UWB Waveform Set Design using Löwdin's Orthogonalization with Hermite Rodriguez Functions

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***Abstract*—In this paper we design a set of spectrally efficient orthogonal waveforms based on Hermite Rodriguez functions. The design is formulated as a semi-infinite quadratic programming problem and subsequently Löwdin method is used to generate a set of orthogonal waveforms. This approach is able to produce many overlapping orthogonal waveforms with high spectral efficiency.**

I. Introduction

In multiuser communication systems and high data rate applications, there is a need to have a large number of waveforms that are mutually orthogonal. It has been shown that higher capacity can be achieved if each user uses a signal set that consists of signals that are orthogonal to those used by other users [1]. This paper aims to propose a method to design a large number of spectrally efficient orthogonal waveforms for UWB communication systems.

There are essentially two main steps in our approach. First, spectral efficiency of a single UWB waveform is optimised directly by maximising the energy of the waveform in a prescribed passband while complying with the prescribed spectral mask constraint. The design of a spectrally efficient waveform is formulated as a constrained non-linear maximization problem where the objective function is quadratic but the constraints are in general non-linear. The rotation theorem [2] is then used to linearise the non-linear magnitude constraints. This optimization formulation did not impose any constraints on the phase of the waveform.

To design the single UWB waveform, Hermite-Rodriguez (HR) functions will be used as the basis functions. The HR basis functions are attractive because a signal can be represented to a high degree of accuracy by using just a few terms in the HR series expansion, and more importantly HR function is well suited for the design of signal with finite time support [3].

In the second step, Löwdin orthogonalization method is used to transform a finite set of equally spaced shifts of the optimal pulse (designed in the first step) to an orthogonal set of pulses. This method minimizes energy distortion with respect to the optimal pulse, thus producing orthogonal waveforms occupying the same spectrum. Besides, the generated orthogonal pulses are not distorted unlike those produced by the sequential approach in [4]. Distorted pulses can not be used directly for pulse position modulation (PPM) in UWB systems [5].

The rest of the paper is organised as follows. Section II briefly describes Hermite filter using orthogonal HR basis functions. In Section III, the single waveform design problem is formulated as a constrained maximisation problem. Then, it is converted into a constrained semi-infinite quadratic programming (QP) problem. In Section IV, the designed waveform (from optimization step) is used in Löwdin orthogolization method to generate shift orthogonal waveform set. In Section V, a numerical example of the design of orthogonal pulses complying with the FCC spectral mask is presented. Section VI concludes the paper.

II. Hermite-Rodriguez Filter

In this section, we shall summarize the relevant background material of Hermite Rodriguez functions. For more details, see [3] and the references therein.

The *Hermite-Rodriguez filter* of order N is defined as

$$g(t) = \sum_{k=0}^{N-1} x_k \omega_{\lambda,k}(t) = \mathbf{x}^T \mathbf{w}_\lambda(t) \tag{1}$$

where

$$\begin{aligned} \mathbf{x} &= [x_0, x_1, \ldots, x_{N-1}]^T, \\ \mathbf{w}_\lambda(t) &= [\omega_{\lambda,0}(t), \omega_{\lambda,1}(t), \ldots, \omega_{\lambda,N-1}(t)]^T, \end{aligned}$$

x_k is the k^{th} filter coefficient and $\omega_{\lambda,k}(t)$ is the k^{th} order Hermite Rodriguez function given by

$$\omega_{\lambda,k}(t) = \frac{1}{\sqrt{2^k k!}} H_k\left(\frac{t}{\lambda}\right) \frac{1}{\sqrt{\pi}\lambda} e^{-t^2/\lambda^2} \quad k \in [0, \infty). \tag{2}$$

λ is a scaling parameter and $H_k(t)$ is the Hermite polynomials defined recursively as

$$H_k(t) = \begin{cases} 1 & k = 0 \\ 2t & k = 1 \\ 2tH_{k-1}(t) - 2(k-1)H_{k-2}(t) & k \geq 2. \end{cases} \tag{3}$$

The Fourier transform of $g(t)$ is given by

$$G(f) = \int_{-\infty}^{+\infty} g(t)e^{-j2\pi ft}dt = \sum_{k=0}^{N-1} x_k W_{\lambda,k}(f)$$

where $W_{\lambda,k}(f)$ is the Fourier transform of the Hermite-Rodriguez function given by

$$\mathcal{F}\{\omega_{\lambda,k}(t)\} = W_{\lambda,k}(f) = \frac{(-j)^k}{\sqrt{2^k k!}}(2\pi f\lambda)^k e^{-(2\pi f\lambda/2)^2}. \quad (4)$$

Let $\mathbf{W}(f) = [W_{\lambda,0}(f), W_{\lambda,1}(f), \ldots, W_{\lambda,N-1}(f)]^T$. Then $G(f)$ can be written as

$$G(f) = \mathbf{x}^T\mathbf{W}(f) \quad (5)$$

and the power spectral density of $g(t)$ can be expressed as

$$\Psi_g(f) = |G(f)|^2 = \mathbf{x}^T\Phi(f)\mathbf{x} \quad (6)$$

where $\Phi(f) = \mathbf{W}(f)\mathbf{W}^T(f)$ is an N by N non-negative definite matrix.

III. UWB Waveform Design Problem

We will use HR filter represented by (1) to represent the basic pulse commonly used in a UWB communication system. Thus, UWB basic pulse $p(t)$ can be expressed as

$$p(t) = \sum_{k=0}^{N-1} x_k \omega_{\lambda,k}(t) \quad (7)$$

where x_k will be the design parameters. The PSD of $p(t)$ is $|P(f)|^2 = |G(f)|^2 = \mathbf{x}^T\Phi(f)\mathbf{x}$ where $\Phi(f) = \mathbf{W}(f)\mathbf{W}^T(f)$.

Now, let Ω_p denote the sets of passband frequencies and let $S(f)$ denote the spectral masks. Then the optimal waveform design problem can be formulated as the following constrained optimization problem

$$\max_{\mathbf{x}} \int_{\Omega_p} |P(f)|^2 df \quad (8)$$

subject to

$$|P(f)|^2 \leq S(f), \ \forall f \quad (9)$$

i.e., maximizing the signal power while satisfying the given spectral mask constraint.

The waveform design problem (8) and (9) is formulated as direct maximization of a quadratic function of the filter coefficients subject to linearizable inequality constraints. Our approach does not use the autocorrelation of the filter coefficients unlike the convex optimization approach, where spectral factorization is required to obtain the designed filter coefficients [6], [7]. To proceed, the waveform design problem (8) and (9) is converted to the following simplified non-linear optimization problem

$$\max_{\mathbf{x}} \mathbf{x}^T A\mathbf{x} \quad (10)$$

subject to

$$|P(f)| = |\mathbf{x}^T\mathbf{W}(f)| \leq \sqrt{S(f)}, \ \forall f, \quad (11)$$

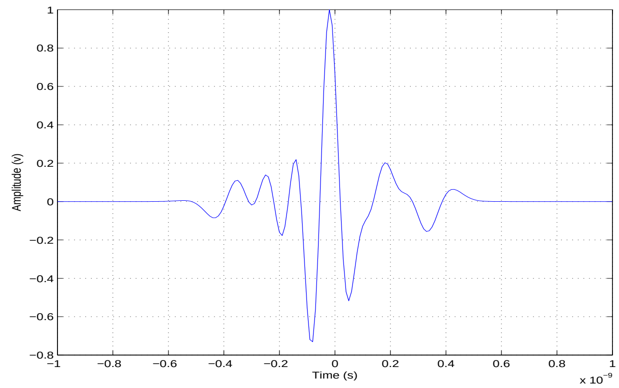

Fig. 1. The optimal waveform $p(t)$.

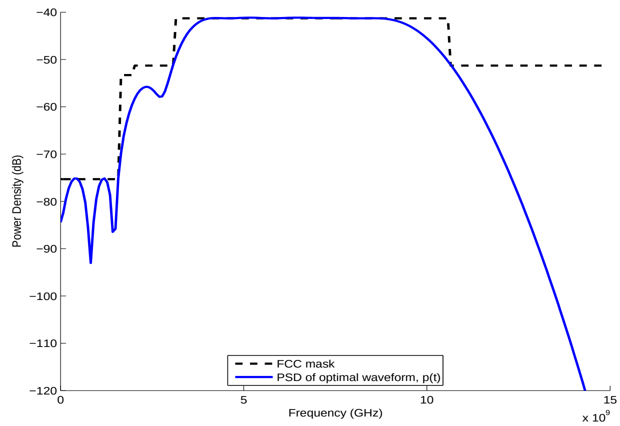

Fig. 2. The PSD of the optimal waveform $p(t)$.

where

$$A = \int_{\Omega_p} \Phi(f)df \quad (12)$$

is a positive definite matrix. Note that (10) and (11) define a constrained non-linear optimization problem, where the objective function is quadratic but the constraint in general is non-linear and non-convex.

Next, using the rotation theorem [2], the magnitude inequality constraint (11) can be linearized as

$$\max_{0\leqslant\theta<2\pi} \Re\{\mathbf{x}^T\mathbf{W}(f)e^{j\theta}\} \leqslant \sqrt{S(f)}. \quad (13)$$

Note that the number of constraints in (13) increases due to the introduction of the new parameter θ. As can be seen that (10) and (13) specify a semi-infinite quadratic programming problem where the number of variables to be optimized is finite but the number of inequality constraints is infinite (depend on θ and f). In this paper we discretize θ and f, and solve the optimization problem (10) and (13) using a simple MATLAB program.

IV. Waveform Orthogonalization

Löwdin orthogonalization method is applied to the optimal waveform designed from the solution of the semi-infinite quadratic programming problem described in Section III.

Consider a finite set of $\{p(.-kT)\}_{k=-M}^{M}$ of $2M+1$ equally spaced translates (shifts) of the optimal waveform $p(t)$. Then, we apply Löwdin orthogonalization method [8] to transform this set of shifts to an orthonormal basis $\{p_k^o\}_{k=-M}^{M}$ such that the average energy distortion $\sum_{k=-M}^{M}||p(.-kT)-p_k^o||_2^2$ is minimized. The Löwdin transform is given by

$$p_n^o = \sum_{m=-M}^{M} [G^{-\frac{1}{2}}]_{nm} p(.-nT) \qquad (14)$$

where $G^{-\frac{1}{2}}$ denotes the inverse square root of $2M+1 \times 2M+1$ dimensional matrix with the elements given by

$$[G]_{nm} = \int_{-\infty}^{+\infty} p(t-mT)p^*(t-nT)dt. \qquad (15)$$

The Löwdin orthogonalization method can be well approximated using the Zak transform allowing for an efficient implementation using the discrete Fourier transform [9].

V. Design Examples

We first solve (10) and (13) to design optimal $p(t)$ with $\lambda = 1.25 \times 10^{-10}$ and $N = 20$. The plots of $p(t)$ and its power spectral density (PSD) are plotted in Fig. 1 and 2 respectively. Fig. 3 and 4 show the set of orthogonal waveforms generated using the Löwdin transform (14) for $M = 2$ and $M = 7$ respectively. It is important to note that the waveforms generated occupy 3 times of the original waveform duration (see fig. 1) independent of the number of waveforms generated. This is important as it is able to design many overlapping orthogonal signals.

Since Löwdin orthogonalization method minimizes average energy distortion with respect to the optimal $p(t)$, the generated waveforms may not satisfy the FCC spectral mask anymore. Thus, we adjust them to ensure their PSD are equal or below the spectral mask. This adjustment is similar to the approaches in [9], [10]. The PSD of the orthogonal waveforms for $M = 2$ and $M = 7$ are plotted in Fig. 5 and 6 respectively. It can be seen clearly that the waveforms occupy the same spectrum. Fig. 7 shows the cross-correlations of a waveform with all the waveforms in the set. There are 5 waveforms in this particular set ($M = 2$).

VI. Conclusion

In this paper we have applied orthogonal Hermite Rodriguez basis functions to design a basic pulse for UWB systems. We formulated the pulse design problem as direct maximization of quadratic function of filter coefficients, thus avoiding the spectral factorization step normally used in convex optimization approaches. We have used the rotation theorem to convert the nonlinear and nonconvex constraints into linear constraints with the addition of a new parameter. The designed optimal pulse is then used in Löwdin orthogonalization method to generate a set of orthogonal and overlapping spectrally efficient waveforms. Design examples have demonstrated the proposed approach is able to produce many overlapping orthogonal waveforms which efficiently utilizes a given spectral mask.

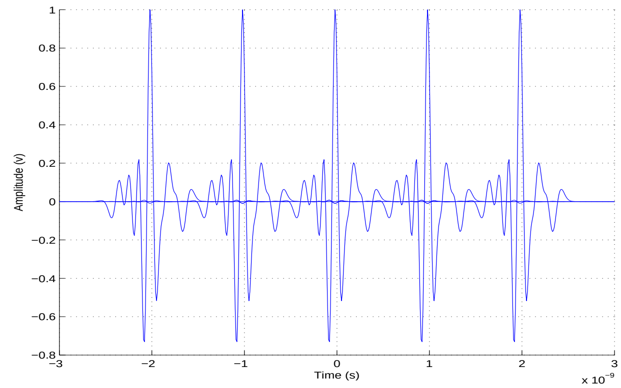

Fig. 3. The generated orthogonal waveforms $\{p_k^o\}_{k=-M}^{M}$ with $M = 2$.

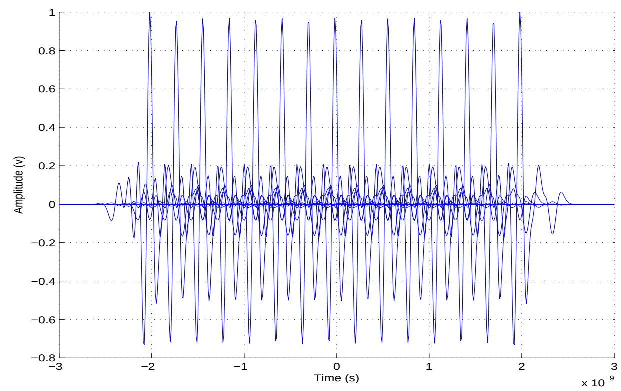

Fig. 4. The generated orthogonal waveforms $\{p_k^o\}_{k=-M}^{M}$ with $M = 7$.

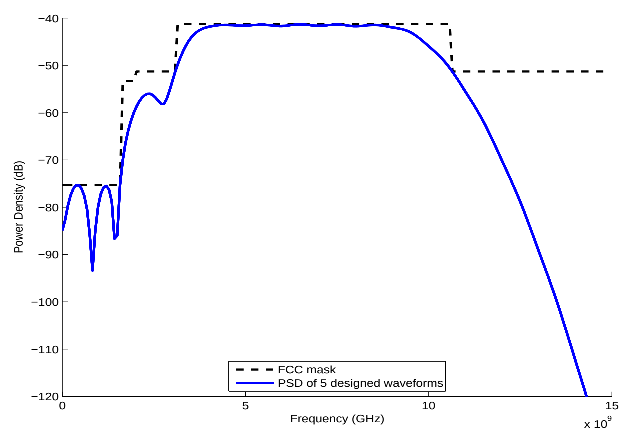

Fig. 5. The PSD of the 5 orthogonal waveforms ($M = 2$).

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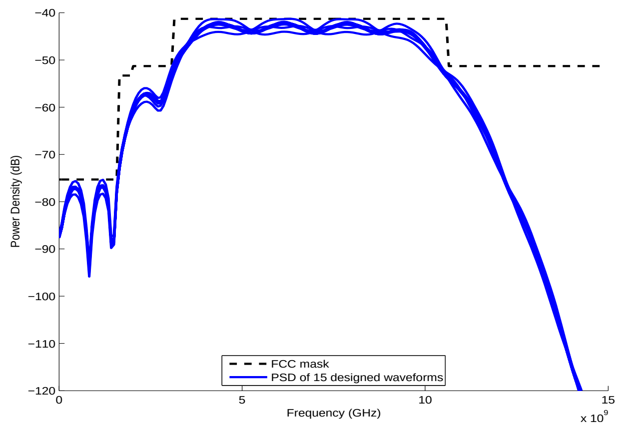

Fig. 6. The PSD of the 15 orthogonal waveforms ($M = 7$).

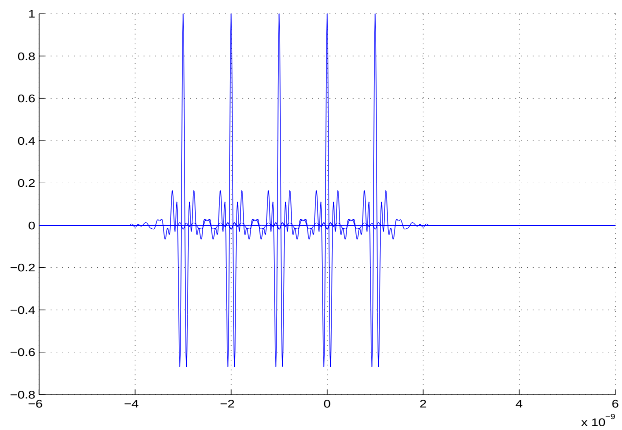

Fig. 7. The cross-correlations of the orthogonal waveforms ($M = 2$).

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