega)v(x), \quad (4.5)$$

where the functions $\hat{f}(x, t, \omega)$, $\hat{\lambda}(x, t, \omega)$, and $\beta_i(x, t, \omega)$ are bounded.

Theorem 4.2 Assume that the following holds:

(i) $\bar{\beta}_i \equiv 0$;

(ii) $\hat{\lambda}(x, t, \omega) \leq 0$ a.e.;

(iii) the functions $\hat{f}(x, t, \omega)$, $\hat{\lambda}(x, t, \omega)$, and $\beta_i(x, t, \omega)$ in (4.5) are differentiable in x for a.e. t, ω , and the corresponding derivatives are bounded;

(iv) $b \in \mathcal{X}_c^3$, $\hat{f} \in \mathcal{X}_c^2$, $\hat{\lambda} \in \mathcal{X}_c^1$, $\beta_i \in \mathcal{X}_c^3$.

(v) $\Gamma u = \kappa u(\cdot, 0)$, where $\kappa \in [-1, 1]$, i.e, boundary condition (4.3) is

$$u(\cdot, T) - \kappa u(\cdot, 0) = \xi. \quad (4.6)$$

Then problem (4.1)-(4.2),(4.6) has a unique solution $(u, \chi_1, \dots, \chi_N)$ in the class $Y^1 \times (X^0)^N$ for any $\varphi \in X^{-1}$ and $\xi \in Z_T^0$. In addition, (4.4) holds with $C > 0$ that does not depend on φ and ξ .

Periodic type conditions (4.6) were introduced in [9] for deterministic parabolic equations.

Theorem 4.2 implies the following result for deterministic parabolic equation with non-local boundary condition [11], Theorem 2.2.

Corollary 4.1 *Under the assumptions of Theorem 4.2, for any $k \in [-1, 1]$, the deterministic boundary value problem*

$$\frac{\partial u}{\partial t} + \mathcal{A}u = -\varphi, \quad u|_{\partial D} = 0, \quad u(x, T) - ku(x, 0) \equiv \Phi(x)$$

has a unique solution $u \in C([0, T]; H^0) \cap L_2([0, T], \mathcal{B}_1, \ell_1, H^1)$ for any $\Phi \in H^0$, $\varphi \in L_2(Q)$, and

$$\|u(\cdot, t)\|_{Y^1} \leq C(\|\Phi\|_{H^0} + \|\varphi\|_{L_2(Q)}),$$

where $C > 0$ does not depend on Φ and φ .

The classical result about well-posedness of the Cauchy condition at initial time corresponds to the special case of $k = 0$.

Remark 4.1 In the literature, the distinction between forward and backward SDEs is usually based solely on the placement of the Cauchy condition: an equation is deemed to be a forward if the condition is at the initial time, and an equation is deemed to be a backward if the condition is at the terminal time. Theorem 4.2 shows that this this criterion can be insufficient for the case of SPDEs. Consider, for example, the condition $u(x, 0) = u(x, T)$ covered by this theorem. The values at the initial time and the terminal time are presented symmetrically here. However, the underlying SPDE is definitely backward, with a diffusion coefficient as a part of the solution, as is typical for BSPDEs.

5 Some applications: portfolio selection problems

Theorem 4.2 can be applied to portfolio selection for continuous time diffusion market model, where the price dynamic is described by Ito stochastic differential equations. Examples of these models can be found in, e.g., [Karatzas and Shreve(1998)].

We consider the following model of a securities market consisting of a risk free bond or bank account with the price $B(t)$, $t \geq 0$, and a risky stock with the price $S(t)$, $t \geq 0$. The prices of the stocks evolve as

$$dS(t) = S(t) (a(t)dt + \sigma(t)dw(t) + \hat{\sigma}(t)d\hat{w}(t)), \quad t > 0, \quad (5.1)$$

where $(w(t), \widehat{w}(t))$ is a Wiener process, $a(t)$ is an appreciation rate, $(\sigma(t), \widehat{\sigma}(t))$ is a vector of volatility coefficients. The initial price $S(0) > 0$ is a given deterministic constant. The price of the bond evolves as

$$B(t) = e^{rt}B(0),$$

where $B(0)$ is a given constant, $r \geq 0$ is a short rate. For simplicity, we assume that $r = 0$ and $B(t) \equiv B(0)$.

We assume that $(w(\cdot), \widehat{w}(t))$ is a standard Wiener process on a given standard probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where Ω is a set of elementary events, \mathcal{F} is a complete σ -algebra of events, and \mathbf{P} is a probability measure.

Let \mathcal{F}_t be the filtration generated by $w(t)$, and let $\widehat{\mathcal{F}}_t$ be the filtration generated by $(w(t), \widehat{w}(t))$. In particular, we assume that \mathcal{F}_0 and $\widehat{\mathcal{F}}_0$ are trivial σ -algebras, i.e., they are the \mathbf{P} -augmentations of the set $\{\emptyset, \Omega\}$.

We assume that the processes $a(t)$, $\sigma(t)$, $\widehat{\sigma}(t)$, $\sigma(t)^{-1}$, and $\widehat{\sigma}(t)^{-1}$ are measurable, bounded and \mathcal{F}_t -adapted.

Strategies for bond-stock-options market

The rules for the operations of the agents on the market define the class of admissible strategies where the optimization problems have to be solved.

Let $X(0) > 0$ be the initial wealth at time $t = 0$ and let $X(t)$ be the wealth at time $t > 0$.

We assume that the wealth $X(t)$ at time $t \in [0, T]$ is

$$X(t) = \beta(t)B(t) + \gamma(t)S(t). \quad (5.2)$$

Here $\beta(t)$ is the quantity of the bond portfolio, $\gamma(t)$ is the quantity of the stock portfolio, $t \geq 0$. The pair $(\beta(\cdot), \gamma(\cdot))$ describes the state of the bond-stocks securities portfolio at time t . Each of these pairs is called a strategy.

A pair $(\beta(\cdot), \gamma(\cdot))$ is said to be an admissible strategy if the processes $\beta(t)$ and $\gamma(t)$ are progressively measurable with respect to the filtration $\widehat{\mathcal{F}}_t$.

In particular, the agents are not supposed to know the future (i.e., the strategies have to be adapted to the flow of current market information).

Let \mathbf{P}_* be an equivalent probability measure such that $S(t)$ is a martingale under \mathbf{P}_* . By the assumptions on $(a, \sigma, \widehat{\sigma})$, this measure exists and is unique.

A pair $(\beta(\cdot), \gamma(\cdot))$ is said to be an admissible self-financing strategy, if $\mathbf{E}_*X(T)^2 < +\infty$ and

$$X(t) = X(0) + \int_0^t \gamma(s)dS(s).$$

A portfolio selection problem

In portfolio theory, a typical problem is constructing a portfolio strategy with certain desirable properties. It will be demonstrated below that Theorem 4.2 can be applied to this problem.

Let us consider the following example.

Let $s_L \in (0, S(0))$, $s_U \in (S(0), +\infty)$.

Let $D = (s_L, s_U)$. and $\xi \in L_\infty(\Omega, \mathcal{F}_T, \mathbf{P}, C_0(\bar{D}))$ be given.

Let us consider problem (4.1)-(4.3) with $n = N = 1$, $D = (s_L, s_U)$, $\varphi \equiv 0$, and with

$$\begin{aligned} \mathcal{A}v &= \frac{1}{2}(\sigma(t)^2 + \hat{\sigma}(t)^2)x^2 \frac{\partial^2 v}{\partial x^2}(x), & B_1 v &\triangleq x\sigma(t) \frac{dv}{dx}(x), \\ (\Gamma u)(x) &= u(x, 0). \end{aligned}$$

In other words, we consider the following problem for $(x, t) \in (s_L, s_U) \times [0, T]$

$$d_t u(x, t) + \frac{1}{2}(\sigma(t)^2 + \hat{\sigma}(t)^2)x^2 \frac{\partial^2 u}{\partial x^2}(x, t) + \sigma(t)x \frac{d\chi}{dx}(x, t) = \chi(x, t)dw(t), \quad (5.3)$$

$$u(s_L, t) = u(s_U, t) = 0, \quad (5.4)$$

$$u(x, T) = u(x, 0) + \xi(x). \quad (5.5)$$

The assumptions of Theorem 4.2 are satisfied for this problem. By this theorem, there exists a unique solution $u(x, t, \omega) : [s_L, s_U] \times [0, T] \times \Omega \rightarrow \mathbf{R}$ of problem (5.3)-(5.5) such that $u \in Y^1$.

Let $\tau = \inf\{t > 0 : S(t) \notin D\}$.

Theorem 5.1 *Let*

$$X(t, x) = u(S(t \wedge \tau), t \wedge \tau), \quad (5.6)$$

where $x \in D$, $t \in [0, T]$ and where $S(t)$ is defined by (5.1) given that $S(0) = x$. Then $\mathbf{E}_* X(T, x)^2 < +\infty$, and the process $X(t, x)$ represents the wealth generated by some self-financing strategy given that $S(0) = x$. In addition,

$$X(T, y) = X(0, x) + \xi(x) \quad \text{if } \tau > T, \quad S(T)y = S(0)x. \quad (5.7)$$

The portfolio described in Theorem 5.1 has the following attractive feature: with a positive ξ , it ensures a systematic gain when $\tau > T$ for the case of stagnated marked prices. The event $\tau < T$ can be considered as an extreme event if s_L is sufficiently small and s_U is sufficiently large.

Note that the assumption that the process $(a(t), \sigma(t))$ is \mathcal{F}_t -adapted was used to ensure existence of u . A more general model where this process is allowed to be $\hat{\mathcal{F}}_t$ -adapted leads to a degenerate SPDE in a bounded domain where Condition 2.1 is not satisfied. This case is not covered by Theorem 4.2. An example of portfolio selection based on a degenerate backward SPDE in the entire space was considered in [25].

6 Proofs

Proof of Theorems 3.1-3.3 can be found in Dokuchaev (2008). The remaining part of the paper contains proofs for Theorems 4.1, 4.2, and 5.1.

Let $s \in (0, T]$, $\varphi \in X^{-1}$ and $\Phi \in Z_s^0$. Consider the problem

$$\begin{aligned} d_t u + (\mathcal{A}u + \varphi) dt + \sum_{i=1}^N B_i \chi_i(t) dt &= \sum_{i=1}^N \chi_i(t) dw_i(t), \quad t \leq s, \\ u(x, t, \omega)|_{x \in \partial D}, & \\ u(x, s, \omega) &= \Phi(x, \omega). \end{aligned} \tag{6.1}$$

The following lemma represents an analog of the so-called "the first energy inequality", or "the first fundamental inequality" known for deterministic parabolic equations (see, e.g., inequality (3.14) in [23], Chapter III).

Lemma 6.1 *Problem (6.1) has a unique solution $(u, \chi_1, \dots, \chi_N)$ in the class $Y^1 \times (X^0)^N$ for any $\varphi \in X^{-1}(0, s)$, $\Phi \in Z_s^0$, and*

$$\|u\|_{Y^1(0,s)} + \sum_{i=1}^N \|\chi_i\|_{X^0} \leq C (\|\varphi\|_{X^{-1}(0,s)} + \|\Phi\|_{Z_s^0}), \tag{6.2}$$

where $C = C(\mathcal{P})$ does not depend on φ and ξ .

(See, e.g., [8] Dokuchaev (1992) or Theorem 4.2 in [14]).

Note that the solution $u = u(\cdot, t)$ is continuous in t in $L_2(\Omega, \mathcal{F}, \mathbf{P}, H^0)$, since $Y^1(0, s) = X^1(0, s) \cap \mathcal{C}^0(0, s)$.

Introduce operators $L_s : X^{-1}(0, s) \rightarrow Y^1(0, s)$ and $\mathcal{L}_s : Z_s^0 \rightarrow Y^1(0, s)$, such that $u = L_s \varphi + \mathcal{L}_s \Phi$, where $(u, \chi_1, \dots, \chi_N)$ is the solution of problem (6.1) in the class $Y^2 \times (X^1)^N$. By Lemma 6.1, these linear operators are continuous.

Introduce operators $\mathcal{Q} : Z_T^0 \rightarrow Z_T^0$ and $\mathcal{T} : X^{-1} \rightarrow Z_T^0$ such that $\mathcal{Q}\Phi + \mathcal{T}\varphi = \Gamma u$, where u is the solution in Y^1 of problem (6.1) with $s = T$, $\varphi \in X^{-1}$, and $\Phi \in Z_T^0$. It is easy to see that these operators are linear and continuous.

Clearly, $u \in Y^1$ is the solution of problem (4.1)-(4.3), if

$$\begin{aligned} u &= L_T \varphi + \mathcal{L}_T u(\cdot, T), \\ u(\cdot, T) - \Gamma u &= \xi. \end{aligned}$$

Since $\Gamma u = \mathcal{Q}u(\cdot, T) + \mathcal{T}\varphi$, we have

$$u(\cdot, T) - \mathcal{Q}u(\cdot, T) - \mathcal{T}\varphi = \xi.$$

If the operator $(I - \mathcal{Q})^{-1} : Z_T^0 \rightarrow Z_T^0$ is continuous, then

$$u(\cdot, T) = (I - \mathcal{Q})^{-1}(\xi + \mathcal{T}\varphi),$$

and

$$\begin{aligned} u &= L_T\varphi + \mathcal{L}_T u(\cdot, T) \\ &= L_T\varphi + \mathcal{L}_T(I - \mathcal{Q})^{-1}(\xi + \mathcal{T}\varphi). \end{aligned} \quad (6.3)$$

Since the operator $\mathcal{Q} : Z_T^0 \rightarrow Z_T^0$ is continuous, the operator $(I - \mathcal{Q})^{-1} : Z_T^0 \rightarrow Z_T^0$ is continuous for small enough $\|\mathcal{Q}\|$, where $\|\mathcal{Q}\|$ is the norm of the operators $\mathcal{Q} : Z_T^0 \rightarrow Z_T^0$. Clearly, this holds for small enough $\|\Gamma\|$, since $\|\mathcal{Q}\| \leq \|\Gamma\|\|\mathcal{L}_T\|$, where $\|\Gamma\|$, and $\|\mathcal{L}_T\|$, are the norms of the operators $\Gamma : Y^1 \rightarrow Z_T^0$, and $\mathcal{L}_0 : Z_T^0 \rightarrow Y^1$, respectively. We consider below the case where $\|\mathcal{Q}\|$ is not necessary small.

Starting from now, we assume that Condition 4.1 is satisfied, in addition to Conditions 2.1-2.3.

The following lemma represents an analog of the so-called "the second energy inequality", or "the second fundamental inequality" known for the deterministic parabolic equations (see, e.g., inequality (4.56) in [23], Chapter III).

Lemma 6.2 *Problem (6.1) has a unique solution $(u, \chi_1, \dots, \chi_N)$ in the class $Y^2 \times (X^1)^N$ for any $\varphi \in X^0$, $\Phi \in Z_T^1$, and*

$$\|u\|_{Y^2} + \sum_{i=1}^N \|\chi_i\|_{X^1} \leq C \left(\|\varphi\|_{X^0} + \|\Phi\|_{Z_T^1} \right), \quad (6.4)$$

where $C > 0$ does not depend on φ and Φ ; it depends on \mathcal{P} and on the supremums of the derivatives listed in Condition 4.1(ii).

The lemma above represents a reformulation of Theorem 3.1 from [17] or Theorem 3.4 in [14], or Theorem 4.3 in [16]. In the cited papers, this result was obtained under some strengthened version of Condition 2.1; this was restrictive. In [17], this result was obtained without this restriction, i.e., under Condition 2.1 only.

Remark 6.1 *Thanks to Theorem 3.1. from [17], Condition 3.5 from [15] and Condition 4.1 from [16] can be replaced by less restrictive Condition 2.1; all results in [15, 16] are still valid.*

Lemma 6.3 *The operator $\mathcal{Q} : Z_T^0 \rightarrow Z_T^0$ is compact.*

Proof of Lemma 6.3. Let $u = \mathcal{L}_0\Phi$, where $\Phi \in Z_T^0$. By the semi-group property of backward SPDEs from Theorem 6.1 from [14], we obtain that $u|_{t \in [0, s]} = \mathcal{L}_s u(\cdot, s)$ for all $s \in (0, T]$. By Lemmas 6.1 and 6.2, we have for $\tau \in \{t_1, \dots, t_m\}$ that

$$\begin{aligned} \|\mathbf{E}\bar{\Gamma}_i u(\cdot, \tau)\|_{W_2^1(D)}^2 &\leq C_0 \|u(\cdot, \tau)\|_{Z_0^1}^2 \leq C_1 \inf_{t \in [\tau, T]} \|u(\cdot, t)\|_{Z_t^1}^2 \\ &\leq \frac{C_1}{T - \tau} \int_{\tau}^T \|u(\cdot, t)\|_{Z_t^1}^2 dt \leq \frac{C_2}{T - \tau} \|\Phi\|_{Z_T^0}^2 \end{aligned}$$

and

$$\|\mathbf{E}\bar{\Gamma}_0 u\|_{W_2^1(D)}^2 \leq C_3 \mathbf{E} \int_0^T \|u(\cdot, t)\|_{Z_t^1}^2 dt \leq C_4 \|\Phi\|_{Z_T^0}^2,$$

for $C_i > 0$ which do not depend on Φ . Hence the operator $\mathcal{Q} : Z_T^0 \rightarrow W_2^1(D)$ is continuous. Since the embedding of $W_2^1(D)$ to H^0 and in Z_T^0 is a compact operator, the proof of Lemma 6.3 follows. \square

Proof of Theorem 4.1. By the assumptions, the equation $\mathcal{Q}\Phi = \Phi$ has the only solution $\Phi = 0$ in H^0 . By Lemma 6.3 and by the Fredholm Theorem, the operator $(I - \mathcal{Q})^{-1} : H^0 \rightarrow H^0$ is continuous. Then the proof of Theorem 4.1 follows from representation (6.3). \square

Let us introduce operators

$$\mathcal{A}^* v \triangleq \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} (b_{ij}(x, t) v(x)) - \sum_{i=1}^n \frac{\partial}{\partial x_i} (\widehat{f}_i(x, t) v(x)) + \widehat{\lambda}(x, t) v(x)$$

and

$$B_i^* v \triangleq - \sum_{k=1}^n \frac{\partial}{\partial x_k} (\beta_{ik}(x, t, \omega) v(x)) + \bar{\beta}_i(x, t, \omega) v(x), \quad i = 1, \dots, N.$$

Here b_{ij} , x_i , β_{ik} are the components of b , β_i , and x .

Let $\rho \in Z_s^0$, and let $p = p(x, t, \omega)$ be the solution of the problem

$$\begin{aligned} d_t p &= \mathcal{A}^* p dt + \sum_{i=1}^N B_i^* p dw_i(t), \quad t \geq s, \\ p|_{t=s} &= \rho, \quad p(x, t, \omega)|_{x \in \partial D} = 0. \end{aligned}$$

By Theorem 3.4.8 from [34], this boundary value problem has a unique solution $p \in Y^1(s, T)$. Introduce an operator $\mathcal{M}_s : Z_s^0 \rightarrow Y^1(s, T)$ such that $p = \mathcal{M}_s \rho$, where $p \in Y^1(s, T)$ is the solution of this boundary value problem.

Proof of Theorem 4.2. By Theorem 3.1 from [12], problem (6.1) has a unique solution $p \in Y^2$ for any $\rho \in Z_s^1$, and

$$\|p\|_{Y^2(s, T)} \leq C \|\rho\|_{Z_s^1}, \quad (6.5)$$

where $C > 0$ does not depend on ρ . This C depends on \mathcal{P} and on the supremums of the derivatives in Condition 4.1.

By Theorem 4.2 from [14], we have that $\kappa p(\cdot, T) = \mathcal{Q}^* \rho$, i.e.,

$$(\rho, \mathcal{Q}\Phi)_{Z_0^0} = (\rho, \kappa v(\cdot, 0))_{Z_0^0} = (p(\cdot, T), \kappa v(\cdot, T))_{Z_T^0} = (\kappa p(\cdot, T), \Phi)_{Z_T^0} \quad (6.6)$$

for $v = \mathcal{L}_T \Phi$. (See also Lemma 6.1 from [8] and related results in [38]).

Suppose that there exists $\Phi \in Z_T^0$ such that $\kappa v(\cdot, 0) = v(\cdot, T)$ for $v = \mathcal{L}_T \Phi$, i.e., $v(\cdot, 0) = \mathcal{Q}\Phi = \Phi$. Let us show that $\Phi = 0$ in this case.

Since $\mathcal{Q}\Phi \in Z_0^0$, it follows that $\Phi \in H^0 = Z_0^0$. Let $p = \mathcal{M}_0 \rho$ and $\bar{p}(x, t, 0) = \mathbf{E}p(x, t, \omega)$ (meaning the projection from Z_T^0 on $H^0 = Z_0^0$). Introduce an operator $\mathbf{Q} : H^0 \rightarrow H^0$ such that $\kappa \bar{p}(\cdot, T) = \mathbf{Q}\rho$. By (6.6), the properties of Φ lead to the equality

$$(\rho - \kappa p(\cdot, T), \Phi(\cdot, T))_{Z_T^0} = (\rho - \kappa \bar{p}(\cdot, T), \Phi(\cdot, T))_{H_0} = 0 \quad \forall \rho \in H^0. \quad (6.7)$$

It suffices to show that the set $\{\rho - \kappa \bar{p}(\cdot, T)\}_{\rho \in H^0}$ is dense in H^0 . For this, it suffices to show that the equation $\rho - \mathbf{Q}\rho = z$ is solvable in H^0 for any $z \in H^0$.

Let us show that the operator $\mathbf{Q} : H^0 \rightarrow H^0$ is compact. Let p be the solution of (6.5). This means that $\kappa \mathbf{E}p(\cdot, T) = \mathbf{Q}\rho$. By Lemma 6.2, it follows that

$$\|p(\cdot, \tau)\|_{Z_\tau^1} \leq C_* \|p(\cdot, s)\|_{Z_s^1}, \quad \tau \in [s, T], \quad (6.8)$$

where $C_* > 0$ does not depend on p , s , and τ .

We have that $p|_{t \in [s, T]} = \mathcal{M}_s p(\cdot, s)$ for all $s \in [0, T]$, and, for $\tau > 0$,

$$\begin{aligned} \|\bar{p}(\cdot, T)\|_{W_2^1(D)}^2 &\leq C_0 \|p(\cdot, T)\|_{Z_T^1}^2 \leq C_1 \inf_{t \in [0, T]} \|p(\cdot, t)\|_{Z_t^1}^2 \\ &\leq \frac{C_1}{T} \int_0^T \|p(\cdot, t)\|_{Z_t^1}^2 dt \leq \frac{C_2}{T} \|p\|_{X^1}^2 \leq \frac{C_3}{T} \|\Phi\|_{H^0} \end{aligned}$$

for $C_i > 0$ that do not depend on Φ . Hence the operator $\mathbf{Q} : H^0 \rightarrow H^1$ is continuous. The embedding of H^1 into H^0 is a compact operator.

Similar to the proof of Theorem 3.4 from [13], pp. 574–575, we obtain that if $p = \mathcal{M}_0 \rho$, $\bar{p}(x, t, 0) = \mathbf{E}p(x, t, \omega)$, $\rho \in H^0$, and $\rho \neq 0$, then

$$\int_D |\bar{p}(x, T)| dx < \int_D |\bar{p}(x, 0)| dx.$$

It follows that if

$$\kappa \bar{p}(\cdot, T) = \kappa \mathbf{E}p(\cdot, T) = \mathbf{Q}\rho = p(\cdot, 0) \quad (6.9)$$

for some $\rho \in H^0$ then $\rho = 0$; this proof will be omitted here. We had proved also that the operator \mathbf{Q} is compact. By the Fredholm Theorem, it follows that the equation $\rho - \mathbf{Q}\rho = z$ is solvable in H^0 for any $z \in H^0$. By (6.7), it follows that $\Phi = 0$. Therefore, the condition $\kappa u(\cdot, 0) = u(\cdot, T)$ fails to be satisfied for $u \neq 0$, $\xi = 0$, and $\varphi = 0$. Thus, $u = 0$ is the unique solution of problem (4.1)-(4.3) for $\xi = 0$ and $\varphi = 0$. Then the proof of Theorem 4.2 follows from Theorem 4.1. \square

Without a loss of generality, we assume that there exist functions $\tilde{\beta}_i : Q \times \Omega \rightarrow \mathbf{R}^n$, $i = 1, \dots, M$, such that

$$2b(x, t, \omega) = \sum_{i=1}^N \beta_i(x, t, \omega) \beta_i(x, t, \omega)^\top + \sum_{j=1}^M \tilde{\beta}_j(x, t, \omega) \tilde{\beta}_j(x, t, \omega)^\top,$$

and $\tilde{\beta}_i$ has the similar properties as β_i . (Note that, by Condition 2.1, $2b \geq \sum_{i=1}^N \beta_i \beta_i^\top$).

Let $\tilde{w}(t) = (\tilde{w}_1(t), \dots, \tilde{w}_M(t))$ be a new Wiener process independent on $w(t)$. Let $a \in L_2(\Omega, \mathcal{F}, \mathbf{P}; \mathbf{R}^n)$ be such that $a \in D$ a.s. We assume also that a is independent from $(w(t) - w(t_1), \hat{w}(t) - \hat{w}(t_1))$ for all $t > t_1 > s$. Let $s \in [0, T)$ be given. Consider the following Ito equation

$$\begin{aligned} dy(t) &= \hat{f}(y(t), t) dt + \sum_{i=1}^N \beta_i(y(t), t) dw_i(t) + \sum_{j=1}^M \tilde{\beta}_j(y(t), t) d\tilde{w}_j(t), \\ y(s) &= x. \end{aligned} \quad (6.10)$$

Let $y(t) = y^{x,s}(t)$ be the solution of (6.10), and let $\tau^{x,s} \triangleq \inf\{t \geq s : y^{x,s}(t) \notin D\}$. For $t \geq s$, set

$$\gamma^{x,s}(t) \triangleq \exp\left(-\int_s^t \hat{\lambda}(y^{x,s}(t), t) dt\right). \quad (6.11)$$

Lemma 6.4 *Let $\Phi \in L_\infty(\Omega, \mathcal{F}_T, \mathbf{P}, C_0(\bar{D}))$, and let $u \in Y^1$ be solution of (6.1) with $\varphi = 0$. Then the process $\gamma^{x,s}(t \wedge \tau)u(y^{x,s}(t \wedge \tau), t \wedge \tau)$ is a martingale.*

Proof of Lemma 6.4. For the case where $u \in \mathcal{X}_c^2$ and $\chi_j \in \mathcal{X}_c^1$, this lemma follows from the proof of Lemma 4.1 from [15] (see Remark 6.1).

Let us consider the general case. Let $\rho \in Z_s^0$ be such that $\rho \geq 0$ a.e. and $\int_D \rho(x) dx = 1$ a.s. Let $a \in L_2(\Omega, \mathcal{F}, \mathbf{P}; \mathbf{R}^n)$ be such that $a \in D$ a.s. and a has the conditional probability density function ρ given \mathcal{F}_s . We assume that a is independent from $(w(t_1) - w(t_0), \hat{w}(t_1) - \hat{w}(t_0))$, $s < t_0 < t_1$. Let $p = \mathcal{M}_s \rho$, and let $y^{a,s}(t)$ be the solution of Ito equation (6.10) with the initial condition $y(s) = a$.

To prove the theorem, it suffices to show that

$$\gamma(t \wedge \tau^{x,s})u(y^{x,s}(t \wedge \tau^{x,s}), t \wedge \tau^{x,s}) = \mathbf{E}_t \gamma(T \wedge \tau^{x,s})u(y^{x,s}(T \wedge \tau^{x,s}), T \wedge \tau^{x,s}) \quad \text{a.s.}$$

for any t . For this, it suffices to prove that

$$\begin{aligned} & \mathbf{E} \int_D \rho(x) \gamma(t \wedge \tau^{x,s}) u(y^{x,s}(t \wedge \tau^{x,s}), t \wedge \tau^{x,s}) \\ &= \mathbf{E} \int_D \rho(x) \mathbf{E}_t \gamma(T \wedge \tau^{x,s}) u(y^{x,s}(T \wedge \tau^{x,s}), T \wedge \tau^{x,s}) \end{aligned} \quad (6.12)$$

for any $\rho \in Z_s^0$ such as described above.

By Theorem 6.1 from [15] and Remark 6.1, we have that

$$\int_D p(x, t) u(x, t) dx = \mathbf{E}_t \gamma^{a,s}(t \wedge \tau^{a,s}) u(y^{a,s}(t \wedge \tau^{a,s}), t \wedge \tau^{a,s})$$

and

$$\int_D p(x, T) u(x, T) dx = \mathbf{E}_T \gamma(T \wedge \tau^{a,s}) u(y^{a,s}(T \wedge \tau^{a,s}), T \wedge \tau^{a,s}).$$

By the duality established in Theorem 3.3 from [15] and Remark 6.1, it follows that

$$\mathbf{E} \int_D p(x, t) u(x, t) dx = \mathbf{E} \int_D p(x, T) u(x, T) dx.$$

This means that $\mathbf{E}(\mathbf{E}_t q(a, s, t)) = \mathbf{E}(\mathbf{E}_T q(a, s, T))$, where

$$q(a, s, t) = \gamma^{a,s}(t \wedge \tau^{a,s}) u(y^{a,s}(t \wedge \tau^{a,s}), t \wedge \tau^{a,s}).$$

Hence

$$\mathbf{E}(\mathbf{E}_t q(a, s, t)) = \mathbf{E}(\mathbf{E}_T q(a, s, T)). \quad (6.13)$$

Without loss of generality, we shall assume that a is a random vector on the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}})$, where $\tilde{\Omega} = \Omega \times \Omega'$, where $\Omega' = D$, $\tilde{\mathcal{F}} = \overline{\mathcal{F}_s \otimes \mathcal{B}_D}$, where \mathcal{B}_D is the set of Borel subsets of D , and

$$\tilde{\mathbf{P}}(S_1 \times S_2) = \int_{S_1} \mathbf{P}(d\omega) \mathbf{P}'(\omega, S_2), \quad \mathbf{P}'(\omega, S_2) = \int_{S_2} \rho(x, \omega) dx,$$

for $S_1 \in \mathcal{F}$ and $S_2 \in \mathcal{B}_D$. The symbol $\tilde{\mathbf{E}}$ denotes the expectation in $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}})$. We suppose that $\tilde{\omega} = (\omega, \omega')$, $\tilde{\Omega} = \{\tilde{\omega}\}$, and $a(\tilde{\omega}) = \omega'$.

We have that

$$\begin{aligned} \mathbf{E}(\mathbf{E}_t q(a, s, t)) &= \mathbf{E} \int_{\tilde{\Omega}} \tilde{\mathbf{P}}(d\omega | \mathcal{F}_t) q(\omega', s, t, \omega) \\ &= \mathbf{E} \int_D d\omega' \rho(\omega') \int_{\Omega} \mathbf{P}(d\omega | \mathcal{F}_t) q(\omega', s, t, \omega) = \mathbf{E} \int_D \mathbf{E}_t \rho(\omega') q(\omega', s, t, \omega) d\omega' \\ &= \mathbf{E} \int_D \rho(\omega') q(\omega', s, t, \omega) d\omega' \\ &= \mathbf{E} \int_D \rho(x) \gamma^{x,s}(t \wedge \tau^{x,s}) u(y^{x,s}(t \wedge \tau^{x,s}), t \wedge \tau^{x,s}) dx \end{aligned}$$

and

$$\begin{aligned}
\mathbf{E}(\mathbf{E}_t q(a, s, T)) &= \mathbf{E} \int_{\tilde{\Omega}} \tilde{\mathbf{P}}(d\omega | \mathcal{F}_t) q(\omega', s, T, \omega) \\
&= \mathbf{E} \int_D d\omega' \rho(\omega') \int_{\Omega} \mathbf{P}(d\omega | \mathcal{F}_t) q(\omega', s, T, \omega) \\
&= \mathbf{E} \int_D \rho(\omega') \mathbf{E}_t q(\omega', s, T, \omega) d\omega' \\
&= \mathbf{E} \int_D \rho(x) \gamma^{x,s}(T \wedge \tau^{x,s}) \mathbf{E}_t u(y^{x,s}(T \wedge \tau^{x,s}), T \wedge \tau^{x,s}) dx.
\end{aligned}$$

Since the choices of α and ρ are arbitrary, it follows from (6.13) that (6.12) holds. This completes the proof of Lemma 6.4. \square

Proof of Theorem 5.1. Without a loss of generality, we assume that \mathbf{P} is a martingale probability measure, i.e., $S(t)$ is a martingale and $dS(t) = \sigma(t)S(t)dw(t) + \hat{\sigma}(t)S(t)d\hat{w}(t)$.

We have that $u \in Y_1$. Hence $u(\cdot, T) = Z_T^1$. Similar to the proof of Lemma 6.3, we obtain that $u(\cdot, 0) \in Z_0^1$, and $\|u(\cdot, 0)\|_{Z_0^1} \leq C\|u(\cdot, T)\|_{Z_T^0}$, where $C > 0$ does not depend on $u(\cdot, T)$. Since the embedding of H^1 to $C_0(\bar{D})$ is continuous for $n = 1$, we obtain that $u(\cdot, 0) \in L_2(\Omega, \mathcal{F}, \mathbf{P}, C_0(\bar{D}))$. By (5.7) and by the assumptions on ξ , we obtain that $u(\cdot, T) \in L_2(\Omega, \mathcal{F}, \mathbf{P}, C(\bar{D}))$. Hence $\mathbf{E}\zeta^2 \leq \mathbf{E}(\sup_x |u(x, T)|^2) < +\infty$.

Let $\zeta = u(S(T \wedge \tau), T \wedge \tau)$. Since the market is complete, there exists admissible $\gamma(t)$ such that ζ can be represented as

$$\zeta = u(S(T \wedge \tau), T \wedge \tau) = \mathbf{E}u(S(T \wedge \tau), T \wedge \tau) + \int_0^T \gamma(t) dS(t).$$

Therefore, $\mathbf{E}\{u(S(T \wedge \tau), T \wedge \tau) | \mathcal{F}_t\}$ is the wealth for the self-financing strategy such that replicates ζ .

By Lemma 6.4, it follows that $\mathbf{E}\{u(S(T \wedge \tau), T \wedge \tau) | \mathcal{F}_t\} = u(S(t \wedge \tau), t \wedge \tau)$. Hence (5.6) is the wealth for the self-financing strategy that replicates ζ .

Let $S_1(t) = S(t)/S(0)$. Further, if $\tau > T$ and $S(T)y = S(0)x$, then, by the definitions,

$$X(T, y) - X(0, x) = u(S_1(T)y, T) - u(S_1(0)x, t) = u(x, T) - u(x, 0) = \xi(x).$$

Hence condition (5.7) is satisfied. This completes the proof of Theorem 5.1. \square

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